ANNALES SCIENTIFIQUES DE L'É.N.S.

C. J. REGO

The compactified Jacobian

Annales scientifiques de l'É.N.S. 4e série, tome 13, nº 2 (1980), p. 211-223
<http://www.numdam.org/item?id=ASENS_1980_4_13_2_211_0>

Ann. scient. Éc. Norm. Sup.
4e série, t. 13, 1980, p. 211 à 223.

THE COMPACTIFIED JACOBIAN

By C. J. REGO

Let X be a reduced and irreducible curve over an algebraically closed field k. For X singular the generalized Jacobian variety of X i.e. the group variety parametrising line bundles of degree zero, on X, is an extension of an Abelian variety by a commutative affine group. In particular it is not complete. In [11] Mumford and Mayer proposed a natural compactification of the Jacobian consisting of torsion free O_X modules of rank 1 with Euler characteristic equal to $\chi(O_X)$. The construction of this compact scheme was settled in D'Souza's thesis where more was proved. The main results of [6] are:

(i) For any integer d let P_d be defined as follows. Fix a regular point "y" $\in$ X so for any k-scheme S we get a section defined by $\sigma_S(S)=(y)\times S$.

$$\overline{P}_d(S)=\{\text{isomorphism classes of coherent } O_{X\times S} \text{ modules } F_S, \text{ flat over S, inducing on the geometric fibres of } f_S : X\times S \to S, \text{ torsion free sheaves } F_{s_0} \text{ of rank 1 and } \chi(F_{s_0})=d, \text{ plus isomorphisms } \sigma_S^* F_S \approx O_S\}.$$

Then $\overline{P}_d$ is a representable functor.

(ii) The morphism of functors

$$\Phi_d : \quad \mathrm{Hilb}^{-d} \to \overline{P}_d$$

[obtained by considering an ideal sheaf $I_S \subset O_{X\times S}$ flat on S as an element of $\overline{P}_d(S)$] is smooth at points $F\in\overline{P}_d(k)$, where F is an O_X module of Gorenstein dimension zero, whenever $-d \gg 0$. In particular Φ_d is smooth when X is Gorenstein (and $-d\gg 0$). (Recall that a module M over a local ring A has Gorenstein dimension zero if:

(i) M is reflexive;

(ii) $\mathrm{Ext}^1(M, A)=\mathrm{Ext}^1(M^*, A)=0$.

We say F is of Gorenstein dimension zero if each stalk satisfies the above conditions.)

(iii) If at each point $x \in X$ the δ invariant at x i.e. length [normalization $(O_{X,x})/O_{X,x}$] is less than or equal to one then $\overline{P}_d$ is reduced and irreducible. If the singularities of X have multiplicity at most two then $\overline{P}_d$ is irreducible.

See [2] for related material.

It is observed in [6] that (ii) implies the method of Chow-Matsusaka-Grothendieck for the construction of the Picard scheme extends to represent $\overline{P}_d$ in the Gorenstein case. In general (ii) is false and the equidimensionality of Φ_d, $-d \gg 0$, implies that X is Gorenstein, as is verified in [12].

The main results of this article are:

THEOREM A. — *If the singularities of* X *have embedding dimension two then* $\overline{P}$ *is irreducible. If* X *has a singularity of embedding dimension* $\geqq 3$ *then* $\overline{P}$ *is reducible.*

THEOREM B. — *The boundary* $\overline{P} - \mathrm{Pic}^0(X)$ *of* $\overline{P}$, *when* X *has planar singularities, is a union of m irreducible, codimension one subsets of* $\overline{P}$ *where*

$$m = \sum_{Q \in X} (\textit{multiplicity}\ O_{X,Q} - 1).$$

The first statement of Theorem A is deduced in [1] from Iarrobino's calculation of the dimension of the Punctual Hilbert scheme of k [X, Y] (*see* [10]). We give a short self contained proof by induction on the multiplicity of a singular point of X. The induction works because the "polar is an adjoint curve of lower multiplicity than the given curve". We find it convenient to work with the scheme E of paragraph 2 rather than $\overline{P}$. Since Iarrobino's estimate appears as a Corollary of our method the treatment may be viewed as an application of curves to punctual Hilbert schemes of smooth surfaces. The proof of Theorem B utilizes Briançon's recent result [4] that the Punctual Hilbert scheme of k [X, Y] is irreducible. It seems likely that Briançon's Theorem may be provable using the method of Theorem A.

The scheme E of paragraph 2 is useful also in describing the boundary of $\overline{P}$ when X has singularities of module type in the sense of [14].

An amusing aspect of the techniques used here is the amount of mileage one can get from the use of the fact that $\mathfrak{a}^{**} = \mathfrak{a}$ when $\mathfrak{a}$ is an ideal in a one dimensional Gorenstein ring.

1. Preliminaries and Notation

We write $\overline{P}$ for $\overline{P}_d$,

$$d = \chi(O_X) = \text{rank } H^0(X, O_X) - \text{rank } H^1(X, O_X).$$

The functor $\overline{P}$ is identified with the scheme representing it. As $\overline{P}$ can be constructed for a family $X_S \to S$ we sometimes write $\overline{P}(X)$ or $\overline{P}(X_S \mid S)$. Note that the algebraic group $\mathrm{Pic}^0(X)$ is contained as an open subset in $\overline{P}$ but $\overline{\mathrm{Pic}^0(X)} \neq \overline{P}$ in general. The morphism $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(\overline{X})$ obtained by pulling back line bundles to the normalization $\overline{X}$ is surjective with kernel G. One can think of G as O_X submodules L of K = the function field of X, with $L_y = O_{X,y}$, for smooth points y and $L_{x_i} = u_i \cdot O_{X,x_i}$, for x_i singular points and where u_i is a unit in the normalization of O_{X,x_i}. Hence dimension $G = \delta = \mathrm{rank}\ H^0(X, O_{\overline{X}}/O_X)$. Note that $\mathrm{Pic}^0(X)$ and hence G acts on $\overline{P}$ by tensoring. Suppose $F \in \overline{P}(k)$ and $L \in G(k)$, $L_{x_i} = u_i \cdot O_{X,x_i}$ then if $F \otimes L = F'$, $F \neq F'$ if and only if $u_i \in \mathrm{End}(F_{x_i})$ for some i. Hence the dimension of the G orbit through F is equal to rank $H^0(O_{\overline{X}}/\mathrm{End}(F))$. Remembering that if two fractional ideals over a domain are isomorphic then one is a multiple of the other by an element of the quotient field, we see immediately that the two torsion free O_X modules which are locally isomorphic "differ" by a line bundle.

DEFINITION 1.0. — We say $F \in \overline{P}(k)$ is a boundary point if F is not locally free and there is a coherent module $\mathscr{F}$ on $X \times \mathrm{Spec}\, k\,[t]$ flat over $\mathrm{Spec}\, k\,[t]$ with $\mathscr{F}/t.\mathscr{F} \approx F$ and $\mathscr{F} \otimes k((t))$ on $X \times \mathrm{Spec}\, k((t))$ a locally free rank one $O_{X \times \mathrm{Spec}\, k((t))}$ module.

Remark 1.1. — For an arbitrary flat deformation of F as above we have $\mathscr{F}$ to be of maximal depth, hence principal, at all smooth points of $X \times \mathrm{Spec}\, k\,[t]$. Hence the property of being a boundary point is local around the singular points $\{x_i\}$ — and depends only on the O_{X,x_i} modules F_{x_i}. If the modules F_{x_i}, for every i, can be deformed (flatly) on $O_{X,x_i} \otimes_k k\,[t]$ to a (generically) locally principal module then F is a boundary point. To see this assume for simplicity that X has one singular point (x_0) and write $S = \mathrm{Spec}\, k\,[t]$. The deformation of F_{x_0} defines a torsion free module $\mathscr{F}_V$ on $V \times S$, for an affine open neighbourhood V of x_0, with the property $\mathscr{F}_V \mid (V \times S) - (x_0) \times$ (closed point of S), is locally free. Extend $\mathscr{F}_V$ as a coherent sheaf to $X \times S$ and double dualize to get $\mathscr{F}'$. Now $\mathscr{F}'$, being reflexive and rank one, $\mathscr{F}'$ is flat over S. Put $\mathscr{F}'/t.\mathscr{F}' = F'$ and note that $F'_{x_0} \approx F_{x_0}$, so $F'_{x_0} = f.F_{x_0}$, where f is a rational function on X. Tensoring by a suitable line bundle L we get $L \otimes F' \approx F$. Then $L \otimes_k k[t] \otimes \mathscr{F}' = \mathscr{F}$ has F for special fibre and exhibits F as a boundary point. The case of several singular points is left to the reader. We will usually speak of boundary points as being modules over the local ring O_{X,x_0}.

The simplest non-trivial example of a boundary point is the maximal ideal. Write $0 = O_{X,x_0}$ and look at the diagonal ideal $I \subset O \otimes_k O$ and consider one O as parameter. The generic fibre of I is supported at smooth points, hence is locally principal, and the special fibre is just the maximal ideal. Since boundary points form a closed subset of $\overline{P}$ the limit of boundary points is a boundary point.

In the study of boundary points it suffices for most purposes to work with the points in the closure of G in $\overline{P}$. This is because of the:

PROPOSITION 1.2. — *If* $F \in \overline{P}$ *is a limit of line bundles then there is a line bundle* L *such that* $F \otimes L$ *is a limit of line bundles belonging to* G *i.e.*: $F \otimes L \in \overline{G}$.

Proof. — Suppose $\mathscr{F}$ is an $O_{X \times \operatorname{Spec} k[t]}$ module expressing F an a boundary point so $\mathscr{F}/t.\mathscr{F} \approx F$ and defines a morphism $h : \operatorname{Spec} k[t] \to \overline{P}$ with generic point of $h(\operatorname{Spec} k[t])$ in $\operatorname{Pic}^0(X)$. By composition with the morphism $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(\overline{X})$ we have a morphism $p' : \operatorname{Spec}\ k((t)) \to \operatorname{Pic}^0(\overline{X})$ and since $\operatorname{Pic}^0(\overline{X})$ is complete p' can be extended to $p : \operatorname{Spec} k[t] \to \operatorname{Pic}^0(\overline{X})$. By smoothness of $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(\overline{X})$ we can lift $p(\operatorname{Spec} k[t])$ to a curve T in $\operatorname{Pic}^0(X)$ and we have a morphism $p_0 : \operatorname{Spec} k[t] \to \operatorname{Pic}^0(X)$ with image T. Write $\mathscr{L}^{-1}$ for the line bundle on $X \times \operatorname{Spec} k[t]$ defined by p_0. By construction $\mathscr{L} \otimes \mathscr{F}$ is a family of O_X modules with the generic member a point in $G(k((t)))$ and with limit equal to $L \otimes F$, $L \approx \mathscr{L}/t.\mathscr{L}$. This proves the proposition.

Remark 1.3. — One may try to prove $\overline{P}$ irreducible as follows. Let $I \subset O_{X,x_0}$, length $(O_{X,x_0}/I) = n$. If I can be deformed to an ideal with non-trivial support at smooth points of X so that its colength at x_0 is less than n, then by induction on n, I is a limit of boundary points hence is a boundary point. In general this argument fails because the Punctual Hilbert scheme $H_0^n(X)$ of ideals in O_X supported at x_0 and of colength n, is a component of $\operatorname{Hilb}^n(X)$. Let X be (locally at x_0) embedded in a smooth surface S. Iarrobino has shown that the dimension of $H_0^n(S)$ is equal to $(n-1)$ so $H_0^n(X) \subsetneq H_0^n(S)$ has dimension less than or equal to $(n-1)$. To prove the irreducibility of $\overline{P}$ in this case it thus suffices to show that the components of $\operatorname{Hilb}^n(X)$ have dimension greater than or equal to n. This can be checked as follows. Suppose $f \in O_S$ defines X at x_0 and $f \in I$ with length $(O_S/I) = n$. By [8] $\operatorname{Hilb}^n(S)$ is smooth with a dense open subset defined by n distinct points on S. Let $\mathscr{I} \subset O_S \otimes k[t]$ define a deformation of O_S/I into "n distinct points" and $f \in \mathscr{I}$ map to f in $\mathscr{I}/t.\mathscr{I} = I$. Then, locally, f defines a family of curves over $\operatorname{Spec} k[t]$ and gives a section of

$$\operatorname{Hilb}^n(O_S \otimes k[t]/(f) \mid k[t]) \to \operatorname{Spec} k[t].$$

Look at the generic fibre of the relative Hilbert scheme; it has an n-dimensional component defined by the collection of "n-distinct points on the generic curve". By construction the point of $\operatorname{Hilb}^n(X)$ defined by O_X/I is in the limit of these n dimensional components of "nearby fibres". Since I was arbitrary $\operatorname{Hilb}^n(X)$ is of dimension greater than or equal to n at every point. In [1] this fact was verified as follows. The Poincare sheaf $M = O_H \otimes O_S/\mathscr{I}$ is a rank n vector bundle on $H = \operatorname{Hilb}^n(S)$. Then the section of M given by $1 \otimes f \in O_H \otimes O_S$ vanishes exactly on $\operatorname{Hilb}^n(X) \subset \operatorname{Hilb}^n(S)$. By [8] $\dim H = 2n$ so $\dim \operatorname{Hilb}^n(X) \geqq n$ at every point. In paragraph 3 we will prove that any extra component of $\overline{P}$, when X has planar singularities, has smaller dimension that $\operatorname{Pic}^0(X)$. By D'Souza's Theorem this would yield a component of $\operatorname{Hilb}^d(X)$, $d \gg 0$, of dimension less than d which is impossible. As a Corollary we derive Iarrobino's estimate for dimension $H_0^n(S)$.

One final remark: if a Gorenstein curve has irreducible $\overline{P}$ it has irreducible Hilb^n for every n. To see this take $I \subset O_{X,x_0}$, where I is the stalk at x_0 of $\mathscr{I}$, a sheaf of ideals on X, with $H^0(X, O_X/\mathscr{I})$ of dimension d, $d \gg 0$. By D'Souza's Theorem $\overline{P}$ irreducible $\Rightarrow \operatorname{Hilb}^d(X)$ irreducible. So $\mathscr{I}$ can be deformed to a product of maximal ideals. Restricting this deformation to a neighbourhood of x_0 shows that I is in the closure of the open subset of Hilb defined by n distinct points of X. Hence $\operatorname{Hilb}^n(X)$ is irreducible.

2. The Functor E

Let $\mathscr{C}$ be the sheaf of conductors on X and write $U = X - \{x_i\}$ for the open subset of smooth points of X. Denote by $\mathscr{C}_1$ a subsheaf of $\mathscr{C}$ with $\mathscr{C}_1$ an $O_{\overline{X}}$ module. Let A be the semi local ring of functions regular at the $\{x_i\}$ and C, C_1 the ideals in A corresponding to $\mathscr{C}$ and $\mathscr{C}_1$. For $d \leqq \text{rank } H^0(\overline{X}, O_{\overline{X}}/\mathscr{C}_1) = \text{length } (\overline{A}/C_1)$, $\overline{A}$ the normalization of A, we define the functor $E(d, \mathscr{C}_1)$ by

$$E(d, \mathscr{C}_1)(S) = \{ F_S \mid F_S \in \overline{P}_q(S),$$
$$q = \chi(O_{\overline{X}}) - d, \ \mathscr{C}_1 \otimes_k O_S \subset F_S \subset O_{\overline{X}} \otimes_k O_S$$
$$\text{and } O_{\overline{X}} \otimes O_S / F_S \text{ is a locally free } O_S \text{ module of rank } d \}.$$

Since $\mathscr{C}_1 \otimes O_S = O_{\overline{X}} \otimes O_S$ on $U \times S$ the functor $E(d, \mathscr{C}_1)$ may be identified with the functor $E(d, C_1)$:

$$E(d, C_1)(S) = \{ I_S \mid C_1 \otimes_k O_S \subset I_S \subset \overline{A} \otimes_k O_S,$$
$$I_S \text{ an } A \otimes_k O_S \text{ module and } \overline{A} \otimes_k O_S / I_S \text{ a locally free}$$
$$O_S \text{ module of rank } d \}.$$

PROPOSITION 2.1. — *$E(d, \mathscr{C}_1)$ is representable by a projective scheme.*

Proof. — It is more convenient to check that $E(d, C_1)$ is representable. Look at the Grassmanian of vector subspaces of $\overline{A}/C_1$ of codimension d. For a subspace V to be an A module it suffices (and is necessary) that V be closed under the action of the group of units of A/C. In fact an S valued point of the Grassmanian is a locally free O_S module $\overline{I}_S$ where $\overline{I}_S$ comes from I_S, $C_1 \otimes O_S \subset I_S \subset \overline{A} \otimes O_S$. For I_S to be an $A \otimes_k O_S$ module, I_S must be invariant by multiplication by sections of $A \otimes O_S$ and as I_S is an O_S module it is enough that I_S is closed under multiplication by units of A. Since

$$C_1 . I_S \subset C_1 . (\overline{A} \otimes O_S) \subset C_1 \otimes O_S,$$

the finite dimensional algebraic group $(A/C_1)^*$ acts on Grass $(\overline{A}/C_1, d)$ and I_S defines a point of $E(C_1, d)$ iff it is a fixed point for the action of $(A/C_1)^*$. We may therefore apply the results of Fogarty [7] to conclude that E is representable by a closed subscheme of Grass $(A/C_1, d)$.

Remark 2.2. — There is an obvious morphism

$$e = e(C_1, d) : \ E(C_1, d) \to \overline{P}_q, \qquad q = \chi(O_{\overline{X}}) - d,$$

which is proper as E is projective. Note that $E(d, C_1)$ is defined by A/C_1 so we get the same scheme for two curves with analytically isomorphic singularities. In particular, E is not sensitive to the birational character of the curve.

THEOREM 2.3. — (*a*) *Given* $\mathscr{C}_2 \subset \mathscr{C}_1 \subset \mathscr{C}$ *there is an injective, proper morphism*

$$q(\mathscr{C}_1, \mathscr{C}_2, d) : \quad E(\mathscr{C}_1, d) \to E(\mathscr{C}_2, d).$$

(*b*) *The morphism* $e(\mathscr{C}_1, \delta) : E(\mathscr{C}_1, \delta) \to \overline{P}$ *has image containing*

$$G = \ker(\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(\overline{X}))$$

and is contained in the set of F *with* $F \mid U \approx O_U$. *In particular, putting* $\mathscr{C}_1 = \mathscr{C}$, *every boundary point defines an element of* $E(\mathscr{C}, \delta)$. *For* $\mathscr{C}_1$ *"sufficiently small" every isomorphism class of fractional ideals modulo multiplication by a line bundle is represented in* $E(\mathscr{C}_1, \delta)$.

(*c*) *The morphism* $e(\mathscr{C}_1, d)$ *is finite* $\forall d$ *and is injective if* $O_{\overline{X}}/\mathscr{C}$ *is local. In general* $e(\mathscr{C}_1, \delta)$ *restricted to* $e^{-1}(G)$ *is injective.*

(*d*) X *is Gorenstein* $\Leftrightarrow$ *every isomorphism class of fractional ideals modulo multiplication by a line bundle is represented in* $E(\mathscr{C}, \delta)$. *In particular if* X *is not Gorenstein then* $\overline{P}$ *is reducible.*

Proof. — The proof of (*a*) is immediate. To verify (*b*) let $F \in E(\mathscr{C}_1, \delta)$ so there is an exact sequence

$$0 \to F \to O_{\overline{X}} \to O_{\overline{X}}/F \to 0,$$

with $\chi(O_{\overline{X}}/F) = \mathrm{rank}\ H^0(O_{\overline{X}}/F) = \delta$. Hence $\chi(F) = \chi(O_{\overline{X}}) - \delta = \chi(O_X)$ so image of e is in $\overline{P}_{\chi(O_X)} = \overline{P}$. Let L be a line bundle with $L \otimes_{O_X} O_{\overline{X}}$ trivial on $\overline{X}$ i.e. : L is defined by $u \in \overline{A}$. Then L can be embedded in $O_{\overline{X}}$ so that $L \mid U = O_{\overline{X}} \mid U$ and $L_{x_i} = u . O_{X, x_i}$. Hence $H^0(O_{\overline{X}}/L)$ has rank δ and as $u . O_{X, x_i} \supset u . \mathscr{C}_{1, x_i} = \mathscr{C}_{1, x_i}$ we find L defines an element of $E(\mathscr{C}_1, \delta)$. This shows that $G \subset e(E(\mathscr{C}_1, \delta))$. It remains to prove the last assertion of (*b*). Let I be an ideal in A. Since $\overline{A}$ is a P.I.D., $I . \overline{A} = (y) . \overline{A}$ and it is easy to verify that y can be chosen in I. Then we have $1 \in y^{-1} . I$ so

$$A \subset y^{-1} . I \subset y^{-1} . y . \overline{A} = \overline{A}.$$

Let $z_1, z_2, \ldots, z_r$ generate the maximal ideals of $\overline{A}$. Any x in the quotient field of A can be written $x = u . \prod z_i^{s_i}$, u a unit in $\overline{A}$ and $s_i \in \mathbb{Z}$. Put $v_i(x) = s_i$. If $x . I \subset \overline{A}$ one checks easily that

$$\mathrm{length}\ (\overline{A}/x . I) = \mathrm{length}\ (\overline{A}/I) + \sum_{i=1}^{r} v_i(x).$$

Choose $C_1 = z_1^{\delta} . C$. Given an A module isomorphic to say an ideal I we can get an isomorphic copy $y^{-1} . I$ between A and $\overline{A}$, as above. Then $z_1^p . y^{-1} . I$ with $p = \mathrm{length}(y^{-1} . I/A)$ contains $z_1^p . C$ and is contained in $\overline{A}$ with length $(\overline{A}/z_1^p . y^{-1} . I) = \delta$. Further as $p \leqq \delta$ we have $z_1^p . C \supset C_1$. So with the above choice of C_1 every fractional ideal is represented in $E(C_1, \delta)$. It is now easy to globalize this fact; given an arbitrary O_X module torsion free of rank one we may assume after tensoring with a line bundle that it contains O_X and is contained in $O_{\overline{X}}$. Now the above argument can be applied. This proves (*b*).

To verify (*c*) suppose J_1, J_2, are A modules contained in $\overline{A}$ representing two points of $E(C_1, d) \equiv E(\mathscr{C}_1, d)$. If $J_1 \approx J_2$ then there is an x in the quotient field with $J_1 = x . J_2$. If $v_i(x)$ is too large or too small for some i then $x . J_2 \not\supset C_1$ or $x . J_2 \not\subset \overline{A}$ so $\forall i$, $v_i(x)$ is bounded above and below. Hence modulo multiplication by elements of $\overline{A}$ there are finitely many x satisfying $x . J_2 = J_1$. But for a unit $u \in \overline{A}$ with $u . J_1 \neq J_1$ we have J_1 and $u . J_1$ mapping to different points in $\overline{\mathrm{P}}_q$, $q=\chi(\mathrm{O}_{\overline{\mathrm{X}}})-d$. On the other hand if $\overline{\mathrm{A}}$ has only one maximal ideal the above considerations show that $\mathrm{J}_1 \approx \mathrm{J}_2$ imply $\mathrm{J}_1=u.\mathrm{J}_2$, $u \in \overline{\mathrm{A}}^*$, so $e(\mathrm{C}_1, d)$ is injective. Finally if $e(\mathrm{C}_1, \delta)(\mathrm{J}_1) \in \mathrm{G}$ i.e.: $\mathrm{J}_1 \approx \mathrm{A}$ then $x.\mathrm{J}_1 \subset \overline{\mathrm{A}}$ implies $x.\mathrm{A} \subset \overline{\mathrm{A}}$ so that x is in $\overline{\mathrm{A}}$ and $v_i(x) \geqq 0$, $\forall i$. But as length $(\overline{\mathrm{A}}/\mathrm{J}_1)=$ length $(\overline{\mathrm{A}}/x.\mathrm{J}_1)$ we have $\Sigma v_i(x)=0$ so x is a unit. This proves (c).

From the preceeding it follows that to prove (d) we must verify that A is Gorenstein $\Leftrightarrow$ every A submodule of the quotient field is represented by an element of $\mathrm{E}(\mathrm{C}, \delta)$. So suppose A is Gorenstein and let $\mathrm{C} \subset \mathrm{J} \subset \mathrm{A}$ with $y \in \mathrm{A}$ and $\mathrm{J}.\overline{\mathrm{A}}=y.\overline{\mathrm{A}}$, $y=u.\prod.z_i^{s_i}$. We claim $\sum s_i \geqq$ length (A/J). To see this look at the picture

$$\begin{array}{ccc} y.\overline{\mathrm{A}} & \subsetneq & \overline{\mathrm{A}} \\ \cup & & \cup, \\ y.\mathrm{A} \subset & \mathrm{J} & \subset \mathrm{A} \end{array} \qquad s=\sum s_i,$$

which shows that

length $(\mathrm{A}/\mathrm{J}) \leqq$ length $(y.\overline{\mathrm{A}}/y.\mathrm{A})+$ length $(\overline{\mathrm{A}}/y.\overline{\mathrm{A}})-$ length $(\overline{\mathrm{A}}/\mathrm{A})=\delta+s-\delta=s$.
Hence $\exists(l_1, l_2, \ldots, l_r)$, $l_i \leqq s_i$, $\forall i$ and $\mathrm{J}_1=\prod z_i^{-l_i}.\mathrm{J} \subset \overline{\mathrm{A}}$ with $\Sigma l_i=$ length (A/J). But as length $(\overline{\mathrm{A}}/\mathrm{J}_1)=\delta$ and $\mathrm{C} \subset \prod z_i^{-l_i}.\mathrm{C} \subset \mathrm{J}_1$, J_1 defines an element of $\mathrm{E}(\mathrm{C}, \delta)$. We must now show that every isomorphism class is represented by an ideal between C an A. But if J is an arbitrary fractional ideal then by Gorenstein duality we can write $\mathrm{J}=\mathrm{N}^{-1}$ and embed N in $\overline{\mathrm{A}}$ so $\mathrm{A} \subset \mathrm{N} \subset \overline{\mathrm{A}}$. Then $\mathrm{J} \approx \mathrm{N}^{-1}$ is isomorphic to an ideal of A containing C.

To complete the proof of (d) we will verify that for A not Gorenstein there is a module J with $\mathrm{A} \subset \mathrm{J}$ and length $(\mathrm{J}/\mathrm{A})=1$; but no multiple of J defines an element of $\mathrm{E}(\mathrm{C}, \delta)$. We may assume that A is local. Let $\mathrm{A} \subset \mathrm{J} \subset \overline{\mathrm{A}}$ with length $(\mathrm{J}/\mathrm{A})=1$ and suppose there is a y with $y.\mathrm{J} \subset \overline{\mathrm{A}}$, length $(\overline{\mathrm{A}}/y.\mathrm{J})=\delta$. Since length $(\overline{\mathrm{A}}/\mathrm{J})=\delta-1$, $y=u.z_i$ for some i and u a unit in $\overline{\mathrm{A}}$. If $\mathrm{C}=\prod z_j^{c_j}.\overline{\mathrm{A}}$ then $z_i^{-1}.\mathrm{C} \supset \mathrm{C}$ so if $\mathrm{C} \subset z_i.\mathrm{A}$ we get $z_i^{-1}.\mathrm{C} \subset \mathrm{A}$ which contradicts the definition of C as the largest $\overline{\mathrm{A}}$ ideal in A. Hence $\mathrm{C} \not\subset z_i.\mathrm{A}$ and $\mathrm{C}+z_i.\mathrm{A} \supset z_i.\mathrm{A}$ which gives

$$u.z_i.\mathrm{A}+u.\mathrm{C}=u.z_i.\mathrm{A}+\mathrm{C} \underset{\neq}{\supset} u.z_i.\mathrm{A} \subset y.\mathrm{J}.$$

Length considerations give $\mathrm{J}=\mathrm{A}+z_i^{-1}.\mathrm{C}$. So any point of $\mathrm{E}(\mathrm{C}, \delta)$ defined by a J with $\mathrm{J} \supset \mathrm{A}$ and length $(\mathrm{J}/\mathrm{A})=1$ must be of the above type for some i. But if A is non Gorenstein length (End $(\mathfrak{m})/\mathrm{A}) > 1$, $\mathfrak{m}$ the maximal ideal of A. Further every one dimensional subspace of End $(\mathfrak{m})/\mathrm{A}$ defines an A module of the required type and since k is infinite (algebraically closed) there are infinitely many such. Hence for A non-Gorenstein there is a fractional ideal not represented in $\mathrm{E}(\mathrm{C}, \delta)$ and we are through.

Remark 2.5. – If J defines an element of $\mathrm{E}(\mathrm{C}_1, d)$, $d>\delta$ we have length $(\overline{\mathrm{A}}/\mathrm{J})>\delta$ so J cannot contain a unit of $\overline{\mathrm{A}}$. Hence $\mathrm{J}.\overline{\mathrm{A}}=\prod z_i^{r_i}.\overline{\mathrm{A}}$, $r_i \geqq 0$, some $r_j>0$. If say $r_1>0$ then $\mathrm{C}_1 \subset z_1^{-1}.\mathrm{C}_1 \subset z_1^{-1}\mathrm{J} \subset \overline{\mathrm{A}}$ which defines an element of $\mathrm{E}(\mathrm{C}_1, d-1)$. If $\overline{\mathrm{A}}$ is local there is only one z_i and we get a map $\mathrm{E}(\mathrm{C}_1, d) \rightarrow \mathrm{E}(\mathrm{C}_1, d-1)$. It is easily checked (using the fact that every A module in $\overline{\mathrm{A}}$ is represented by one between A and $\overline{\mathrm{A}}$) that the $\mathrm{E}(\mathrm{C}, d)$, $d<\delta$ "cover" $(\mathrm{E}(\mathrm{C}_1, \delta)$-G) for C_1 sufficiently small. Here a map is defined by multiplying J by an element of $\overline{\mathrm{A}}$ of suitable valuation.

Given a divisor $\sum n_P . P$ on a smooth curve $\overline{X}$ with $n_P \geqq 0$ there corresponds a curve X with one singular point and with $\overline{X}$ its normalization [14]. Given an affine open neighbourhood of the P with $n_P > 0$ having coordinate ring R then X is defined by the subring of R equal to $k + \mathfrak{m}_p^{n_P}$, $\mathfrak{m}_P$ the maximal ideal of $O_{X,P}$. These singularities are characterized by property that the maximal ideal is the conductor. For these singularities we have $E(C, \delta) \approx \mathbb{P}^\delta$ and as G is of dimension δ we have $E(C, \delta) = \overline{G}$. Hence in this case E yields exactly the boundary points of $\overline{P}$. We leave it to the reader to verify that there are only finitely many G orbits in this case. For example if X is defined by $\operatorname{Spec} k[x^n, x^{n+1}, \ldots, x^{2n}]$ the points in $E(C, \delta)$, $\delta = n-1$ are defined by $J_m = (x^n, \ldots, x^m, x^{m+2}, \ldots, x^{2n})$. There are therefore δ G orbits in $\overline{G} - G$ and these are of decreasing dimension.

PROPOSITION 2.6. — *For* X *rational with one unibranched singularity* $\overline{P}$ *is simply connected.*

Proof. — By the above $\overline{P}$ is bijective with $E(C_1, \delta)$ for C_1 sufficiently small. Now E is defined as a fixed point subset of a Grassmanian under the action of the group of units of A/C_1, A the singular local ring. As $k^* \subset \text{units}(A/C_1)$ acts trivially we have an action of an additive group on Grass. By [7] :

$$\pi_1(E(C_1, \delta)) \approx \pi_1(\text{Grass}) = (e),$$

which proves the proposition.

For an arbitrary family of curves $\varphi : X_S \to S = \operatorname{Spec} k[t]$ it is not clear how to define a relative E functor. Suppose however that the normalization $\overline{X}_S$ is smooth and the induced mapping $\varphi : \overline{X}_S \to S$ has smooth fibres. Also assume that if C is the conductor of X_S then O_{X_S}/C is S flat and $C/t.C$ is the conductor of $\varphi^{-1}(0)$. Then the relative E functor can be defined in an obvious way and is representable. This is because it can be interpreted as a fixed point set in Grass $(O_{\overline{X}_S}/C, d)$ of the group of units of O_{X_S}/C. Note that as O_{X_S}/C is S flat Fogarty's results [8] apply.

PROPOSITION 2.7. — *Dimension* $\overline{P} \leqq$ *genus* $(\overline{X}) + (\delta/2+1)^2$.

Proof. — Dimension $\overline{P}$ = dimension $(\operatorname{Pic}^0(\overline{X}))$ + dimension $E(C_1, \delta)$, C_1 sufficiently small, so we have to estimate the dimension of E. The constructions of [13] show that given any curve singularity X there is a family

$$\varphi : \quad X_S \to S = \operatorname{Spec} k[t]$$

with

$$X_S \otimes k((t)) \approx X \otimes_k k((t)) \qquad \text{and} \qquad X_0 = X_S \otimes_{k[t]} k$$

a singularity associated to a divisor $\sum n_P$ as described above. Further, the family φ satisfies the conditions given above which enable us to construct a relative E scheme over S which yields the E schemes of the fibres. By upper semi-continuity it suffices to obtain the estimate

$$\dim E \leqq (\delta+1)^2/4,$$

for a singularity associated to a divisor $\sum n_P . P$. But as the maximal ideal is the conductor, all the $E(C, d)$'s are Grassmanians and they cover $E(C_1, \delta)$. As dim $E(C, d)=d(\delta+1-d)$, We get the required estimate.

3. Main Theorems

THEOREM A. — $\overline{P}$ *is irreducible* $\Leftrightarrow$ *the embedding dimension of* X *at every point is less than or equal to two.*

Proof. — Let X have planar singularities. By paragraph 1 the property of an O_X module $\mathscr{F}$ being a boundary point is local around the singular points $x_i \in X$. So let there be one singular point x_0. Then it suffices by Theorem 2.3 to show that $E(C, \delta)$ is irreducible (since X is Gorenstein). Finally, the E scheme depends only on $O_{X,x_0}/C$ so we can as we can as well study the completion $\hat{O}_{X,x_0} \approx k[X, Y]/(f)=A$. Put $v=\text{ord}\, f$ and suppose the initial form of f is not X^v. Then if the characteristic of k is zero one checks easily (or *see* [3]) that $g=f_Y$ is an adjoint i.e.: g defines an element of the conductor C of $\overline{A}$ in A and ord $g=v-1$. More generally we have the:

LEMMA. — *In any characteristic there is a "g" in* C *of order* $(v-1)$.

Proof. — Let A_1 be the blow up of the maximal ideal $\mathfrak{m}$ of A and C_1 the conductor of A_1 in $\overline{A}$. Recall that $\mathfrak{m}^{v-1}$ is the conductor of A in A_1 and $C=C_1 . \mathfrak{m}^{v-1}$. Also by the definition of blowing up there is a Z in $\mathfrak{m}$ satisfying $Z.A_1=\mathfrak{m}.A_1$ so that $\mathfrak{m}^{v-1}.A_1=Z^{v-1}.A_1$.

As $C=C_1.\mathfrak{m}^{v-1}$, $C \subset \mathfrak{m}^{v-1}$ and we have to show that $C \not\subset \mathfrak{m}^v$. Suppose not, then

$$(3.1.0) \qquad C_1 . \mathfrak{m}^{v-1} \subset \mathfrak{m}^v$$

implies

$$(3.1.1) \quad C_1 \subset \text{Hom}\,(\mathfrak{m}^{v-1}, \mathfrak{m}^v) = \text{Hom}\,(Z^{v-1}.A_1, Z^{v-1}.Z.A_1) = \text{Hom}\,(Z^{-1}.A_1, A_1) = Z.A_1.$$

This says that $Z^{-1}.C_1 \subset A_1$, Z a non-unit in $\overline{A}$ and contradicts the definition of C_1 as the largest $\overline{A}$ ideal in A_1. The Lemma is thereby proved.

Remark. — We refer to any such "g" as a polar of "f".

To continue with the proof assume $\overline{P}$ is irreducible for plane curves of multiplicity less than ν. By the final remark of paragraph 1 this means that the punctual Hilbert scheme $\text{Hilb}_0^n(k[X, Y]/(g))$ has dimension less than or equal to $(n-1)$. As $\text{Hilb}_0^n(A/C) \subsetneq \text{Hilb}_0^n(k[X, Y]/(g))$ we have dim $\text{Hilb}_0^n(A/C) \leqq n-1$. For $d>\delta$ write $E'(d)$ for the closure of the subscheme of $E(C, d)$ generated by $\text{Hilb}_0^{d-\delta}(A/C) \subsetneq E(C, d)$ *via* translation by elements of $G=\overline{A}^*/A^*$. As noted in Remark 2.5 we do not have morphisms $E(C, d) \to E(C, \delta)$ when $\overline{A}$ is not local and $d>\delta$. However working with $e(E(C, d))$ we see easily that if Z is a closed G-stable subset of $e(E(C, d)) \subset \overline{P}_q$ then "tensoring by a line bundle" of suitable degree defines a bijection $Z \to Z_0 \subset \overline{P}_{\chi(O_X)} = \overline{P}$. In this sense we note that as A is Gorenstein and every fractional ideal lies between C and A, we can cover $e(E(C, \delta)-G)$ by $e(E'(d))$, $\delta < d \leqq 2\delta$. Hence $\overline{P}\text{-Pic}^0(X)$ is covered by $\bigcup_d \text{Pic}^0(X).e(E'(d))$. As

$$\dim \text{Pic}^0(X).e(E'(d)) = \dim E'(d) + \dim \text{Pic}^0(X)$$

and by paragraph 1 the dimension of every component of $\overline{P}$ is greater or equal to dim $\text{Pic}^0(X)$ it suffices to prove:

$$(3.1.2) \qquad \dim E'(d) < \delta \quad \text{for} \quad \delta < d \leqq 2\delta.$$

Let $W_d \subset E'(d)$ be an irreducible open subset satisfying the property that the G orbits in W_d are of the same dimension s, where automatically, s is the maximal dimension of the G orbits in the closure of $W_d \equiv \overline{W}_d \subset E'(d)$. Then taking a generic quotient by G we have:

$$(3.1.2) \qquad \dim \{\text{isomorphism classes of modules in } W_d\} = \dim W_d - s.$$

Let J define a point in W_d so $C \subset J \subset A$ and length $(A/J) = d - \delta$. The intersection of the G orbit through J with $\text{Hilb}_0^{d-\delta}(A/C)$ is identified with $\{u.J \mid u \in G, u.J \subset A\}$ so we have:

$$(3.1.3) \qquad \dim ((G.J) \cap \text{Hilb}_0^{d-\delta}(A/C)) = \text{length } (J^{-1}/\text{End } (J)).$$

Further for J in W_d,

$$(3.1.4) \qquad \text{length } (\text{End } (J)/A) = \text{length } (\overline{A}/A) - \text{length } (\overline{A}/\text{End } (J)) = \delta - s.$$

Hence we get,

$$(3.1.5) \quad \dim ((G.J) \cap \text{Hilb}_0^{d-\delta}(A/C))$$
$$= \text{length } (\overline{A}/A) - \text{length } (\overline{A}/J^{-1}) - \text{length } (\text{End}(J)/A)$$
$$(\text{by duality}) = \delta - \text{length } (J/C) - \delta + s = d + s - 2\delta.$$

Outside a proper closed subset of W_d every J has $(G.J) \cap \text{Hilb}_0^{d-\delta}(A/C) \neq \emptyset$ and hence we get

$$(3.1.6) \quad \dim \text{Hilb}^{d-\delta}(A/C)$$
$$= \dim (\text{generic moduli of isomorphism classes in } W_d)$$
$$+ \dim (G.J) \cap \text{Hilb}_0^{d-\delta}(A/C),$$

which by the above yields,

$$(3.1.7)\quad (d-\delta)-1 \geqq \dim \mathrm{Hilb}_0^{d-\delta}(k[X, Y]/(g)) \quad (g \text{ a polar}) \geqq \dim \mathrm{Hilb}_0^{d-\delta}(A/C) \geqq \dim W_d - s + (d+s-2\delta).$$

Since $\overline{W}_d$ is an arbitrary irreducible component of $E'(d)$ we get dim $E'(d) < \delta$ and so (3.1.2) is proved. Hence $\overline{P}$ is irreducible. For the other implication note that if A is not Gorenstein the result is contained in Theorem 2.3; so let A be Gorenstein.

We must show that A has embedding dimension two. If not the vector space $\mathfrak{m}/\mathfrak{m}^2$ with $\mathfrak{m}$ the maximal ideal of A, is of rank greater than or equal to 3. Note that every subspace of $\mathfrak{m}/\mathfrak{m}^2$ yields an ideal so that the projective space of codimension 1 subspaces yields a closed subscheme of $\mathrm{Hilb}_0^2(A)$ of dimension greater than or equal to 2. But for X Gorenstein we have noted in the final remark of paragraph 1 that for $\overline{P}(X)$ to be irreducible every $\mathrm{Hilb}^n(X)$ must be irreducible. In order that $\mathrm{Hilb}^2(X)$ be irreducible it must have the same dimension as the second symmetric product of X i.e. : equal to two. Now $\mathrm{Hilb}_0^2(A)$ is a closed subscheme of $\mathrm{Hilb}^2(X)$ not equal to the whole of it so dim $\mathrm{Hilb}_0^2(A) \geqq 2$ implies dim $\mathrm{Hilb}^2(X) \geqq 3$ which proves $\overline{P}(X)$ is reducible.

Remark 3.2. — Essential use is made of D'Souza's smoothness theorem in the last paragraph of the above proof *via* the remark "for X Gorenstein, $\overline{P}$ irreducible $\Leftrightarrow$ $\mathrm{Hilb}^n(X)$ is irreducible $\forall n$".

COROLLARY. — *Dimension* $\mathrm{Hilb}_0^n(k[X, Y]) = n-1$.

Proof. — Let $f \in k[X, Y]$ define a reduced and irreducible curve through (0, 0) with multiplicity n at the origin and Y its projective closure. Now $\mathrm{Hilb}_0^n(k[X, Y]/f)$ being a proper closed subscheme of $\mathrm{Hilb}^n(X)$ (which by the Theorem and paragraph 1 is of dimension n) has dimension less than or equal to $(n-1)$. But

$$\mathrm{Hilb}_0^n(k[X, Y]) = \mathrm{Hilb}_0^n(k[X, Y]/f)$$

as $f \in (X, Y)^n$ and every ideal of length n contains $(X, Y)^n$. It remains only to exhibit a component of dimension $(n-1)$. This is given by the family of ideals

$$g \in (X^n, Y + a_1 X + a_2 X^2 + \ldots + a_{n-1} X^{n-1})$$

Recently Briançon [4] has proved that $\mathrm{Hilb}_0^n(k[X, Y])$ is irreducible so the above family is dense open. The above discussion quickly yields.

THEOREM B. — *The boundary of* $\dot{\overline{P}}$ *for a curve with planar singularities has m irreducible components each of codimension one in* $\overline{P}$, *where*

$$(3.3)\qquad m = \sum_{Q \in X} [\textit{multiplicity}\ (Q) - 1].$$

Proof. — It is easily seen and left to the reader to check that the irreducible components of the boundary are "generated" by $\mathrm{Pic}^0(X)$ action by the corresponding subsets of $\overline{G} - G$. It therefore suffices to work with the E scheme of $A=\hat{O}_{X, x_0}$ where x_0 is a typical singular point of X. Let

$$A=k[X, Y]/f, \qquad v=\text{ord } f=\text{mult }(x_0),$$

C the conductor of A. As recalled earlier $\mathfrak{m}_A^{v-1}\supset C$ and in fact $\mathfrak{m}_A^{v-1}$ is the conductor of A in its first blow up. On the one hand, the polar is an adjoint curve of multiplicity $v-1$ and is contained in C. We have

$$\text{Hilb}_0^n(A)=\text{Hilb}_0^n(A/C) \qquad \text{for} \quad n<v-1.$$

On the other hand,

$$\text{Hilb}_0^n(A)\supsetneqq\text{Hilb}_0^n(A/C) \qquad \text{for} \quad n\geqq v.$$

This is because if g is the polar of f then

$$g\in(X^n, Y+a_1X+a_2X^2+\ldots+a_{n-1}X^{n-1})$$

for generic choice of a_i since $(g, Y+a_1X+\ldots)$ will have length $v-1$ for almost all a_i. By Briançon's Theorem dim $\text{Hilb}_0^n(A/C)<n-1, n\geqq v$. The calculation of Theorem A shows that $E'(d)$ is irreducible of dimension $\delta-1$ for $d\leqq\delta+v-1$ and dimension $E'(d)<\delta-1$ for $d>\delta+v-1$. Now the $e(E'(d))$ cover $e(E(C, \delta))-G$ in the sense outlined in the proof of Theorem A. Further, since $\overline{P}$ is irreducible (i. e.: every fractional ideal is a boundary point) we have $e(E(C, \delta))=\overline{G}$ by Theorem 2.3. As G is affine $\overline{G}-G$ is a union of codimension one subsets. These are defined by the $E'(d)$ for $\delta<d\leqq\delta+v-1$. This proves the Theorem.

Remark 3.4. — It is likely that Briançon's Theorem is provable by the methods introduced here.

Acknowledgement

Thanks are due to C. S. Seshadri and C. D'Souza for introducing me to the subject of this article.

Addendum

This paper dates from 1977 and was to appear elsewhere. As publication was delayed it is being published in *Annales scientifiques de l'École Normale Supérieure*.

REFERENCES

[1] A. Altman, A. Iarrobino and S. Kleiman, *Irreducibility of the Compactified Jacobian, Singularities of Real and Complex Maps* (*Proceedings of the Nordic Summer School*, Oslo, Vol. 76, 1977, pp. 1-12, P. Holm, Ed., Sijthoff and Noordhoff).

[2] A. Altman and S. Kleiman, *Algebraic Systems of Linearly Equivalent Divisor-like Subschemes* (*Composito Math.*, Vol. 29, 1974, pp. 113-139).

[3] H. Bass, *On the Ubiquity of Gorenstein Rings* (*Math. Zeit.*, Vol. 82, 1963, pp. 8-28).

[4] J. BRIANÇON, *Description of* $\mathrm{Hilb}^n \mathbb{C}\{X, Y\}$ (*Inventiones Math.*, Vol. 41, 1977, pp. 45-89).

[5] J. L. COOLIDGE, *A Treatise on Algebraic Plane Curves*, Dover Publications, 1959.

[6] C. D'SOUZA, *Compactification of the Generalized Jacobian*, [*Thesis*, Bombay University, 1974 (to appear in *Proc. Ind. Academy of Sciences*)].

[7] J. FOGARTY, *Fixed Point Schemes* (*Amer. J. Math.*, Vol. 188, 1977).

[8] J. FOGARTY, *Families on an Algebraic Surface* (*Amer. J. Math.*, Vol. 90, 1968, pp. 511-521).

[9] G. HORROCKS, *Fixed Point Schemes of Additive Group Actions* (*Topology*, Vol. 8, 1969, pp. 233-242).

[10] A. IARROBINO, *Punctual Hilbert Schemes* (*Mem. A.M.S.*, Vol. 188, 1977).

[11] D. MUMFORD, *Further Comments on Boundary Points*, Preprint, A.M.S. Summer Institute, Woods Hole, 1964.

[12] C. J. REGO, *A Note on One Dimensional Gorenstein Rings* (*J. Pure and Appl. Algebra*, Vol. 12, 1978, pp. 111-115).

[13] C. J. REGO, *Deformation of Singular Curves* (to appear).

[14] J.-P. SERRE, *Groupes algébriques et corps de classes*, Hermann, Paris, 1959.

(Manuscrit reçu le 13 juin 1979,
révisé le 4 août 1979.)

C. J. REGO,
School of Mathematics,
Tata Institute
of Fundamental Research,
Bombay-5, India.