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STRUCTURE OF THE GROUP OF HOMEOMORPHISMS PRESERVING A GOOD MEASURE ON A COMPACT MANIFOLD

By A. FATHI

0. Introduction

We give proofs of the results announced in [FV].

Let M^n be a compact connected manifold without boundary of dimension n. Let μ be a probability measure on M^n without atoms and which is strictly positive on each non empty open subset of M; such a measure is called a good measure. We prove the following results.

THEOREM. — *The group $\mathscr{H}(M^n, \mu)$ of μ-preserving homeomorphisms is locally contractible (for the compact open topology), i.e. there exists a neighborhood $\mathscr{U}$ of Id_{M^n} in $\mathscr{H}(M^n, \mu)$ such that the inclusion $\mathscr{U} \subset \mathscr{H}(M^n, \mu)$ is homotopic to a constant map. Moreover, the inclusion $\mathscr{H}(M^n, \mu) \subset \mathscr{H}(M^n)$, where $\mathscr{H}(M^n)$ is the group of all homeomorphisms, is a weak homotopy equivalence, i.e. it induces isomorphisms on all homotopy groups.*

THEOREM. — *Suppose that M^n is differentiable or PL and that $n \geqq 3$. Let $\mathscr{H}_0(M^n, \mu)$ be the path component of $\mathscr{H}(M^n, \mu)$ which contains the identity. The abelianization of $\mathscr{H}(M^n, \mu)$ is isomorphic to a quotient of $H_1(M^n, \mathbb{R})$ by some discrete subgroup. The commutator subgroup $[\mathscr{H}_0(M^n, \mu), \mathscr{H}_0(M^n, \mu)]$ is a simple group; moreover, it is generated by the elements of $\mathscr{H}_0(M^n, \mu)$ which are supported in topological n-balls.*

In fact, we do not need that M^n is differentiable or PL, the Theorem above is true under more general conditions which are explained in the text. In contrast the condition $n \geqq 3$ is essential for our methods; to our knowledge the case $n=2$ is still unsettled. The case of S^1 is treated by direct examination since $\mathscr{H}_0(S^1, \mu)$ is isomorphic to S^1. Generalizations to the non compact case are also given.

We will describe now the proofs of these Theorems; this will give a fairly good idea of the content of this work.

The first ingredient is the von Neumann-Oxtoby-Ulam Theorem [OU_2]; it says that given two good measures μ and ν on M^n, there exists a homeomorphism h of M^n such that $h_* \mu = \nu$. What we need is in fact a parametrized version of this Theorem; more precisely we would like that h depends continuously on μ and ν. We are not able to prove this fact, but instead we remark that if we restrict to the subset of good measures which have the same subsets of measure 0 as a fixed measure μ_0, then the proof of Oxtoby and Ulam gives us a homeomorphism h which depends continuously on μ and ν. As a consequence, we obtain that $\mathscr{H}(\mathrm{M}^n, \mu)$ is a deformation retract of the group $\mathscr{H}(\mathrm{M}^n, \mu\text{-bireg})$, where $\mathscr{H}(\mathrm{M}^n, \mu\text{-bireg})$ is the group of homeomorphisms h such that h and h^{-1} send subsets of μ-measure 0 to subsets of μ-measure 0. Next we remark, as is well known, that the Cernavskii-Edwards-Kirby technique shows that $\mathscr{H}(\mathrm{M}^n, \mu\text{-bireg})$ is locally contractible; this gives us immediately the local contractibility of $\mathscr{H}(\mathrm{M}^n, \mu)$. Then, we show that $\mathscr{H}(\mathrm{M}^n, \mu) \subsetneq \mathscr{H}(\mathrm{M}^n)$ is a weak homotopy equivalence by showing that $\mathscr{H}(\mathrm{M}^n, \mu\text{-bireg}) \subsetneq \mathscr{H}(\mathrm{M}^n)$ is a weak homotopy equivalence. This last fact is a consequence, *via* a Theorem of Eilenberg and Wilder [EW], of the local contractibility of $\mathscr{H}(\mathrm{M}^n, \mu\text{-bireg})$ and the fact that it is dense in $\mathscr{H}(\mathrm{M}^n)$.

The proof of the second Theorem begins with the construction of the mass flow homomorphism; this is a group homomorphism $\theta : \mathscr{H}_0(\mathrm{M}^n, \mu) \to \mathrm{H}_1(\mathrm{M}^n, \mathbb{R})/\Gamma$, where Γ is some discrete subgroup of $\mathrm{H}_1(\mathrm{M}^n, \mathbb{R})$. The existence of this map was first given by Schwartzman [Sc]; its differentiable version is attributed to Weinstein [Th]. The definition given here is inspired by Herman's definition of the rotation number of a homeomorphism of the circle [He$_2$]. It is more convenient to define first $\tilde{\theta} : \tilde{\mathscr{H}}_0(\mathrm{M}^n, \mu) \to \mathrm{H}_1(\mathrm{M}^n, \mathbb{R})$, where $\tilde{\mathscr{H}}_0(\mathrm{M}^n, \mu)$ is the universal cover of $\mathscr{H}_0(\mathrm{M}^n, \mu)$. Since $\mathscr{H}_0(\mathrm{M}^n, \mu)$ is locally contractible, an element of $\tilde{\mathscr{H}}_0(\mathrm{M}^n, \mu)$ is represented by an isotopy $(h_t)_{t\in[0,\,1]}$ such that $h_0 = \mathrm{id}$ and $h_{t*}\,\mu = \mu$ for each $t \in [0, 1]$; two such isotopies represent the same element if they are homotopic with fixed extremities, the homotopy being through measure preserving isotopies. To define $\tilde{\theta}$, we first remark that $\mathrm{H}_1(\mathrm{M}^n, \mathbb{R}) = \mathrm{Hom}([\mathrm{M}^n, \mathrm{S}^1], \mathbb{R})$, where $[\mathrm{M}^n, \mathrm{S}^1]$ is the set of homotopy classes of maps from M^n to S^1. We will identify S^1 with $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, this allows us to write the group law on S^1 additively. Given $(h_t) \in \tilde{\mathscr{H}}_0(\mathrm{M}_n, \mu)$ we define a homomorphism $\tilde{\theta}(h_t) : [\mathrm{M}^n, \mathbb{T}^1] \to \mathbb{R}$ in the following way:

Let $f : \mathrm{M}^n \to \mathbb{T}^1$ be continuous; the homotopy $fh_t - f : \mathrm{M}^n \to \mathbb{T}^1$ verifies $fh_0 - f = 0$, hence we can lift it to a homotopy $\overline{fh_t - f} : \mathrm{M}^n \to \mathbb{R}$ such that $\overline{fh_0 - f} = 0$:

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ \mathrm{M}^n \arrow[ur, "\overline{fh_t - f}"] \arrow[r, "fh_t - f"'] & \mathbb{T}^1 = \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

By definition, $\tilde{\theta}(h_t)(f) = \displaystyle\int_{\mathrm{M}^n} \overline{fh_1 - f}\, d\mu$. One first shows that this gives a group homomorphism $[\mathrm{M}^n, \mathbb{T}^1] \to \mathbb{R}$, then that $\tilde{\theta}(h_t)$ depends only on the class of (h_t) and finally that $\tilde{\theta} : \tilde{\mathscr{H}}_0(\mathrm{M}^n, \mu) \to \mathrm{H}_1(\mathrm{M}^n, \mathbb{R})$ is a group homomorphism. If we put

$$\Gamma = \tilde{\theta}(\mathrm{Ker}\, \tilde{\mathscr{H}}_0(\mathrm{M}^n, \mu) \to \mathscr{H}_0(\mathrm{M}^n, \mu)),$$

we obtain by passing to the quotient a group homomorphism $\theta : \mathscr{H}_0(\mathrm{M}^n, \mu) \to \mathrm{H}_1(\mathrm{M}^n, \mathbb{R})/\Gamma$. The subgroup Γ is discrete because it is contained in

$H_1(M^n, \mathbb{Z})$. We show that $\tilde{\theta}$ (hence θ) is surjective, if $H_1(M^n, \mathbb{R})$ has a basis represented by imbedded curves having tubular neighborhoods. It remains to show that $\operatorname{Ker}\theta$ is simple, and is generated by the homeomorphisms supported in topological n-balls. We then prove this second fact in the case where M^n has a handle decomposition, the proof is done by induction on the number of handles; moreover, we show that $\operatorname{Ker}\theta$ is generated by homeomorphisms having support in arbitrarily small topological n-balls. This last fact implies, by the classical method of Epstein and Higman ([E], [H]), that the commutator subgroup $[\operatorname{Ker}\theta, \operatorname{Ker}\theta]$ is simple. Then, we prove that $\operatorname{Ker}\theta = [\operatorname{Ker}\theta, \operatorname{Ker}\theta]$ by the same method as in [F] where we proved that the group of bimeasurable Lebesgue measure preserving transformations of [0, 1] is a simple group.

Some more facts and the extension to the non compact case are proven in the different appendices.

The results of this work are of course related to results of Epstein [Ep], Herman [He_1], Mather [Ma] and Thurston [Th] on the simplicity of diffeomorphisms groups, and also to results due to Anderson [A], Černavskii, Edwards and Kirby [EK] on the algebraic and topological structure of the homeomorphisms group of a manifold. Our debt to their work is important, but our greatest debt is to the work of Oxtoby and Ulam; their paper [OU_2] is certainly the most important tool for any study of measure preserving homeomorphisms.

The part of this work centering around the first of the two Theorems mentionned above is a joint work with Yves-Marie Visetti, to whom I am most grateful. This work was done at the instigation of Michel Herman and with his help and constant encouragements. It owes also a great deal to the good will and encouragements of Larry Siebenmann. I want also to thank Lucien Guillou who followed this work step by step and listened to all stupidities without loosing his temper.

LIST OF CONTENTS

1. Some generalities on measures on compact metric spaces

Let X be a compact metric (non empty) space. We will denote by $\mathscr{M}(X)$ the set of probability measures on X. Recall that a probability measure on X is a non negative measure, defined on the σ-algebra of all Borel subsets of X, whose total mass is 1.

We put on $\mathscr{M}(X)$ the weak topology (*see* [DGS], p. 9 for a definition), with this topology $\mathscr{M}(X)$ is a compact metric space. We will have sometimes to consider measures on a locally compact space Y. A Radon measure on Y is a measure defined on the σ-algebra of all Borel subsets, and *which is finite on* each compact subset of Y. Hence, such a measure is σ-finite, if Y is σ-compact, in particular if Y has a countable basis of open sets.

In the remainder of this section, X is a compact metric space.

We recall some facts which are proved in chapter 2 of [DGS].

PROPOSITION 1.1. — *If μ is in $\mathscr{M}(X)$, then μ is regular, i.e. for each Borel set* B *we have:*

$$\mu(B)=\sup\{\mu(C)\mid C \text{ compact} \subset B\},$$
$$\mu(B)=\inf\{\mu(U)\mid U \text{ open} \supset B\}.$$

PROPOSITION 1.2. — *The weak topology on $\mathscr{M}(X)$ is the weakest topology on $\mathscr{M}(X)$ such that each function $\mu \mapsto \mu(U)$, U an open set in X [resp. $\mu \mapsto \mu(C)$, C a closed set in X], is lower semi-continuous (resp. upper semi-continuous).*

DEFINITION 1.3. — A measure on X is good if it is a probability measure with no atoms and whose support is X itself. We will denote by $\mathscr{M}_g(X)$ the set of good measures on X.

Remark. — The support of μ is X if and only if μ is strictly positive on each non empty open set.

PROPOSITION 1.4. — *If X has no isolated point, then $\mathscr{M}_g(X)$ is a dense G_δ subset of $\mathscr{M}(X)$.*

Proof. — By [DGS], 2.16, the set of non atomic measures is a dense G_δ in $\mathscr{M}(X)$. We have now to show that the set $\{\mu\in\mathscr{M}(X)\mid \text{support }\mu = X\}$ is a dense G_δ. Let $(U_n)_{n\in\mathbb{N}}$ be a basis of open sets of X, we can assume $U_n\neq\emptyset$ for each n. It is easy to show that:

$$\{\mu\in\mathscr{M}(X)\mid \text{support }\mu=X\} = \bigcap_{n\in\mathbb{N}}\{\mu\in\mathscr{M}(X)\mid \mu(U_n)>0\}.$$

Now each set $\{\mu\in\mathscr{M}(X)\mid\mu(U_n)>0\}$ is open by 1.2. Moreover, if $x_n\in U_n$ and $\nu\in\mathscr{M}(X)$ then, for each $t>0$, $(1-t)\nu+t\delta_{x_n}$ is in this set and, of course, $\nu=\lim\limits_{t\to 0}(1-t)\nu+t\delta_{x_n}$. Hence, $\{\mu\mid\mu(U_n)>0\}$ is open and dense.

An application of Baire category Theorem finishes the proof. □

We put the compact open topology on $\mathscr{H}(X)$ the space of homeomorphisms of X. Since X is compact metric, this topology is the same as the uniform topology, it is metrizable. In fact, we can define this topology by a complete metric; if d is a metric on X, we define a metric $\bar{d}$ on $\mathscr{H}(X)$ by:

$$\bar{d}(f, g)=\sup_{x\in X} d(f(x), g(x))+\sup_{x\in X} d(f^{-1}(x), g^{-1}(x)).$$

The metric $\overline{d}$ is complete and defines the uniform topology on $\mathscr{H}(X)$.

We recall for future reference that an isotopy between h_0 and h_1, where h_0 and $h_1 \in \mathscr{H}(X)$, is simply a continuous path in $\mathscr{H}(X)$ between h_0 and h_1. It is also given by a continuous map $H : X \times [0, 1] \to X$ such that, for each $t \in [0, 1]$, the map $X \to X, x \mapsto H(x, t)$ is a homeomorphism, which is equal to h_0 for $t=0$ and to h_1 for $t=1$. An isotopy of X is simply an isotopy between the identity of X and some other homeomorphism, i. e. a continuous path in $\mathscr{H}(X)$ starting at the identity of X.

There is a natural map: $\mathscr{H}(X) \times \mathscr{M}(X) \to \mathscr{M}(X)$, $(h, \mu) \mapsto h_* \mu$, where $h_* \mu$ is defined by $h_* \mu(B) = \mu(h^{-1}(B))$ for each Borel set B in X.

PROPOSITION 1.5. — *The map* $\mathscr{H}(X) \times \mathscr{M}(X) \to \mathscr{M}(X)$, $(h, \mu) \mapsto h_* \mu$, *is continuous.*

Proof. — It suffices to show that if C is a closed set in X and α a real number, then the set $\{(h, \mu) \in \mathscr{H}(X) \times \mathscr{M}(X) \mid h_* \mu(C) < \alpha\}$ is open.

Fix h_0 and μ_0 in this set, we have $\mu_0(h_0^{-1}(C)) < \alpha$. We can find a compact set K such that: $h_0^{-1}(C) \subset \operatorname{Int}(K)$, and $\mu_0(K) < \alpha$. The set

$$\{h \in \mathscr{H}(X) \mid h^{-1}(C) \subset \operatorname{Int}(K)\} \times \{\mu \in \mathscr{M}(X) \mid \mu(K) < \alpha\}$$

is a neighborhood of (h_0, μ_0) which is contained in

$$\{(h, \mu) \in \mathscr{H}(X) \times \mathscr{M}(X) \mid h_* \mu(C) < \alpha\}. \quad \square$$

Given a measure μ on X, we define $\mathscr{H}(X, \mu)$ as the set of homeomorphisms of X which preserve μ:

$$\mathscr{H}(X, \mu) = \{h \in \mathscr{H}(X) \mid h_* \mu = \mu\}.$$

COROLLARY 1.6. — *If* μ *is a measure on* X, *the set* $\mathscr{H}(X, \mu)$ *is a closed subgroup of* $\mathscr{H}(X)$.

APPENDIX A.1

ON THE TOPOLOGICAL TYPE OF $\mathscr{M}_g(X)$

THEOREM A.1. — *Let* X *be a compact metric space without isolated points. Then,* $\mathscr{M}_g(X)$ *(with the weak tolopoly) is homeomorphic to the Hilbert space* l^2. *Moreover, the pair* $(\mathscr{M}(X), \mathscr{M}_g(X))$ *is homeomorphic to* $([0, 1]^{\mathbb{N}},]0, 1[^{\mathbb{N}})$.

Proof. — By the Anderson-Kadec Theorem ([BP], p. 189, Th. 5.2), the Hilbert space l^2 is homeomorphic to $]0, 1[^{\mathbb{N}}$, hence we have to prove only the last part of the Theorem.

We will use the apparatus of infinite dimensional topology. We will give reference to [BP] and [To] for the quoted results.

First, $\mathscr{M}(X)$ is a separable compact convex set of infinite dimension, hence by Keller's Theorem ([BP], p. 100, Th. 3.1), it is homeomorphic to the Hilbert cube $Q = [0, 1]^{\mathbb{N}}$. To finish the proof, all we have to do is to prove that $\mathscr{M}(X) - \mathscr{M}_g(X)$ is a Z-skeletoid (*see* [BP], chap. IV and V). Since $\mathscr{M}_g(X)$ is convex and dense, it is easy to see that for each open convex set U the inclusion $U \cap \mathcal{M}_g(X) \subset U \cap \mathcal{M}(X)$ is a homotopy equivalence, hence by [To], Th. 2.3, $\mathcal{M}(X) - \mathcal{M}_g(X)$ is locally homotopically negligible in $\mathcal{M}(X)$. Remark that $\mathcal{M}(X) - \mathcal{M}_g(X)$ is a F_σ since $\mathcal{M}_g(X)$ is a G_δ. So, we have proved that $\mathcal{M}(X) - \mathcal{M}_g(X)$ is a Z_σ-set. To finish the proof, we must show that it contains a Z-skeletoid ([BP], p. 156, Th. 3.2). Now, if $x_0 \in X$, the set $\{ t\delta_{x_0} + (1-t)\mu \mid t \in]0, 1]$ and $\mu \in \mathcal{M}(X) \}$ is an F_σ set which is contained in $\mathcal{M}(X) - \mathcal{M}_g(X)$, so it is a Z_σ-set; it follows then from [BP], p. 157, Prop. 4.2, that this set is a Z-skeletoid. This finishes the proof. $\square$

Remarks. — (1) By using Dirac measures on X, we can imbed naturally X in $\mathcal{M}(X)$ as a Z-set. By using the action of $\mathcal{H}(X)$ on $\mathcal{M}(X)$, we obtain a group imbedding $\mathcal{H}(X) \subset \mathcal{H}(\mathcal{M}(X))$ which gives us canonical extensions of the homeomorphisms of X to homeomorphisms of $\mathcal{M}(X) \simeq Q$. Since two Z-imbeddings of X in Q are ambient homeomorphic, we obtain the following:

If X is a Z-set in Q, we can construct a group homomorphism $\mathcal{H}(X) \to \mathcal{H}(Q)$, $h \to \bar{h}$, such that $\bar{h} \mid X = h$, for each $h \in \mathcal{H}(X)$.

(2) We will see in paragraph 3 that, if M^n is a compact connected closed manifold and $\mu \in \mathcal{M}_g(M^n)$, then $\mathcal{H}(M^n)/\mathcal{H}(M^n, \mu) \simeq \mathcal{M}_g(M^n)$, hence $\mathcal{H}(M^n)/\mathcal{H}(M^n, \mu)$ is homeomorphic to l^2. Applying this fact to $I = [-1, 1]$, we obtain the well-known result $\mathcal{H}(I) \cong l^2 \times \{ \mathrm{Id}, r \}$ where $r : I \to I$ is given by $r(x) = -x$.

2. Some generalities on manifolds and measures on manifolds

We introduce the following notations:

$B^n = \{ x \in \mathbb{R}^n \mid \|x\| \leqq 1 \}$ where $\| \quad \|$ is the usual euclidian norm;

$I^n = [0, 1]^n \subset \mathbb{R}^n$;

$\mathbb{H}^n_+ = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geqq 0 \}$;

$\mathbb{H}^n_{++} = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{n-1} \geqq 0 \text{ and } x_n \geqq 0 \}$.

By a manifold M^n, we mean a Hausdorff topological space which is locally homeomorphic to $\mathbb{H}^n_+$, and which has a countable basis of open sets. Locally homeomorphic to $\mathbb{H}^n_+$ means, of course, each point of M^n has an open neighborhood homeomorphic to some open set of $\mathbb{H}^n_+$. By the definition we have given, our manifolds may have a non empty boundary. If M^n is a manifold, we will note by ∂M^n its boundary and by $\mathring{M}^n$ its "interior" $M^n - \partial M^n$.

To avoid confusion, if A is a subset of a topological space X, we will note its interior as a subset of X by Int (A), and its "boundary" as a subset of X will be called frontier and noted by Fr (A).

If M^n is a manifold, we will denote by $\mathcal{H}^\partial(M^n)$ the subgroup of $\mathcal{H}(M^n)$ defined by:

$$\mathcal{H}^\partial(M^n) = \{ h \in \mathcal{H}(M^n) \mid h \mid \partial M^n = \text{identity} \}.$$

The following Lemma is trivial and its proof is left to the reader.

LEMMA 2.0. — *Let μ by a σ-finite measure defined on some space* X. *Let $\{ A_\lambda \mid \lambda \in \Lambda \}$ be a family of disjoint measurable sets. There is only a countable number of A_λ such that $\mu(A_\lambda) \neq 0$.*

Note the following trivial consequences of Lemma 2.0.

(1) If M^n is a manifold and μ is a (Radon) measure on M^n such that $\mu(\partial M^n)=0$, then we can find (arbitrary small) coverings of M^n by sets $\{B_l\}_{l\in\mathbb{N}}$ such that each B_l is homeomorphic to I^n and $\mu(\partial B_l)=0$.

(2) If μ is a (Borel) measure on the cube I^n such that $\mu(\partial I^n)=0$, then we can find (arbitrary small) subdivisions of I^n by hyperplanes into subcubes $c_1, \ldots, c_2$ such that $\mu(\partial c_i)=0$ for each i.

We will need the following Proposition of Oxtoby and Ulam [OU_1].

PROPOSITION 2.1. — *Let* M^n *be a compact manifold and* μ *a (Borel) measure on* M^n *with* $\mu(\partial M^n)=0$. *If* A *is a closed set of* M^n *with* $\operatorname{int}(A)=\emptyset$, *then there exists a homeomorphism* h *of* M^n *such that* $\mu(h(A))=0$ *and* $h\,|\,\partial M^n=\mathrm{Id}$. *More precisely, the set* $\{h\in\mathscr{H}^\partial(M^n)\,|\,\mu(h(A))=0\}$ *is a dense* G_δ *in* $\mathscr{H}^\partial(M^n)$.

To prove 2.1, we must first give a Lemma.

LEMMA 2.2. — *Suppose* μ *is a measure on* I^n *such that* $\mu(\partial I^n)=0$, *and* S *is a closed subset of* I^n *such that* $\operatorname{int}(S)=\emptyset$. *Then, for each* δ *and* $\varepsilon>0$, *there exists a homeomorphism* h *of* I^n *such that:*

– $h\,|\,\partial I^n=\mathrm{id}$;

– *h is δ-close to the identity;*

– $\mu(h(S))<\varepsilon$.

Proof. — Let $c_1, \ldots, c_k$ be a subdivision of I^n by cubes such that $\operatorname{diam}(c_i)<\delta$ and $\mu(\partial c_i)=0$ for each i. We can find an open neighborhood U of $\sum_{i=1}^{k}\partial c_i$, such that $\mu(U)<\varepsilon$. Since S has empty interior, we can find, for each i, a small subcube $c_i'\subset\mathring{c}_i$ with faces parallel to the faces of c_i and such that $S\cap c_i'=\emptyset$.

For each i, we can define a homeomorphism h_i of c_i such that:

– $h_i\,|\,\partial c_i=\mathrm{Id}$;

– $h_i(c_i-c_i')\subset U\cap c_i$.

Piecing together the h_i gives the desired homeomorphism h. □

Proof of 2.1. — Let $B_1, \ldots, B_k$ be a covering of M^n by sets homeomorphic to I^n and such $\mu(\partial B_i)=0$ for each i.

If $i\in\{1, \ldots, k\}$ and $j\in\mathbb{N}^*=\mathbb{N}-\{0\}$, define a subset $\mathscr{U}(i,j)$ of $\mathscr{H}^\partial(M^n)$ by:

$$\mathscr{U}(i,j)=\left\{h\in\mathscr{H}^\partial(M^n)\,|\,\mu[h(A)\cap B_i]<\frac{1}{j}\right\}.$$

We clearly have:

$$\{h\in\mathscr{H}^\partial(M^n)\,|\,\mu(h(A))=0\} = \bigcap_{i,j}\mathscr{U}(i,j).$$

So we must show that $\mathscr{U}(i,j)$ is open and dense in $\mathscr{H}^\partial(M^n)$. Denseness of $\mathscr{U}(i,j)$ follows easily from Lemma 2.2. We prove now that $\mathscr{U}(i,j)$ is open.

Fix $h_0 \in \mathscr{U}(i, j)$, we can find an open neighborhood U of $h_0(A) \cap B_i$ such that $\mu(U) < 1/j$. If h is close enough to h_0, we have $h(A) \cap B_i \subset U$, which implies that $\mu(h(A) \cap B_i) < 1/j$. $\square$

Remark. — Proposition 2.1 is also valid when M^n is not compact and when the compact-open topology on M^n is replaced by the fine topology.

We will use the following Theorems due to M. Brown.

PROPOSITION 2.3. — *Let M^n be a compact connected n-dimensionnal manifold. There exists a map $\varphi : I^n \to M^n$ such that:*

(i) *φ is surjective;*

(ii) *$\varphi \mid \mathring{I}^n$ is a homeomorphism onto its image;*

(iii) *$\varphi(\partial I^n) \cap \varphi(\mathring{I}^n) = \emptyset$ and in particular $\varphi(\partial I^n)$ has empty interior in M^n.*

PROPOSITION 2.3′. — *Suppose M^n is a connected manifold, with $\partial M^n \neq \emptyset$. There exists an open imbedding $\psi : \mathbb{H}^n_+ \to M^n$, where $\mathbb{H}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geqq 0\}$, such that:*

(i) *$\psi^{-1}(\partial M^n) = \partial \mathbb{H}^n_+$;*

(ii) *$\psi(\mathbb{H}^n_+)$ is dense in M^n; in particular $M^n - \psi(\mathbb{H}^n_+)$ is a closed subset of M^n with empty interior.*

The proof of 2.3 and 2.3′ can be found in [B]. The proof of 2.3 is also [CV], p. 461.

Complement to 2.3 and 2.3′. — *If μ is some (Radon) measure on M^n such that $\mu(\partial M^n) = 0$, then, we can assume that $\mu(\varphi(\partial I^n)) = 0$ in the case on 2.3 and $\mu(M^n - \psi(\mathbb{H}^n_+)) = 0$ in case of 2.3′.*

Proof. — Since $\varphi(\partial I^n)$ [resp. $M^n - \psi(\mathbb{H}^n_+)$] is a closed subset of M^n with empty interior, we can, by 2.1, find a homeomorphism h of M^n such that $\mu[h(\varphi(\partial I^n))] = 0$ (resp. $\mu[h(M^n - \psi(\mathbb{H}^n_+))] = 0$). We can now replace φ by $h\varphi$ (resp. ψ by $h\psi$) to obtain the desired result. Remark that, in the case of 2.3′, we have not assumed M^n compact, we can apply 2.1 in this case also by the remark following the proof of 2.1. $\square$

The Alexander isotopy works also in the case of Lebesgue measure as we will see it now.

PROPOSITION 2.4. — *Let m be the Lebesgue measure on I^n. Then*

$$\mathscr{H}^{\partial}(I^n, m) = \{ h \in \mathscr{H}(I^n, m) \mid h \mid \partial I^n = \mathrm{id} \}$$

is contractible.

Proof. — We can replace, of course, $I^n = [0, 1]^n$ by $J^n = [-1/2, 1/2]^n$. Introduce on $\mathbb{R}^n$ the norm $| \ |$ defined by $|(x_1, \ldots, x_n)| = \max(|x_1|, \ldots, |x_n|)$. In this norm, J^n is the ball of radius 1/2 and center 0.

If $h \in \mathscr{H}^{\partial}(J^n, m)$, define $(h_t)_{t \in [0, 1]}$ in the following way:

$$h_t(x) = \begin{cases} t h\left(\dfrac{x}{t}\right), & \text{if } |x| \leqq \dfrac{1}{2} t, \\ x & \text{otherwise.} \end{cases}$$

We have $h_0 = \mathrm{id}$, $h_1 = h$ and each h_t is a homeomorphism of J^n. Moreover, h_t depends continuously on (h, t). It remains to check that each h_t preserves Lebesgue measure, but this is clear since a homothety in $\mathbb{R}^n$ transforms Lebesgue measure into a scalar multiple of itself. $\square$

3. The von Neumann-Oxtoby-Ulam Theorem and some of its consequences

If M^n is a compact manifold, we define the subset $\mathcal{M}_g^\partial(M^n)$ of $\mathcal{M}_g(M^n)$ by:

$$\mathcal{M}_g^\partial(M^n) = \{ \mu \in \mathcal{M}_g(M^n) \mid \mu(\partial M^n) = 0 \}.$$

Recall that we have a natural map:

$$\mathcal{H}(M^n) \times \mathcal{M}(M^n) \xrightarrow{\pi} \mathcal{M}(M^n),$$
$$(h, \mu) \mapsto h_* \mu.$$

This map defines an action of the group $\mathcal{H}(M^n)$ on $\mathcal{M}(M^n)$. The von Neumann-Oxtoby-Ulam Theorem says that $\mathcal{M}_g^\partial(M^n)$ is an orbit of this action.

THEOREM 3.1 (von Neumann-Oxtoby-Ulam). — *Suppose* M^n *is a compact connected manifold. If* μ_1 *and* $\mu_2 \in \mathcal{M}_g^\partial(M^n)$, *then there exists a homeomorphism* h *of* M^n *such that* $h \mid \partial M^n = \mathrm{id}$, *and* $h_* \mu_1 = \mu_2$.

The proof of this Theorem can be found in [OU_2], section II.

In the following, we fix a compact manifold M^n and a measure $\mu_0 \in \mathcal{M}_g^\partial(M^n)$. We have the map:

$$\mathcal{H}^\partial(M^n) \xrightarrow{\pi_0} \mathcal{M}_g(M^n),$$
$$h \mapsto h_* \mu_0.$$

Theorem 3.1 shows that this map is surjective and, in particular, that the induced map $\mathcal{H}^\partial(M^n)/\mathcal{H}^\partial(M^n, \mu_0) \to \mathcal{M}_g^\partial(M^n)$ is a bijection. In fact, this map is a homeomorphism, we will not prove this fact here, but note only that it can be deduced from Theorem 3.1.

Question 3.2. — Does the (surjective) map $\pi_0 : \mathcal{H}^\partial(M^n) \to \mathcal{M}_g^\partial(M^n)$ have a (continuous) section?

The answer to this question may be negative. The proof of Theorem 3.1, given by Oxtoby and Ulam, can be used to give a partial positive answer to 3.2. We proceed to explain this now.

Define $\mathcal{M}_g^\partial(M^n, \mu_0)$ as the subset of $\mathcal{M}_g^\partial(M^n)$ consisting of the measures which have the same sets of 0 measure as μ_0:

$$\mathcal{M}_g^\partial(M^n, \mu_0) = \{ \mu \in \mathcal{M}_g^\partial(M^n) \mid \forall A \subset M^n \text{ Borel set } \mu(A) = 0 \Leftrightarrow \mu_0(A) = 0 \}.$$

THEOREM 3.3. — *If* M^n *is compact and connected, the surjective map* $\pi_0 : \mathcal{H}^\partial(M^n) \to \mathcal{M}_g^\partial(M^n)$ *has a continuous section above* $\mathcal{M}_g^\partial(M^n, \mu_0)$. *In other words, there exists a continuous map* $\sigma : \mathcal{M}_g^\partial(M^n, \mu_0) \to \mathcal{H}^\partial(M^n)$ *such that* $\pi_0 \sigma = \text{identity}$.

We will first prove this Theorem with $M^n = I^n$ and $\mu_0 = m$ the Lebesgue measure. We formulate Theorem 3.3 in this case in slightly modified terms.

LEMMA 3.4. — *Let* P *be a topological space. Given any two continuous maps:* $P \to \mathcal{M}_g^\partial(I^n, m)$, $p \mapsto \nu_p$, *and* $p \mapsto \mu_p$. *There exists a continuous map:* $P \to \mathcal{H}^\partial(I^n)$, $p \mapsto f_p$, *such that, for each* $p \in P$, $(f_p)_* \nu_p = \mu_p$.

Proof of 3.4. — The proof is divided in 4 steps.

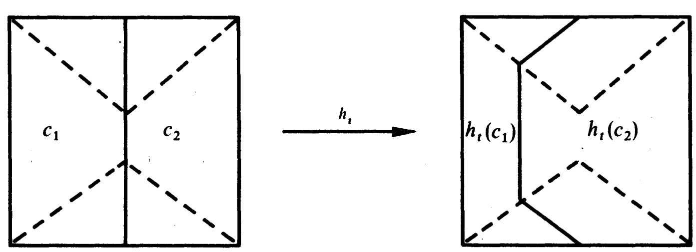
First of all, define $\mathcal{H}^\partial(I^n, \text{bireg})$ as the set of biregular homeomorphisms of I^n (fixing the boundary). A homeomorphism h of I^n is biregular if h and h^{-1} are absolutely continuous with respect to Lebesgue measure. We have:

$$\mathcal{H}^\partial(I^n, \text{bireg}) = \{ h \in \mathcal{H}^\partial(I^n) \mid h_* m \in \mathcal{M}_g^\partial(I^n, m) \}.$$

Step 1. — Suppose I^n is divided into two (closed) cubes c_1, c_2 by a hyperplane parallel to a coordinate hyperplane. There exists a continuous map $P \to \mathcal{H}^\partial(I^n, \text{bireg})$, $p \to h_p$, such that:

- for each p in P, $\mu_p(h_p(c_1)) = \nu_p(c_1)$ and $\mu_p(h_p(c_2)) = \nu_p(c_2)$;
- the family of homeomorphisms $(h_p)_{p \in P}$ is equicontinuous.

Proof of step 1. — Let $(h_t)_{t \in [0, 1]} : I^n \to I^n$ be any (continuous) homotopy such that:
- $\forall t \in]0, 1[$, $h_t \in \mathcal{H}^\partial(I^n, \text{bireg})$;
- $\forall t', t \in [0, 1]$, $t' > t$, $h_{t'}(\text{Int}(c_1)) \supset h_t(c_1)$;
- $h_0(c_1) = c_1 \cap \partial I^n$ and $h_1(c_1) = I^n$.



Remark that if $\mu \in \mathcal{M}_g^\partial(I^n, m)$, the map $t \to \mu(h_t(c_1))$ is a strictly increasing continuous map from $[0, 1]$ onto itself. We can then define a map: $P \to [0, 1]$, $p \mapsto t_p$, by $\mu_p(h_{t_p}(c_1)) = \nu_p(c_1)$. One can check that this map is continuous.

Since $\nu_p \in \mathcal{M}_g^\partial(I^n)$, we have $\nu_p(c_1) \in]0, 1[$ and hence $t_p \in]0, 1[$. This shows that $h_{t_p} \in \mathcal{H}^\partial(I^n, \text{bireg})$. Remark also that the family $(h_{t_p})_{p \in P}$ is equicontinuous because it is contained in the compact family $(h_t)_{t \in [0, 1]}$.

We can define the map $P \to \mathcal{H}^\partial(I^n, \text{bireg})$, $p \mapsto h_p$, by $h_p = h_{t_p}$. $\square$

Step 2. — If $\{c_1, \ldots, c_k\}$ is a subdivision of I^n in (closed) cubes by hyperplanes parallel to coordinate hyperplanes, then there exists a continuous map $P \to \mathscr{H}^\partial(I^n, \text{bireg})$, $p \mapsto h_p$, such that:

– for each p in P and each cube c_i in $\{c_1, \ldots, c_k\}$, $\mu_p(h_p(c_i)) = \nu_p(c_i)$;

– the family of homeomorphisms $(h_p)_{p\in P}$ is equicontinuous.

Proof of step 2. — Induction using step 1.

Step 3. — For each $\varepsilon > 0$, there exists a subdivision ξ of I^n in (closed) cubes by hyperplanes parallel to coordinate hyperplanes, and two continuous maps $P \to \mathscr{H}^\partial(I^n, \text{bireg})$, $p \mapsto h_p$ and $p \mapsto g_p$, such that:

– $\forall c \in \xi$, $\forall p \in P$, $\mu_p(h_p(c)) = \nu_p(g_p(c))$;

– $(h_p)_{p\in P}$ and $(g_p)_{p\in P}$ are equicontinuous;

– $\forall c \in \xi$, $\forall p \in P$, $\operatorname{diam}(c) < \varepsilon$, $\operatorname{diam}(h_p(c)) < \varepsilon$ and $\operatorname{diam}(g_p(c)) < \varepsilon$.

Proof of step 3. — Let $\{c_1, \ldots, c_k\}$ be a subdivision of I^n in cubes of diameter $< \varepsilon$. By step 2, we can find a continuous map: $P \to \mathscr{H}^\partial(I^n, \text{bireg})$, $p \mapsto h_p$, such that:

– $\forall p \in P$, $\forall c_i \in \{c_1, \ldots, c_k\}$, $\mu_p(h_p(c_i)) = \nu_p(c_i)$;

– $(h_p)_{p\in P}$ is equicontinuous.

Since the family $(h_p)_{p\in P}$ of homeomorphisms of I^n is equicontinuous, we can find a subdivision ξ of I^n, finer than $\{c_1, \ldots, c_k\}$ and such that: $\forall p \in P$, $\forall c \in \xi$, $\operatorname{diam}(h_p(c)) < \varepsilon$.

By applying step 2 inside each cube c_i $(i = 1, \ldots, k)$ to the subdivision $\xi | c_i$ and the measures $[\nu_p | c_i]_{p\in P}$ and $[(h_p^{-1})_* \mu_p | c_i]_{p\in P}$, we can obtain a continuous map: $P \to \mathscr{H}^\partial(I^n, \text{bireg})$, $p \mapsto g_p$, such that:

– $\forall c \in \xi$, $\forall p \in P$, $\nu_p(g_p(c)) = \mu_p(h_p(c))$;

– $\forall p \in P$, $\forall i = 1, \ldots, k$, $g_p(c_i) = c_i$ and $g_p | \partial c_i = \text{identity}$;

– $(g_p)_{p\in P}$ is equicontinuous.

It is easily verified that ξ, $(h_p)_{p\in P}$ and $(g_p)_{p\in P}$ have the desired properties. □

Step 4, end of the proof of 3.4. — Using step 3, we can construct by induction on $i \in \mathbb{N}$, subdivisions ξ^i (in closed cubes) of I^n and continuous maps $P = \mathscr{H}^\partial(I^n, \text{bireg})$, $p \mapsto h_p^i$ and $p \mapsto g_p^i$, such that:

(*a*) ξ^{i+1} refines ξ^i;

(*b*) $\forall i \geqq 1$, $\forall p \in P$, $\forall c \in \xi^i$, $\operatorname{diam}(c) < 1/2^i$, $\operatorname{diam}(h_p^1 \ldots h_p^i(c)) < 1/2^i$, and $\operatorname{diam}(g_p^1 \ldots g_p^i(c)) < 1/2^i$;

(*c*) $\forall i \geqq 1$, $\forall p \in P$, $\forall c \in \xi^i$, $g_p^{i+1}(c) = h_p^{i+1}(c) = c$;

(*d*) $\forall i \geqq 1$, $\forall p \in P$, $\forall c \in \xi^i$, $\mu_p(h_p^1 \ldots h_p^i(c)) = \nu_p(g_p^1 \ldots g_p^i(c))$;

(*e*) $\forall i \geqq 1$, $(h_p^i)_{p\in P}$ and $(g_p^i)_{p\in P}$ are two equicontinuous family of homeomorphisms of I^n.

More precisely, ξ^{i+1}, $(h_p^{i+1})_{p\in P}$ and $(g_p^{i+1})_{p\in P}$ are constructed by application of step 3 inside each cube c of the subdivision ξ^i. Condition (*e*) at step i insures that, if the diameter of each cube in ξ^{i+1} is small enough, then condition (*b*) will be realized at step $(i+1)$.

Define now $H_p^i \in \mathscr{H}^\partial(I^n, \text{bireg})$ and $G_p^i \in \mathscr{H}^\partial(I^n, \text{bireg})$ by:

$$H_p^i = h_p^1 \ldots h_p^i \quad \text{and} \quad G_p^i = g_p^1 \ldots g_p^i.$$

By (*b*) and (*c*), we obtain easily:

$$d\,[\mathrm{H}_p^i,\,\mathrm{H}_p^{i+1}]<\frac{1}{2^i},\qquad d\,(\mathrm{G}_p^i,\,\mathrm{G}_p^{i+1})<\frac{1}{2^i},\qquad d\,[(\mathrm{H}_p^i)^{-1},\,(\mathrm{H}_p^{i+1})^{-1}]<\frac{1}{2^i}$$

and

$$d\,[(\mathrm{G}_p^i)^{-1},\,(\mathrm{G}_p^{i+1})^{-1}]<\frac{1}{2^i}.$$

These inequalities imply that $\mathrm{H}_p^\infty=\lim_{i\to\infty}\mathrm{H}_p^i$ and $\mathrm{G}_p^\infty=\lim_{i\to\infty}\mathrm{G}_p^i$ exist (for each $p\in\mathrm{P}$) and belong to $\mathscr{H}^\partial(\mathrm{I}^n)$.

Moreover, the two maps: $\mathrm{P}\to\mathscr{H}^\partial(\mathrm{I}^n)$, $p\mapsto\mathrm{H}_p^\infty$ and $p\mapsto\mathrm{G}_p^\infty$, are continuous, since they are uniform limits of continuous maps.

Using condition (*a*) and (*c*), we obtain:

$$\forall c\in\xi^i,\quad \mathrm{H}_p^\infty(c)=\mathrm{H}_p^i(c)\qquad\text{and}\qquad \mathrm{G}_p^\infty(c)=\mathrm{G}_p^i(c).$$

Hence using (*d*), we have:

$$\forall i\in\mathbb{N},\quad \forall p\in\mathrm{P},\quad \forall c\in\xi^i,\qquad \mu_p(\mathrm{H}_p^\infty(c))=\nu_p(\mathrm{G}_p^\infty(c)).$$

Using the last fact and the fact that $\lim_{i\to\infty}\operatorname{diam}\xi^i=0$, we obtain:

$$(\mathrm{H}_p^\infty)_*^{-1}\,\mu_p=(\mathrm{G}_p^\infty)_*^{-1}\,\nu_p.$$

Define then f_p by $f_p=\mathrm{G}_p^\infty(\mathrm{H}_p^\infty)^{-1}$. This gives us a continuous map: $\mathrm{P}\to\mathscr{H}^\partial(\mathrm{I}^n)$, $p\mapsto f_p$, such that $(f_p)_*\,\mu_p=\nu_p$.

This ends the proof of 3.4. □ □

Proof of Theorem 3.3. — By 2.3 and its complement, there is a surjective map $\varphi:\mathrm{I}^n\to\mathrm{M}^n$ such that $\varphi\,|\,\mathring{\mathrm{I}}^n$ is a homeomorphism onto its image, $\varphi(\mathring{\mathrm{I}}^n)\cap\varphi(\partial\mathrm{I}^n)=\emptyset$ and $\mu_0(\varphi(\partial\mathrm{I}^n))=0$. We can define a measure $\bar{\mu}_0$ on I^n in the following way:

If A is a Borel set in I^n, then $\bar{\mu}_0(\mathrm{A})=\mu_0(\varphi(\mathrm{A}))$.

We have clearly $\bar{\mu}_0\in\mathscr{M}_g^\partial(\mathrm{I}^n)$, hence by theorem 3.1, there exists a homeomorphism g of I^n such that $g_*\,m=\bar{\mu}_0$.

If we put $\psi=\varphi g$, we obtain a surjective map $\mathrm{I}^n\to\mathrm{M}^n$, such that:

– $\psi\,|\,\mathring{\mathrm{I}}^n$ is a homeomorphism onto its image;

– $\mu_0(\psi(\partial\mathrm{I}^n))=0$;

– $\psi_*\,m=\mu_0$.

We define a map $\bar{\psi}:\mathscr{M}_g^\partial(\mathrm{M}^n,\,\mu_0)\to\mathscr{M}_g^\partial(\mathrm{I}^n,\,m)$ by:

If $\mu\in\mathscr{M}_g^\partial(\mathrm{M}^n,\,\mu_0)$ and A is a Borel set in I^n, then $\bar{\psi}(\mu)(\mathrm{A})=\mu(\psi(\mathrm{A}))$.

The reader will check, using the properties of ψ, that $\bar{\psi}$ is well defined and continuous (in fact, it is a homeomorphism but we do not need that here). Moreover, we have $\bar{\psi}(\mu_0)=m$.

We can also define a continuous map $\tilde{\psi}: \mathscr{H}^{\partial}(\mathrm{I}^n) \to \mathscr{H}(\mathrm{M}^n)$ by:
If $h \in \mathscr{H}^{\partial}(\mathrm{I}^n)$,

$$\tilde{\psi}(h) = \begin{cases} \psi h \psi^{-1} & \text{on } \psi(\mathring{\mathrm{I}}^n), \\ \mathrm{id} & \text{on } \psi(\partial \mathrm{I}^n). \end{cases}$$

By 3.4, we can find a continuous section $\theta: \mathscr{M}_g^{\partial}(\mathrm{I}^n, m) \to \mathscr{H}^{\partial}(\mathrm{I}^n)$ of $\mathscr{H}^{\partial}(\mathrm{I}^n) \to \mathscr{M}_g^{\partial}(\mathrm{I}^n)$, $h \mapsto h_* m$.

It is easy to check that the composite map:

$$\mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0) \xrightarrow{\bar{\psi}} \mathscr{M}_g^{\partial}(\mathrm{I}^n, m) \xrightarrow{\theta} \mathscr{H}^{\partial}(\mathrm{I}^n) \xrightarrow{\tilde{\psi}} \mathscr{H}^{\partial}(\mathrm{M}^n)$$

is a (continuous) section of $\mathscr{H}^{\partial}(\mathrm{M}^n) \xrightarrow{\pi_0} \mathscr{M}_g^{\partial}(\mathrm{M}^n)$ above $\mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0)$. □

Define $\mathscr{H}(\mathrm{M}, \mu_0\text{-bireg})$ as the set of homeomorphisms $h \in \mathscr{H}(\mathrm{M}^n)$ such that h and h^{-1} are absolutely continuous with respect to μ_0. We have:

$$\mathscr{H}(\mathrm{M}^n, \mu_0\text{-bireg}) = \{ h \in \mathscr{H}(\mathrm{M}^n) \mid h_* \mu_0 \in \mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0) \}.$$

Define also $\mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0\text{-bireg})$ by:

$$\mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0\text{-bireg}) = \mathscr{H}(\mathrm{M}^n, \mu_0\text{-bireg}) \cap \mathscr{H}^{\partial}(\mathrm{M}^n).$$

As an immediat consequence of Theorem 3.3, we have:

COROLLARY 3.5. — *If* M^n *is compact and connected, then:*

(i) $\mathscr{H}(\mathrm{M}^n, \mu_0\text{-bireg}) \cong \mathscr{H}(\mathrm{M}^n, \mu_0) \times \mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0)$;

(ii) $\mathscr{H}(\mathrm{M}^n, \mu_0)$ *is a retract by deformation of* $\mathscr{H}(\mathrm{M}^n, \mu_0\text{-bireg})$, *in particular the inclusion* $\mathscr{H}(\mathrm{M}^n, \mu_0) \subsetneq \mathscr{H}(\mathrm{M}^n, \mu_0\text{-bireg})$ *is a homotopy equivalence;*

(iii) $\mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0\text{-bireg}) \cong \mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0) \times \mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0)$;

(iv) $\mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0)$ *is a retract by deformation of* $\mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0\text{-bireg})$; *in particular, the inclusion* $\mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0) \subsetneq \mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0\text{-bireg})$ *is a homotopy equivalence.*

Proof. — By 3.3, the surjective map: $\mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0\text{-bireg}) \xrightarrow{\pi_0} \mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0)$ has a continuous section σ. A homeomorphism of $\mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0\text{-bireg})$ on $\mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0) \times \mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0)$ can be defined by:

$$\begin{gathered} \mathscr{H}^{\partial}(\mathrm{M}, \mu_0\text{-bireg}) \to \mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0) \times \mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0), \\ h \mapsto ([\sigma(\pi_0(h))]^{-1} h, \pi_0(h)). \end{gathered}$$

This proves (iii). — Remark that under this homeomorphism, $\mathscr{H}^{\partial}(\mathrm{M}, \mu_0)$ goes to the subset $\mathscr{H}^{\partial}(\mathrm{M}, \mu_0) \times \{ \mu_0 \}$. Since $\mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0)$ is convex, $\{ \mu_0 \}$ is a retract by deformation of $\mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0)$. If we put the preceeding facts together, we obtain a proof of (iv).

We can prove (i) and (ii) in the same way since any section of π_0:

$$\mathscr{H}^{\partial}(\mathrm{M}^n, \mu_0\text{-bireg}) \to \mathscr{M}_g^{\partial}(\mathrm{M}^n, \mu_0)$$

is also a section of $\pi_0: \mathscr{H}(\mathrm{M}_n, \mu_0\text{-bireg}) \to \mathscr{M}_g^{\partial}(\mathrm{M}_n, \mu_0)$. □

We give now another application of the von Neumann-Oxtoby-Ulam Theorem.

PROPOSITION 3.6. — *Let* M^n *be a compact connected manifold and* $\mu_0 \in \mathcal{M}_g^\partial(M^n)$. *Suppose* $N^n \subset M^n$ *is a compact connected codimension* 0 *submanifold, such that* $\overline{M^n - N^n}$ *is a connected codimension* 0 *submanifold. If* $h : M^n \to M^n$ *is a homeomorphism such that* $h \mid N^n$ *is measure preserving* [*i.e. for each Borel subset* A *of* N^n, $\mu_0(h(A)) = \mu_0(A)$], *then there exists a measure preserving homeomorphism* $\overline{h} : M^n \to M^n$, *such that* $\overline{h} \mid N^n = h \mid N^n$.

Proof. — We define two measures ν_1 and ν_2 on the compact connected manifold $V^n = \overline{M^n - N^n}$.

If A is a Borel subset of V^n, then:

$$\nu_1(A) = \mu_0(A \cap \mathring{V}^n) \qquad \text{and} \qquad \nu_2(A) = \mu_0(h(A \cap \mathring{V}^n)).$$

Clearly, $\mu_i = (1/\nu_i(V^n))\,\nu_i \in \mathcal{M}_g^\partial(V^n)$, hence, by 3.1, there exists $f \in \mathcal{H}^\partial(V^n)$ such that $f_* \mu_1 = \mu_2$. We have:

$$\begin{aligned}\nu_1(V^n) = \mu_0(\mathring{V}^n) = \mu_0(M^n - N^n) &= \mu_0(M^n) - \mu_0(N^n)\\ &= \mu_0(h(M^n)) - \mu_0(h(N^n)) = \mu_0(h(M^n - N^n)) = \nu_2(V^n),\end{aligned}$$

i.e.

$$\nu_1(V^n) = \nu_2(V^n).$$

We conclude from this, that $f_* \nu_1 = \nu_2$. Since $f \mid \partial V^n = \text{identity}$, we can extend f by the identity on N^n to a homeomorphism $\overline{f}$ of M^n. It suffices to define $\overline{h}$ by $\overline{h} = h \circ \overline{f}$. □

The proof of 3.6 is typical of most of the proofs given in this work.

Before stating the next Proposition, we recall the generalized Schoenflies Theorem.

Let C^n be a parallelotope in $\mathbb{R}^n$ $\left(\text{i.e. } C^n = \prod_{i=1}^{n} [a_i, b_i],\ a_i \neq b_i\right)$ or an Euclidian ball of (arbitrary) finite radius (more generally C^n can be any locally flat ball in $\mathbb{R}^n$ such that the Lebesgue measure of ∂C^n is 0). We have $\partial C^n \cong S^{n-1}$. If j is an embedding, $j : \partial C^n \hookrightarrow \mathbb{R}^n$, then, by the generalized Jordan Theorem, $\mathbb{R}^n - j(\partial C^n)$ has two connected components one bounded and the other unbounded. We will note the bounded component by $B(j)$. In general, $B(j)$ is not a ball ([Ru], p. 47 and 69 or [CV], p. 417 and 461), but if j can be extended to an imbedding $\tilde{j} : U \hookrightarrow \mathbb{R}^n$, where U is a neighborhood of ∂C^n in $\mathbb{R}^n$, then $B(j)$ is homeomorphic to C^n and we can extend j to a homeomorphism $\overline{j}$ of C^n onto $B(j)$. This is the generalized Schoenflies Theorem; for a proof, *see* [Ru], p. 48 or [CV], p. 461. We now give the "measure preserving" version of this Theorem. Recall that m is the Lebesgue measure on $\mathbb{R}^n$.

PROPOSITION 3.7. — *Suppose* $j : \partial C^n \hookrightarrow \mathbb{R}^n$ *is an imbedding which can be extended to an imbedding of a neighborhood of* ∂C^n *in* $\mathbb{R}^n$. *Suppose, furthermore, that we have:* $m(j(\partial C^n)) = 0$ *and* $m(B(j)) = m(C^n)$. *Then,* j *can be extended to a* (*Lebesgue*) *measure preserving homeomorphism* $\overline{j} : C^n \xrightarrow{\sim} B(j)$.

Proof. — We can extend j to a homeomorphism $\tilde{j}: C^n \xrightarrow{\simeq} B(j)$. We have two measures on C^n, $\nu_1 = (m \mid C^n)$ and $\nu_2 = (\tilde{j}^{-1})_* (m \mid B(j))$. It is easily verified that we can apply to ν_1 and ν_2 the von Neumann-Oxtoby-Ulam Theorem. We obtain $h \in \mathscr{H}^\partial(C^n)$ such that $h_* \nu_1 = \nu_2$. If we define $\bar{j}$ by $\bar{j} = \tilde{j} \circ h$, we obtain the desired homeomorphism. □

PROPOSITION 3.8. — *If* $\mu \in \mathscr{M}_g^\partial(I^n)$, *then* $\mathscr{H}^\partial(I^n, \mu)$ *is contractible. In particular, if* M^n *is a compact manifold,* B *is a subset of* M^n *homeomorphic to* I^n *and* $\mu \in \mathscr{M}_g^\partial(M^n)$, *then each* μ*-preserving homeomorphism of* M^n, *whose support is contained in* $\mathring{B}$, *is isotopic to the identity via a* μ*-preserving isotopy, whose support is contained in* $\mathring{B}$.

An isotopy $(h_t)_{t \in [0, 1]}$ is μ-preserving, if each h_t is μ-preserving. The support of h is $\operatorname{supp}(h) = \overline{\{ x \in X \mid x \neq h(x) \}}$, and the support of $(h_t)_{t \in [0, 1]}$ is

$$\operatorname{supp}(h_t) = \overline{\{ x \in X \mid \exists\, t \in [0, 1]\, h_t(x) \neq x \}}.$$

Proof of 3.8. — The first part is an immediate consequence of 3.1 and 2.4. The second part is a trivial consequence of the first. □

4. Local contractibility of $\mathscr{H}(M^n, \mu)$ and related results

We will have to use the Černavskii-Edwards-Kirby results ([EK] or [R]). In fact, Černavskii and Edwards-Kirby worked with topological imbeddings, but we will need their results for biregular topological imbeddings, i. e. topological imbeddings preserving sets of measure 0. This is not a restriction because the Edwards-Kirby method works in this case also; the author learned this fact from M. Rogalski.

Before giving the statements of the Theorems that we will need, we give some definitions and some notations.

In the following, we fix M^n a compact manifold and $\mu_0 \in \mathscr{M}_g^\partial(M^n)$.

If A is a subset of M^n, by an imbedding k of A in M^n, we mean an injective (continuous) map $k: A \hookrightarrow M^n$, such that k is a homeomorphism of A onto $k(A)$ and $k^{-1}(\partial M^n)$ is $A \cap \partial M^n$. Remark that according to the general terminology a map $k: A \to M^n$ is called an imbedding if k is a homeomorphism of A onto $k(A)$, and an imbedding is said to be clean (or proper) if we have $k^{-1}(\partial M^n) = A \cap \partial M^n$. As we will always work with clean imbeddings, the word imbedding means in fact clean imbedding. The space of imbeddings of A into M^n will be denoted by $\mathscr{I}(A; M^n)$. If $k: A \to M^n$ is an imbedding and A is a Borel subset of M^n, we can define a measure $k^* \mu_0$ on A by $k^* \mu_0(B) = \mu_0(k(B))$ for each Borel subset $B \subset A$. We will say that an imbedding $k: A \to M^n$ is biregular (with respect to μ_0), if $k^* \mu_0$ and $\mu_0 \mid A$ have the same sets of measure 0. We will denote the set of biregular imbeddings of A in M^n by $\mathscr{I}(A; M^n, \mu_0\text{-bireg})$. We will say that an imbedding $k: A \to M^n$ preserves the measure μ_0 if the measures $k^* \mu_0$ and $\mu_0 \mid A$ are equal. We will denote the set of measure preserving imbeddings by $\mathscr{I}(A; M^n, \mu_0)$.

Suppose B is a subset of M^n, we define $\mathscr{I}(A, B; M^n)$ by:

$$\mathscr{I}(A, B; M^n) = \{ k \in \mathscr{I}(A; M^n) \mid k \mid B \cap A = \text{identity} \}.$$

Similarly, we define $\mathscr{I}(A, B; M^n, \mu_0)$ and $\mathscr{I}(A, B; M^n, \mu_0\text{-bireg})$.

All spaces of imbeddings will be endowed with the compact open topology.

We can now state the Černavskii-Edwards-Kirby Theorem for biregular imbeddings.

THEOREM 4.1. — *Let* U *be an open subset of* M^n, $C \subset U$ *a compact set, and* D *and* D′ *two closed subsets of* M^n *such that* $D \subset \operatorname{Int}(D') \subset D'$. *There exists a neighborhood* $\mathscr{U}$ *of the inclusion* $i : U \subset M^n$ *in* $\mathscr{I}(U, D'; M^n, \mu_0\text{-bireg})$ *and a continuous map*

$$\varphi: \quad \mathscr{U} \times [0, 1] \to \mathscr{I}(U; M^n, \mu_0\text{-bireg})$$

such that:

(1) $\varphi_0(k) = k$, $\forall k \in \mathscr{U}$;

(2) $\varphi_1(k) \,|\, C = \textit{identity}$, $\forall k \in \mathscr{U}$;

(3) $\varphi_t(k) \,|\, D \cap U = \textit{identity}$, $\forall k \in \mathscr{U}$, $\forall t \in [0, 1]$, *i.e.*

$$\varphi(\mathscr{U} \times [0, 1]) \subset \mathscr{I}(U, D; M^n, \mu_0\text{-bireg});$$

(4) $\varphi_t(k) = k$ *outside some compact neighborhood* K *of* C *in* U, *independent of* t *and* k; *i.e. there exists a compact set* K *such that:* $C \subset \operatorname{Int}(K) \subset K \subset U$ *and* $\varphi_t(k) \,|\, U - K = k \,|\, U - K$, $\forall k \in \mathscr{U}$, $\forall t \in [0, 1]$;

(5) *moreover, if* $k \,|\, U \cap \partial M^n = \textit{identity}$, *then* $\varphi_t(k) \,|\, U \cap \partial M^n = \textit{identity}$;

(6) $\varphi_t(i) = i$, $\forall t \in [0, 1]$.

The proof of Theorem is the same as the one given in [EK] for Theorem 5.1. See appendix B.4 for some indications.

Remark that if φ_t is given by Theorem 4.1, then $k\varphi_1(k)^{-1} : \varphi_1(k)(U) \to M^n$ is equal to k on C and is the identity outside $\varphi_1(k)(K)$, hence we can extend it by the identity to a homeomorphism of M^n. This gives:

COROLLARY 4.2. — *With the same notations as in* 4.1, *if* $k \in \mathscr{I}(U, D'; M^n, \mu_0\text{-bireg})$ *is close to* i, *then we can extend* $k \,|\, C$ *to* $\bar{k} \in \mathscr{H}(M^n, \mu_0\text{-bireg})$. *Moreover, we have:*

– $\bar{k}$ *depends continuously on* k;

– $\bar{i} = \textit{identity}$;

– $\bar{k} \,|\, D \cap U = \textit{identity}$;

– if $k \,|\, U \cap \partial M^n$ *is the identity, then* $\bar{k} \,|\, \partial M^n$ *is the identity;*

– *moreover, if* U′ *is some given neighborhood of* C, *we can suppose that* $\bar{k}$ *is the identity outside* U′.

If we apply 4.1 in the case $C = U = M^n$, $D = D' = \emptyset$, we obtain:

COROLLARY 4.3. — *The group* $\mathscr{H}(M^n, \mu_0\text{-bireg})$ *and* $\mathscr{H}^{\partial}(M^n, \mu_0\text{-bireg})$ *are locally contractible.*

Since, by 3.5, $\mathscr{H}(M^n, \mu_0)$ and $\mathscr{H}^{\partial}(M^n, \mu_0)$ are retracts of $\mathscr{H}(M^n, \mu_0\text{-bireg})$ and $\mathscr{H}^{\partial}(M^n, \mu_0\text{-bireg})$, we obtain:

THEOREM 4.4. — *If* M^n *is a compact manifold and* $\mu_0 \in \mathscr{M}_g^{\partial}(M^n)$, *then* $\mathscr{H}(M^n, \mu_0)$ *and* $\mathscr{H}^{\partial}(M^n, \mu_0)$ *are locally contractible.*

In fact to apply 3.5, we must assume that M^n is connected. But, if M^n is not connected and $M_1^n, \ldots, M_l^n$ are its connected components, then the group $\mathscr{H}(M^n, \mu_0)$ is locally homeomorphic to the product $\prod_{i=1}^{l} \mathscr{H}(M_i^n, \mu_0)$. This fact legitimate the proof we have given of 4.4.

We now want to prove the following Theorem:

THEOREM 4.5. — *If* M^n *is a compact connected manifold and* $\mu_0 \in \mathscr{M}_g^\partial(M^n)$, *then the inclusions* $\mathscr{H}(M^n, \mu_0) \subsetneq \mathscr{H}(M^n)$ *and* $\mathscr{H}^\partial(M^n, \mu_0) \subsetneq \mathscr{H}^\partial(M^n)$ *are weak homotopy equivalences, i.e. induce isomorphisms on all homotopy groups.*

We will prove 4.5 in the case $\mathscr{H}(M^n, \mu_0) \subset \mathscr{H}(M^n)$. The proof of the other case is exactly the same.

We know that $\mathscr{H}(M^n, \mu_0) \subsetneq \mathscr{H}(M^n, \mu_0\text{-bireg})$ is a homotopy equivalence, so 4.5 follows easily from the following Lemma:

LEMMA 4.6. — *If* M^n *is compact connected and* $\mu_0 \in \mathscr{M}_g^\partial(M^n)$, *then the inclusion* $\mathscr{H}(M^n, \mu_0\text{-bireg}) \subsetneq \mathscr{H}(M^n)$ *is a weak homotopy equivalence.*

To prove 4.6, we will apply a Theorem of Eilenberg and Wilder ([EW] § 2, pp. 615-617) or ([To] § 2, pp. 98-102), which we recall now. We explain this theorem in terms suitable to our needs.

Suppose X is a metric space with metric d, $A \subset X$ is some subset. Suppose furthermore that, given $\varepsilon > 0$, there exists $\delta > 0$ such that each continuous map $\varphi : S^r \to A$ (S^r the r-sphere, r arbitrary) with $\operatorname{diam}(\varphi(S^r)) < \delta$ can be extended to a continuous map $\overline{\varphi} : B^{r+1} \to A$ with $\operatorname{diam}(\overline{\varphi}(B^{r+1})) < \varepsilon$. Then, the Eilenberg-Wilder Theorem says, in particular, the inclusion $A \subsetneq \overline{A}$ is a weak homotopy equivalence.

Suppose now that G is a metrizable group with metric d; we can suppose that d is right (or left) invariant. Suppose that H is a subgroup of G which is locally contractible. Then, given $\varepsilon > 0$, we can find a $\delta > 0$ such that $N_\delta(e, H)$, the δ-neighborhood in H [i.e. $N_\delta(e, H) = \{ h \in H \mid d(h, e) < \delta \}$], can be contracted to a point in $N_\varepsilon(e, H)$. Now, using the invariance of the metric d, it is easy to show that each continuous map $\varphi : S^r \to H$ with $\operatorname{diam}(\varphi(S^r)) < \delta$ can be extended to a continuous map $\overline{\varphi} : B^{k+1} \to H$ with $\operatorname{diam}(\overline{\varphi}(B^{r+1})) < \varepsilon$. Hence, the inclusion $H \subsetneq \overline{H}$ is a weak homotopy equivalence. In particular, if H is dense in G, then $H \subsetneq G$ is a weak homotopy equivalence.

Applying what we said above and 4.3, we have reduced Lemma 4.6 to the following:

LEMMA 4.7. — *The subgroup* $\mathscr{H}(M^n, \mu_0\text{-bireg})$ *is dense in* $\mathscr{H}(M^n)$.

The proof of Lemma 4.7 follows from another Lemma, which we prove first.

LEMMA 4.8. — *Given any measure* $\mu \in \mathscr{M}_g^\partial(M^n)$, *then there exists a homeomorphism* $h \in \mathscr{H}^\partial(M^n)$, *as close to the identity as we want, such that* $h_* \mu \in \mathscr{M}_g^\partial(M^n, \mu_0)$.

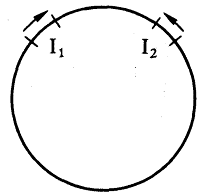
Proof of 4.8. — We prove it in the case $M^n = I^n$ and $\mu_0 = m =$ Lebesgue measure. The general case follows from this one, using a method similar to that given in the proof of 3.3.

Let $\mu \in \mathcal{M}_g^\partial(\mathrm{I}^n)$. We can find a subdivision of I^n by hyperplanes into subcubes $c_1, \ldots, c_k$ such that $\mu(\partial c_i)=0$ and $\operatorname{diam}(c_i)<\varepsilon$. Using the von Neumann-Oxtoby-Ulam Theorem (3.1), in each cube c_i, we can find a homeomorphism $h \in \mathcal{H}^\partial(\mathrm{I}^n)$ such that $h \big| \bigcup_{i=1}^k \partial c_i = \text{identity}$, $h(c_i)=c_i$ and $h_* \mu \,|\, c_i = (\mu(c_i)/m(c_i))\, m$.

The homeomorphism h is ε-close to the identity and $h_* \mu \in \mathcal{M}_g^\partial(\mathrm{M}^n, \mu_0)$. □

Proof of 4.7. — Let g be in $\mathcal{H}(\mathrm{M}^n)$, consider the measure $\mu = g_* \mu_0$. Applying 4.8, we obtain $h \in \mathcal{H}^\partial(\mathrm{M}^n)$, as close to the identity as we want, such that $h_* \mu \in \mathcal{M}_g^\partial(\mathrm{M}^n, \mu_0)$. The homeomorphism hg is a as close to g as we want, and $(hg)_* \mu_0 \in \mathcal{M}_g^\partial(\mathrm{M}^n, \mu_0)$, which means $hg \in \mathcal{H}(\mathrm{M}^n, \mu_0\text{-bireg})$. □

The analogue of Theorem 4.1 is false for measure preserving imbeddings. We give an example which contradicts its Corollary 4.2. Take two small intervals in S^1, I_1 and I_2, and push them one towards the other using rotations. This small push cannot be extended to a measure preserving homeomorphism of S^1.



We give analogs of 4.2 for the measure preserving case when C is a locally flat codimension 0 submanifold.

Recall that $\mathrm{N}^n \subset \mathrm{M}^n$ is a locally flat codimension 0 submanifold if, for each x in N^n, we can find a chart $\mathrm{U} \to \mathrm{M}^n$, with $0 \in \mathrm{U} \subset \mathbb{R}^n$ and $\varphi(0)=x$, such that:

— if $x \in \mathring{\mathrm{M}}^n$, then $\mathrm{U}=\mathbb{R}^n$, and either $\varphi(\mathbb{R}^n) \subset \mathrm{N}^n$ or $\varphi(\mathbb{H}^n_+) = \mathrm{N}^n \cap \varphi(\mathbb{R}^n)$, where $\mathbb{H}^n_+ = \{x=(x_1, \ldots, x_n) \mid x_n \geqq 0\}$;

— if $x \in \partial \mathrm{M}^n$, then $\mathrm{U}=\mathbb{H}^n_+$ and either $\varphi(\mathbb{H}^n_+) \subset \mathrm{N}^n$ or $\varphi(\mathbb{H}^n_{++}) = \mathrm{N}^n \cap \varphi(\mathbb{H}^n_+)$, where $\mathbb{H}^n_{++} = \{x=(x_1, \ldots, x_n) \mid x_{n-1} \geqq 0 \text{ and } x_n \geqq 0\}$.

PROPOSITION 4.9. — *Let* N^n *be a compact locally flat submanifold of the compact connected manifold* M^n, *such that* $\mathrm{M}^n - \mathrm{N}^n$ *is connected. Let* $\mu_0 \in \mathcal{M}_g^\partial(\mathrm{M}^n)$, *and* U *be an open neighborhood of* N^n *in* M^n.

For each $k \in \mathcal{I}(\mathrm{U}; \mathrm{M}^n, \mu_0)$ *which is close enough to the inclusion* $i : \mathrm{U} \subset \mathrm{M}^n$, *we can find a homeomorphism* $\bar{k} \in \mathcal{H}(\mathrm{M}^n, \mu_0)$ *such that:*

— $k \,|\, \mathrm{N}^n = \bar{k} \,|\, \mathrm{N}^n$;

— $\bar{k}$ *depends continuously on* k;

— $\bar{i} = $ *identity;*

— *if* $k \,|\, \mathrm{U} \cap \partial \mathrm{M}^n = $ *identity, then* $\bar{k} \,|\, \partial \mathrm{M}^n = $ *identity;*

— *furthermore, if* P *and* Q *are closed subsets of* $\partial \mathrm{M}^n$ *such that* Q *is a neighborhood of* P *in* $\partial \mathrm{M}^n$, *we can insist that* $\bar{k} \,|\, \mathrm{P}$ *be the identity if* $k \,|\, \mathrm{Q}$ *is the identity.*

Proof. — Since N^n is locally flat, $V^n=\overline{M^n-N^n}$ is a locally flat submanifold of M^n, it is also connected being the closure of the connected set M^n-N^n. By 4.2, we can, for each $k\in\mathscr{I}(U; M^n, \mu_0)$ close enough to the identity, find a homeomorphism k_1 of M^n, such that:

- $k_1\in\mathscr{H}(M^n, \mu_0\text{-bireg})$;
- $k_1|N^n=k|N^n$;
- k_1 depends continuously on k;
- $i_1=\text{identity}$;
- if $k|U\cap\partial M^n=\text{identity}$, then $k_1|\partial M^N=\text{identity}$;
- $k_1|P$ is the identity, if $k_1|Q$ is the identity.

Now remark that, since $\mu_0(k(N^n))=\mu_0(N^n)$, a stupid substraction argument shows that $v=\mu_0(k_1(\mathring{V}^n))$ is independent of k_1.

Define, for each k_1, a measure $\mu(k_1)$ on V^n by:

$$\mu(k_1)(A)=\frac{1}{v}\mu_0(k_1(A\cap\mathring{V}^n)),$$

if $A\subset V^n$ is a Borel subset.

It is easily verified that $\mu(k_1)\in\mathscr{M}_g^\partial(V^n, \bar{\mu}_0)$, where $\bar{\mu}_0=(1/v)(\mu_0|V^n)=\mu(i_1)$. Moreover, $\mu(k_1)$ depends continuously on k_1, and hence on k.

By Theorem 3.3, for each $\mu(k_1)$, we can find a homeomorphism $k_2\in\mathscr{H}^\partial(V^n)$ such that $k_{2*}\bar{\mu}_0=\mu(k_1)$, moreover k_2 depends continuously on $\mu(k_1)$, and hence on k. Since $k_2|\partial V^n=\text{identity}$, we can extend, by the identity, each k_2 to a homeomorphism of M^n, which we still call k_2.

One can verify that $\bar{k}=k_1 k_2$ is the desired homeomorphism of M^n. □

Let V^{n-1} be a compact connected manifold, and consider the manifold $V^{n-1}\times[0, 1]$. Consider four arbitrary but fixed numbers: $0<c<a<b<d<1$. If $k:V^{n-1}\times]c, d[\subsetneq V^{n-1}\times]0, 1[$ is an (open) imbedding, then $V^{n-1}\times[0, 1]-k(V^{n-1}\times[a, b])$ has two connected components; we call $\mathscr{C}_j(k)$ the connected component containing $V^{n-1}\times\{j\}$, $j=0$ or 1. With these notations, we can now state:

PROPOSITION 4.10. — *Let* $\mu_0\in\mathscr{M}_g^\partial(V^{n-1}\times[0, 1])$. *For each*

$$k\in\mathscr{I}(V^{n-1}\times]c, d[; V^{n-1}\times]0, 1[, \mu_0)$$

close enough to the inclusion $i:V^{n-1}\times]c, d[\subsetneq V^{n-1}\times]0, 1[$ *and verifying* $\mu_0(\mathscr{C}_0(k))=\mu_0(V^{n-1}\times[0, a[)$ *[or equivalently* $\mu_0(\mathscr{C}_1(k))=\mu_0(V^{n-1}\times]b, 1])$*], we can find a homeomorphism* $\bar{k}$ *of* $V^{n-1}\times[0, 1]$ *such that:*

- $\bar{k}$ *preserves* μ_0, *i.e.* $\bar{k}\in\mathscr{H}(V^{n-1}\times[0, 1], \mu_0)$;
- $\bar{k}|V^{n-1}\times[a, b]=k|V^{n-1}\times[a, b]$;
- $\bar{k}$ *depends continuously on* k;
- $\bar{k}|V^{n-1}\times\{0\}\cup V^{n-1}\times\{1\}=\textit{identity}$;
- *if* $k|\partial V^{n-1}\times]c, d[$ *is the identity, so is* $\bar{k}|\partial V^{n-1}\times[0, 1]$.

The proof of 4.10 is almost the same as the proof of 4.9. We leave it to the reader. Remark that since k preserves the measure, we have $\mu_0(\mathscr{C}_0(k))=\mu_0(V^{n-1}\times[0, a[)$ if and only if $\mu_0(\mathscr{C}_1(k))=\mu_0(V^{n-1}\times]b, 1])$.

In fact, 4.9 and 4.10 are two particular cases of a more general Theorem, which we will now explain. We will use only the particular cases given in 4.9 and 4.10.

Suppose N^n is a compact locally flat codimension 0 submanifold of the compact connected manifold M^n, then M^n-N^n has a finite number of connected components $C_1, \ldots, C_q$. Moreover, each closure $\overline{C}_h$ is a compact connected locally flat codimension 0 submanifold of M^n. Suppose U is an open neighborhood of N^n in M^n and $\mu_0\in\mathscr{M}^\partial_g(M^n)$. Let k be an imbedding $U\subset M^n$ preserving μ_0, then by 4.2, if k is close enough to the inclusion $U\subset M^n$, we can find a homeomorphism $\overline{k}\in\mathscr{H}_{\text{bireg}}(M^n, \mu_0)$ extending $k|N^n$. We claim that the set $\overline{k}(C_h)$ is independent of the particular extension of $k|N^n$ to M^n, this follows easily from the Lemma:

LEMMA 4.11. — *If $f: M^n\to M^n$ is a homeomorphism of M^n such that $f|N^n=$ identity then $f(C_h)=C_h$, $h=1, \ldots, q$.*

Proof. — Let $p\in\overline{C}_h-C_h$ (such a point exists because we assumed M^n connected). Then, of course, $p\in\mathrm{Fr}(N^n)$. Since we assumed N^n locally flat, we can find a neighborhood V of p in M^n such that $V\underset{\varphi}{\xrightarrow{\sim}} W\times]-1, 1[$ with W some connected open subset of $\mathbb{H}^{n-1}_+$, such that $V\cap N^n=\varphi^{-1}(W\times]-1, 0])$. Since $W\times]0, 1[$ is connected, the set $\varphi(W\times]0, 1[)$ is contained in a unique component of M^n-N^n, which must be of course C_h. This shows that $V\cap C_l=\emptyset$ if $l\neq h$. And as a consequence, C_h is the only component of M^n-N^n such that p belongs to its frontier. Now, since $f|N^n=\mathrm{id}$, the set $f(C_h)$ is a connected component of M^n-N^n, such that $p=f(p)$ belongs to its frontier. Hence $C_h=f(C_h)$. □

We can now state the generalization of 4.9 and 4.10.

THEOREM 4.12. — *Keeping the notations given above, let $k\in\mathscr{I}(U; M^n, \mu_0)$ be an imbedding close enough to the inclusion. Suppose moreover that if $\overline{k}\in\mathscr{H}(M^n)$ is an extension of $k|N^n$, then $\mu_0(\overline{k}(C_h))=\mu_0(C_h)$ for each connected component $C_1, \ldots, C_q$ of M^n-N^n.*

There exists a homeomorphism $\tilde{k}$ of M^n such that:

- *$\tilde{k}$ preserves μ_0; i.e. $\tilde{k}\in\mathscr{H}(M^n, \mu_0)$;*
- *$\tilde{k}|N^n=k|N^n$;*
- *$\tilde{k}$ depends continuously on k;*
- *if $k|\partial M^n\cap U$ is the identity, so is $\tilde{k}|\partial M^n$.*

The proof of 4.12 is the same as the one given for 4.9.

We conclude the section with the extension of isotopies.

THEOREM 4.13. — *Let N^n be a locally flat compact codimension 0 submanifold of the compact manifold M^n. Let U be an open neighborhood of N^n and $\mu_0\in\mathscr{M}^\partial_g(M^n)$. Suppose $(k_t)_{t\in[0,1]}$ is an isotopy of open imbeddings: $U\hookrightarrow M^n$ such that k_0 is the inclusion $i: U\hookrightarrow M$.*

A necessary and sufficient condition for the extension of $(k_t \mid \mathrm{N}^n)_{t\in[0,1]}$ *to* M^n *preserving* μ_0 *is the following condition:*

($\star$) *If* $(\tilde{k}_t)_{t\in[0,1]}$ *is an extension of* $(k_t \mid \mathrm{N}^n)_{t\in[0,1]}$ *to an isotopy of* M^n, *then, for each connected component* C *of* $\mathrm{M}^n-\mathrm{N}^n$, *we have* $\mu_0(\tilde{k}_t(\mathrm{C}))=\mu_0(\mathrm{C})$, $t\in[0, 1]$.

Proof. — Remark that if ($\star$) is satisfied by an extension of $k_t \mid \mathrm{N}^n$, it is satisfied by all. This proves the necessity of ($\star$).

The proof of the sufficiency of ($\star$) is the same (up to minor change) as the proof of 4.9, once we know the existence of an extension $(\tilde{k}_t)_{t\in[0,1]}$ of $(k_t \mid \mathrm{N}^n)_{t\in[0,1]}$ with each $\tilde{k}_t\in\mathscr{H}(\mathrm{M}^n, \mu_0\text{-bireg})$. But the existence of such an extension follows from 4.1 (*see* [EK], p. 79, proof of Corollary 1.2). □

APPENDIX A.4

Is $\mathscr{H}(\mathrm{M}^n, \mu)$ an ANR

We recall that a metric space X is an ANR (=absolute neighborhood retract) if it can be embedded as a closed retract of some open set in a normed space, *see* [BP], chapt. 2, for definition and some properties of ANR's. The ANR's are of interest because they form a fairly large category of spaces which are "well-behaved".

If M^n is a compact connected manifold with $n\geqq 3$, it is an open question whether $\mathscr{H}(\mathrm{M}^n)$ is an ANR or not. It is known in dimension two [LM] that $\mathscr{H}(\mathrm{M}^n)$ is an ANR. One can ask the same question for $\mathscr{H}(\mathrm{M}^n, \mu)$ where μ is a good measure on M^n. This second question can be trivially reduced to the first one, as we will see now.

Proposition A.4.1. — *Let* M^n *be a compact manifold and let* μ *be a good measure on* M^n *with* $\mu(\partial \mathrm{M}^n)=0$. *If* $\mathscr{H}(\mathrm{M}^n)$ *is an ANR, then* $\mathscr{H}(\mathrm{M}^n, \mu\text{-bireg})$ *and* $\mathscr{H}(\mathrm{M}^n, \mu)$ *are ANR's.*

Proof. — We will use the results and terminology of [To].

By 4.3 and 4.7, $\mathscr{H}_{\text{bireg}}(\mathrm{M}^n, \mu)$ is a locally contractible dense subgroup of $\mathscr{H}(\mathrm{M}^n)$; this implies [To], remark 2.9, that $\mathscr{H}(\mathrm{M}^n)-\mathscr{H}(\mathrm{M}^n, \mu\text{-bireg})$ is locally homotopically negligible in $\mathscr{H}(\mathrm{M}^n)$. Hence, by [To], Th. 3.1, if $\mathscr{H}(\mathrm{M}^n)$ is an ANR so is $\mathscr{H}(\mathrm{M}^n, \mu\text{-bireg})$. On the other hand $\mathscr{H}(\mathrm{M}^n, \mu)$ is a retract of $\mathscr{H}(\mathrm{M}^n, \mu\text{-bireg})$, and a retract of an ANR is an ANR. □

Corollary A.4.2. — *If* M^2 *is a compact 2-dimensionnal manifold and* $\mu\in\mathscr{M}_g^{\partial}(\mathrm{M}^2)$, *then* $\mathscr{H}_{\text{bireg}}(\mathrm{M}^2, \mu)$ *and* $\mathscr{H}(\mathrm{M}^2, \mu)$ *are ANR's.*

APPENDIX B.4

Outline of the proof of the Černavskii-Edwards-Kirby Theorem in the biregular case

We outline, with few details, the proof of the Černavskii-Edwards-Kirby Theorem for biregular imbeddings.

First, Theorem 4.1 is proved in the case where M^n is an open set of $\mathbb{R}^n$, and the measure is Lebesgue measure. The main Lemma is the following handle Lemma.

HANDLE LEMMA B.4.1. — *Consider* $B^k \times \mathbb{R}^{n-k}$ *endowed with* m *the Lebesgue measure. There is a neighborhood* $\mathcal{U}$ *of the inclusion* $i : B^k \times 4\mathring{B}^{n-k} \subset B^k \times \mathbb{R}^{n-k}$ *in* $\mathscr{I}(B^k \times 4\mathring{B}^{n-k}, \partial B^k \times 4\mathring{B}^{n-k}; B^k \times \mathbb{R}^{n-k}, m\text{-bireg})$ *and a deformation* φ_t *of* $\mathcal{U}$ *inside* $\mathscr{I}(B^k \times 4\mathring{B}^{n-k}, \partial B^k \times 4\mathring{B}^{n-k}; B^k \times \mathbb{R}^{n-k}, m\text{-bireg})$ *such that:*

- $\varphi_1(k) \mid B^k \times B^{n-k} = \mathrm{Id}$, $\forall k \in \mathcal{U}$;
- $\varphi_t(i) = i$, $\forall t \in [0, 1]$;
- $\varphi_t(k) = k$ *outside* $B^k \times 3B^{n-k}$, $\forall k \in \mathcal{U}$, $\forall t \in [0, 1]$.

The proof of this Lemma is exactly the same as the proof given for Lemma 4.1 in [EK], once we replace the immersed punctured torus argument by the furling argument explained in [EK], § 8. We have only to remark that the furling is done via some composition with some PL (hence biregular) homeomorphisms, and that the final compression argument can be done using a PL compression.

Once, we have this Lemma, we can obtain Theorem 4.1 in the case where M^n is an open set of $\mathbb{R}^n$ by using a small triangulation like in pages 71-73 of [EK]. Now, if M^n is without boundary and $\mu_0 \in \mathscr{M}_g(M^n)$, a simple application of the von Neumann-Oxtoby-Ulam Theorem shows that there exists an atlas $\{(h_i, U_i) \mid U_i \text{ open in } \mathbb{R}^n \text{ (in fact a ball)}\}$ such that $h_i^* \mu_0$ is the Lebesgue measure (up to scaling). Using this, we can apply the usual chart by chart argument (*see* [EK]) to prove Theorem 4.1 in this case. Of course this also proves Theorem 4.1 in the case where $\partial M^n \neq \emptyset$ but $C \cap \partial M^n = \emptyset$.

It remains then to prove Theorem 4.1 in the case where $C \cap \partial M^n \neq \emptyset$. Here we have to change a little the argument used in [EK]. In [EK], they apply the Theorem in the empty boundary case, to the triple ∂M^n, $C \cap \partial M^n$, $U \cap \partial M^n$ and then they use a small collar to extend the deformation and after that it remains to rectify the imbeddings in the interior. We cannot do that here because during this kind of deformation, we are not sure to obtain biregular imbeddings, for example if $h : B^n \to B^n$ is even a measure preserving homeomorphism, $h \mid \partial B^n$ might be very bad!

We get rid of this minor point by the following handle Lemma.

LEMMA B.4.2. — *Consider* $[0, 1] \times B^k \times \mathbb{R}^{n-k-1}$ *endowed with the product of Lebesgue measures. There is a neighborhood* $\mathcal{U}$ *of the inclusion* i:

$$[0, 1] \times B^k \times 4\mathring{B}^{n-k-1} \subset [0, 1] \times B^k \times \mathbb{R}^{n-k-1}$$

in

$$\mathscr{I}([0, 1] \times B^k \times 4\mathring{B}^{n-k-1}, \{1\} \times B^k \times \mathbb{R}^{n-k-1} \cup [0, 1] \times \partial B^k \times \mathbb{R}^{n-k-1}; \quad [0, 1] \times B^k \times \mathbb{R}^{n-k-1}, m\text{-bireg})$$

and a deformation φ_t *of* $\mathcal{U}$ *inside*

$$\mathscr{I}([0, 1] \times B^k \times 4\mathring{B}^{n-k-1}, \{1\} \times B^k \times \mathbb{R}^{n-k-1} \cup [0, 1] \times \partial B^k \times \mathbb{R}^{n-k-1}; \quad [0, 1] \times B^k \times \mathbb{R}^{n-k-1}, m\text{-bireg})$$

such that:

– $\varphi_1(k)\,|\,[0, 1]\times B^k\times B^{n-k-1}=\mathrm{Id}$, $\forall\, k\in\mathscr{U}$;

– $\varphi_t(i)=i$, $\forall\, t\in[0, 1]$;

– $\varphi_t(k)=k$ *outside* $[0, 1]\times B^k\times 3\,B^{n-k-1}$, $\forall\, k\in\mathscr{U}$, $\forall\, t\in[0, 1]$.

This Lemma can be proved along the same lines as the one above.

Once we have this Lemma, we can prove, using the standard techniques of [EK], the following Proposition.

PROPOSITION B.4.3. – *Let* N^{n-1} *be a manifold without boundary and with a good measure* ν. *Let* K *be a compact set,* V *an open neighborhood of* K *in* N^{n-1} *and let* $F\subset F'$ *be two closed subsets of* N^{n-1}, *with* F′ *a neighborhood of* F. *We endow* $[0, 1]\times N^{n-1}$ *with the product measure* $m\times\nu$ *where* m *is a Lebesgue measure on* $[0, 1]$. *There exists a neighborhood* $\mathscr{U}$ *of the inclusion* $i:[0, 1]\times V\subset[0, 1]\times N^{n-1}$ *in*

$$\mathscr{I}([0, 1]\times V, [0, 1]\times F'\cup\{1\}\times V; [0, 1]\times N^{n-1}, m\text{-bireg})$$

and a deformation φ_t *of* $\mathscr{U}$ *in*

$$\mathscr{I}([0, 1]\times V, [0, 1]\times F\cup\{1\}\times V; [0, 1]\times N^{n-1}, m\text{-bireg})$$

such that:

– $\varphi_1(k)\,|\,[0, 1]\times K=\mathrm{Id}$, $\forall\, k\in\mathscr{U}$;

– $\varphi_t(i)=i$, $\forall\, t\in[0, 1]$;

– $\varphi_t(k)=k$ *outside some compact neighborhood of* $[0, 1]\times K$ *in* $[0, 1]\times V$, $\forall\, k\in\mathscr{U}$, $\forall\, t\in[0, 1]$.

Now, we can finish the proof in the following way. The boundary ∂M^n of M^n has a small neighborhood homeomorphic to $[0, 1]\times\partial M^n$. We can apply the without-boundary case to deform the imbeddings so they become the identity on a compact neighborhood of $C-[0, 1[\times\partial M^n$ and then finish to deform them to the identity by using Proposition B.4.3.

5. The mass flow homomorphism

Let X be a compact metric space. We recall that an isotopy of X is a continuous map $h:X\times[0, 1]\to X$ such that each map $h_t:X\to X$, $x\to h(x, t)$, is a homeomorphism and $h_0=\mathrm{id}_X$. We will use the notation $(h_t)_{t\in[0, 1]}$ or simply (h_t) for an isotopy. In fact, an isotopy is the same as a continuous path in $\mathscr{H}(X)$ starting at the identity. We will note by $\mathscr{I}\mathscr{S}(X)$ the space of isotopies of X. Of course, $\mathscr{I}\mathscr{S}(X)$ is endowed with the compact open topology.

We have, of course, a continuous surjective map:

$$\mathscr{I}\mathscr{S}(X)\xrightarrow{p}\mathscr{H}_0(X),\qquad (h_t)\mapsto h_1,$$

where $\mathscr{H}_0(X)$ is the path component of id_X in $\mathscr{H}(X)$.

We say that two isotopies (h_t) and (g_t) are isotopic with fixed extremities, which we note $(h_t)\sim(g_t)$, if $h_1=g_1$ and there exists a continuous map $\mathrm{I}\times\mathrm{I}\to\mathscr{H}(\mathrm{X})$, $t, s\mapsto \mathrm{H}_{t,s}$, such that $\mathrm{H}_{0,s}=\mathrm{id}_{\mathrm{X}}$, $\mathrm{H}_{1,s}=h_1=g_1$, $\mathrm{H}_{t,0}=h_t$ and $\mathrm{H}_{t,1}=g_t$. The relation $\sim$ is an equivalence relation on $\mathscr{IS}(\mathrm{X})$. We note the set of equivalence classes by $\widetilde{\mathscr{H}}_0(\mathrm{X})$. The equivalence class of (h_t) will be noted by $\{h_t\}$. The set $\widetilde{\mathscr{H}}_0(\mathrm{X})$ carries a topology, namely the quotient topology obtained from the compact open topology on $\mathscr{IS}(\mathrm{X})$.

The map $\mathscr{IS}(\mathrm{X})\xrightarrow{p}\mathscr{H}_0(\mathrm{X})$ induce a continuous map $\widetilde{\mathscr{H}}_0(\mathrm{X})\xrightarrow{p}\mathscr{H}_0(\mathrm{X})$. Remark also that $\mathscr{IS}(\mathrm{X})$ is a group, where the composition law is given by $(h_t)_{t\in[0,1]}\circ(g_t)_{t\in[0,1]}=(h_t g_t)_{t\in[0,1]}$. This group structure induces a group structure on $\widetilde{\mathscr{H}}_0(\mathrm{X})$. The two maps $\mathscr{IS}(\mathrm{X})\to\mathscr{H}_0(\mathrm{X})$, and $\widetilde{\mathscr{H}}_0(\mathrm{X})\to\mathscr{H}_0(\mathrm{X})$ are group homomorphisms.

In fact, if $\mathscr{H}(\mathrm{X})$ is locally contractible (or even 1-semi locally connected), then $\widetilde{\mathscr{H}}_0(\mathrm{X})$ is the universal covering space of $\mathscr{H}_0(\mathrm{X})$.

Now, if μ is a measure on X, we define:

$$\mathscr{IS}(\mathrm{X},\mu)=\{(h_t)\in\mathscr{IS}(\mathrm{X})\mid\forall t\in[0,1],\ h_t\in\mathscr{H}(\mathrm{X},\mu)\},$$
$$\mathscr{H}_0(\mathrm{X},\mu)=\{h\mid\exists(h_t)\in\mathscr{IS}(\mathrm{X},\mu),\ h_1=h\}.$$

We can also define the equivalence relation $\sim$ on $\mathscr{IS}(\mathrm{X},\mu)$ in the same way as above, and define $\widetilde{\mathscr{H}}_0(\mathrm{X},\mu)$ as the set of equivalence classes. If $\mathscr{H}(\mathrm{X},\mu)$ is locally contractible, then $\widetilde{\mathscr{H}}_0(\mathrm{X},\mu)$ is the universal cover of $\mathscr{H}_0(\mathrm{X},\mu)$.

We now consider the topological group $\mathbb{T}^1=\mathbb{R}/\mathbb{Z}$, which is isomorphic to the topological group $\mathrm{S}^1=\{z\in\mathbb{C}\mid |z|=1\}$.

We note by $\mathscr{C}(\mathrm{X},\mathbb{T}^1)$ the set of continuous maps from X to $\mathbb{T}^1$; it is an Abelian group under addition:

$$(f+g)(x)=f(x)+g(x),\qquad \forall x\in\mathrm{X}.$$

Recall that f, $g\in\mathscr{C}(\mathrm{X},\mathbb{T}^1)$ are called homotopic, if there exists a continuous map $\mathrm{X}\times[0,1]\to\mathbb{T}^1$, $(x,t)\mapsto f_t(x)$, such that $f_0=f$ and $f_1=g$.

Homotopy is an equivalence relation. We note by $[\mathrm{X},\mathbb{T}^1]$ the set of homotopy classes; it is an Abelian group, its addition is induced from the one defined above on $\mathscr{C}(\mathrm{X},\mathbb{T}^1)$.

If X is some good space, for example a manifold or a polyhedron, then $[\mathrm{X},\mathbb{T}^1]$ is isomorphic to $\mathrm{H}^1(\mathrm{X},\mathbb{Z})$ the first singular cohomology group of X. In fact, $\mathrm{H}^1(\mathbb{T}^1,\mathbb{Z})\simeq\mathbb{Z}$, and a preferred generator $\sigma\in\mathrm{H}^1(\mathbb{T}^1,\mathbb{Z})$ is given by the natural orientation of the circle. We can define a natural map:

$$\begin{aligned} a_{\mathrm{X}}:\ &[\mathrm{X},\mathbb{T}^1]\to\mathrm{H}^1(\mathrm{X},\mathbb{Z}),\\ &f\mapsto f^*\sigma.\end{aligned}$$

This map is a group homomorphism. If X is a polyhedron, obstruction theory shows that a_{X} is an isomorphism ([Sp], chap. 8). This result is then extended to ANR's, and in particular to manifolds, since an ANR has the homotopy type of a polyhedron ([BP], Cor. 6.6, p. 80 and [LW], Th. 3.8, p. 127).

Remark now that $H^1(X, \mathbb{Z}) = \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z})$. So, there is a natural map:

$$b_X: \quad H_1(X, \mathbb{Z}) \to \mathrm{Hom}(H^1(X, \mathbb{Z}), \mathbb{Z}).$$

Composing b_X with $a_X^*: \mathrm{Hom}(H^1(X, \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{Z})$ gives us a map:

$$\beta_X: \quad H_1(X, \mathbb{Z}) \to \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{Z}).$$

There is also the Hurewicz map ([Sp], p. 387]:

$$h_X: \quad \pi_1(X, x_0) \to H_1(X, \mathbb{Z}),$$

where $\pi_1(X, x_0)$ is the fundamental group of X with base point x_0. We will call α_X the composite $\beta_X h_X$:

$$\alpha_X: \quad \pi_1(X, x_0) \to \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{Z}).$$

We can describe α_X in the following terms: a loop l of X based at x_0 can be considered as a map $l: \mathbb{T}^1 \to X$ with $l(0) = x_0$. Now, if $f: X \to \mathbb{T}^1$ is a continuous map, then: $\alpha_X(l)(f) = \deg(f \circ l)$, where $f \circ l: \mathbb{T}^1 \xrightarrow{l} X \xrightarrow{f} \mathbb{T}^1$. We will use this later.

Suppose that X is a good space and that $H_1(X, \mathbb{Z})$ is a finitely generated group; we note here for future reference that the map β_X tensored by $\mathbb{R}$ gives us an isomorphism:

$$\begin{tikzcd} H_1(X, \mathbb{R}) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R}) \arrow[d, equal] \\ H_1(X, \mathbb{Z}) \otimes \mathbb{R} \arrow[r, "\beta_X \otimes \mathbb{R}"] & \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{Z}) \otimes \mathbb{R}. \end{tikzcd}$$

In particular, if M^n is a compact manifold, $H_1(M^n, \mathbb{R})$ is isomorphic to $\mathrm{Hom}([M^n, \mathbb{T}^1], \mathbb{R})$.

Definition of the mass flow homomorphism. — If μ is a measure on X, we will define a group homomorphism, noted $\tilde{\theta}_{X, \mu}$ or simply $\tilde{\theta}$, and called the mass flow homomorphism:

$$\tilde{\mathcal{H}}_0(X, \mu) \xrightarrow{\tilde{\theta}} \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R}).$$

Suppose first $(h_t) \in \mathcal{IS}(X, \mu_0)$; we define a map $\tilde{\theta}(h_t): \mathcal{C}(X, \mathbb{T}^1) \to \mathbb{R}$ in the following way:

If $f \in \mathcal{C}(X, \mathbb{T}^1)$, then $(fh_t - f): X \to \mathbb{T}^1$ is a homotopy such that $fh_0 - f = f - f = 0$; hence, *see* [CV], chap. 13, we can lift it, in a unique way, to a homotopy: $\overline{(fh_t - f)}: X \to \mathbb{R}$, with $\overline{fh_0 - f} = 0$:

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ X \arrow[ur, "\overline{(fh_t - f)}"] \arrow[r, "(fh_t - f)"'] & \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}. \end{tikzcd}$$

We define:

$$\tilde{\theta}(h_t)(f)=\int_X \overline{(fh_1-f)}\,d\mu.$$

We have $(f+g)h_t-(f+g)=(fh_t-f)+(gh_t-g)$; this equality imply the same one for the liftings, and we obtain:

$$\tilde{\theta}(h_t)(f+g)=\tilde{\theta}(h_t)(f)+\tilde{\theta}(h_t)(g).$$

So $\tilde{\theta}(h_t)\in \mathrm{Hom}(\mathscr{C}(X, \mathbb{T}^1), \mathbb{R})$. Suppose now that $f\in\mathscr{C}(X, \mathbb{T}^1)$ is homotopic to a constant map, then we can lift it to a map $\overline{f}: X\to\mathbb{R}$. It is easy to verify that $\overline{fh_t-f}=\overline{f}h_t-\overline{f}$. This gives us

$$\tilde{\theta}(h_t)(f)=\int_X (\overline{f}h_1-\overline{f})\,d\mu=\int_X \overline{f}h_1\,d\mu-\int_X \overline{f}\,d\mu=0,$$

since h_1 preserves the measure μ. This fact shows that $\tilde{\theta}(h_t)$ induces a homomorphism $[X, \mathbb{T}^1]\to\mathbb{R}$, which we still note $\tilde{\theta}(h_t)$.

Up to now we have defined a map:

$$\mathscr{I}\mathscr{S}(X, \mu)\xrightarrow{\tilde{\theta}}\mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R}).$$

We verify that this map is a group homomorphism. Let $(h_t)\in\mathscr{I}\mathscr{S}(X, \mu)$, $(g_t)\in\mathscr{I}\mathscr{S}(X, \mu)$ and $f\in\mathscr{C}(X, \mathbb{T}^1)$, we have:

$$fh_tg_t-f=(fh_t-f)g_t+(fg_t-f),$$

hence:

$$(\overline{fh_tg_t-f})=(\overline{fh_t-f})g_t+(\overline{fg_t-f}),$$

which in turn implies:

$$\tilde{\theta}(h_tg_t)(f)=\int_X (\overline{fh_1-f})g_1\,d\mu+\int_X \overline{fg_1-f}\,d\mu=\tilde{\theta}(h_t)(f)+\tilde{\theta}(g_t)(f),$$

since g_1 preserves μ.

Now we check that, in fact, $\tilde{\theta}$ is well defined on $\tilde{\mathscr{H}}_0(X, \mu)$. If $(h_t)\sim(g_t)$, let $H_{t,s}$ be a homotopy between them. We have now a two parameters family $fH_{t,s}-f: X\to\mathbb{T}^1$, lifting this two parameters family to $\overline{fH_{t,s}-f}: X\to\mathbb{T}^1$ shows easily that we have $\overline{fh_1-f}=\overline{fg_1-f}$, which implies $\tilde{\theta}(h_t)(f)=\tilde{\theta}(g_t)(f)$.

So we have shown the existence of the map $\tilde{\theta}:\tilde{\mathscr{H}}_0(X, \mu)\to\mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R})$.

The map $\tilde{\theta}$ is continuous in the following sense: if $f\in\mathscr{C}(X, \mathbb{T}^1)$, then the map: $\tilde{\mathscr{H}}_0(X, \mu)\to\mathbb{R}$, $\{h_t\}\mapsto\tilde{\theta}(h_t)(f)$, is continuous. In other words, if we endow $\mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R})$ with the weak topology, then $\tilde{\theta}$ is continuous.

If X is a compact manifold, then $\mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R}) \cong H_1(X, \mathbb{R})$, and each $\tilde{\theta}(h_t)$ can be interpreted as a cycle on X.

Example. — $X=\mathbb{T}^n$, μ = Haar measure. We identify $\mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R})$ with $\mathbb{R}^n$, by taking the n projections $p_i : \mathbb{T}^n \to \mathbb{T}^1$, $(x_1, \ldots, x_n) \mapsto x_i$, as a basis (over $\mathbb{Z}$) of $[X, \mathbb{T}^1]$. Now, if $\alpha \in \mathbb{R}^n$, we can define an isotopy $R_\alpha^t : \mathbb{T}^n \to \mathbb{T}^n$, $x \mapsto x + t\alpha$. A simple calculation shows that $\tilde{\theta}(R_\alpha^t) = \alpha$.

The kernel of the natural map $\tilde{\mathscr{H}}_0(X, \mu) \to \mathscr{H}_0(X, \mu)$ is the set

$$\mathscr{N}(X, \mu) = \{\{h_t\} \in \tilde{\mathscr{H}}_0(X, \mu) \mid h_1 = \mathrm{id}\};$$
$$\mathscr{N}(X, \mu) \subsetneq \tilde{\mathscr{H}}_0(X, \mu) \to \mathscr{H}_0(X, \mu).$$

We define $\Gamma \subset \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R})$ by:

$$\Gamma = \tilde{\theta}(\mathscr{N}(X, \mu)).$$

Passing to the quotient, $\tilde{\theta}$ gives us a homomorphism θ:

$$\theta : \mathscr{H}_0(X, \mu) \to \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R})/\Gamma:$$

$$\begin{tikzcd}
\mathscr{N}(X, \mu) \arrow[r, "\tilde{\theta}"] \arrow[d] & \Gamma = \tilde{\theta}(\mathscr{N}(X, \mu)) \arrow[d] \\
\tilde{\mathscr{H}}_0(X, \mu) \arrow[r, "\tilde{\theta}"] \arrow[d] & \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R}) \arrow[d] \\
\mathscr{H}_0(X, \mu) \arrow[r, "\theta"] & \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R})/\Gamma.
\end{tikzcd}$$

D. Sullivan brought to our attention the following fact:

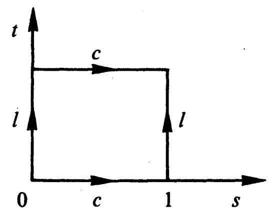
PROPOSITION 5.1. — *If* X *is connected and* $\mu(X)=1$, *the group* Γ *is contained in the image of the center of* $\pi_1(X, x)$ *under the map* $\alpha_X : \pi_1(X, x) \to \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{Z}) \subset \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R})$.

LEMMA 5.2. — *If* X *is connected and* $\mu(X)=1$ *and* $\{h_t\} \in \mathscr{N}(X, \mu)$, *then* $\tilde{\theta}(h_t)$ *is the image under* α_X *of the loop* $\{h_t(x) \mid t \in [0, 1]\}$.

Proof. — Let $f : X \to \mathbb{T}^1$, consider the lifting $\overline{fh_t - f} : X \to \mathbb{R}$. Since $h_1 = \mathrm{id}$ and X is connected, $\overline{fh_1 - f}$ is a constant map whose value is an integer n. So $\tilde{\theta}(h_t)(f) = \int_X n\, d\mu = n$. It is easy to show that n is the degree of the map $\mathbb{T}^1 \to \mathbb{T}^1$, $t \mapsto f(h_t(x))$. According to the description of the map α_X given before, we have proved the Lemma. $\square$

Proof of 5.1. — Let $\{h_t\} \in \mathscr{N}(X, \mu)$. Define the loop $l : [0, 1] \to X$ by: $t \mapsto h_t(x)$. We know that, by 5.2, $\alpha(l) = \tilde{\theta}(\{h_t\})$. So it suffices to show that l is in the center of $\pi_1(X)$.

Let $c : [0, 1] \to X$ be a loop based at x, $c(0)=c(1)=x$. Define a map $[0, 1] \times [0, 1] \xrightarrow{H} X$, $(s, t) \mapsto h_t(c(s))$. Looking at the restriction of H to the four sides of $[0, 1] \times [0, 1]$, we obtain the following picture:



So, $l^{-1} c^{-1} lc$ is homotopic to a constant map, and in $\pi_1(X, x)$, we have $lc=cl$. □

Remark as a consequence of 5.1, that if X is a compact connected manifold, then Γ is contained in the integral part of $H_1(X, \mathbb{R}) = \mathrm{Hom}([X, \mathbb{T}^1], \mathbb{R})$. This shows that Γ is discrete in the natural topology of $H_1(X, \mathbb{R})$ [this natural topology is given by the fact that $H_1(X, \mathbb{R})$ is a finite dimensional real vector space].

Example. — $X = \mathbb{T}^n$: μ = Haar measure. If we identify $H_1(\mathbb{T}^n, \mathbb{R})$ with $\mathbb{R}^n$ as before, then $\Gamma = \mathbb{Z}^n \subset \mathbb{R}^n$,

LEMMA 5.3. — *Let* M^n *be a manifold,* B *be a subset of* M^n *homeomorphic to the n-ball* B^n *and* $\mu \in \mathcal{M}^{\partial}_g(M^n)$. *Suppose* $h \in \mathcal{H}(M^n, \mu)$ *is isotopic to the identity by a* μ*-preserving isotopy having its support in* B, *then* $\theta(h)=0$. *If* $h \in \mathcal{H}(M^n, \mu)$ has is support in $\mathring{B}$, *then n is isotopic to* Id_{M^n} *and* $\theta(h)=0$.

Proof. — Let (h_t) be an isotopy such that $h_1 = h$. We will show that $\theta(h_t)=0$. Let f be in $\mathscr{C}(M^n, \mathbb{T}^1)$. Remark that $fh_t - f$ is identically 0 outside B, this implies that the lifting $\overline{fh_t - f} : M^n \to \mathbb{R}$ is also identically 0 outside B, hence $\tilde{\theta}(h_t)(f) = \int_B \overline{fh_t - f}\, du$. Now $f \mid B : B \to \mathbb{T}^1$ can be lifted to a map $\bar{f} : B \to \mathbb{R}$, since $B \cong B^n$ is contractible. It is easy to see that $\overline{fh_t - f} \mid B = \bar{f}h_t \mid B - \bar{f}$, hence we obtain $\tilde{\theta}(h_t)=0$. Since $h_t(B)=B$ and h_1 is μ-preserving.

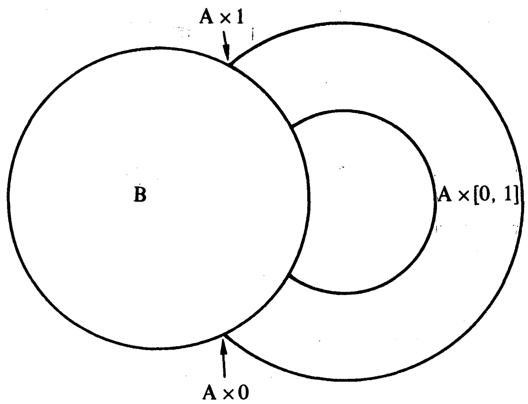
The remaining part of the Lemma is a consequence of 3.8. □

If $\mathcal{H}_0(X, \mu)$ is locally contractible, then the map θ is continuous. This follows immediately from the continuity of $\tilde{\theta}$ and the fact that in this case $\tilde{\mathcal{H}}_0(X, \mu) \to \mathcal{H}_0(X, \mu)$ is a covering map.

Remark. — The existence of the mass flow homomorphism is well known; *see* [Sc] for the case of a flow and [Th] for the case of volume preserving diffeomorphisms. The definition we gave is inspired by Michel Herman's definition of a rotation number for homeomorphisms of $\mathbb{T}^1$ ([He$_2$], chap. 2).

WHY θ IS CALLED THE MASS FLOW HOMOMORPHISM ? — Let A and B be compact spaces and $A \times \{0, 1\} \xrightarrow{\varphi} B$ be some imbedding. We define X by gluing $A \times [0, 1]$ to B using the map φ:

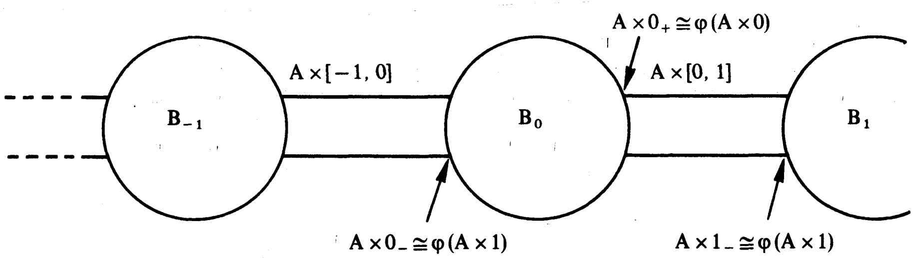
$$X = (A \times [0, 1] \coprod B)/\varphi(a, 0) \sim (a, 0), \quad \varphi(a, 1) \sim (a, 1),$$



We can define a natural map $f : X \to \mathbb{T}^1$ by:

$$f(x) = \begin{cases} t \bmod 1, & \text{if } x(a, t) \in A \times [0, 1], \\ 0, & \text{if } x \in B. \end{cases}$$

We obtain a covering $\overline{X} \xrightarrow{\pi} X$ as the pull-back by f of the covering $\mathbb{R} \to \mathbb{T}^1$. The space $\overline{X}$ can also be defined as in the figure below:



where each B_i is a copy of B.

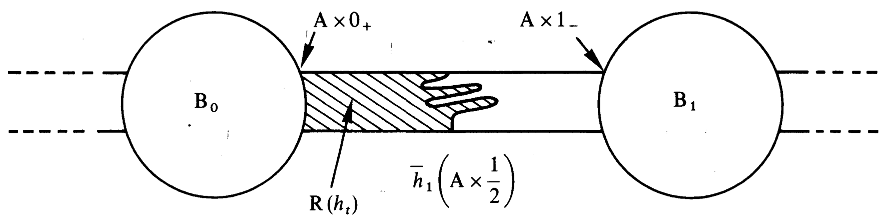
The map $\bar{f} : \overline{X} \to \mathbb{R}$ covering f may be defined by:

$$\bar{f}(\bar{x}) = \begin{cases} n, & \text{if } \bar{x} \in B_n, \quad \forall n \in \mathbb{Z}, \\ t, & \text{if } \bar{x} = (a, t) \in A \times [m, m+1], \quad \forall m \in \mathbb{Z}. \end{cases}$$

Moreover, the covering transformations of $\overline{X} \to X$ are generated by the map $\tau : \overline{X} \to \overline{X}$, defined by:

$$\tau(\bar{x}) = \begin{cases} \text{“}\bar{x}\text{”} \in B_{n+1}, & \text{if } \bar{x} \in B_n, \\ (a, t+1), & \text{if } \bar{x} = (a, t) \in A \times [m, m+1]. \end{cases}$$

Suppose now that μ is some measure on X, such that $\mu | A \times [0, 1] = \nu \times dt$, where ν is a measure on A and dt is Lebesgue measure on [0, 1]. We can "lift" μ to a measure $\bar{\mu}$ of $\bar{X}$ [of course $\bar{\mu}(\bar{X}) = +\infty$]. Let $(h_t) \in \mathscr{I}\mathscr{S}(X, \mu)$, we can lift (h_t) in a unique way to $(\bar{h}_t) \in \mathscr{I}\mathscr{S}(\bar{X}, \bar{\mu})$. Moreover, $(\bar{h}_t)$ depends continuously on (h_t) and $(\bar{h}_t)$ commutes with the covering transformations of $\bar{X} \to X$. Suppose that (h_t) is close enough to the identity, then we can define the region $R(h_t) \subset \bar{X}$ which consists of the points between $A \times 0_+$ and $\bar{h}_1(A \times 1/2)$.



We also define $R'(h_t)$ as the region between $\bar{h}_1(A \times 1/2)$ and $A \times 1$.

PROPOSITION 5.4. — *We have* $\tilde{\theta}(h_t)(f) = \bar{\mu}[R(h_t)] - \bar{\mu}(A \times [0, 1/2])$.

Proof. — Since $F = A \times [-1/2, 0] \cup B_0 \cup A \times [0, 1/2]$ is a fundamental domain of $\bar{X} \to X$, we have:

$$\tilde{\theta}(h_t)(f) = \int_F (\bar{f}\bar{h}_1 - \bar{f})\, d\bar{\mu} = \int_F \bar{f}\bar{h}_1\, d\bar{\mu} - \int_F \bar{f}\, d\bar{\mu}.$$

We compute first

$$\int_F \bar{f}\, d\bar{\mu} = \int_{A \times [-1/2, 0]} \bar{f}\, d\bar{\mu} + \int_{B_0} \bar{f}\, d\bar{\mu} + \int_{A \times [0, 1/2]} \bar{f}\, d\bar{\mu}.$$

Since $\bar{f} | B_0 = 0$, we obtain $\int_{B_0} \bar{f}\, d\bar{\mu} = 0$.

We compute then $\int_{A \times [-1/2, 0]} \bar{f}\, d\bar{\mu}$ and $\int_{A \times [0, 1/2]} \bar{f}\, d\bar{\mu}$:

$$\int_{A \times [-1/2, 0]} \bar{f}\, d\bar{\mu} = \int_{A \times [-1/2, 0]} t\, d\nu\, dt = \nu(A) \int_{-1/2}^{0} t\, dt = -\frac{1}{8}\nu(A),$$

$$\int_{A \times [0, 1/2]} f\, d\bar{\mu} = \int_{A \times [0, 1/2]} t\, d\nu\, dt = \nu(A) \int_{0}^{1/2} t\, dt = \frac{1}{8}\nu(A).$$

This gives: $\int_F \bar{f}\, d\bar{\mu} = 0$.

Now we compute $\int_F \bar{f}\bar{h}_1 \, d\bar{\mu}$:

$$\int_F \bar{f}\bar{h}_1 \, d\bar{\mu} = \int_{\bar{h}_1(F)} \bar{f} \, d\bar{\mu},$$

but

$$\bar{h}_1(F) = \tau^{-1}[R'(h_t)] \cup B_0 \cup R(h_t),$$

hence

$$\int_F \bar{f}\bar{h}_1 \, d\bar{\mu} = \int_{\tau^{-1}[R'(h_t)] \cup B_0} \bar{f} \, d\bar{\mu} + \int_{R(h_t)} \bar{f} \, d\bar{\mu}.$$

Since τ preserves $\bar{\mu}$, we have:

$$\int_{R(h_t)} \bar{f} \, d\bar{\mu} = \int_{\tau^{-1}[R(h_t)]} \bar{f}\tau \, d\bar{\mu} = \int_{\tau^{-1}[R(h_t)]} (t+1) \, d\nu \, dt = \int_{\tau^{-1}[R(h_t)]} \bar{f} \, d\bar{\mu} + \bar{\mu}[R(h_t)].$$

Since $\tau^{-1}[R(h_t)] \cup \tau^{-1}[R'(h_t)] \cup B_0 = A \times [-1, 0] \cup B_0$, we obtain:

$$\begin{aligned}\int_F \bar{f}\bar{h}_1 \, d\bar{\mu} &= \bar{\mu}[R(h_t)] + \int_{A \times [-1, 0] \cup B_0} \bar{f} \, d\bar{\mu} \\ &= \bar{\mu}[R(h_t)] + \nu(A) \int_{-1}^{0} t \, dt = \bar{\mu}[R(h_t)] - \frac{1}{2}\nu(A) \\ &= \bar{\mu}[R(h_t)] - \bar{\mu}\left(A \times \left[0, \frac{1}{2}\right]\right).\end{aligned}$$

Finally:

$$\tilde{\theta}(h_t)(f) = \bar{\mu}[R(h_t)] - \bar{\mu}\left(A \times \left[0, \frac{1}{2}\right]\right). \quad \square$$

Remarks. — (1) Proposition 5.4 can be interpreted by saying that $\tilde{\theta}(h_t)$ is the mass that has passed algebraically through the "membrane" $A \times 1/2 \subset X$. If we imagine $A \times [0, 1]$ as a pipe, this explains the name of the mass flow homomorphism.

(2) Let $a, b, c, d \in]0, 1[, c < a < b < d$. Suppose that A is a connected manifold V^{n-1}, and that $\mu \mid V^{n-1} \times [0, 1] = \nu \times dt$, where ν is (up to normalisation) a good measure on V^{n-1}. If we define $\mathscr{C}_0(h_t)$ as the connected component of $V^{n-1} \times [0, 1] - h_1(V^{n-1} \times [a, b])$ which contains $V^n \times 0$, then we have, by 5.4, $\tilde{\theta}(h_t)(f) = \mu(\mathscr{C}_0(h_t)) - \mu(V^{n-1} \times [0, a[)$. Hence we can apply Proposition 4.10 to $h_1 \mid V^{n-1} \times]c, d[$, if and only if $\tilde{\theta}(h_t)(f) = 0$.

APPENDIX A.5

On the different forms of the mass flow homomorphism

Suppose M^n is a compact C^∞ differentiable manifold with a volume form Ω, such that $\int_{M^n} \Omega = 1$. Let $\mathcal{IS}^\infty(M^n, \Omega)$ denote the set of C^∞ isotopies $(h_t)_{t\in[0,1]}$ with $h_0 = \mathrm{Id}$, which preserve Ω. There is a homomorphism $\tilde{V}_{M^n,\Omega} = \tilde{V}$, $\mathcal{IS}^\infty(M, \Omega) \to H^{n-1}(M^n, \mathbb{R})$, *see* [Th]. We will show that, up to Poincaré' duality, $\tilde{V}$ is equal to $\tilde{\theta}_{M^n, \mu_\Omega}$, where μ_Ω is the measure obtained from Ω.

First, we recall the definition of $\tilde{V}$. If $(h_t)_{t\in[0,1]}$ is in $\mathcal{IS}^\infty(M^n, \Omega)$, we consider the time dependent vector field X_t on M^n defined by:

$$X_t(m) = \frac{\partial h_t}{\partial t}[h_t^{-1}(m)].$$

Since h_t preserves Ω, for each t, the $(n-1)$-form $i(X_t)\Omega$ is closed, hence so is the $(n-1)$-form $\int_0^1 i(X_t)\Omega\, dt$. By definition, $\tilde{V}(h_t)$ is the cohomology class of $\int_0^1 i(X_t)\Omega\, dt$.

Proposition A.5.1. — *The cohomology class $\tilde{V}(h_t)$ is the Poincaré dual of the homology class $\tilde{\theta}(h_t)$.*

Proof. — Let ω be the canonical volume form on $\mathbb{T}^1$.

If $f: M^n \to \mathbb{T}^1$, we have to show that:

$$\int_{M^n} \left[\int_0^1 i(X_t)\Omega\, dt\right] \wedge f^*\omega = \tilde{\theta}(h_t)(f).$$

We will denote the left hand side of the equality above by $\langle \tilde{V}(h_t), f \rangle$. By Fubini Theorem, we have:

$$\langle \tilde{V}(h_t), f \rangle = \int_{M^n\times[0,1]} [i(X_t)\Omega] \wedge dt \wedge f^*\omega.$$

Remark that $\Omega \wedge f^*\omega = 0$, since it is an $(n+1)$-form on an n-manifold. We obtain then:

$$0 = i(X_t)[\Omega \wedge f^*\omega] = [i(X_t)\Omega] \wedge f^*\omega + (-1)^n \Omega \wedge [i(X_t) f^*\omega].$$

Now:

$$\langle \tilde{V}(h_t), f \rangle = \int_{M^n\times[0,1]} -[i(X_t)\Omega] \wedge (f^*\omega) \wedge dt = (-1)^n \int_{M^n\times[0,1]} \Omega \wedge [(f^*\omega)(X_t)\, dt]$$

$$= \int_{M^n\times[0,1]} (f^*\omega)(X_t)\, dt \wedge \Omega = \int_{M^n\times[0,1]} h_t^*[(f^*\omega)(X_t)\, dt \wedge \Omega)], \quad \text{by naturality of integration,}$$

$$= \int_{M^n\times[0,1]} (f^*\omega)(X_t \circ h_t)\, dt \wedge \Omega, \quad \text{since } h_t^*\Omega = \Omega;$$

$$= \int_{M^n} \left[\int_0^1 f^*\omega(X_t \circ h_t)\, dt\right] \Omega.$$

We have:

$$(f^*\omega)(X_t \circ h_t(m)) = (f^*\omega)\left(\frac{\partial h_t}{\partial t}(m)\right) = \omega\left(\frac{\partial f \circ h_t}{\partial t}(m)\right).$$

It follows easily that $u \to \int_0^u f^*\omega(X_t \circ h_t(m))\,dt$ is a lift in $\mathbb{R}$ of the map $[0, 1] \to \mathbb{T}^1$, $u \to fh_u(m) - f(m)$.

This means that the lift $\overline{fh_t - f}$ of $fh_t - f$ is just $\int_0^t f^*\omega(X_u \circ h_u)\,du$.

Hence we obtain:

$$\langle \tilde{V}(h_t), f \rangle = \int_{M^n} \overline{fh_1 - f}\,\Omega = \int_{M^n} \overline{fh_1 - f}\,d\mu_\Omega,$$

which is, by definition, $\tilde{\theta}(h_t)(f)$. $\square$

In [Sc], Schwartzman gave for a flow another definition of the mass flow homomorphism which we recall now.

Suppose that $(h_t)_{t \in \mathbb{R}}$ is a flow on a compact space X, and suppose that (h_t) preserves μ. Given a function $f : X \to T^1$, consider the lift $\overline{fh_t - f} : X \to \mathbb{R}$ of $fh_t - f$ with $\overline{fh_0 - f} = 0$, Schwartzman (using Birkhoff ergodic Theorem) shows that $\lim_{t \to \infty} (1/t)[\overline{fh_t - f}](x)$ exists for μ-almost every x, then he defines the "asymptotic cycle" associated to $(h_t)_{t \in \mathbb{R}}$ and μ by:

$$\int_X \lim_{t \to \infty} \frac{1}{t}[\overline{fh_t - f}](x)\,d\mu(x).$$

PROPOSITION A.5.2. — *The limit* $\lim_{t \to \infty} 1/t\,[\overline{fh_t - f}](x)$ *exists for* μ*-almost every* x *and its* μ *integral is* $\tilde{\theta}(h_t)(f)$.

Proof. — First remark that:

$$\overline{fh_{t+t'} - f} = [\overline{fh_{t'} - f}]\,h_t + \overline{fh_t - f},$$

which can also be written as:

$$[\overline{fh_{t'} - f}]\,h_t = \overline{fh_{t+t'} - f} - \overline{fh_t - f} \quad (\star).$$

Using the compactness of X, we obtain a constant K such that:

$$\forall t' \in [0, 1], \quad \forall t \in \mathbb{R}, \quad \forall x \in X, \qquad \left|[\overline{fh_{t+t'} - f}](x) - [\overline{fh_t - f}](x)\right| < K \quad (\star\star).$$

Now, by Birkhoff ergodic Theorem,

$$f^*(x) = \lim_{t \to \infty} \frac{1}{t}\int_0^t [\overline{fh_1 - f}]\,h_u(x)\,du$$

exists for μ-almost every x and moreover

$$\int_X f^* \, d\mu = \int_X \overline{fh_1 - f} \, d\mu.$$

By ($\star$), we have:

$$\int_0^t [\overline{fh_1 - f}] \, h_u(x) \, du = \int_0^t \{ [\overline{fh_{u+1} - f}](x) - [\overline{fh_u - f}](x) \} \, du$$

$$= \int_t^{t+1} \{ [\overline{fh_u - f}](x) \, du - \int_0^1 [\overline{fh_u - f}](x) \} \, du.$$

This gives for μ-almost all x:

$$f^*(x) = \lim_{t \to \infty} \frac{1}{t} \int_t^{t+1} [\overline{fh_u - f}](x) \, du.$$

Using ($\star\star$), it is easy to see that:

$$\lim_{t \to \infty} \frac{1}{t} \int_t^{t+1} [\overline{fh_u - f}](x) \, du = \lim_{t \to \infty} \frac{1}{t} [\overline{fh_t - f}](x).$$

So the computations done above show that $\lim\limits_{t \to \infty} 1/t \, [\overline{f \circ h_t - f}](x)$ exists for μ-almost every x, and also that the integral of this function is $\int_X \overline{fh_1 - f} \, d\mu = \tilde{\theta}(h_t)(f)$. $\square$

6. Study of the Kernel of θ in the case of a compact manifold

We will suppose that M^n is a compact manifold and that μ is in $\mathscr{M}_g^\partial(M^n)$.

We first consider the surjectivity of the map $\tilde{\theta} : \tilde{\mathscr{H}}_0(M^n, \mu) \to H_1(M^n, \mathbb{R})$.

Examples. — (1) $M^n = \mathbb{T}^1 \times B^{n-1}$ where B^{n-1} is the Euclidian ball of dimension $n-1$. We suppose that μ is the product of the Haar measure dt on $\mathbb{T}^1$, by the Lebesgue measure m on B^{n-1}. Remark that $H_1(M^n, \mathbb{Z}) \cong \mathbb{Z}$ with generator $[\mathbb{T}_1 \times 0]$.

Choose some continuous function $\varphi : B^{n-1} \to \mathbb{R}$, such that $\int_{B^{n-1}} \varphi \, dm = 1$ and $\varphi \mid \partial B^{n-1} = 0$. If $\alpha \in \mathbb{R}$ define $(h_t^\alpha)_{t \in [0, 1]}$ by:

$$h_t^\alpha : \mathbb{T}^1 \times B^{n-1} \to \mathbb{T}^1 \times B^{n-1},$$

$$(u, x) \mapsto (u + \alpha\varphi(x) \, t, x).$$

Then $h_t^\alpha \mid \partial B^{n-1} = \mathrm{id}$, $(h_t^\alpha)_{t \in [0, 1]} \in \tilde{\mathscr{H}}^\partial(M^n, \mu)$ and $\tilde{\theta}(h_t^\alpha) = \alpha[\mathbb{T}^1 \times 0]$. Moreover $(h_t^\alpha)_{t \in [0, 1]}$ depends continuously on α. This shows that $\tilde{\theta}$ is surjective and, in fact, has a continuous section. Remark that this section is also a group homomorphism.

(2) $M^n=[[0, 1]\times B^{n-1}/(0, x)\sim(1, \tau x)]$ where τ is the orthogonal symmetry with respect to some hyperplan of $\mathbb{R}^{n-1}$. In fact, M^n is the total space of the disc bundle obtained from the non orientable $(n-1)$-vector bundle over $\mathbb{T}^1$, the 0 section being given by $[[0, 1]\times\{0\}/(0, 0)\sim(1, 0)]=\mathbb{T}^1$. The Lebesgue measure on $[0, 1]\times B^{n-1}$ gives in a natural way a measure μ on M^n. We can show, as in example 1, that $\tilde{\theta}$ is surjective (we only have to replace the function φ by one which is τ invariant). In this case also the construction shows that $\tilde{\theta}$ has a continuous (group) section. Moreover, all our isotopies are the identity on ∂M^n.

Recall now that, if M^n is a manifold, an element $v\in H_1(M^n, \mathbb{R})$ is represented by an imbedded curve, if there exists an imbedding $k:\mathbb{T}^1\to M^n$, such that $[k(\mathbb{T}^1)]=v$. We say that a curve $k:\mathbb{T}^1\to M^n$ has a tubular neighborhood if there exists a vector bundle over $\mathbb{T}^1: E\xrightarrow{p}\mathbb{T}^1$, and an *open* imbedding $\bar{k}: E\to M^n$, such that $\bar{k}$ restricted to the 0 section is the map $k:\mathbb{T}^1\to M^n$. Remark that there are only two vector bundles of a given dimension on $\mathbb{T}^1$, the total spaces of their disc bundles are given in examples 1 and 2 above.

PROPOSITION 6.1. — *Let M^n be a compact manifold, and $\mu\in\mathscr{M}_g^\partial(M^n)$. Suppose that we can find a basis of $H_1(M^n, \mathbb{R})$ which is represented by imbedded curves having tubular neighborhoods. Then, the map $\tilde{\theta}:\tilde{\mathscr{H}}(M^n, \mu)\to H_1(M^n, \mathbb{R})$ is surjective and has a continuous section. Moreover, if $n\geqq 3$, this continuous section can also be choosen to be a group homomorphism.*

Proof. — First, we assume $n\geqq 3$. Let $p_i: E_i\to\mathbb{T}^1$, $i=1, \ldots, q$, be vector bundles and $\varphi_i: E_i\subset M^n$ be open imbeddings such that the images C_i under the φ_i of the 0 sections represent a basis of $H_1(M^n, \mathbb{R})$. Since $n\geqq 3$, by a general position argument, we can assume that the C_i's are disjoint, then, by shrinking down the E_i's, we can assume that $\varphi_i(E_i)\cap\varphi_j(E_j)=\emptyset$, $i\neq j$. If $D(E_i)$ is the disc bundle in E_i, then $D(E_i)$ is homeomorphic to one of the two manifolds given in the examples above. We can also assume, by the von Neumann-Oxtoby-Ulam Theorem, that the measure μ restricted to $D(E_i)$ is taken, under this homeomorphism, to a scalar multiple of the measure defined in these examples. Using the examples, we find for each i, $i=1, \ldots, q$, and each $\alpha\in\mathbb{R}$ an isotopy $(h_t^{i,\alpha})_{t\in[0,1]}$ such that:

- support $(h_t^{i,\alpha})\subset\varphi_i(D(E_i))$;
- $\tilde{\theta}(h_t^{i,\alpha})=\alpha[C_i]$;
- $\alpha\mapsto(h_t^{i,\alpha})$ is a continuous group homomorphism.

We define the section $S: H_1(M, \mathbb{R})\to\tilde{\mathscr{H}}_0(M^n, \mu)$ by:

$$S(\alpha_1[C_1]+\alpha_2[C_2]+\ldots+\alpha_q[C_q])=(h_t^{1,\alpha_1}\circ\ldots\circ h_t^{q,\alpha_q})_{t\in[0,1]}.$$

The map S is clearly a continuous section of $\tilde{\theta}$. Moreover, it is a group homomorphism since $h_t^{i,\alpha}$ commutes with $h_t^{j,\beta}$, their supports being disjoint.

We leave the case $n=2$ to the reader. We loose the group homomorphism property because we cannot assume the C_i's disjoint; by general position, all we obtain is that C_i and C_j intersects transversally in a finite number of points. □

Remark that the hypothesis of Proposition 5.1: "There is a basis of $H_1(M^n, \mathbb{R})$ represented by imbedded curves having tubular neighborhood" is satisfied if M^n is differentiable [Hr] or PL [Hu]. It is also satisfied if $n \neq 4$ since this is a consequence of the stable homeomorphism Theorem, which is known in dimension $\neq 4$ [K].

As a Corollary, we obtain:

COROLLARY 6.2. — *Under the same hypothesis as* 5.1, *there exists a homeomorphism:*

$$\tilde{\mathscr{H}}_0(M^n, \mu) \cong (\operatorname{Ker} \tilde{\theta}) \times H_1(M^n, \mathbb{R}).$$

In particular, $\operatorname{Ker} \tilde{\theta}$ *is connected and locally contractible.*

Proof. — The existence of the homeomorphism follows from the existence of a continuous section. Remark that $\tilde{\mathscr{H}}_0(M^n, \mu)$ is connected and locally contractible being a covering of $\mathscr{H}_0(M^n, \mu)$, which is connected and locally contractible. Hence, the second part of the Corollary follows from the first. □

COROLLARY 6.3. — *Under the same hypothesis as* 6.1, *the subgroup* $\operatorname{Ker} \theta \, (\subset \mathscr{H}_0(M^n, \mu))$ *is connected and locally contractible.*

Proof. — First, we show that $p(\operatorname{Ker} \tilde{\theta}) = \operatorname{Ker} \theta$ where $p : \tilde{\mathscr{H}}_0(M^n, \mu) \to \mathscr{H}_0(M^n, \mu)$. Let $h \in \operatorname{Ker} \theta$ and $\tilde{h} \in \tilde{\mathscr{H}}_0(M^n, \mu)$ such that $p(\tilde{h}) = h$. Since $\theta(h) = 0$, we have $\tilde{\theta}(\tilde{h}) \in \Gamma$. By definition of Γ, there exists $\tilde{g} \in \operatorname{Ker} p$ such that $\tilde{\theta}(\tilde{h}) = \tilde{\theta}(\tilde{g})$. Remark now that $p(\tilde{h}\tilde{g}^{-1}) = h$ and $\tilde{\theta}(\tilde{h}\tilde{g}^{-1}) = 0$. Hence $p(\operatorname{Ker} \tilde{\theta}) = \operatorname{Ker} \theta$.

Now, we show that $p : \operatorname{Ker} \tilde{\theta} \to \operatorname{Ker} \theta$ is a covering. Suppose S is a local section of $p : \tilde{\mathscr{H}}_0(M^n, \mu) \to \mathscr{H}_0(M^n, \mu)$. This section is defined on some open set $\mathscr{U} \ni \mathrm{Id}_{M^n}$, $S : U \to \tilde{\mathscr{H}}_0(M^n, \mu)$, $p \circ S = \mathrm{id}_{\mathscr{U}}$ and $S(\mathrm{id}_{M^n}) = \{ \mathrm{id}_{M^n} \}$ the constant isotopy on M^n. We have of course $\tilde{\theta}[S(\mathscr{U} \cap \operatorname{Ker} \theta)] \subset \Gamma$. Since Γ is discrete and $\tilde{\theta}$ and S are continuous, if $\mathscr{U}$ is small enough then $\tilde{\theta} S(\mathscr{U} \cap \operatorname{Ker} \theta) = \{ 0 \}$. This implies that $S(\mathscr{U} \cap \operatorname{Ker} \theta) \subset \operatorname{Ker} \tilde{\theta}$. Hence $p : \operatorname{Ker} \tilde{\theta} \to \operatorname{Ker} \theta$ has a local section, which implies that it is a covering. Using this fact and 6.2, it follows that $\ker \theta$ is connected and locally contractible. □

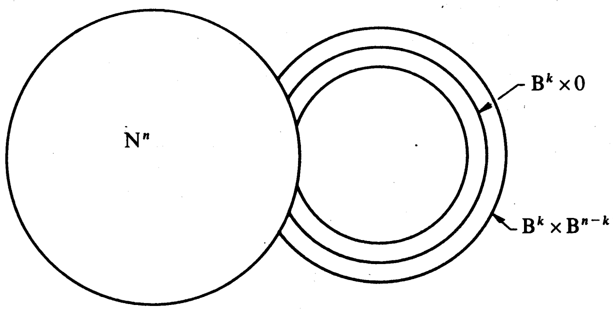
We now explain what is a handle decomposition of a manifold.

First, a n-dimensional handle is simply a space homeomorphic to a product $B^k \times B^{n-k}$, where B^k and B^{n-k} are the Euclidian balls of dimension k and $n-k$. The number k is called the index of the handle. If N^n be a manifold and φ a locally flat [1] imbedding $\varphi : \partial B^k \times B^{n-k} \to \partial N^n$, we can form the topological space $M^n = N^n \cup_\varphi (B^k \times B^{n-k})$. This space is obtained from the disjoint union $N^n \coprod (B^k \times B^{n-k})$ by identifying x with $\varphi(x)$ for each x in $\partial B^k \times B^{n-k}$. In fact, M^n is a manifold whose boundary is

$$\partial M^n = [\partial N^n - \varphi(\partial B^k \times \mathring{B}^{n-k})] \cup (B^k \times \partial B^{n-k}).$$

[1] Locally flat, in this case means that φ extends to an (open) imbedding of $\partial B^k \times$ (neighborhood of B^{n-k} in $\mathbb{R}^{n-k}$) into ∂N^n.

We say that M^n is obtained from N^n by adding an n-handle of index k. Remark that $N^n \cup B^k \times 0$ is a retract by deformation of M^n.



We will say that a manifold M^n has a handle decomposition if M^n can be obtained from a ball B^n by adding n-handles. Each differentiable or PL manifold has a handle decomposition ([Mi], [Ru]). If M^n is a topological manifold, the same is true if $n \leqq 3$ (because each manifold of dimension $\leqq 3$ can be given a PL structure, *see* [Mo]); if $n \geqq 6$, then M^n has also a handle decomposition [KS].

PROPOSITION 6.4. — *Let* M^n *be a compact manifold having a handle decomposition, and such that there exists a basis of* $H_1(M, \mathbb{R})$ *represented by embedded curves having tubular neighborhood. Let* $\mu \in \mathcal{M}_g^{\partial}(M^n)$. *If* $h \in \mathcal{H}_0(M^n, \mu)$ *is in* $\operatorname{Ker} \theta$, *then* h *can be written as a composition* $h = h_1 \ldots h_q$, *with each* $h_i \in \mathcal{H}_0(M^n, \mu)$ *and support* (h_i) *is contained in a topological n-ball* (*a topological n-ball is a subset of* M^n *homeomorphic to* B^n).

Proof. — We will use the fact that $H_1(M, \mathbb{R})$ has a basis represented by curves having tubular neighborhood, only to apply 6.3 and obtain that $\operatorname{Ker} \theta$ is connected. Since $\operatorname{Ker} \theta$ is connected, it is generated, as a group, by any neighborhood of id_{M^n}. Hence, we can assume that h is close to the identity.

We will prove 6.4 by induction on the number of handles in a handle decomposition of M^n. We have to prove then the following thing:

If

$$M^n = N^n \cup B^k \times B^{n-k},$$

with

$$N^n \cap B^k \times B^{n-k} = \partial N^n \cap B^k \times B^{n-k} = \partial B^k \times B^{n-k}$$

and if 6.4 is true for N^n, then it is true for M^n.

We will denote by B_0^k some concentric ball contained in $\mathring{B}^k$, for example:

$$B_0^k = \left\{ x \in \mathbb{R}^n \;\middle|\; \|x\| \leqq \frac{1}{2} \right\} \subset \{ x \in \mathbb{R}^n \mid \|x\| \leqq 1 \}.$$

Remark that $N_0^n = M^n - (\mathring{B}_0^k \times B^{n-k})$ is homeomorphic to N^n. We can assume by 2.1 and the von Neumann-Oxtoby-Ulam Theorem that $\mu \,|\, B^k \times B^n$ is, up to normalization, equal to

Lebesgue measure on $B^k \times B^{n-k} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$. Since $\partial N_0^n \subset \partial M^n \cup \partial B_0^k \times B^{n-k}$, we have $\mu(\partial N_0^n) = 0$, hence $\mu_0 = (1/\mu(N_0^n))\mu \mid N_0^n \in \mathcal{M}_g^\partial(N_0^n)$. We will show that if h is in $\operatorname{Ker} \theta_{M^n}$, where $\theta_{M^n} : \mathcal{H}_0(M^n, \mu) \to H_1(M^n, \mathbb{R})/\Gamma_M$, and is close to id_{M^n}, then it can be written in $\mathcal{H}_0(M^n, \mu)$ as a composition $h = fg$ with support $(f) \subset B^k \times B^{n-k} \cong B^n$, support $(g) \subset N_0^n$, and $\theta_{N_0^n}(g \mid N_0^n) = 0$, where $\theta_{N_0^n} : \mathcal{H}_0(N_0^n, \mu_0) \to H_1(N_0^n, \mathbb{R})/\Gamma_N$. Since N_0^n is homeomorphic to the manifold N^n, for which 6.4 is true, this will finish the proof.

We distinguish four cases depending on k the index of the handle.

First case: $k = 0$. – The manifold M^n is the disjoint union of $N^n = N_0^n$ and B^n. We define f by $f \mid B^n = h$, $f \mid N^n = \mathrm{id}$, and g by $g \mid B^n = \mathrm{id}$, $g \mid N^n = h$.

Second case: $k = 1$. – We have $B^1 = [-1, 1]$, which we can of course replace by $[0, 1]$. Hence $M^n = N^n \cup [0, 1] \times B^{n-1}$ with

$$N^n \cap ([0, 1] \times B^{n-1}) = \partial N^n \cap ([0, 1] \times B^{n-1}) = \{0, 1\} \times B^{n-1}.$$

In this case, $N_0^n = N^n \cup [0, 1/4] \times B^{n-1} \cup [3/4, 1] \times B^{n-1}$. If $h \in \operatorname{Ker} \theta_{M^n}$ is close to the identity, and (h_t) is a small isotopy with $h_1 = h$, then $\tilde{\theta}_{M^n}(h_t) = 0$. We can then apply remark (2) after Proposition 4.4, and obtain $f \in \mathcal{H}(M^n, \mu_0)$ such that support $(f) \subset [0, 1] \times B^{n-1}$ and $h = f$ on $[1/4, 3/4] \times B^{n-1}$; moreover, if h is the identity so is f, and f depends continuously on h. If we define $g = f^{-1}h$, then g is the identity on $[1/4, 3/4] \times B^{n-1}$, it depends continuously on h and is the identity if h is the identity. Clearly, $g \mid N_0^n$ preserves μ_0 and is isotopic to the identity, since it is close to the identity if h is close enough to the identity.

We must now show that $\theta_{N_0^n}(g \mid N_0^n)$ is 0. Let (g_t) be a small isotopy, preserving μ, with support in N_0^n and such that $g_1 = g$. Remark that if $\varphi : M^n \to \mathbb{T}^1$ is continuous, we have:

$$\tilde{\theta}_{M^n}(g_t)(\varphi) = \mu_0(N_0^n)\tilde{\theta}_{N_0^n}(g_t \mid N_0^n)(\varphi \mid N_0^n).$$

Since $N_0^n \cup [1/4, 3/4] \times 0$ is a retract of M^n and since $\mathbb{T}^1$ is path connected, each continuous map $N_0^n \to \mathbb{T}^1$ can be extended to a continuous map $M^n \to \mathbb{T}^1$. So we have to show that $\tilde{\theta}_{M^n}(g_t) = 0$. Using that (g_t) is small, this follows from

$$\theta_{M^n}(g) = \theta_{M^n}(f^{-1}) + \theta_{M^n}(h) = 0 + 0 = 0.$$

We have $\theta(f^{-1}) = 0$, because f is isotopic to the identity by an isotopy having support in $[0, 1] \times B^{n-1} \cong B^n$.

Third case: $k = 2$. – Since $B^2 \times B^{n-2} - B_0^2 \times B^{n-2}$ is connected by 4.9 for each $h \in \mathcal{H}(M^n, \mu)$ which is close enough to the identity, we can find $f \in \mathcal{H}(M^n, \mu)$ with support $(f) \subset B^2 \times B^{n-2}$ and $h = f$ on $B_0^2 \times B^{n-2}$. Moreover, f depends continuously on h, and is the identity if h is. We define then $g = f^{-1}h$, and take a small isotopy $(g_t) \in \mathcal{IS}(M^n, \mu)$ with $g = g_1$ supported in N_0^n. We have to show that if $h \in \operatorname{Ker} \theta$ then $\tilde{\theta}_{N_0^n}(g_t \mid N_0^n) = 0$. Remark that if a continuous map $\varphi : N_0^n \to \mathbb{T}^1$ can be extended to M^n, then, as in the second case above $\tilde{\theta}_{N_0^n}(g_t \mid N_0^n)(\varphi) = 0$. Now, since $N_0^n \cup B_0^2 \times 0$ is a retract of M^n, a map $\varphi : N_0^n \to \mathbb{T}^1$ can be extended to M^n if and only if $\varphi \mid \partial B_0^2 \times 0$ is null homotopic, which is equivalent to $\deg(\varphi \mid \partial B_0^2 \times 0) = 0$, where $\varphi \mid \partial B_0^2 \times 0 : \partial B_0^2 \times 0 \cong \mathbb{T}^1 \to \mathbb{T}^1$. Hence, if each map $\varphi : N_0^n \to \mathbb{T}^1$ verifies $\deg(\varphi \mid \partial B_0^2 \times 0) = 0$, we obtain $\tilde{\theta}_{N_0^n}(g_t) = 0$.

It is possible that some map $\varphi : N_0^n \to \mathbb{T}^1$ does not verify $\deg(\varphi \mid \partial B_0^2 \times 0) = 0$. In this case, let $\varphi_0 : N_0^n \to \mathbb{T}^1$ be such that:

$$\deg(\varphi_0 \mid \partial B_0^2 \times 0) = \min \{ \deg(\varphi \mid \partial B_0^2 \times 0) \mid \varphi : N_0^n \to \mathbb{T}^1, \text{ and } \deg(\varphi \mid \partial B_0^2 \times 0) > 0 \}.$$

Then as a group, we have:

$$[N_0^n, \mathbb{T}^1] = \mathbb{Z}\varphi_0 + \{ \varphi : N_0^n \to \mathbb{T}^1 \mid \deg(\varphi \mid \partial B_0^2 \times 0) = 0 \}.$$

This implies that $\tilde{\theta}_{N_0^n}(g_t \mid N_0^n) = 0$ if and only if $\tilde{\theta}_{N_0^n}(g_t \mid N_0^n)(\varphi_0) = 0$. Of course, we have to modify (g_t) to obtain this condition.

Using example 1 given in the beginning of 6, we can find an isotopy $(s_t) \in \mathscr{I}\mathscr{S}(M^n, \mu)$, with support $(s_t) \subset B^2 \times B^{n-2} - \mathring{B}_0^2 \times B^{n-2} \cong \mathbb{T}^1 \times B^{n-1}$, and such that

$$\tilde{\theta}_{N_0^n}(s_t \mid N_0^n)(\varphi_0) = -\tilde{\theta}_{N_0^n}(g_t \mid N_0^n)(\varphi_0).$$

This implies that $\tilde{\theta}_{N_0^n}(s_t g_t \mid N_0^n) = 0$. Hence, if we define $g' = s_1 g$ and $f' = f s_1^{-1}$, we obtain $h = f' g'$ with support $(f') \subset B^2 \times B^{n-2}$, support $(g') \subset N_0^n$ and $\theta(g' \mid N_0^n) = 0$.

Fourth case: $k \geqq 3$. — Using 4.9 as in the third case, we can write each $h \in \mathscr{H}_0(M^n, \mu)$, close to the identity, as $h = fg$, with f and $g \in \mathscr{H}_0(M^n, \mu)$, support $(f) \subset B^k \times B^{n-k}$, and support $(g) \subset N_0^n$. To prove that $\theta_{N_0^n}(g \mid N_0^n) = 0$, if $h \in \operatorname{Ker} \theta_{M^n}$, it suffices to show that each continuous map $N_0^n \to \mathbb{T}^1$ extends to a continuous map $M^n \to \mathbb{T}^1$. Using the fact that $N_0^n \cup B_0^k$ is a retract of M^n, it suffices to remark that each continuous map $\partial B_0^k \cong S^{k-1} \to \mathbb{T}^1$ extends to a continuous map $B_0^k \cong B^k \to \mathbb{T}^1$, since $k - 1 \geqq 2$. □

Remark. — Using in the proof of 6.4, a handle decomposition with handles of small diameter, we see that we can add in Proposition 6.4 that the supports of the h_i are as small as we want. This follows also from the next Lemma.

LEMMA 6.5. — *Let m be Lebesgue measure on* I^n *and* $\varepsilon > 0$. *Given any* $h \in \mathscr{H}_0(I^n, m)$, *we can write* $h = h_1 h_2 \dots h_q$, *with* $h_i \in \mathscr{H}_0(I^n, m)$ *and* diam [support (h_i)] $< \varepsilon$.

Proof. — We will show how to write each $h \in \mathscr{H}_0(I^n, m)$ as a product $h = h_1 \dots h_q$, with support (h_i) contained in either $[0, 3/4] \times I^{n-1}$ or $[1/4, 1] \times I^{n-1}$, moreover

$$h_i \left| \left[0, \frac{3}{4}\right] \times I^{n-1} \in \mathscr{H}_0\left(\left[0, \frac{3}{4}\right] \times I^{n-1}, m\right)\right.$$

or

$$h_i \left| \left[\frac{1}{4}, 1\right] \times I^{n-1} \in \mathscr{H}_0\left(\left[\frac{1}{4}, 1\right] \times I^{n-1}, m\right).\right.$$

Of course, 6.5 will follow by "applying the preceding result in smaller and smaller cubes", we leave these details to the reader.

Since $\mathscr{H}_0(\mathrm{I}^n, m)$ is (by definition!) connected, we can assume h close to the identity. Since

$$\left(\left[0, \frac{3}{4}\right] \times \mathrm{I}^{n-1} - \left[0, \frac{1}{4}\right] \times \mathrm{I}^{n-1}\right) = \left[\frac{1}{4}, \frac{3}{4}\right] \times \mathrm{I}^{n-1}$$

is connected we can apply 4.9 and obtain for each $h \in \mathscr{H}_0(\mathrm{I}^n, m)$, close to the identity, a homeomorphism h_1 of I^n preserving the measure such that support

$$(h_1) \subset \left[0, \frac{3}{4}\right] \times \mathrm{I}^{n-1} \quad \text{and} \quad h_1 = h \quad \text{on} \quad \left[0, \frac{1}{4}\right] \times \mathrm{I}^{n-1},$$

moreover h_1 depends continuously on h and is the identity if h is. We conclude from this that, if f is close enough to the identity,

$$h_1 \Bigg| \left[0, \frac{3}{4}\right] \times \mathrm{I}^{n-1} \in \mathscr{H}_0\left(\left[0, \frac{3}{4}\right] \times \mathrm{I}^{n-1}, m\right),$$

moreover $h_2 = h_1^{-1} h$ has support in

$$\left[\frac{1}{4}, 1\right] \times \mathrm{I}^{n-1} \quad \text{and} \quad h_2 \Bigg| \left[\frac{1}{4}, 1\right] \times \mathrm{I}^{n-1} \in \mathscr{H}_0\left(\left[\frac{1}{4}, 1\right] \times \mathrm{I}^{n-1}, m\right).$$

since $h = h_1 h_2$, this finishes the proof. □

If $\mathscr{U}$ is some open covering of M^n, a homeomorphism of M^n is called $\mathscr{U}$-small if its support is contained in some element of $\mathscr{U}$.

We restate the results obtained above.

THEOREM 6.6. — *Let* M^n *be a compact manifold having a handle decomposition and such that there exists a basis of* $\mathrm{H}_1(\mathrm{M}^n, \mathbb{R})$ *represented by imbedded curves having tubular neighborhoods; moreover let* $\mu \in \mathscr{M}_g^\partial(\mathrm{M}^n)$. *The map* $\theta : \mathscr{H}_0(\mathrm{M}^n, \mu) \to \mathrm{H}_1(\mathrm{M}^n, \mathbb{R})/\Gamma$ *is surjective. The kernel of* θ *is generated as a group by its elements having support in n-balls. Moreover given any open covering* $\mathscr{U}$ *of* M^n, *we can write each element of* $\operatorname{Ker}\theta$ *as a composition of* (μ*-preserving) homeomorphisms, which are* $\mathscr{U}$*-small.*

APPENDIX A.6

THE CASE WHERE THE HOMEOMORPHISMS ARE THE IDENTITY ON THE BOUNDARY AND THE NON COMPACT CASE

Let M^n be a compact connected manifold with $\partial \mathrm{M}^n \neq \emptyset$ and $\mu \in \mathscr{M}_g^\partial(\mathrm{M}^n)$. Consider

$$\mathscr{I}\mathscr{S}^\partial(\mathrm{M}^n, \mu) = \{(h_t) \in \mathscr{I}\mathscr{S}(\mathrm{M}^n, \mu) \mid h_t \mid \partial \mathrm{M}^n = \mathrm{id} \text{ for each } t\},$$

$$\mathscr{H}_0^\partial(\mathrm{M}^n, \mu) = \{h \mid \exists (h_t) \in \mathscr{I}\mathscr{S}^\partial(\mathrm{M}^n, \mu) \text{ with } h_1 = h\}.$$

We have a natural map $\mathscr{I}\mathscr{S}^{\partial}(\mathrm{M}^n, \mu) \to \mathscr{H}_0^{\partial}(\mathrm{M}^n, \mu)$, $(h_t) \mapsto h_1$. The restriction of

$$\tilde{\theta} : \quad \mathscr{I}\mathscr{S}(\mathrm{M}^n, \mu) \to \mathrm{H}_1(\mathrm{M}^n, \mathbb{R})$$

gives a map noted $\tilde{\theta}^{\partial} : \mathscr{I}\mathscr{S}^{\partial}(\mathrm{M}^n, \mu) \to \mathrm{H}_1(\mathrm{M}^n, \mathbb{R})$.

LEMMA A.6.1. – *If $\partial \mathrm{M}^n \neq \emptyset$, the map $\tilde{\theta}^{\partial} : \mathscr{I}\mathscr{S}^{\partial}(\mathrm{M}^n, \mu) \to \mathrm{H}_1(\mathrm{M}^n, \mathbb{R})$ is zero on the kernel of $\mathscr{I}\mathscr{S}^{\partial}(\mathrm{M}^n, \mu) \to \mathscr{H}_0(\mathrm{M}^n, \mu)$. Hence it defines a homomorphism:*

$$\theta^{\partial} : \quad \mathscr{H}_0^{\partial}(\mathrm{M}^n, \mu) \to \mathrm{H}_1(\mathrm{M}^n, \mathbb{R}).$$

Proof. – Let $(h_t)_{t \in [0, 1]}$ be in the kernel of $\mathscr{I}\mathscr{S}^{\partial}(\mathrm{M}^n, \mu) \to \mathscr{H}_0^{\partial}(\mathrm{M}^n, \mu)$. This means that $h_0 = h_1 = \mathrm{id}$ and $h_t | \partial \mathrm{M}^n = \mathrm{id}$ for each $t \in [0, 1]$. By 5.2, if $x \in \mathrm{M}^n$, the loop $\{ h_t(x) | t \in [0, 1] \}$ represents $\tilde{\theta}^{\partial}(h_t)$, but, for $x \in \partial \mathrm{M}^n$, this loop is constant. □

Remark that, if $\mathrm{H}_1(\mathrm{M}^n, \mathbb{R})$ has a basis represented by imbedded curves having tubular neighborhoods, the section of $\tilde{\theta}$ constructed in Proposition 6.1 has image in $\mathscr{I}\mathscr{S}^{\partial}(\mathrm{M}^n, \mu)$, this shows in particular that θ^{∂} is surjective.

The following Theorem can be proved along the same lines as Theorem 6.6.

THEOREM A.6.2. – *Let M^n be a compact manifold with $\partial \mathrm{M}^n \neq \emptyset$ and $\mu \in \mathscr{M}_g^{\partial}(\mathrm{M}^n)$. Suppose that M^n has a handle decomposition and that there exists a basis of $\mathrm{H}_1(\mathrm{M}^n, \mathbb{R})$ represented by imbedded curves having tubular neighborhoods. The kernel of the surjective map $\theta^{\partial} : \mathscr{H}_0^{\partial}(\mathrm{M}^n, \mu) \to \mathrm{H}_1(\mathrm{M}^n, \mathbb{R})$ is generated as a group by its elements having support in n-balls. Moreover given any open covering $\mathscr{U}$ of M^n, we can write each element of $\mathrm{Ker}\,\theta$ as a composition of elements of $\mathrm{Ker}\,\theta$ which are $\mathscr{U}$-small.*

Let V^n be a non compact connected manifold and let μ be a Radon measure (finite on every compact set) which is strictly positive on each open set. We denote by $\mathscr{H}_0^c(\mathrm{V}^n, \mu)$ the group of homeomorphisms h, such that h preserves μ, has compact support and moreover is isotopic to the identity by a μ-preserving isotopy having compact support. We will denote by $\mathscr{I}\mathscr{S}^c(\mathrm{V}^n, \mu)$ the set of isotopies which have compact support and preserve μ. We have a natural surjective map $\mathscr{I}\mathscr{S}^c(\mathrm{V}^n, \mu) \xrightarrow{p} \mathscr{H}_0^c(\mathrm{V}^n, \mu)$, $(h_t)_{t \in [0, 1]} \to h_1$. The space $\mathscr{I}\mathscr{S}^c(\mathrm{V}^n, \mu)$ has a group structure defined by: $(h_t)(g_t) = (h_t g_t)$. Of course, p is a group homomorphism.

We will suppose that each compact set $\mathrm{K} \subset \mathrm{V}^n$ is contained in some *compact* codimension zero locally flat submanifold of V^n. This condition is realized, for example, if V^n is differentiable, or PL, or have a handle decomposition.

We now define a homomorphism: $\theta_{\mathrm{V}^n, \mu} : \mathscr{H}_0^c(\mathrm{V}^n, \mu) \to \mathrm{H}_1(\mathrm{V}^n, \mathbb{R})$. We first define: $\tilde{\theta}_{\mathrm{V}^n, \mu} : \mathscr{I}\mathscr{S}^c(\mathrm{V}^n, \mu) \to \mathrm{H}_1(\mathrm{V}^n, \mathbb{R})$.

If $(h_t)_{t \in [0, 1]} \in \mathscr{I}\mathscr{S}^c(\mathrm{V}^n, \mu)$, we choose a compact submanifold $\mathrm{N}^n \subset \mathrm{V}^n$ which contains the support of h_t and we define $\tilde{\theta}_{\mathrm{V}^n, \mu}(h_t)$ as the image of $\tilde{\theta}_{\mathrm{N}^n, \mu | \mathrm{N}^n}(h_t | \mathrm{N}^n)$ under the map $\mathrm{H}_1(\mathrm{N}^n; \mathbb{R}) \to \mathrm{H}_1(\mathrm{V}^n, \mathbb{R})$ induced by the inclusion $\mathrm{N}^n \subset \mathrm{V}^n$. The fact that $\tilde{\theta}_{\mathrm{V}^n, \mu}(h_t)$ is well defined (i. e. independent from the choice of N^n) is an easy consequence of the naturality of the definition of $\tilde{\theta}_{\mathrm{N}^n, \mu}$. It is easy to check that $\tilde{\theta}_{\mathrm{V}^n, \mu}$ is a group homomorphism.

LEMMA A.6.3. – *Let $(h_t)_{t \in [0, 1]} \in \mathscr{I}\mathscr{S}^c(\mathrm{V}^n, \mu)$ verify $h_1 = \mathrm{id}_{\mathrm{V}^n}$, then $\tilde{\theta}_{\mathrm{V}^n, \mu}(h_t) = 0$.*

Proof. — Let M^n be a compact submanifold which contains the support of h_t, we can suppose N^n connected. By 5.2, the loop $\{h_t(x) \mid t \in [0, 1]\}$ represent $\tilde{\theta}_{N^n, \mu|N^n}(h_t \mid N^n)$ for each $x \in N^n$, but if x is in the frontier of N^n in V^n the loop $\{h_t(x) \mid t \in [0, 1]\}$ is constant. □

By the above Lemma, the map $\tilde{\theta}_{V^n, \mu}$ gives a well defined map:

$$\theta_{V^n, \mu} : \mathscr{H}_0^c(V^n, \mu) \to H_1(V^n, \mathbb{R}).$$

LEMMA A.6.4. — *Let $h \in \mathscr{H}_0^c(V^n, \mu)$ be in the kernel of $\theta_{V^n, \mu}$, and let let $(h_t)_{t \in [0, 1]} \in \mathscr{I}\mathscr{S}^c(V^n, \mu)$ be such that $h_1 = h$. There exists a compact submanifold M^n containing the support of $(h_t)_{t \in [0, 1]}$ and such that $\tilde{\theta}_{M^n, \mu|M^n}(h_t \mid M^n) = 0$.*

Proof. — Let N^n be a compact submanifold containing the support of $(h_t)_{t \in [0, 1]}$. By hypothesis, the image of $\tilde{\theta}_{M^n, \mu|N^n}(h_t \mid N^n)$ in $H_1(V, \mathbb{R})$ is 0. This implies that there exists a compact set $K \supset N^n$ such that the image of $\tilde{\theta}_{N^n, \mu|N^n}(h_t \mid N^n)$ in $H_1(K, \mathbb{R})$ is 0. We can then take for M^n, any compact submanifold containing K. □

We will suppose now that $\partial V^n = \emptyset$ and each compact set of V^n is contained in a compact submanifold which verifies the hypothesis of Theorem A.6.2. Using what was said above and Theorem A.6.2, it is easy to prove the following Theorem:

THEOREM A.6.5. — *Under the hypothesis above on V^n, the map $\theta : \mathscr{H}_0^c(V^n, \mu) \to H_1(V^n, \mathbb{R})$ is surjective. The kernel of θ is generated by its elements having support in n-balls. Moreover, given any open covering $\mathscr{U}$ of V^n, any element $h \in \operatorname{Ker} \theta$, can be written as a composition of $\mathscr{U}$-small elements of $\mathscr{H}_0^c(V^n, \mu)$.*

7. The algebraic structure of the kernel of θ in the case of a manifold

We will suppose in this section that M^n is a compact *connected* manifold *without boundary* and that μ is a good measure on M^n, i.e. $\mu \in \mathscr{M}_g(M^n)$.

The following Lemma is proved essentially in [OU$_2$], p. 895.

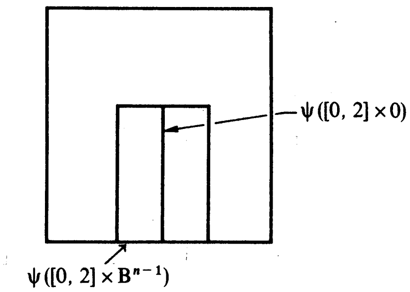
LEMMA 7.1. — *Given any two points $x, y \in M^n$, there exists $h \in \mathscr{H}_0(M^n, \mu)$ such that $h(x) = y$; moreover we can assume that h is a composition of (μ-preserving) homeomorphisms supported by (topological) n-balls. In particular,* $\operatorname{Ker} \theta$ *operates transitively on the points of* M^n.

Sketch of proof. — First prove that given two points x, y in $\mathring{I}^n$ there is a Lebesgue measure preserving homeomorphism h of I^n such that $h(x) = y$ and $h \mid \partial I^n = \mathrm{id}$. Then extend this result to M^n using its connectedness and the fact that its boundary is empty. □

We will use the well known fact that in I^n we can find a locally flat arc with positive measure. We prove this fact now.

LEMMA 7.2. — *There exists an imbedding $\varphi : [0, 2] \times B^{n-1} \to I^n$ such that $\varphi^{-1}(\partial I^n) = 0 \times B^{n-1}$ and $\alpha = \varphi([0, 1] \times 0)$ has Lebesgue measure equal to 1/4.*

Proof. — We can always find an imbedding $\psi : [0, 2] \times B^{n-1} \to I^n$ such that $\psi^{-1}(\partial I^n) = 0 \times B^{n-1}$, and such that the Lebesgue measure of $\psi([0, 1] \times 0)$ is 0.



Let m denote Lebesgue measure on I^n and $\bar{m}$ denote Lebesgue measure on [0, 1]. Define a measure μ on I^n by:

$$\mu = \frac{3}{4} m + \frac{1}{4} (\psi \mid [0, 1] \times 0)_* \bar{m}.$$

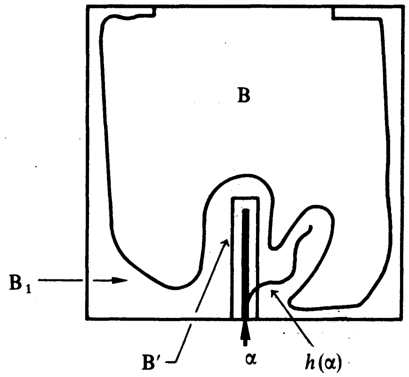
The measure μ is good, moreover $\mu(\partial I^n) = 0$ and $\mu[\psi([0, 1] \times 0)] = 1/4$. By the von Neumann-Oxtoby-Ulam Theorem, we can find a homeomorphism h of I^n verifying $h_* \mu = m$. The imbedding $\varphi = h\psi$ has the desired properties. □

In the next Lemma, we will have to assume that the dimension is $\geqq 3$. This will put the same dimensional restriction on our final result.

LEMMA 7.3. — *Suppose h is a homeomorphism of I^n, $n \geqq 3$, which preserves the Lebesgue measure m and such that $h \mid \partial I^n = \mathrm{Id}$. We can write $h = h_1 h_2$ with:*

- *h_1 and h_2 are homeomorphisms of I^n which preserve m;*
- *for $i = 1$ or 2, $h_i \mid \partial I^n = \mathrm{Id}$;*
- *for $i = 1$ or 2, the support of h_i is contained in a locally flat (topological) n-ball B_i such that $m(B_i) < 3/4$, $m(\partial B_i) = 0$ and $B_i \cap \partial I^n = \partial B_i \cap \partial I^n$ is a $(n-1)$-ball locally flat in ∂B_i and in ∂I^n.*

Proof. — Let α be the arc given in 7.2. Since $n \geqq 3$, the set $I^n - (\alpha \cup h(\alpha))$ is connected; we can, by 2.3′, find a locally flat n-ball $B \subset I^n - (\alpha \cup h(\alpha))$ such that $m(B) > 1/4$, $m(\partial B) = 0$ and $B \cap \partial I^n = \partial B \cap \partial I^n$ is a $(n-1)$-ball locally flat in ∂B and ∂I^n. By the generalized Schöenflies Theorem (*see* section 3 before 3.7), $B_1 = \overline{I^n - B}$ is a locally flat n-ball in I^n, such that $B_1 \cap \partial I^n = \partial B_1 \cap \partial I^n$ is a $(n-1)$-ball locally flat in ∂B_1 and ∂I^n. By construction, $\alpha \cup h(\alpha)$ is contained in $\mathrm{Int}(B_1)$. Let B' be a locally flat n-ball in I^n such that $\alpha \subset B'$, $B' \cup h(B') \subset \mathrm{Int}\, B_1$, $m(\partial B') = 0$ and $B' \cap \partial I^n = \partial B' \cap \partial I^n$ is locally flat in $\partial B'$ and ∂I^n, we can take (for example) for B' the image under φ (*see* 7.2) of $[0, 1+\varepsilon] \times$ (small neighborhood of 0 in B^{n-1}). Using the generalized Schöenflies Theorem, we can construct a homeomorphism h_1 of B_1 such that $h_1 \mid \partial B_1 =$ identity and $h_1 \mid B' = h \mid B'$. Applying 3.7, we see that we can add that h_1 preserves m. We extend h_1 by the identity to I^n. We define $h_2 = h_1^{-1} h$, the support of h_2 is contained in $B_2 = \overline{I^n - B'}$, we have $m(B_2) < 3/4$ since B' contains the arc α. □



Remark. — In the above situation, if U is a neighborhood of I^n in $\mathbb{R}^n$, for $i=1, 2$, we can find a homeomorphism with support in U and sending B_i on I^n. This is a consequence of the generalized Schöenflies Theorem, since B_i is a locally flat n-ball in I^n such that $B_i \cap \partial I^n = \partial B_i \cap \partial I^n$ is a $(n-1)$-ball locally flat in ∂B_i and ∂I^n.

Construction. — We can find two sequences of n-balls $(C_i)_{i \geqq 1}$, $(D_i)_{i \geqq 1}$ contained in $\mathring{I}^n$ and such that:

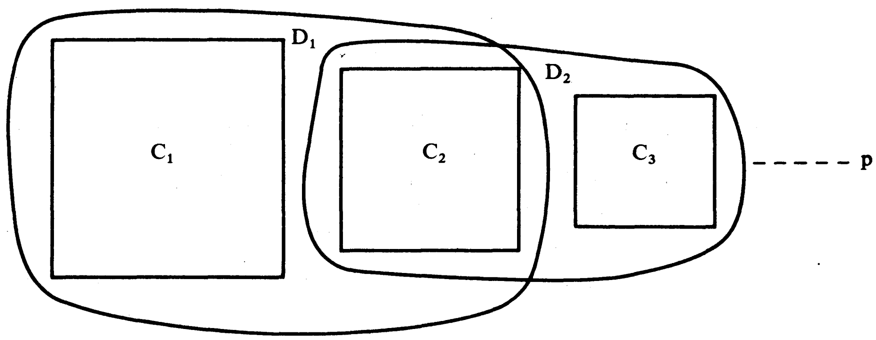
— the C_i's are all disjoint;

— D_i intersects only D_{i-1} and D_{i+1}, in particular the D_{2i}'s the disjoint and the D_{2i+1}'s are disjoint;

— $C_i \cup C_{i+1}$ is contained in $\mathring{D}_i$, hence $D_i \cap C_j = \emptyset$ if $j \neq i, i+1$;

— the D_i's and the C_i's converge to some point p in I^n, this means that each neighborhood of p contains all but a finite number of the C_i's and the D_i's;

— $m(\partial D_i) = m(\partial C_i) = 0$, and $m(C_{i+1}) = (3/4)\, m(C_i)$.



With this notations, we now prove:

THEOREM 7.4. — *Let n be $\geqq 3$. If $f \in \mathscr{H}^\partial(I^n, m)$ has its support in $\mathring{I}^n$, then f is in the commutator subgroup of $\mathscr{H}^\partial(I^n, m)$. More precisely, we can write $f = [k_1, k_2] \ldots [k_{2q-1}, k_{2q}]$ with $k_i \in \mathscr{H}^\partial(I^n, m)$, support $(k_i) \subset \mathring{I}^n$.*

By definition, here and in the following, $[s, t] = sts^{-1}t^{-1}$.

Proof. — *First case:* support $(f) \subset C_1$ introduced above.

We apply 7.3 to f and C_1 instead of h and I^n. We can write $f = h_1 h_2$, with $h_i \in \mathscr{H}^\partial (C_1, m)$, support $(h_i) \subset B_i$ with $m(B_i) < (3/4) m(C_1)$. Moreover by the remark following 7.3, we can find a homeomorphism with support in D_1 sending B_i onto C_1. Since there exists a homeomorphism with support in D_1 sending C_1 to a subcube of C_2 having measure equal to $m(B_i) < (3/4) m(C_1) = m(C_2)$, we obtain, by 3.6, two homeomorphisms g_1 and $g_2 \in \mathscr{H}^\partial (D_1, m)$ such that $g_i(B_i) \subset C_2$. We extend g_1 and g_2 to I^n by the identity. Define now $f_2 = g_1 h_1 g_1^{-1} g_2 h_2 g_2^{-1}$, we have support $(f_2) \subset C_2$, and

$$ff_2^{-1} = h_1 h_2 g_2 h_2^{-1} g_2^{-1} g_1 h_1^{-1} g_1^{-1} = h_1 [h_2, g_2] h_1^{-1} [h_1, g_1] = [h_1 h_2 h_1^{-1}, h_1 g_2 h_1^{-1}] [h_1, g_1].$$

Hence we can write $f = [s_1, t_1][s'_1, t'_1] f_2$ with support $(f_2) \subset C_2$ and support (s_1), support (s'_1), support (t_1) and support $(t'_1) \subset D_1$.

Using the above procedure, it is easy, by induction, to construct sequences of elements in $\mathscr{H}^\partial (I^n, m) (f_i)_{i \geqq 1}, (s_i)_{i \geqq 1}, (s'_i)_{i \geqq 1}, (t_i)_{i \geqq 1}$ and $(t'_i)_{i \geqq 1}$ such that:

- $f_1 = f$ and support $(f_i) \subset C_i$;
- support (s_i), support (s'_i), support (t_i) and support $(t'_i) \subset D_i$;
- $f_i = [s_i, t_i][s'_i, t'_i] f_{i+1}$.

Define now k_i by $k_i = [s_i, t_i][s'_i, t'_i] = f_i f_{i+1}^{-1}$. The support of k_i is contained in D_i. Since the D_{2i}'s are disjoint, the infinite composition $k_{\text{even}} = k_2 k_4 k_6 k_8 \ldots k_{2i} \ldots$ has a meaning, it is also clear that this composition preserves Lebesgue measure, moreover it is a homeomorphism because the D_i's converge to a point. In the same way, $k_{\text{odd}} = k_1 k_3 k_5 \ldots k_{2i+1} \ldots$ is a homeomorphism preserving Lebesgue measure. Since $k_i = f_i f_{i+1}^{-1}$ and the support of the f_i's are disjoint, we obtain:

$$k_{\text{odd}} = f_1 f_2^{-1} f_3 f_4^{-1} \ldots = (f_1 f_3 f_5 \ldots)(f_2^{-1} f_4^{-1} f_6^{-1} \ldots),$$
$$k_{\text{even}} = f_2 f_3^{-1} f_4 f_5^{-1} \ldots = (f_2 f_4 f_6 \ldots)(f_3^{-1} f_5^{-1} \ldots).$$

This implies:

$$k_{\text{odd}} \, k_{\text{even}} = f_1 = f.$$

Remark also since $k_i = [s_i, t_i][s'_i, t'_i]$ that:

$$k_{\text{odd}} = [s_{\text{odd}}, t_{\text{odd}}][s'_{\text{odd}}, t'_{\text{odd}}],$$
$$k_{\text{even}} = [s_{\text{even}}, t_{\text{even}}][s'_{\text{even}}, t'_{\text{even}}].$$

$$f = [s_{\text{odd}}, t_{\text{odd}}][s'_{\text{odd}}, t'_{\text{odd}}][s_{\text{even}}, t_{\text{even}}][s'_{\text{even}}, t'_{\text{even}}].$$

Hence, f is a product of commutators.

General case. — Since support $(f) \subset \mathring{I}^n$, by 4.5, we can write $f = f_1 \cdot \ldots \cdot f_q$, where $f_i \in \mathscr{H}^\partial (I^n, m)$ has support contained in a cube $K_i^n \subset \mathring{I}^n$ with measure $\leqq m(C_1)$. Since $m(K_i^n) \leqq m(C_1)$, there exists $g_i \in \mathscr{H}^\partial (I^n, m)$ such that $g_i(K_i^n) \subset C_1$. By the first case, the homeomorphism $g_i f_i g_i^{-1}$ is a product of commutators, hence, by conjugation, f_i is also a product of commutators. This implies that f is a product of commutators. □

Using 7.4 and 6.6, we obtain immediately:

THEOREM 7.5. — *Let* M^n, $n \geqq 3$, *be a compact manifold without boundary, having a handle decomposition and such that there exists a basis of* $H_1(M^n, \mathbb{R})$ *represented by imbedded curves having tubular neighborhoods, and let* μ *be a good measure on* M^n. *The kernel of the map* $\theta : \mathscr{H}_0(M^n, \mu) \to H_1(M^n, \mathbb{R})/\Gamma$ *is perfect, i.e. equal to its commutator subgroup. Hence it is also equal to the commutator subgroup of* $\mathscr{H}_0(M^n, \mu)$, *since* $H_1(M^n, \mathbb{R})$ *is abelian.*

We now explain the Epstein Higman argument ([Ep], [Hi]). This is a general argument showing essentially that perfectness implies simplicity for a group of geometric transformations satisfying some mild hypothesis. We will explain this in our context.

Suppose that M^n and μ are as in Theorem 7.5 and suppose furthermore that M^n is connected, we will show that $\operatorname{Ker}\theta$ is simple, i.e. has no non trivial normal subgroup.

Let H be a normal subgroup of $\operatorname{Ker}\theta$, suppose $H \neq \{\mathrm{Id}\}$. Choose then $f \in H$, with $f \neq \mathrm{Id}$. By this last hypothesis, we can find some non empty open subset U of M^n, such that $U \cap f(U) = \emptyset$.

Suppose that h, $g \in \operatorname{Ker}\theta$ have both support in U. Remark that $[h, f] = hfh^{-1}f^{-1} = (hfh^{-1})f^{-1}$ is in H since it is a normal subgroup, remark also that $[h, f] = h(fh^{-1}f^{-1})$ and that $fh^{-1}f^{-1}$ has support in $f(U)$ which is disjoint from U. Since g has support in U, the last fact implies that g commutes with $fh^{-1}f^{-1}$, hence:

$$[[h, f], g] = h(fh^{-1}f^{-1})g(fhf^{-1})h^{-1}g^{-1}$$
$$= hg(fh^{-1}f^{-1})(fhf^{-1})h^{-1}g^{-1} = hgh^{-1}g^{-1} = [h, g].$$

Since $[h, f]$ is in H, the commutator $[h, g] = [[h, f], g]$ is in H.

So we have showed that if h and $g \in \operatorname{Ker}\theta$, have support in U, then $[h, g] \in H$.

By 7.1, since M^n is connected, $\mathscr{U} = \{k^{-1}(U) \mid k \in \operatorname{Ker}\theta\}$ is an open covering of M^n. Choose some metric defining the topology of M^n and let $\varepsilon > 0$ be a Lebesgue number, with respect to that metric, for the open covering $\mathscr{U}$, this means that each set of diameter less than ε is contained in some member of $\mathscr{U}$. We will show now that if $h, g \in \operatorname{Ker}\theta$ have their supports $\varepsilon/2$-small, then $[h, g] \in H$. We consider two cases. The first one is support $(h) \cap$ support $(g) = \emptyset$, in this case $[h, g] = \mathrm{Id} \in H$. The second case is support $(h) \cap$ support $(g) \neq \emptyset$, hence support $(h) \cup$ support (g) has diameter less than ε, which implies that it is contained in some member of $\mathscr{U}$, say $k^{-1}(U)$. The homeomorphisms khk^{-1} and kgk^{-1} have both their supports in U, hence $k[h, g]k^{-1} = [khk^{-1}, kgk^{-1}]$ is in H, since H is normal this implies that $[h, g] \in H$.

Up to now, we have shown, in particular, that if h and $g \in \operatorname{Ker}\theta$ have their supports $\varepsilon/2$-small, then they commute in $\operatorname{Ker}\theta/H$. By 6.6, $\operatorname{Ker}\theta$ is generated by its elements having support $\varepsilon/2$-small, hence $\operatorname{Ker}\theta/H$ is abelian. We conclude that $[\operatorname{Ker}\theta, \operatorname{Ker}\theta]$ is contained in H; since $\operatorname{Ker}\theta$ is perfect, we obtain $\operatorname{Ker}\theta = H$. Hence $\operatorname{Ker}\theta$ is simple. Remark also that the same argument proves that $\operatorname{Ker}\theta$ is the smallest normal subgroup of $\mathscr{H}_0(M^n, \mu)$.

We have just proved the following Theorem.

THEOREM 7.6. — *Let* M^n, $n \geqq 3$, *be a compact connected manifold without boundary, having a handle decomposition and such that there exists a basis of* $H_1(M^n, \mathbb{R})$ *represented by imbedded closed curves having tubular neighborhoods. Let* μ *be a good measure on* M^n. *The* *kernel of the map* $\theta : \mathscr{H}_0(M^n, \mu) \to H_1(M^n, \mathbb{R})/\Gamma$ *is a simple group. Moreover it is the smallest normal subgroup of* $\mathscr{H}_0(M^n, \mu)$ *and it is also equal to the commutator subgroup of* $\mathscr{H}_0(M^n, \mu)$.

Question. — What happens in the case of a compact surface (i. e. $n=2$) ? Remark that, for $n=2$, Theorem 4.4 is false for C^∞ diffeomorphisms preserving m; in fact [Ba], there exists a surjective homomorphism $R : \mathrm{Diff}_c^\infty(\mathring{I}^2, m) \to \mathbb{R}$ and the kernel of R is simple. As it is defined R has no meaning for homeomorphisms, and to our knowledge the existence (or non existence) of an extension of R to $\mathscr{H}_c(\mathring{I}^2, m)$ is still an open question.

APPENDIX A.7

THE NON COMPACT AND THE NON EMPTY BOUNDARY CASES

Suppose that M^n is either non compact or $\partial M^n \neq \emptyset$ (or both). Let μ be a Radon measure which is >0 on non empty open sets and has no atoms. Consider the group $\mathscr{H}_0^{c, \partial}(M^n, \mu)$ of μ-preserving homeomorphisms isotopic to the identity by a μ-preserving isotopy which has *compact support contained in* $\mathring{M}^n$ (*the interior of* M^n). We have (*see* appendix A.6) a homomorphism $\theta : \mathscr{H}_0^{c, \partial}(M^n, \mu) \to H_1(M^n, \mathbb{R})$.

THEOREM A.7.1. — *Suppose* $n \geqq 3$ *and suppose that each compact set of* M^n *is contained in a compact codimension* 0 *submanifold* N^n *having a handle decomposition and such that there exists a basis of* $H_1(N^n, \mathbb{R})$ *represented by imbedded curves having tubular neighborhoods. Then, the map* $\theta : \mathscr{H}_0^{c, \partial}(M^n, \mu) \to H_1(M^n, \mathbb{R})$ *is surjective and its kernel is a simple group which is equal to the commutator subgroup of* $\mathscr{H}_0^{c, \partial}(M^n, \mu)$. *Moreover*, $\mathrm{Ker}\,\theta$ *is the smallest non trivial normal subgroup of* $\mathscr{H}_0^{c, \partial}(M^n, \mu)$.

The proof of this Theorem is the same as the proof of Theorem 7.6, once we have the results of appendix A.6.

Index of notations

B^n,	Euclidian n-ball in $\mathbf{R}^n$;
I^n,	n-cube $[0, 1]^n$;
$\mathbb{H}^n_+$,	half space in $\mathbf{R}^n$;
$\mathbb{H}^n_{++}$,	first quadrant in $\mathbf{R}^n$;
$\mathring{M}^n$,	interior of the manifold M^n;
∂M^n,	boundary of the manifold M^n;
$\mathrm{Int}(A)$,	interior of the A as a subset of a topological space;
$\mathrm{Fr}(A)$,	frontier, boundary of A as a subset of a topological space;
$\mathscr{H}(X)$,	homeomorphisms group of X;
$\mathscr{H}_0(X)$,	path component of Id_X in $\mathscr{H}(X)$;
$\mathscr{H}(X, \mu)$,	group of μ-preserving homeomorphisms;
$\mathscr{H}^\partial(M^n)$,	homeomorphisms fixing ∂M^n;
$\mathscr{H}(M^n, \mu\text{-bireg})$,	group of homeomorphisms biregular for μ;
$\mathscr{H}^\partial(M^n, \mu\text{-bireg})$,	$\mathscr{H}^\partial(M^n) \cap \mathscr{H}(M^n, \mu\text{-bireg})$;
$\tilde{\mathscr{H}}_0(X)$,	set of homotopy classes of isotopies of X, also the universal cover of $\mathscr{H}_0(X)$ when it exists;

$\mathscr{I}(A; M^n)$,	set of imbeddings of A in M^n;
$\mathscr{I}(A, B; M^n)$,	$\{ j \in \mathscr{I}(A; M^n) \mid j \mid B = \mathrm{id} \}$;
$\mathscr{I}(A; M^n, \mu)$,	set of μ-preserving imbeddings of A in M^n;
$\mathscr{I}(A; M^n, \mu\text{-bireg})$,	set of μ-biregular imbeddings of A in M^n;
$\mathscr{I}(A, B; M^n, \mu)$	$\mathscr{I}(A; M^n, \mu) \cap \mathscr{I}(A, B; M^n)$;
$\mathscr{I}(A, B; M^n, \mu\text{-bireg})$,	$\mathscr{I}(A, B; M^n, \mu\text{-bireg}) \cap \mathscr{I}(A, B; M^n)$;
$\mathscr{I}\mathscr{S}(X)$,	set of isotopies of X;
$\mathscr{I}\mathscr{S}(X, \mu)$,	set of μ-preserving isotopies of X;
$\mathscr{M}(X)$,	set of probability measures on X;
$\mathscr{M}_g(X)$,	set of good measures on X;
$\mathscr{M}_g^\partial(M^n)$,	set of good measures μ on M^n such that $\mu(\partial M^n) = 0$;
$\mathscr{M}_g^\partial(M^n, \mu_0)$,	set of $\mu \in \mathscr{M}_g^\partial(M^n, \mu_0)$ which have the same sets of measure 0 as μ_0.

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