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ANDREI N. TODOROV

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SURFACES OF GENERAL TYPE WITH $p_g=1$ AND $(K, K)=1$. I

By Andrei N. TODOROV

Introduction

The aim of this article is to describe all surfaces with $p_g=1$ and $(K, K)=1$. The first examples of such surfaces were constructed by Kunev in [Ku]. Here we give the following description of all surfaces with $p_g=1$ and $(K, K)=1$: every such surface is a complete intersection of two quasi-homogeneous polynomials in $\mathbb{P}^4(1, 2, 2, 3, 3)$. This fact was conjectured by M. Reid and I learned it from I. Dolgacev. From this description it follows that the moduli space of surfaces with $p_g=1$ and $(K, K)=1$ consists of one component. These surfaces are interesting because they are simply connected and the local Torelli theorem is not true for some of them. Thus surfaces with $p_g=1$ and $(K, K)=1$ that are canonical Galois coverings of $\mathbb{P}^2$ give counter examples to a conjecture of P. Griffiths, which states that the local Torelli theorem is true for all simply-connected surfaces of general type with $p_g \geqq 1$. Even more the auther recently proved that these surfaces give counter examples to global Torelli theorem. We give a complete description of all Galois coverings of $\mathbb{P}^2$ with $p_g=1$ and $(K, K)=1$. For surfaces with $p_g=1$ and $(K, K)=1$ that are not a canonical Galois coverings of $\mathbb{P}^2$ the local Torelli theorem is true.

The auther wants to express his gratitute to his sudent and friend V. Kunev for many valuable conversations during the preparation of this article. This resulted in improvements of some of the proofs. Part of these results were reported in the Mathematische Arbeitstagung 1978 in Bonn. The auther wants to express his gratitude to the organizers of this conference for the extremely stimulating atmosphere created during the conference.

1. A description of all surfaces with $p_g=1$ and $(K, K)=1$

We need some definitions in order to formulate Theorem 1.

Definition 1. — An weighted projective space of type $(w_0, w_1, \ldots, w_n)$, where w_i are positive integers, is defined as Proj $\mathbb{C}(w_0, \ldots, w_n)$, where $\mathbb{C}(w_0, \ldots, w_n)$ is the polynomial ring with the following graduation, $\deg x_i = w_i$.

Definition 2. — We will say that $f(x_0, \ldots, x_n) = \sum_k a_k x^k$ is a quasi-homogeneous polynomial of type $(w_0, \ldots, w_n)$ of deg m iff $k = (k_0, \ldots, k_n)$ and $k_0 w_0 + \ldots + k_0 w_n = m$.

Definition 3. — An weighted complete intersection in $\mathbb{P}^n(w_0, \ldots, w_n)$ we will call a variety V, whose ideal in the graded ring $\mathbb{C}(x_0, \ldots, x_n)$ is generated by a regular sequence of quasi-homogeneous polynomials $f_{d_0}, \ldots, f_{d_k}$, where d_i is the degree of f_{d_i}.

Theorem 1. — *Every surface with an ample canonical class,* $p_g = 1$ *and* $(K, K) = 1$ *is a complete intersection of type* $(6, 6)$ *in* $\mathbb{P}^4(1, 2, 2, 3, 3)$.

Proof. — First I will give the reason for choosing $\mathbb{P}^4(1, 2, 2, 3, 3)$ as a space of embedding surfaces with $p_g = 1$ and $(K, K) = 1$. First I will recall some facts proved by V. Kunev for surfaces with $p_g = 1$ and $(K, K) = 1$:

Theorem (*see* [Ku]). — *Let* S *be a minimal model of a surface with* $p_g = 1$ *and* $(K, K) = 1$. *Then* (*a*) *the complete linear system* $|2K_S|$ *gives a holomorphic map* $f_{|2K_S|} : S \to P^2$, (*b*) *the complete linear system* $|3K_S|$ *gives a holomorphic birational map.*

Bombieri proved in [Bom] the following lemma: Let S be a minimal model of a surface with $p_g = 1$ and $(K, K) = 1$, then the general element of $|2K_S|$ is irreducible and nonsingular.

From the definition we know that $\dim H^0(S, \Omega_S^2) = 1$. Let $H^0(S, \Omega_S^2)$ be generated by s_0. From Riemann-Roch we get that $\dim H^0(S, O(2K_S)) = 3$. Let $H^0(S, O(2K_S))$ be generated by s_0^2, s_1, s_2. From Kunev's theorem and Bombieri's lemma, it follows that we can choose s_1 and s_2 in the following way; let C_1 and C_2 be the divisors of s_0 and s_1, then we may suppose that C_1 and C_2 are nonsingular curves intersecting each other transversally. From Riemann-Roch theorem it follows that $\dim H^0(S, O(3K_S)) = 5$. Let $H^0(S, O(3K_S))$ be generated by $s_0^3, s_0 s_1, s_0 s_2, s_3, s_4$. From Kunev's theorem it follows that we can choose the divisors of s_3 and s_4, C_3 and C_4, so that C_3 intersects C_4 transversally and both of them intersect C_1 and C_2 transversally. The theorem of Kunev gives us a hope that S can be embedded in $\mathbb{P}^4(1, 2, 2, 3, 3)$, i.e. in $\operatorname{Proj} \mathbb{C}(s_0, s_1, s_2, s_3, s_4)$, where $\deg s_0 = 1$, $\deg s_1 = 2$, $\deg s_2 = 2$, $\deg s_3 = 3$ and $\deg s_4 = 3$.

Remark. — From now on all curves C_i will be fixed, where C_i is the divisor of s_i for all $i > 0$ and C_i are nonsingular and have the properties described above.

In order to prove Theorem 1, we need the following construction:

The construction of X_4. — From the fact $C_1 \in |2K_S|$ and the results of Wawrik [W] we can construct a $\mathbb{Z}_2$ cyclic covering $p_1 : X_1 \to S$ ramified over C_1. Let me denote by $|H_1|$ the complete linear system $|p_1^* K_S|$. Let $p_2 : X_2 \to X_1$ be a $\mathbb{Z}_2$ covering of X_1 ramified over $p_1^* C_2$. Let me denote by $H_2 = p_2^* H_1$. It is clear that $(p_1 p_2)^* C_3$ belongs to $|3H_2|$ and we can construct a cyclic $\mathbb{Z}_3$ covering $p_3 : X_3 \to X_2$ ramified over $(p_2 p_1)^* C_3$. Again I will denote by $H_3 = p^*{}_3 H_2$. We see immediately that $(p_3 p_2 p_1)^* C_4$ belongs to $|3H_3|$ so that we can construct a cyclic $\mathbb{Z}_3$ covering $p_4 : X_4 \to X_3$ ramified over $(p_3 p_2 p_1)^* C_4$. From the fact that all C_i are nonsingular and transect each other transversally, we conclude that all X_i are nonsingular surfaces, $i = 1, 2, 3, 4$. If we can prove that X_4 can be embedded as a complete intersection of type (6, 6) in $\mathbb{P}^4$, Theorem 1 will be proved, because $\mathbb{P}^4(1, 2, 2, 3, 3)=\mathbb{P}^4/G$, where G is a group which acts in the following way

$$(g, (x_0 : x_1 : x_2 : x_3 : x_4)=(x_0 g_0 : x_1 g_1 : x_2 g_2 : x_3 g_3 : x_4 g_4)$$
$$g_i=\exp(2\pi b_i/w_i), \qquad 0 \leqq b_i < w_i.$$

The equivalence of these two definitions is proved in [D]. Thus our aim is to prove that X_4 is a complete intersection of type (6, 6) in $\mathbb{P}^4$.

LEMMA 1. — (*a*) $\dim H^0(X_1, O(H_1))=2$, (*b*) $|H_1|$ *does not have fixed components*, (*c*) $(H_1, H_1)=2$.

Proof. — The proof is based on the following remark: $\mathbb{Z}_2=(1, s)$ acts on $H^0(X_1, O(H_1))$ and so $H^0(X_1, O(H_1))=H^0(O(H_1))^+ \oplus H^0(O(H_1))^-$, where $H^0(O(H_1))^+$ is the invariant and $H^0(O(H_1))^-$ is the anti-invariant subspace. It is a well-known fact that $H^0(O(H_1))^+=p_1^* H^0(O(K_S))$ and thus $\dim H^0(O(H_1))^+=1$. Now we must compute $\dim H^0(O(H_1))^-$. Notice that $O(H_1)=p_1^* O(K_S)$ and it follows that the cocycle defining $O(H_1)$ is of the form $f_{ij}=p_1^*(g_{ij})$. Let U_i be a covering of X_1 by polycylinders. If $f \in H^0(O(H_1))^-$ then it follows that $f^s=-f$ and $f_i^s=-f_i$, where $f_i=f_{U_i}$. Indeed, from the definition of f it follows that $f_i=f_{ij} f_j$ and so from $f_{ij}^s=f_{ij}$ it follows that $f_i^s=-f_i$. Now let U_i contains the branch locus of p_1, C_1'. It is a well-known fact that we can choose the local coordinate system (x_i, y_i) in U_i in such a manner that $x_i^s=x_i$ and $y_i^s=-y_i$, where y_i is the local equation of C_1' in U_i. Now let

(1.1) $$f_i(x_i, y_i)=\sum a_{mn} x_i^m y_i^n \quad\text{and}\quad f_i^s=\sum(-1)^n a_{mn} x_i^m y_i^n,$$

(1.2) $$f_i^s=-f_i \quad\text{iff } f_i=\sum a_{mn} x_i^m y_i^{2n+1}, \qquad \text{where } m \text{ and } n>0.$$

So

(1.3) $$f_i=-f_i \quad\text{iff } f_i=y_i g_i(x_i, y_i^2).$$

From (1.3) it follows that if $f^s=-f$ then $(f)=C_1'+D$, where D is an effective divisor on X_1. If we can prove that C_1' is rationally equivalent to H_1, then from (1, 3) it will follows that $\dim H^0(O(H_1))^-=1$.

PROPOSITION 1.1. — *The branch locus of* C_1' *is rationally equivalent to* H_1.

Proof. — See [W].

Q.E.D.

Proposition 1.1 proves (*a*) of Lemma 1.

Q.E.D.

$|H_1|$ does not have fixed components because $C_1' \in |H_1|$ and it is a nonsingular curve. Thus (*b*) is proved.

On S we have $(K_S, C_1)=(K_S, 2K_S)=2$ and on X_1 we have

$$(p_1^* K_S, p_1^* C_1)=\deg(p_1)\times(K_S, C_1)=4=(H_1, 2C_1')=(H_1, 2H_1).$$

So we obtain that $(H_1, H_1)=2$.

Q.E.D.

LEMMA 2. — $\dim H^0(O(H_2))=3$. *We can choose* C_2 *in such a manner that the linear system* $|H_2|$ *gives a holomorphic map* $X_2 \to \mathbb{P}^2$, $(H_2, H_2)=4$.

Proof. — We know that $\mathbb{Z}_2=(1, s)$ acts on X_2 and $X_2/s=X_1$ and so we can repeat the arguments of Lemma 1 and conclude that $H^0(O(H_2))=H^0(O(H_2))^+ + H^0(O(H_2))^-$, where $H^0(O(H_2))^+ = p_2^*(H^0(O(H_1))$ and $H^0(O(H_2))^-$ is generated by f, where $(f)=C'_2$ is the branch locus of p_2. From all these facts and Lemma 1 we get that $\dim H^0(O(H_2))=3$.

(*b*) If $|H_1|$ has base points, these points can be at most two because of $(H_1, H_1)=2$. Let these two points be P_1 and P_2. From Kunev's theorem it follows that we can choose C_2 in a such a manner that C_2 does not contain the images of P_1 and P_2 on S. Now our result follows from the decomposition

$$H^0(O(H_2))=H^0(O(H_2))^+ \oplus H^0(O(H_2))^- = p_2^* H^0(O(H_1))+\mathbb{C} f,$$

where $(f)=C'_2$ the branch locus of p_2 and $(p_2 p_1)^* C_2=2C'_2$.

(*c*) The proof of $(H_2, H_2)=4$ is the same as the proof of $(H_1, H_1)=2$.

Q.E.D.

LEMMA 3. — (*a*) $\dim H^0(X_3, O(H_3))=4$, (*b*) *the complete system* $|H_3|$ *gives a holomorphic map* $g_3 : X_3 \to Y \subsetneq \mathbb{P}^3$, Y *is a hypersurface of degree* 6, X_3 *is a double covering of* Y *ramified over a curve rationally equivalent to* 6H, H *is the hypersurface section on* Y. (*c*) $(H_3, H_3)=12$.

Proof. — The proof is based on several steps.

STEP 1. — $\dim H^0(O(H_3))=4$.

Proof. — $\mathbb{Z}_3=(1, s, s^2)$ acts on X_3 and thus on $H^0(O(H_3))$. From here it follows that

$$H^0(O(H_3))=H^0(O(H_3))^+ \oplus H^0(O(H_3))^{\varepsilon} \oplus H^0(O(H_3))^{\varepsilon^2},$$

where $H^0(O(H_3))^+$ is the invariant subspace and $H^0(O(H_3))^{\varepsilon}$ and $H^0(O(H_3))^{\varepsilon^2}$ are eigen subspaces with eigen values ε and ε^2, where $\varepsilon^3=1$ and $\varepsilon\neq 0$. From $H^0(O(H_3))^+ = p_3^* H^0(O(H_2))$ follows that $\dim H^0(O(H_3))^+ =3$ (this is Lemma 2).

PROPOSITION 3.1. — $\dim H^0(O(H_3))^{\varepsilon}=1$ *and* $\dim H^0(O(H_3))^{\varepsilon^2}=0$.

Proof. — Let U_i be a covering of X_3. Let f and g be elements of $H^0(O(H_3))^{\varepsilon}$ and $H^0(O(H_3))^{\varepsilon^2}$ respectively. Let me denote by f_i and g_i, $f|_{U_i}$ and $g|_{U_i}$. If $U_i \cap C'_3 \neq \emptyset$, where C'_3 is the branch locus of p_3, then we can choose the coordinates in U_i in the following manner: $x_i^s=x_i$ and $y_i^s=\varepsilon y_i$, where y_i is the local equation of C'_3 in U_i. Repeating the same arguments as in Lemma 1 we get

$$(3.2)\qquad f_i^s=\varepsilon f_i \quad\text{and}\quad g_i^s=\varepsilon^2 g_i \qquad \text{if}\quad f^s=\varepsilon f \quad\text{and}\quad g^s=\varepsilon^2 g.$$

Let

$$(3.3)\qquad f_i=\sum a_{mn} x_i^m y_i^n \qquad\text{and}\qquad g_i=\sum b_{mn} x_i^m y_i^n.$$

From (3.2) and (3.3) we obtain:

$$(3.4)\qquad f_i^s=\varepsilon f_i \quad \text{iff } f_i=y_i f'_i(x_i, y_i^3) \qquad\text{and}\qquad g_i=\varepsilon^2 g_i \quad \text{iff } g_i=y_i^2 g'_i(x_i, y_i^3).$$

From (3.4) it follows that if f and g are elements of $H^0(O(H_3))^{\varepsilon}$ and $H^0(O(H_3))^{\varepsilon^2}$ respectively, then $(f)=C'_3+D$ and $(g)=2C'_3+D_1$. Proposition 3.1 follows from the fact that C'_3 is rationally equivalent to H_3. For the proof of this fact, *see* [W].

Q.E.D.

Remark. — Notice that we have proved that $H^0(O(H_3))=p_3^* H^0(O(H_2))+\mathbb{C}\, y$, where $\mathbb{C}$ is the complex number field and $(y)=C'_3$.

STEP 2. — $(H_3, H_3)=12$.

Proof. — The proof is the same as the proof for $(H_1, H_1)=2$.

Q.E.D.

STEP 3. — $\deg g_3(X_3)$ is one of the following numbers: 2, 3, 4, 6 and 12.

Proof. — It follows from Lemma 2 and the remark after Step 1 that the complete linear system $|H_3|$ gives a holomorphic map $g_3 : X_3 \to Y \subsetneq P^3$. Now Step 3 follows from the following formula: $(H_3, H_3)=\deg g_3 \times (H, H)_Y$, where $(H, H)_Y$ is the selfintersection number of the hyperplane section on Y.

Q.E.D.

STEP 4. — Let x_1, x_2, x_3 and x_4 be sections of $H^0(O(H_3))$, which are linearly independent and generate $H^0(O(H_3))$. Then all monomials formed from x_1, x_2, x_3 and x_4 and having degree 4 are linearly independent in $H^0(O(4H_3))$. We suppose that $\deg x_i=1$ for all i.

Proof. — The proof is based on several propositions.

PROPOSITION 3.2 :

$$H^0(O(4H_3))=p_3^* H^0(O(H_2))+x_4\, p_3^* H^0(O(3H_2))+x_4^2\, p_3^* H^0(O(2H_2)),$$

where x_4 is such that $(x_4)=C'_3$, the branch locus of p_3.

Proof. — From the way we constructed X_3 we know that $\mathbb{Z}_3$ acts on X_3. From here it follows that $\mathbb{Z}_3$ acts on $H^0(O(H_3))$. From this action we get the following decomposition

$$H^0(O(4H_3))=H^0(O(4H_3))^{+}+H^0(O(4H_3))^{\varepsilon}+H^0(O(4H_3))^{\varepsilon^2},$$

where $H^0(O(4H_3))^{+}$ is the invariant subspace, $H^0(O(4H_3))^{\varepsilon}$ and $H^0(O(4H_3))^{\varepsilon^2}$ are eigen subspaces with eigen values ε and ε^2. Repeating the same arguments as in Step 1, we get

$$(3.5) \qquad \begin{cases} f\in H^0(O(4H_3))^{\varepsilon} & \text{iff } f|_{U_i}=f_i(x_i, y_i)=y_i f'_i(x_i, y_i^3), \\ g\in H^0(O(4H_3))^{\varepsilon^2} & \text{iff } g|_{U_i}=g_i(x_i, y_i)=y_i^2 g'_i(x_i, y_i^3), \end{cases}$$

where (x_i, y_i) is a local coordinate system in U_i such that $x_i^{\varepsilon}=x_i\, yn\, dy_i^{\varepsilon}=y_i$, y_i is the local equation of C'_3 in U_i. From (3.5) and the fact that C'_3 is rationally equivalent to H_3 we obtain:

$$(3.6) \quad \begin{cases} f\in H^0(O(4H_3))^{\varepsilon} \ \text{iff } f=x_4 f', \quad \text{where} \ (x_4)=C'_3 \ \text{and} \ f'\in p_3^* H^0(O(3H_2)), \\ g\in H^0(O(4H_3))^{\varepsilon^2} \ \text{iff} \ g=x_4^2 g', \ \text{where} \ g'\in p_3^* H^0(O(2H_2)). \end{cases}$$

PROPOSITION 3.2. — *Follows from* (3.6) *and the fact* $H^0(O(4H_3))^+ = p_3^* H^0(O(4H_2))$.

Q.E.D.

PROPOSITION 3.3:

$$H^0(O(4H_2)) = (p_2 p_1)^* H^0(O(4K_S)) + y_2 (p_2 p_1)^* H^0(O(3K_S)) + y_1 (p_2 p_1)^* H^0(O(3K_S)) + y_2 y_1 (p_2 p_1)^* H^0(O(2K_S)),$$

where $(y_2) = C'_2$ *(the branch locus of* p_2*) and* $y_1 = (p_2)^* z_1$, $(z_1) = C'_1$, *the branch locus of* p_1.

Proof. — $\mathbb{Z}_2$ acts on X_2 and so it acts on $H^0(O(4H_2))$. Thus we have

$$H^0(O(4H_2)) = H^0(O(4H_2))^+ + H^0(O(4H_2))^-.$$

We know that

$$H^0(O(4H_2))^+ = p_2^* H^0(O(4H_1)).$$

Repeating the same arguments as in Proposition 4.1 we will get that

$$H^0(O(4H_2))^- = y_2 p_2^* H^0(O(3H_1)),$$

where $(y_2) = C'_2$. So we get

$$H^0(O(4H_2)) = p_2^* H^0(O(4H_1)) + y_2 p_2^* H^0(O(3H_1)). \tag{3.7}$$

We know that X_1 is a double covering of S ramified over C_1. $\mathbb{Z}_2$ acts on X_1. From this action we get

$$H^0(O(4H_1)) = p_2^* H^0(O(4H_1))^+ + H^0(O(4H_1))^-. \tag{3.8}$$

Repeating the arguments of Remark 2 we obtain:

$$H^0(O(4H_1))^+ = p_1^* H^0(O(4K_S)), \tag{3.9}$$

$$H^0(O(4H_1))^- = z_1 p_1^* H^0(O(3K_S)), \tag{3.10}$$

where $(z_1) = C'_1$ the branch locus of p_1 and $z_1 \in H^0(O(H_1))$.

Repeating the same discussion for $H^0(O(3H_1))$ we get that

$$H^0(O(3H_1)) = p_1^* H^0(O(3K_S)) + z_1 p_1^* H^0(O(2K_S)). \tag{3.11}$$

Combining (3.8), (3.9) and (3.10) we get

$$H^0(O(4H_1)) = p_1^* H^0(O(4K_S)) + z_1 p_1^* H^0(O(3K_S)). \tag{3.12}$$

Putting (3.11) and (3.12) in (3.7) leads us to

$$H^0(O(4H_2)) = (p_2 p_1)^* H^0(O(4K_S)) + p_2(z_1)(p_2 p_1)^* H^0(O(3K_S)) + y_2 (p_2 p_1)^* H^0(O(2K_S)) + y_2 p_2^*(z_1) H^0(O(2K_S)). \tag{3.13}$$

(3.13) Proves Proposition 3.3 if we take into account that $y_1=p_2^*(z_1)$.

Q.E.D.

Remark. — We can choose x_1, x_2, x_3 and x_4 (a basis of $\mathrm{H}^0(\mathrm{X}_3, \mathrm{O}(\mathrm{H}_3))$) in a such way that $x_1^2=(p_3 p_2 p_1)^*(s_1)$, $x_2^2=(p_3 p_2 p_1)^*(s_2)$, $x_3=(p_3 p_2 p_1)^*(s_0)$ and $x_3^4=(p_3 p_2 p_1)^*(s_3)$, where $(s_0)=\mathrm{K}_\mathrm{S}$, $(s_1)=\mathrm{C}_1$, $(s_2)=\mathrm{C}_2$ and $(s_3)=\mathrm{C}_3$.

Proof. — In Lemma 1 we proved that $\mathrm{H}^0(\mathrm{O}(\mathrm{H}_1))=p_1^*\mathrm{H}^0(\mathrm{O}(\mathrm{K}_\mathrm{S}))+\mathbb{C}z_1$, where $(z_1)=\mathrm{C}_1'$, the branch locus of p_1. From the fact that X_1 is a double covering of S ramified over C_1, it follows that $p_1^*(s_1)=z_1^2$. In Lemma 2 we proved that

$$\mathrm{H}^0(\mathrm{O}(\mathrm{H}_2))=p_2^*\mathrm{H}^0(\mathrm{O}(\mathrm{H}_1))+\mathbb{C}y_2,$$

where $(y_2)=\mathrm{C}_2'$, the branch locus of p_2. From the fact that X_2 is a double covering of X_1 ramified over $p_1^*(\mathrm{C}_2)$, it follows that $(p_2 p_1)^*(s_2)=y_2^2$. In Lemma 3 we proved that

$$\mathrm{H}^0(\mathrm{O}(\mathrm{H}_3))=p_3^*\mathrm{H}^0(\mathrm{O}(\mathrm{H}_2))+\mathbb{C}x_4,$$

where $(x_4)=\mathrm{C}_3'$. From the fact that X_3 is a cyclic $\mathbb{Z}_3$ covering of X_2 ramified over $(p_2 p_1)^*(\mathrm{C}_3)$ it follows that $(p_3 p_2 p_1)^*(s_3)=x_4^3$. Combining all these facts we conclude that

$$\mathrm{H}^0(\mathrm{O}(\mathrm{H}_3))=(p_3 p_2 p_1)^*\mathrm{H}^0(\mathrm{O}(\mathrm{K}_\mathrm{S}))+\mathbb{C}(p_2 p_1)^*(z_1)+\mathbb{C}p_3^*(y_2)+\mathbb{C}x_4.$$

Now taking into acount that $\mathrm{H}^0(\mathrm{O}(\mathrm{K}_\mathrm{S}))=\mathbb{C}s_0$ and denoting by $x_1=(p_3 p_2)^*(z_1)$, $x_2=p_3^*(y_2)$, $x_3=(p_3 p_2 p_1)^*(s_0)$ we can state that $\mathrm{H}^0(\mathrm{O}(\mathrm{H}_3))$ is generated by x_1, x_2, x_3 and x_4.

Q.E.D.

Proposition 3.4:

(*a*) $$\mathrm{H}^0(\mathrm{O}(3\mathrm{H}_2))=(p_2 p_1)^*\mathrm{H}^0(\mathrm{O}(3\mathrm{K}_\mathrm{S}))+y_2(p_2 p_1)^*\mathrm{H}^0(\mathrm{O}(2\mathrm{K}_\mathrm{S})) +y_1(p_2 p_1)^*\mathrm{H}^0(\mathrm{O}(2\mathrm{K}_\mathrm{S}))\,y_1 y_2(p_2 p_1)^*\mathrm{H}^0(\mathrm{O}(\mathrm{K}_\mathrm{S})).$$

y_1 *and* y_2 *have the same meaning as in Proposition* 3.4.

(*b*) $$\mathrm{H}^0(\mathrm{O}(2\mathrm{H}_2))=(p_2 p_1)^*\mathrm{H}^0(\mathrm{O}(2\mathrm{K}_\mathrm{S}))+y_1(p_2 p_1)^*\mathrm{H}^0(\mathrm{O}(\mathrm{K}_\mathrm{S})) +y_2(p_2 p_1)^*\mathrm{H}^0(\mathrm{O}(\mathrm{K}_\mathrm{S}))+\mathbb{C}y_1 y_2.$$

Proof. — Repeat the proof of Proposition 3.3.

Q.E.D.

Proposition 3.5. — $\mathrm{H}^0(\mathrm{O}(4\mathrm{K}_\mathrm{S}))$ *is generated by* s_0^4, $s_0^2 s_1$, $s_0^2 s_2$, $s_0 s_3$, $s_0 s_4$, s_1^2, s_2^2 *and* $s_1 s_2$. *The* s_i *are chosen in the way pointed out on Paragraph* 1.

Proof. — From the exact sequence

$$0\to\mathrm{O}(3\mathrm{K}_\mathrm{S})\xrightarrow{\otimes s_0}\mathrm{O}(4\mathrm{K}_\mathrm{S})\to\mathrm{O}(4\mathrm{K}_\mathrm{S})|_{\mathrm{K}_\mathrm{S}}\to 0$$

we get the following inclusion

$$0\to\mathrm{H}^0(\mathrm{O}(3\mathrm{K}_\mathrm{S}))\xrightarrow{\otimes s_0}\mathrm{H}^0(\mathrm{O}(4\mathrm{K}_\mathrm{S})).$$

From this inclusion it follows that $s_0^4, s_0^2 s_1, s_0^2 s_2, s_0 s_3, s_0 s_4$ are linearly independent. Let me denote the vector space spanned by these linearly independent vectors by V_1. The subspace V_1 has dimension 5. Let me denote the subspace spanned by s_1^2, s_2^2 and $s_1 s_2$ by V_2. We will show that $\dim V_2=3$. If $\dim V_2<3$ then we will have $a_1 s_1^2+a_2 s_2^2+a_3 s_1 s_2=0$. From this equation we get $a_1 s_1^2=s_2(a_2 s_2+a_3 s_1)$. From the last equation it follows that C_2 is contained in C_1. This is impossible. If $V_1 \cap V_2=\emptyset$, then Proposition 3.5 will be proved. Suppose that $V_1 \cap V_2 \neq \emptyset$ and let $v \in V_1 \cap V_2$ and $v \neq 0$. Thus

$$v=b_1 s_1+b_2 s_2+b_3 s_1 s_2=s_0(c_1 s_0^3+c_2 s_0 s_1+c_3 s_3+c_4 s_0 s_2+c_5 s_4).$$

From this formula we obtain:

$$b_1 s_1^2+b_2 s_2^2+b_3 s_1 s_2 \equiv 0 \quad \text{on } K_S. \tag{3.14}$$

Notice that it is impossible. Indeed, let U be a neighborhood of a point on K_S. Let $s_1|_U=f_1$ and $s_2|_U=f_2$. From the definition of s_1 and s_2 and Kunev's theorem it follows that we can find a point $P \in K_S \cap U$ such that $f_1(P) \neq 0$ and $f_2(P) \neq 0$. This fact contradicts (3.14). Proposition 3.5 is thus proved.

Q.E.D.

The end of the proof of Step 4. – Let me denote by $P_3=p_3 p_2 p_1$. Combining Propositions 3.2, 3.3 and 3.4 and taking into account the remark after Proposition 3.3, we will obtain the following formula

$$\begin{aligned} \text{(3.15)}\quad H^0(O(4H_3)) &= P_3^* H^0(O(4K_S))+x_1 P_3^* H^0(O(3K_S))+x_2 P_3^* H^0(O(3K_S)) \\ &+x_1 x_2 P_3^* H^0(O(2K_S))+x_4 x_1 P_3^* H^0(O(2K_S))+x_4 x_2 P_3^* H^0(O(2K_S)) \\ &+x_4 P_3^* H^0(O(3K_S))+x_4 x_1 x_2 P_3^* H^0(O(K_S))+x_4^2 P_3^* H^0(O(K_S)) \\ &+x_4^2 x_1 P_3^* H^0(O(K_S))+x_4^2 x_2 P_3^* H^0(O(K_S))+x_4^2 x_1 x_2 \mathbb{C}. \end{aligned}$$

Note that this is a decomposition into a direct sum. From (3.15) we come to:

PROPOSITION 3.6. – *The basis of* $H^0(X_3, O(3H_3))$ *consists of all monomials of degree* 4 *formed of* x_1, x_2, x_3 *and* x_4 *plus* $x_1 P_3(s_4)$, $x_2 P_3(s_4)$, $x_3 P(s_4)$ *and* $x_4 P_3(s_4)$.

STEP 5. – $\deg g_3(X_3)=6$, i.e. $g_3(X_3)$ is a hypersurface of degree 6 in $\mathbb{P}^3$.

Proof. – From Step 4 we see that $Y=g_3(X_3)$ cannot be a hypersurface of degree less or equal to 6. From Step 3 it follows that deg Y is either 6 or 12. Suppose that $\deg Y=12$. From this fact it follows that all monomials of degree 6 formed of x_1, x_2, x_3 and x_4 are linearly independent in $H^0(X_3, O(6H_3))$. It is clear that we have the following inclusion; $P_3^*: H^0(S, O(6K_S)) \subsetneq H^0(X_3, O(6H_3))$. From this inclusion and the fact that all monomials of degree 6 formed of x_1, x_2, x_3 and x_4 are linearly independent it follows that s_0^6, $s_0 s_1 s_3$, $s_0 s_2 s_3$, $s_0^4 s_1$, $s_0^4 s_2$, $s_0^3 s_3$, $s_0^2 s_1^2$, $s_0^2 s_2^2$, $s_0^2 s_1 s_2$, s_3^2, s_1^3, s_2^3, $s_1^2 s_2$ and $s_1 s_2^2$ are linearly independent vectors in $H^0(S, O(6K_S))$ and spanned a vector subspace V of dimension 14. (Formally the proof of the fact that $\dim V=14$ follows from the remark on Paragraph 7 and the above inclusion.) We have the following standart exact sequence

$$0 \to H^0(O(3K_S)) \xrightarrow{\otimes s_4} H^0(O(6K_S)) \xrightarrow{r} H^0(O(6K_S)|_{C_4}).$$

From this exact sequence it follows that if $v\neq 0$ and $v\in V$, then $r(v)\neq 0$, so $V\cap s_4\otimes H^0(O(3K_S))=\emptyset$. From this fact we obtain that

$$\dim H^0(O(6K_S))\geqq \dim V+\dim H^0(O(3K_S))=19.$$

From Kodaira vanishing theorem for surfaces of general type, i. e. $\dim H^i(S, O(nK_S))=0$ it i and n are greater then 0, and Riemann-Roch theorem, we get that $\dim H^0(O(6K_S))=17$. This contradiction proves Proposition 3.6.

Q.E.D.

STEP 6. — Suppose that K_S is an ample divisor. Then Y is a nonsingular variety.

Proof. — Mumford proved that $\mathrm{Proj}(\oplus H^0(S, O(nK_S)))$ is a nonsingular model of a surface of general type S if K_S is an ample divisor. K_S is the canonical class of S. From this result it follows that:

PROPOSITION 3.7. — $\mathrm{Proj}(\oplus H^0(X_3, O(nH_3)))$ *is a nonsingular model of* X_3.

Proof. — From Lemma 3 it follows that $q(X_4)=Y$ is a surface of degree 6 in $\mathbb{P}^3$. If X_1. Let me denote by R_1 the ring $\oplus H^0(X_1, O(nH_1))$ and by R the canonical ring of S, i. e. $R=H^0(S, O(nK_S))$. We must prove that for any maximal ideal m in R_1, the local ring $R_{1\,(m)}$ is regular. Notice that $R_1=R[X]/(X^2-s_1)$, so $m'=m\cap R$ is a maximal ideal in R if m is maximal one in R_1. For the proof of this fact look at Zariski and Samuel book *Commutative Algebra*. If the ideal m' does not contain the ideal (s_0) then $\hat{R}_{(m')}\cong\hat{R}_{1\,(m)}$. For the proof of this see Zariski and Samuel (the sign $\wedge$ means the completion in the m-adic topology). Now it is a standart fact from the local algebra that if the completion of a local ring is a regular one then the local ring is also regular. So in this case Proposition 3.7 is proved. Now suppose that $(s_1)\subset m'$. It is clear that we have the following isomorphism: $R_{1\,(m)}=R_{(m')}[X]/(X^2-s_1)$. The ring $R_{(m')}[X]/(X^2-s_1)$ is regular iff $s_1\neq 0 \bmod m'^2$, i. e. s_1 is a local parameter in $R_{(m')}$. The last condition is fulfilled because the divisor of s_1, C_1, is a nonsingular curve. So from here Proposition 6.1 follows. The criterium we used is proved in Serre book *Local Algebra* in *Springer Lecture Notes*. If we repeat the same arguments for the rings $R_i=H^0(X_i, O(nH_i))$, $i=2$, 3 and 4 we will get that $\mathrm{Proj}(R_i)$ is a nonsingular model of X_i.

Q.E.D.

PROPOSITION 3.8. — Y *is a nonsingular hypersurface in* P^3.

Proof. — First we will prove that $Z_2=(1, s)$ acts on X_3 and $X_3/s=Y$, i. e. $g_3: X_3\to Y$ is the natural map $X_3\to X_3/s$. Let me denote by $K(X_3)$ the field of rational functions on X_3 and by $K(Y)$ the field of rational functions on Y. We have the natural inclusion: $K(Y)\subsetneq K(X_3)$. From Step 5, i.e. $\deg g_3=2$, we get that $\deg(K(X_3):K(Y))=2$. So $K(X_3)$ is a Galois extension of $K(Y)$ with a Galois group $G=Z_2$, i.e. $K(Y)=K(X_3)^G$. From the fact that $K(Y)$ is the quotion field of the subring $R'\subsetneq R_3$ generated by x_1, x_2, x_3 and x_4, it follows that $K(Y)\cap R_3=R'$. From this fact we get immediately that $R'=R_3^G$. That G acts on R_3 follows from the following theorem: Every birational automorphism is a biregular one on the minimal model of a surface of general type. From the definition of R' it follows that $\mathrm{Proj}(R')=Y=g_3(X_3)$. Now it is clear that

$Y = X_3/s$ and since Y is a factor of a nonsingular surface X_3 by the action of a group Z_2, it follows that Y is a normal hypersurface in $\mathbb{P}^3$. This fact leads us to conclude that Y can have at most isolated singular points. These singular points can be ordinary double points because $Y = X_3/\mathbb{Z}_2$ and their number is equal to the number of the fixed points by the action of Z_2. Let me denote the fixed points by the action of $\mathbb{Z}_2$ by p_i. To obtain a nonsingular model Y of Y, we first blow X_3 at all fixed points p_i and obtain a surface $\hat{X}_3$. It is easy to see that the involution s can be lifted to an involution $\hat{s}$ on $\hat{X}_3$. Let p be the canonical map $p : \hat{X}_3 \to X_3$. Let $E'_i = p^{-1}(p_i)$, then $s|_{E'_i} = \mathrm{id}$. This implies that the quotient space $\hat{Y}$ of $\hat{X}_3$ by the involution s is nonsingular. Moreover, the morphism p induces a morphism $\hat{p} : \hat{X}_3 \to \hat{Y}$ which gives a resolution of singularities of Y. From this whole discussion it follows that we have a map $\hat{g}_3 : \hat{X}_3 \to \hat{Y}$, where $\hat{X}_3$ and $\hat{Y}$ are nonsingular varieties and $\hat{Y} = \hat{X}_3/\mathbb{Z}_2$. These facts shows us that the ramification divisor of $\hat{g}_3$ consists of the disjoint union of nonsingular curves. Now let me compute the canonical class of X_3. We will use the following lemma proved in [M] on p. 110.

LEMMA. — *Let* $f : X^r \to Y^r$ *be a regular dominating map of smooth r-dimensional varieties with a branch locus* B. *Then for all rational r-forms* w *on* Y:

$$(f^* w) = B + f^{-1}((w)). \tag{3.16}$$

From this formula we immediately get

$$K_{X_3} = 5K_3. \tag{3.17}$$

Note that $K_Y = 2H$. Let the branch locus of g_3 be $C + \sum E'_i$. It is a standart fact that $K_{X_3} = p^* K_{X_3} + \sum E'_i$. From formula (3.16) we obtain:

$$5H_3 + \sum E'_i = 2H_3 + C + \sum E'_i. \tag{3.18}$$

From (3.18) we deduce that C is rationally equivalent to $3H_3$. Let R be the ramification divisor of g_3. From the fact that X_3 is a double covering of Y ramified over R, it follows that $g_3^*(R) = 2C \sim 6H_3$, where $\sim$ means rationally equivelent. Thus we get

$$R \sim 6H. \tag{3.19}$$

Next we will prove that Y is a nonsingular surface. If we prove that $\mathbb{Z}_2$ acts without isolated fixed points, then Y will automatically be nonsingular. Let me denote by n the number of fixed points on X_3. The proof of the fact that Y is a nonsingular surface is based on the following formula, connecting the topological Euler characteristics of X and Y, where X is a Z_n cyclic covering of X ramified over R:

$$\chi(X) = n\chi(Y) - (n-1)\chi(R), \tag{3.20}$$

where are the topological Euler characteristics.

Using (3.20) it is very easy to compute $\chi(X_3)$ and we will get that

$$\chi(X_3) = 504. \tag{3.21}$$

Notice that

(3.21) $$\chi(\hat{X}_3)=\chi(X_3)+n,$$

where n is the number of fixed points of the action of $\mathbb{Z}_2$ on X_3.

From (3.20) we obtain:

(3.23) $$\chi(X_3)=2\chi(Y)-\chi(R),$$

where R is the ramification divisor of g_3.

Let us compute $\chi(Y)$ and $\chi(R)$. Because Y has only ordinary double points, then from the results of Briescorn it follows that the minimal nonsingular model Y of Y is diffeomorphic to a nonsingular hypersurface of degree 6 in $\mathbb{P}^3$. Let Z be a hypersurface of degree 6 in P^3. Then from the well known formula: $12(p_g-q+1)=(K_Z, K_Z)+\chi(Z)$ we can conclude that

(3.24) $$\chi(Z)=108.$$

Notice that $R=C+\sum E_i$, where C is rationally equivalent to 6 H and E_i is an exceptional curve of the second type and as all E_i are P^1 we get that $\chi(E_i)=2$. From the adjunctional formula on Y we get that $2p_g(C)-2=(C, C+K_Y)=(6H, 6H+2H)=6\times 6\times 8=288$. So $\chi(C)=-288$:

(3.25) $$\chi(R)=\chi(C)+\sum\chi(E_i)=-288+2n.$$

From (3.22), (3.23), (3.24) and (3.25) we get

(3.26) $$\chi(X_3)+n=2\times 108+288-2n.$$

Combining (3.21) and (3.26) we see that $n=0$, so thus proving Step 6 and Lemma 3.

Q.E.D.

LEMMA 4. — *Let* X_4 *be a* Z_3 *cyclic covering of* X_3 *ramified over* $(p_3\, p_2\, p_1)^*(C_4)$, *where* $C_4=(s_4)\in|3K_S|$. *Then:* (*a*) $\dim H^0(H_4, O(H_4))=5$ *and* $(H_4, H_4)=36$, (*b*) *the complete linear system* $|H_4|$ *gives a map* $g_4: X_4\to P^4$, $\deg g_4=1$ *and* $g_4(X_4)$ *is a nonsingular variety, which is a complete intersection of type* (6, 6).

Proof. — The proof of (*a*) is the same as proof of Lemma 3. Notice that we have $H^0(O(H_4))=p_4^* H^0(O(H_3))+\mathbb{C}x_5$, where $(x_5)=C_4'$, the branch locus of $p_4: X_4\to X_3$.

The proof of (*b*).

PROPOSITION 4.1. — $\deg g_4=1$.

Proof. — Let me consider the composition of maps $X_4\overset{p_4}{\to}X_3\overset{g_3}{\to}Y$ and let me denote this composition by q, i.e. $q: X_4\to Y$. Notice that q is given by the linear system $p_4^* H^0(O(H_3))\subset H^0(X_4, O(H_4))$. Let x_1, x_2, x_3 and x_4 be a basis for $p_4^* H^0(O(H_3))$. From condition (*a*) it follows that x_1, x_2, x_3, x_4 and x_5 is a basis of $H^0(O(H_4))$, where $(x_5)=C_4'$, the branch locus of p_4. Suppose that x and $y \in X_4$ and $q(x) \neq q(y)$, then it follows that $g_4(x) \neq g_4(y)$. Now suppose that a point $P \in Y$, $P \notin R$ (the ramification divisor of g_3) and $P \notin g_3(D_4)$ (the image of the ramification divisor of p_4). From these two conditions, it follows that (*a*) $g_3^{-1}(P) = (Q_1, Q_2)$ and $Q_1 \neq Q_2$, (*b*) $p_4^{-1}(Q_1) = (P_{11}, P_{12}, P_{13})$, $p_4^{-1}(Q_2) = (P_{21}, P_{22}, P_{23})$, where $P_{1i} \neq P_{1j}$ for $1 \leqq i, j \leqq 3$ and $P_{2i} \neq P_{2j}$ for all $1 \leqq i, j \leqq 3$. Note that $q(P_{ki}) = P$ for all k and i. First we will prove that $s_4(Q_1) \neq s_4(Q_2)$. If for all $(Q_1, Q_2) = g_3^{-1}(P)\, s_4(Q_1) = s_4(Q_2)$, then it will follows that s_4 is invariant under the action of $\mathbb{Z}_2$ ($\mathbb{Z}_2$ acts on X_3 and $X_3/\mathbb{Z}_2 = g_3(X_3) = q(X_4)$). On the other hand because $H_3 = g_3^*(H)$, it follows that $\mathbb{Z}_2$ acts on $H^0(X_3, O(H_3))$ and thus

$$H^0(O(H_3)) = H^0(O(H_3))^+ \oplus H^0(O(H_3))^-.$$

From the fact that

$$(a)\ H^0(O(H_3))^+ = g_3^* H^0(Y, O(H)) \qquad \text{and} \qquad (b)\ H^0(O(3H_3))^+$$

is generated by x_1, x_2, x_3 and x_4 (this is Step 4 of Lemma 3) it follows that $H^0(O(3H_3))^+$ is generated by all monomials of degree 3 formed by x_1, x_2, x_3 and x_4. Now it is clear that $p_3^*(s_4) \in H^0(O(3H_3))^+$. Here we use the notations of remark after proposition 3.3. Next we have to prove that $x_4(P_{1i}) \neq x_4(P_{1j})$ for all i and j. This follows from the formula $p_4^*(s_4) = x_4^3$ and the fact that the Galois group $\mathrm{Gal}(X_4/X_3) = (1, s, s^2)$ acts on x_4 in the following manner: $x_4^s = \varepsilon x_4$. (This follows from the fact that $(x_4) = C_4'$ the branch locus of p_4.) From this and the fact that, say $P_{12} = s(P_{11})$ and $P_{13} = s(P_{12}) = s^2(P_{11})$ we obtain $x_4(P_{12}) = \varepsilon x_4(P_{11}) = \varepsilon^2 x_4(P_{13})$, where $\varepsilon \neq 1$ and $\varepsilon^3 = 1$. Thus we get that $x_4(P_{1i}) \neq x_4(P_{1j})$ for all $i \neq j$. The same holds true for P_{2i}. Thus we have proved that for general points x and y such that $x \neq y$, $g_4(x) \neq g_4(y)$. Indeed we have two possibilities (*a*) $q(x) \neq q(y)$: then from the fact $g_4 = (q, x_4)$, we get that $g_4(x) \neq g_4(y)$, (*b*) $q(x) = q(y)$; then we have proved that for general points x and y, $x_4(x) \neq x_4(y)$.

Q.E.D.

PROPOSITION 4.2. — *$g_4(X_4)$ is a nonsingular surface in $\mathbb{P}^4$.*

Proof. — The proof is based on the following sublemma:

SUBLEMMA. — *Let x be any point on X_4 and let U be a neighborhood of x, then we can find two sections s_1 and $s_2 \in H^0(X_4, O(H_4))$ such that the curves (s_1) and (s_2) are nonsingular in U and $x \in (s_1) \cap (s_2)$.*

Proof. — We will consider two different cases: (*a*) $x \notin (x_5) = C_5'$. Let me consider $q(x) \in Y$. For the definition of q, see Proposition 4.6, i.e. $q = g_3 \circ p_4$. From the Bertinni theorem it follows that we can find two hyperplane sections H_1 and H_2 such that (1) H_1 and H_2 are nonsingular curves (2) $q(x) \in H_1 \cap H_2$, (3) H_1 and H_2 transect R, the ramification divisor, transversally. From condition (1) and (3) it follows that $g_3(H_1)$ and $g_3(H_2)$ are nonsingular curves on X_3. From the fact that p_4 is a local isomorphism around y, [this is condition (*a*), i.e. $x \notin C_4'$ the branch locus of p_4], it follows that $p_4^*(g_3^*(H_1))$ and $p_4^*(g_3^*(H_2))$ are nonsingular curves in some neighborhood of x. For this case the sublemma is proved.

(*b*) $x \in C_4'$. Let me consider again $q(x) \in Y$ and $p_4(x)$. For $p_4(x)$ we have two possibilities: (1) $p_4(x) \notin R' \cap D_4$, where D_4 is the ramification divisor of p_4. In a neighborhood of $p_4(x)$, g_3 is a local isomorphism, so that $g_3(D_4)=q(C'_4)$ is a nonsingular curve in some neighborhood of $q(x)$. Now let H be a nonsingular hyperplane section of Y such that H intersects $q(C'_4)$ transversally in $q(x)$. From this review it follows that $q^*(H)$ is a nonsingular curve transecting C'_4 transversally. C'_4 and $q^*(H)$ are then nonsingular curves in some neighborhood of x containing x.

(2) $p_4(x)\in R'\cap D_4$. In this case we have two possibilities (*a*) R' and D_4 intersect each other transversally, then in a neighborhood of $q(x)$, $g_3(D_4)$ is a nonsingular curve. Now let H be a nonsingular hyperplane section intersecting R and $g_3(D_4)$ transversally. Then $q(H)$ is a nonsingular curve such that $q(H)$ intersects C'_4 transversally in x. (*b*) Let D_4 and R' be tangent at $p_4(x)$. Now let H be a nonsingular hyperplane section of Y transversal to R at $q(x)$. Then $q^*(H)$ is a nonsingular curve in a neighborhood of x intersecting transversally C'_4. Thus $q^*(H)$ and C'_4 are the with the required properties.

Q.E.D.

Remark. — We have proved even more, namely, that through any point $x\in X_4$ we can find two sections s_1 and s_2 of $H^0(X_4, O(H_4))$ such that (s_1) and (s_2) are nonsingular curves meeting in x transversally.

Now let me prove Proposition 4.2. The map g_4 is given by

$$x\to(q_0(x), \ldots, q_4(x)).$$

Now we may suppose that in a neighborhood of x, $q_0\neq 0$ and q_1 and q_2 have the properties stated in the remark after the sublemma. From this remark it follows that q_1/q_0 and q_2/q_0 are local coordinates in U. Let me denote these local coordinates by x and y. The map $g_{4U}: U\to\mathbb{C}^4=(t_1, t_2, t_3, t_4)$, where $t_1=x$ and $t_2=y$, $t_3=q_3/q_0$ and $t_4=q_4/q_0$. From the fact that x and y are local coordinates in U it follows that $q_3/q_0=F(x, y)$ and $q_4/q_0=G(x, y)$, so the image of U in $\mathbb{C}^4$, i.e. $g_4(U)$ in $\mathbb{C}^4$ is given by the following equations: $t_3=F(t_1, t_2)$ and $t_4=G(t_1, t_2)$. From these two equations immediately come to the conclusion that $g_4(X_4)$ is a nonsingular variety.

Q.E.D.

PROPOSITION 4.3. — *$g_4(X_4)$ is a complete intersection of type* (6, 6) *in* P^4.

Proof. — From Lemma 3 it follows that $q(X_4)=Y$ is a surface of degree 6 in $\mathbb{P}^3$. If x_1, x_2, x_3 and x_4 is a basis of $p_4^* H^0(X_3, O(H_3))=q^* H^0(Y, O(H))$, then there is a relation of degree 6 among x_1, x_2, x_3, x_4, i.e. $h_6(x_1, x_2, x_3, x_4)=O$ in $H^0(X_4, O(H_4))$. From here it follows that $g_4^*(X_4)$ is contained in a hypersurface of degree 6 in $\mathbb{P}^4$. In Step 4 of lemma 3 we proved that $H^0(X_3, O(3H_3))$ is generated by x_1, x_2, x_3, x_4 and s_4, i.e. from all monomials of degree 3 formed from x_1, x_2, x_3, x_4 and $(p_3 p_2 p_1)^* s_4$. Notice that the branch locus of $g_3 R'$ is an element of $|3H_3|$. It follows Lemma 3. Let $(z)=R'$, where $z\in H^0(O(3H_3))$, so that $z=g(x_1, x_2, x_3, x_4, s_4)$. On the band, we have $R\sim 6H$ and $g_3^*(R)=2R'$, so from $R\sim 6H$ it follows that R is given by the equation $f(x_1, x_2, x_3, x_4)$. From $g_3^* R=2R'$ we get

$$z^2=g^2(x_1, x_2, x_3, x_4, s_4)=f(x_1, x_2, x_3, x_4).$$

From this equation we obtain a second relation of deg=6 in $H^0(X_4, O(6H_4))$ among monomials of deg=6 formed from x_1, x_2, x_3, x_4 and x_5. From here we conclude that $g_4(X_4)$

is contained in the intersection of two hypersurfaces of degree 6 in $\mathbb{P}^4$. From the fact that $(H_4, H_4)=36$ we immediately understand that $g_4(X_4)$ is a complete intersection of type (6, 6) in $\mathbb{P}^4$.

Q.E.D.

Theorem 1 is proved.

Q.E.D.

Remarks. — (1) From Theorem 1 it follows that the moduli space of all surfaces with $p_g=1$ and $(K, K)=1$ consists of one component. (2) All surfaces with $p_g=1$ and $(K, K)=1$ are simply connected. (3) The moduli space of surfaces with $p_g=1$ $(K, K)=1$ is a rational variety.

2. Deformation theory of surfaces with $p_g=1$ and $(K, K)=1$

THEOREM 2. — *Let* S *be a surface with* $p_g=1$ *and* $(K, K)=1$ *for which* K_S *is an ample divisor. Then* $H^2(S, \Theta_S)=0$ *and* $\dim H^1(S, \Theta_S)=18$, *where* Θ_S *is the tangent bundle sheaf.*

Proof. — From the Serre duality it follows that $H^2(S, \Theta_S)^*=H^0(S, \Omega_S^1(K_S))$. If we can prove that $H^0(S, \Omega_S^1(K_S))=0$, then we will get that $H^2(S, \Theta_S)=0$. That $\dim H^1(S, \Theta_S)=18$ follows directly from Riemann-Roch-Hirzebruch theorem and the fact that for surfaces of general type we have $H^0(S, \Theta_S)=0$. Our theorem then will be proved. In Theorem 1 we have proved that a surface X can be constructed, which is a complete intersection of type (6, 6) in $\mathbb{P}^4$ and on X there ares a group $G=\mathbb{Z}_6\oplus\mathbb{Z}_6$ in such a way that $X/G=S$. From this fact we can deduce that $H^0(S, \Omega_S^1(K_S))=H^0(X, \Omega_X^1(H))^G$. Notice that we have proved that $p^*(K_S)=H$, the hyperplane section of X. where $p: X\to X/G=S$. So if we prove that $H^0(X, \Omega_X^1(H))=0$, then Theorem 2 will be proved.

LEMMA 2.1. — $H^0(X, \Omega_X^1(H))=0$.

Proof. — The proof will be given in several steps.

STEP 1. — $H^0(X, \Omega_X^1(H))=H^0(X, \Omega_{P^4}^1(H)|_X)$.

Proof. — We have the following exact sequence

$$0\to\Theta_X\to\Theta_{P^4}|_X\to N_{P^4/X}\to 0. \tag{2.2}$$

We will take the dual of (2.2), multiply it by $O_X(H)$ and take into account that $N_{P^4/X}^*=O_X(-6H)\oplus O_X(-6H)$, thus obtaining:

$$0\to O_X(-6H)\oplus O_X(-6H)\to\Omega_{P^4}^1(H)|_X\to\Omega_X^1(H)\to 0. \tag{2.3}$$

From (2.3) we get

$$0\to H^0(X, \Omega_{P^4}^1(H)|_X)\to H^0(X, \Omega_X^1(H))\to H^1(X, O_X(-5H)). \tag{2.4}$$

PROPOSITION 2.5. — $H^1(X, O(-5H))=0$.

Proof. — This follows immediately from Mumford vanishing theorem. *See* [M].

Q.E.D.

From (2.4) and (2.5) we get Step 1.

Q.E.D.

STEP 2. — $H^0(X, \Omega^1_{\mathbb{P}^4}(H)|_X)=0$.

Proof. — From the Serre duality we get that $H^0(X, \Omega^1_{\mathbb{P}^4}(H)|_X)^*=H^2(X, \Theta_{\mathbb{P}^4}(6H))$. We must prove then that $H^2(\Theta_{\mathbb{P}^4}(6H)|_X)=0$. From the fact that X is a complete intersection in $\mathbb{P}^4$ we receive the following exact sequence

$$0 \to J_X = O_{\mathbb{P}^4}(-6H) \oplus O_{\mathbb{P}^4}(-6H) \to O_{\mathbb{P}^4} \to O_X \to 0, \tag{2.9}$$

J_X the sheaf of ideals that define X in $\mathbb{P}^4$. Let us multiply (2.9) by $\Theta_{\mathbb{P}^4}(6H)$ then

$$0 \to \Theta_{\mathbb{P}^4} \oplus \Theta_{\mathbb{P}^4} \to \Theta_{\mathbb{P}^4}(6) \to \Theta_{\mathbb{P}^4}(6)|_X \to 0. \tag{2.10}$$

From (2.10) we obtain:

$$H^2(P^4, \Theta_{\mathbb{P}^4}(6)) \to H^2(X, \Theta_{\mathbb{P}^4}(6)|_X) \to \oplus H^3(P^4, \Theta_{\mathbb{P}^4}). \tag{2.11}$$

From Bott's results we get that $H^2(P^4, \Theta_{\mathbb{P}^4}(6))=H^3(P^4, \Theta_{\mathbb{P}^4})=0$, *see* [B]. From here it follows that $H^2(X, \Theta_{\mathbb{P}^4}(6H))|_X)=0$.

Q.E.D.

From Step 1 and Step 2 we get that $H^0(X, \Omega^1_X(H))=0$ and, as we have seen, Theorem 2 follows from here.

Q.E.D.

3. Canonical Galois coverings of $\mathbb{P}^2$, that are surfaces with $p_g=1$ and $(K, K)=1$

The aim of this chapter is to describe all surfaces with $p_g=1$ and $(K, K)=1$ for which the map $f_{|2K_S|}: S \to \mathbb{P}^2$ is a Galois covering with the following additional properties: (1) K_S is an ample divisor (2) K_S is a nonsingular curve.

THEOREM 3. — *Let* S *be a surface with* $p_g=1$ *and* $(K, K)=1$ *with the properties described above, i.e.* K_S *is an ample divisor,* K_S *is a nonsingular curve and* $f_{|2K_S|}: S \to \mathbb{P}^2$ *is a Galois covering. Then:*

(*a*) $\mathrm{Gal}(S/\mathbb{P}^2)=\mathbb{Z}_2 \oplus \mathbb{Z}_2$;

(*b*) *one of the involutions, say* s_1, *restricted to* K_S *is the identity map.*

Proof. — Proof of (*a*).

PROPOSITION 3.1. — $\deg f_{|2K_S|}=4$.

Proof. — Let p be a point outside the ramification divisor of $f_{|2K_S|}$. Let L_1 and L_2 be two lines intersecting in p. By the definition of deg of a map we have

$$\deg f_{|2K_S|}=(f^{-1}_{|2K_S|}(L_1), f^{-1}_{|2K_S|}(L_2))=(2K_S, 2K_S)=4.$$

Q.E.D.

PROPOSITION 3.2. — $\mathrm{Gal}(S/\mathbb{P}^2)$ *is either* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. — The order of $\mathrm{Gal}(S/\mathbb{P}^2)$ must be 4 because $\deg f_{|2K_S|}=4$. There are only two groups of order 4 and they are $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$.

Q.E.D.

PROPOSITION 3.3. — *$f_{|2K_S|}$ restricted to the canonical divisor K_S is the canonical map $K_S \to P^1$, i.e. $p_g(K_S)=2$ and so $\deg f_{|2K_S|}|_{K_S}=2$.*

Proof. — Let me consider the exact sequence

$$(3.4) \qquad 0 \to \Omega_S^2 \to \Omega_S^2(K_S) \xrightarrow{\text{res}} \Omega_{K_S}^1 \to 0.$$

Res is the Poincaré residue map. From (3.4) we have

$$(3.5) \qquad 0 \to H^0(\Omega_S^2) \to H^0(\Omega_S^2(K_S)) \to H^0(\Omega_{K_S}^1) \to H^1(\Omega_S^2)=0\,(q(S)=0).$$

From (3.5) we get that the restriction of $f_{|2K_S|}$ on K_S is the canonical map. Note that K_S is a nonsingular curve of genus 2 and so it is a hyperelliptic curve and so the canonical map has degree 2.

Q.E.D.

PROPOSITION 3.4. — $\mathrm{Gal}(S/\mathbb{P}^2)=\mathbb{Z}_2\times\mathbb{Z}_2$.

Proof. — Suppose that $\mathrm{Gal}(S/\mathbb{P}^2)=\mathbb{Z}_4$.

SUBLEMMA. — *Let s be the generator of $\mathbb{Z}_4$, then $s(K_S)=K_S$.*

Proof. — Notice that $\mathbb{Z}_4$ acts on $H^0(S, \Omega_S^2)$. Which leads to the following possibilities: (*a*) $w^s=\pm w$, (*b*) $w^s=\pm iw$, where w is an element of $H^0(S, \Omega_S^2)=\mathbb{C}w$. From these two possibilities and the fact that K_S is the divisor of w, we get what is necessary.

Q.E.D.

From this sublemma it follows that we can find 6 different points on K_S such that $s(p_i)=p_i$; $i=1, 2, \ldots, 6$ and p_i are the Weierstrass points on the hyperelliptic curve of genus two K_S. From the fact that $s(p_i)=p_i$ we get a representation of $\mathbb{Z}_4$ to the tangent space at p_i. This means that we have a map $g:\mathbb{Z}_4\to \mathrm{Aut}(T_{p_i, S})$. Let M be the matrix equal to $g(s)$. Notice that $M^4=E$, $M^2\neq E$ and $M^3\neq E$. If $M=E$ or $M^2=E$, it will mean that in a neighborhood of p_i, $f_{|2K_S|}$ will have degree 1 or 2 and this contradicts proposition 3.1. Because $M^3\neq E$ and $M^4=E$ and from the Jordan decomposition of any linear operator it follows that we can find a basis in $T_{p_i, S}$ for which M will be diagonal. The matrix will be one of the following types

$$(1)\begin{pmatrix}1&0\\0&i\end{pmatrix},\quad (2)\begin{pmatrix}1&0\\0&-i\end{pmatrix},\quad (3)\begin{pmatrix}i&0\\0&i\end{pmatrix},$$

$$(4)\begin{pmatrix}i&0\\0&-i\end{pmatrix},\quad (5)\begin{pmatrix}-1&0\\0&-i\end{pmatrix},\quad (6)\begin{pmatrix}-1&0\\0&i\end{pmatrix},\quad (7)\begin{pmatrix}-i&0\\0&-i\end{pmatrix}.$$

All these matrixes correspond to the fact that we can choose the local coordinates (u, v) around p_i such that (1) $u^s=u$, $v^s=iv$, (2) $u^s=u$, $v^s=-iv$, (3) $u^s=iu$, $v^s=iv$, (4) $u^s=iu$, $v^s=-iv$, (5) $u^s=-u$, $v^s=-iv$, (6) $u^s=u$, $v^s=iv$ and (7) $u^s=iu$, $v^s=-iv$. It would be an easy exercise to find the invariants, to see that if $\mathbb{Z}_4$ acts as in cases (3), (4), (5), (6) and (7) then $S/\mathbb{Z}_4$ will have isolated singularities, but this is impossible because we know that $f_{|2K_S|}:S\to S/\mathbb{Z}_4=\mathbb{P}^2$. If $\mathbb{Z}_4$ acts as in cases (1) and (2), then we will see that the action of $\mathbb{Z}_4$ restricted to a curve defined by $v=0$ in a neighborhood of p_i is the identity. So from here we get that the map $f_{|2K_S|}: S \to S/Z_4$ is either one to one or four to one set theoretically. This contradicts the fact that on K_S, which contains p_i, the map $S \to S/\mathbb{Z}_4$ is set theoretically two to one. From here we conclude that $\mathrm{Gal}(S/\mathbb{P}^2)=\mathbb{Z}_2 \times \mathbb{Z}_2$.

Q.E.D.

PROPOSITION 3.5. — *Let s be an element of* $\mathrm{Gal}(S/\mathbb{P}^2)$, *then either* $s|_{K_S}=\mathrm{id}$ *or it has* 6 *fixed points.*

Proof. — From the Hurwitz formula (*see* [H]) it follows that either s has 6 fixed points or s has two fixed points or s is the identity, when we restrict s on K_S. Suppose that s has two fixed points on K_S. From the Hurwitz formula it follows that $K_S/s=C$ is an elliptic curve. Let $s_1 \in \mathrm{Gal}(S/P^2)=\mathbb{Z}_2 \times \mathbb{Z}_2$ and such that $s \circ s_1 \neq \mathrm{id}$. Notice that if $Y=S/s$, then $Y/s_1=\mathbb{P}^2$, i.e. the composition map $S \xrightarrow{g_1} Y \xrightarrow{g_2} \mathbb{P}^2$ is $f_{|2K_S|}$. From the fact that $f_{|2K_S|}(K_S)=P^1$ it follows that $g_2(C)=g_2(g_1(K_S))=\mathbb{P}^1=C/s_1$. The map $g_2: C \to \mathbb{P}^1$ has degree 2 and from here it follows that set theoretically, the map $f_{|2K_S|}: K_S \to P^1$ has degree 4. This contradicts proposition 3.3. So we have the following possibilities: either $s|_{K_S}$ is the identity map or $s|_{K_S}$ is the canonical involution of the hyperelliptic curve.

Q.E.D.

Let s_1 and s_2 be elements of $\mathrm{Gal}(S/P^2)$ such that $s_1 \circ s_2 \neq \mathrm{id}$. Such elements exist because $\mathrm{Gal}(S/\mathbb{P}^2)=\mathbb{Z}_2 \times \mathbb{Z}_2$. From proposition 3.7 it follows that $s_1|_{K_S}$ is either the identity or is the canonical involution.

If both s_1 and s_2 restricted to K_S are the canonical involutions, then $s_1 s_2$ restricted to K_S will be the identity map and $s_1 s_2 \neq \mathrm{id}$ on S. So it follows that one of the involutions of $\mathrm{Gal}(S/\mathbb{P}^2)$ restricted to K_S is the identity map.

Q.E.D.

THEOREM 4 (due to Kunev). — *Suppose that* S *is a surface with* $p_g=1$ *and* $(K, K)=1$ *with* (*a*) *an ample nonsingular canonical class,* (*b*) S *is a canonical Galois covering of* $\mathbb{P}^2$. *Then local Torelli theorem is not true for* S.

Proof. — Griffiths proved the following criterion in [G] for proving the period map to be local isomorphism: The period map is a local isomorphism iff the natural pairing

$$H^1(\Omega_S^1) \otimes H^0(\Omega_S^2) \to H^1(\Omega_S^1 \otimes \Omega_S^2) \tag{4.1}$$

is surjective. We have the following exact sequence

$$0 \to \Omega_S^1 \xrightarrow{\otimes w} \Omega_S^1(K_S) \to \Omega_S^1(K_S)|_{K_S} \to 0 \tag{4.2}$$

and

$$H^1(\Omega_S^1) \xrightarrow{\otimes w} H^1(\Omega_S^1(K_S)) \to H^1(\Omega_S^1(K_S)|_{K_S}) \to H^2(S, \Omega_S^1)=0. \tag{4.3}$$

From (4.3) it follows that if $H^1(\Omega_S^1(K_S)|_{K_S}) \neq 0$, then the local Torelli is not true.

PROPOSITION 4.1. — $H^1(\Omega_S^1(K_S)|_{K_S}) \neq 0$.

Proof. — From Theorem 3 condition (*b*) it follows that there exists $s \in \mathrm{Gal}(S/\mathbb{P}^2)$ such that $s|_{K_S} = \mathrm{id}$. From this fact it follows that s acts on $\Omega_S^1(K_S)|_{K_S}$, so s acts on $\Omega_S^1|_{K_S}$ and $\Omega_S^1|_{K_S} = \Omega_S^1|_{K_S}^+ + \Omega_S^1|_{K_S}^-$, where $\Omega_S^1|_{K_S}^+ = \Omega_{K_S}^1$ and $\Omega_S^1|_{K_S}^- = N_{S/K_S}^*$ (the conormal bundle).

Thus

$$\Omega_S^1|_{K_S}(K_S) = \Omega_S^1|_{K_S} \leftarrow N_{S/K_S} \quad [\text{because } O_S(K_S)_{K_S} = N_{S/K_S}]$$
$$= \Omega_{K_S}^1 \otimes N_{S/K_S} \oplus N_{S/K_S}^* \otimes N_{S/K_S} = \Omega_{K_S}^1 \otimes N_{S/K_S} \oplus O_{K_S}.$$

From this decomposition we get that $H^1(\Omega_S^1(K_S)|_{K_S}) = H^1(O_{K_S}) = \mathbb{C}^2 \neq 0$. This is because K_S is a nonsingular curve of genus two.

Q.E.D.

THEOREM 5. — *Let* S *be a surface with the following properties:* (*a*) $p_g(S) = 1$ *and* $(K_S, K_S) = 1$. (*b*) K_S *is a nonsingular curve.* (*c*) S *is a canonical Galois covering of* $\mathbb{P}^2$.

Suppose that $s \in \mathrm{Gal}(S/\mathbb{P}^2) = \mathbf{Z}_2 \times \mathbf{Z}_2$, such that $s|_{K_S} = \mathrm{id}$ and $s \neq \mathrm{id}$ (such an automorphism exists according to Theorem 3).

Then $S/s = Y$ is a $K-3$ surface, which is a double covering of P^2 and s has 9 fixed points outside K_S.

Proof:

PROPOSITION 5.1. — *s acts on $H^0(\Omega_S^2)$ as the identity.*

Proof. — Let $w \neq 0$ and $w \in H^0(\Omega_S^2)$. We must prove that $w^s = w$. Let U be a neighborhood of a point x on K_S. In U we can choose a local coordinate system (x, y) such that $x^s = x$ and $y^s = -y$. Notice that y is the local equation of K_S in U. From the fact that the divisor of w is K_S, we obtain

$$w_U = y\, dx \wedge dy, \text{ so } w_U^s = -y\, dx \wedge d(-y) = y\, dx \wedge dy.$$

Proposition 5.1 is thus proved.

Q.E.D.

PROPOSITION 5.2. — *s can have only isolated fixed points outside* K_S.

Proof. — Suppose that $s(p) = p$ and $p \notin K_S$. Let U_p be a neighborhood of p. From (5.1) it follows that s preserve w, i.e. $w^s = w$. From this fact it follows that the representation of Z_2 in $T_{p,S}$ must preserve the skewsymmetric form w. This representation must be a $SL_2(\mathbb{C})$ representation. From this fact it follows that we can find a local coordinate system in $U_p(x, y)$ such that $x^s = -x$ and $y^s = -y$. So p must be an isolated fixed point.

Q.E.D.

Let me blow up all isolated fixed points of s. We will denote by S′ the modified S. Let $p: S' \to S$ be the morphism that blows down all exceptional curves of the first kind. It is a well known fact that $K_{S'} = p(K_S) + \sum \mathbb{P}_i^1$. *See* [H]. We can continue the action of s on S′. An easy calculation shows that $s|_{\mathbb{P}_i^1} = \mathrm{id}$ and $S'/s = Y$ is a nonsingular variety. These are standart facts. From Proposition 5.1 we get that $H^0(S, \Omega_S^2)^G = H^0(Y, \Omega_Y^2) \cong \mathbb{C}$, so that $p_g(Y) = 1$. We know that $s|_{K_{S'}} = \mathrm{id}$. Let x be a point on $K_{S'}$. In a neighborhood of $x \in U$, $w|_U = u\, du \wedge dv$, where u is the local equation of $K_{S'}$ in U and $u^s = -u$ and $v^s = v$. Arround $q(x)$ the local coordinates are u^2 and v, $q: S' \to S'/s = Y$. So $w_Y|_{q(U)} = du^2 \wedge dv$ is a globally defined form on Y such that $q^*(w_Y)=w$. Notice that the divisor of w_Y is zero. From the fact that $q(S')=0$ it follows that $q(Y)=0$, so from the classification theory of algebraic surfaces it follows that Y is a K−3 surface. *See* [Š].

Now let me calculate the number of the fixed points of s. The following formula is true

(5.3) $$\chi(S')=2\chi(Y)-\chi(D),$$

where χ is the topological Euler characteristics and D is the ramification divisor of q.

It is a well known fact that the Euler characteristics of a K−3 surface $\chi(Y)=24$, *see* [Š]. From the Noether formula: $12(p_g-q+1)=(K_S, K_S)+\chi(S)$ we get that $\chi(S)=23$. Now let me denote by n the number of blown up points on S. We get

(5.4) $$\chi(S')=23+n, \qquad \chi(Y)=24.$$

Notice that $D=K_S+\sum_{i=1}^{n} P_i^1$, $\chi(K_S)=-2$ and $\chi(P_i^1)=2$. So

(5.5) $$\chi(D)=-2+2n.$$

Now from (5.3), (5.4) and (5.5) we get that $3n=27$ so $n=9$.

Q.E.D.

4. Examples and the description of surfaces with $p_g=1$ and $(K, K)=1$ that are canonical Galois coverings of P^2

THEOREM 6. — *Let* $S \subsetneq \mathbb{P}^4(1, 2, 2, 3, 3)$ *which is complete intersection of type* (6, 6) *with the following properties:* (*a*) *the equations that define* S *contain* s_0 *in even degrees,* (*b*) K_S *is a nonsingular curve.* ($\deg s_0=1$), *where* $\mathbb{P}^4(1, 2, 2, 3, 3) \subseteqq \mathbb{P}^4(s_0, s_1, s_2, s_3, s_4)$. *Then* S *is a canonical Galois covering of* $\mathbb{P}^2$. *The ramification divisor of* $f_{|2K_S|}$ *consists of two nonsingular curves of degree* 3 *in* $\mathbb{P}^2$ *meeting in* 9 *distinct points and a line.*

Proof. — Theorem 1 shows that $p_g(S)=1$ and $(K_S, K_S)=1$. Let β is an automorphism of $\mathbb{P}^4(1, 2, 2, 3, 3)$ and $\beta(s_0, s_1, s_2, s_3, s_4)=(-s_0, s_1, s_2, s_3, s_4)$. From condition (*a*) it follows that β is an involution on S and $\beta|_{K_S}=\mathrm{id}$. It is not difficult to prove that the fixed points of β outside K_S are the points $(s_0, s_1, s_2, 0, 0)$ on S. These points are exactly the intersection points of the curves $(s_3)=C_3$ and $(s_4)=C_4$. We are supposing that we have chosen s_1, s_2, s_3 and s_4 exactly in the same way as in theorem 1, i.e. $(s_1)=C_1$, $(s_2)=C_2$, $(s_3)=C_3$ and $(s_4)=C_4$ are nonsingular curves intersecting each other transversally. From the fact that C_i for $i=3$ and 4 birrationally equivalent to $3K_S$ we get that the number of the fixed points of β outside K_S is equal to 9. Now let me denote by $Y=S/\beta$. From Theorem 5 we know that Y is a K−3 surface.

PROPOSITION 6.1. — Y *is a double covering of* P^2 *ramified over two cubic curves meeting each other in* 9 *distinct points.*

Proof. — It is a well-known fact that if C is a nonsingular curve on a K−3 surface, then the complete linear system $|C|$ gives a holomorphic map if $p_g(C) \geqq 1$. *See* [Š], Chapter 10. Let me denote by C the image of K_S on Y. Because K_S is fixed by the involution β, it follows that C is isomorphic to K_S, so that $p_g(C)=2$. By the theorem mentioned above the complete linear system $|C|$ gives a holomorphic map. From Riemann-Roch theorem it follows that we have a holomorphic map $f_{|C|}: Y \to \mathbb{P}^2$. Because Y is a $K-3$ surface and the map $f_{|C|}$ has degree 2, the ramification divisor of $f_{|C|}$ is rationally equivalent to $6L$, L is a line in $\mathbb{P}^2$. For the proof of this *see* [W]. Now I claim that the branch locus of $f_{|C|}$ consists of the images of C_3 and C_4 in Y. Let me denote these two images by D_3 and D_4. From the definition of β it follows that β leaves C_3 and C_4 invariant. Let me compute the number of the fixed points of the action of β on D_3 and D_4. We have 9 points that are the fixed points of β outside K_S and these 9 points are $C_3 \cap C_4$. On the other hand $\beta|_{K_S}=\text{id}$, so that $K_S \cap C_i$ ($i=3$ and 4) are fixed points on C_3 and C_4. From $(C_i, K_S)=(3K_S, K_S)=3$ for $i=3$ and 4, we get that the number of fixed points of β on C_3 and C_4 is equal to $9+3=12$. Of course we have 12 fixed points on each of C_i, $i=3$ and 4. From the adjunctional formula we get that $p_g(C_i)=7$ for $i=3$ and 4. From the Hurwitz formula it follows that $p_g(D_i)=1$ for $i=3$ and 4. We need to compute (C, C_i) on Y. Let me denote by p the natural map $p: S \to S/\beta = Y$. From the formula $(p^* D_3, p^* C)=(3K_S, 2K_S)=\deg p$. $(D_3, C)=2$. $(D_3, C)=6$ we get $(C, D_3)=(C, D_4)=3$. From these calculations we get that the degree of the line bundle $O_Y(C)$ restricted to both elliptic curves D_3 and D_4 is 3. So $f_{|C|}$ restricted to D_3 and D_4 gives one to one map, i.e. $f_{|C|}: D_i \pm P^2$ for $i=3$ and 4. This a standart fact about elliptic curves. *See* [H]. From this discussion we conclude that the images of D_3 and D_4 are contained in the ramification divisor of $Y \to \mathbb{P}^2$. From the fact that the ramification divisor is rationally equivalent to $6L$, we get what is necessary.

Q.E.D.

Theorem 6 follows from Proposition 6.1 and Propositions (3.1) and (3.3).

Q.E.D.

Remark. — From Theorem 6 we get an explicit description of all Galois (canonical) coverings of $\mathbb{P}^2$ in terms of the equations of S in $\mathbb{P}^4(s_0, s_1, s_2, s_3, s_4)=\mathbb{P}^4(1, 2, 2, 3, 3)$, i.e. these are all surfaces in $P^4(1, 2, 2, 3, 3)$ that are complete intersections of type (6, 6) and the equations that define S must contain s_0 in even degree.

This is one way of describing the canonical Galois coverings of $\mathbb{P}^2$. The other way is the following one and it is due to H. Clemens. Let Y be a double covering of $\mathbb{P}^2$, ramified over two elliptic curves meeting in 9 different points. Let Y′ be the surface obtained by blowing up all 9 double points on Y. Let $E_1, \ldots, E_9$ be the exceptional curves of the second type on Y′. Let C be the praimage of the line L that does not contain any of the intersection points of the ramification divisor. It is not very difficult to prove that $C+E_1+\ldots+E_9$ is divisible by two in $H_2(Y', \mathbb{Z})$. Indeed, it is not difficult to see that $3C \sim 2D_3+E_1+\ldots+E_9$ or $C \sim 2D_3-2C+E_1+\ldots+E_9$ and so $C+E_1+\ldots+E_9 \sim 2D_3-2C+2(E_1+\ldots+E_9)$. Now let me define S′ as a double covering of Y′ ramified over $C+E_1+\ldots+E_9$. It is not very difficult to prove that the minimal model of S′, S is a surface with $p_g=1$ and $(K_S, K_S)=1$. From here we can compute the number of moduli of all canonical Galois coverings of $\mathbb{P}^2$. First one can prove that if Y is a $K-3$ surface which contains 9 exceptional curves of the second type and a curve of genus two not intersecting these 9 projective lines, then the curve of genus two plus the 9 lines are divisible by two in the second homology group. Thus repeating the second construction of Galois coverings of $\mathbb{P}^2$. Note that the number of moduli of $K-3$ surfaces with the above properties is equal to 10. Two more moduli are obtained from the choice of C. So the number of the moduli of all canonical Galois coverings of $\mathbb{P}^2$ is 12.

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Andrei N. TODOROV,
Institute of Mathematics,
Bulgarian Academy of Sciences,
P.B. 373, Sofia 1000,
Bulggaria;
Home address: Sofia 1113,
kv. "Iztok" bl. 43-A ap. 18.

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