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SURFACES OF GENERAL TYPE WITH $p_g=1$ AND $(K, K)=1$. I

BY ANDREI N. TODOROV

Introduction

The aim of this article is to describe all surfaces with $p_g=1$ and $(K, K)=1$. The first examples of such surfaces were constructed by Kunev in [Ku]. Here we give the following description of all surfaces with $p_g=1$ and $(K, K)=1$: every such surface is a complete intersection of two quasi-homogeneous polynomials in $\mathbb{P}^4(1, 2, 2, 3, 3)$. This fact was conjectured by M. Reid and I learned it from I. Dolgacev. From this description it follows that the moduli space of surfaces with $p_g=1$ and $(K, K)=1$ consists of one component. These surfaces are interesting because they are simply connected and the local Torelli theorem is not true for some of them. Thus surfaces with $p_g=1$ and $(K, K)=1$ that are canonical Galois coverings of $\mathbb{P}^2$ give counter examples to a conjecture of P. Griffiths, which states that the local Torelli theorem is true for all simply-connected surfaces of general type with $p_g \geq 1$. Even more the author recently proved that these surfaces give counter examples to global Torelli theorem. We give a complete description of all Galois coverings of $\mathbb{P}^2$ with $p_g=1$ and $(K, K)=1$. For surfaces with $p_g=1$ and $(K, K)=1$ that are not a canonical Galois coverings of $\mathbb{P}^2$ the local Torelli theorem is true.

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1. A description of all surfaces with $p_g=1$ and $(K, K)=1$

We need some definitions in order to formulate Theorem 1.

DEFINITION 1. — An weighted projective space of type $(w_0, w_1, \dots, w_n)$, where w_i are positive integers, is defined as $\text{Proj } \mathbb{C}(w_0, \dots, w_n)$, where $\mathbb{C}(w_0, \dots, w_n)$ is the polynomial ring with the following graduation, $\deg x_i = w_i$.

DEFINITION 2. — We will say that $f(x_0, \dots, x_n) = \sum_k a_k x^k$ is a quasi-homogeneous polynomial of type $(w_0, \dots, w_n)$ of deg m iff $k = (k_0, \dots, k_n)$ and $k_0 w_0 + \dots + k_n w_n = m$.

DEFINITION 3. — An weighted complete intersection in $\mathbb{P}^n(w_0, \dots, w_n)$ we will call a variety V , whose ideal in the graded ring $\mathbb{C}(x_0, \dots, x_n)$ is generated by a regular sequence of quasi-homogeneous polynomials $f_{d_0}, \dots, f_{d_k}$, where d_i is the degree of f_{d_i} .

THEOREM 1. — *Every surface with an ample canonical class, $p_g = 1$ and $(K, K) = 1$ is a complete intersection of type $(6, 6)$ in $\mathbb{P}^4(1, 2, 2, 3, 3)$.*

Proof. — First I will give the reason for choosing $\mathbb{P}^4(1, 2, 2, 3, 3)$ as a space of embedding surfaces with $p_g = 1$ and $(K, K) = 1$. First I will recall some facts proved by V. Kunev for surfaces with $p_g = 1$ and $(K, K) = 1$:

THEOREM (see [Ku]). — *Let S be a minimal model of a surface with $p_g = 1$ and $(K, K) = 1$. Then (a) the complete linear system $|2K_S|$ gives a holomorphic map $f_{|2K_S|} : S \rightarrow \mathbb{P}^2$, (b) the complete linear system $|3K_S|$ gives a holomorphic birational map.*

Bombieri proved in [Bom] the following lemma: Let S be a minimal model of a surface with $p_g = 1$ and $(K, K) = 1$, then the general element of $|2K_S|$ is irreducible and nonsingular.

From the definition we know that $\dim H^0(S, \Omega_S^2) = 1$. Let $H^0(S, \Omega_S^2)$ be generated by s_0 . From Riemann-Roch we get that $\dim H^0(S, \mathcal{O}(2K_S)) = 3$. Let $H^0(S, \mathcal{O}(2K_S))$ be generated by s_0^2, s_1, s_2 . From Kunev's theorem and Bombieri's lemma, it follows that we can choose s_1 and s_2 in the following way; let C_1 and C_2 be the divisors of s_0 and s_1 , then we may suppose that C_1 and C_2 are nonsingular curves intersecting each other transversally. From Riemann-Roch theorem it follows that $\dim H^0(S, \mathcal{O}(3K_S)) = 5$. Let $H^0(S, \mathcal{O}(3K_S))$ be generated by $s_0^3, s_0 s_1, s_0 s_2, s_3, s_4$. From Kunev's theorem it follows that we can choose the divisors of s_3 and s_4 , C_3 and C_4 , so that C_3 intersects C_4 transversally and both of them intersect C_1 and C_2 transversally. The theorem of Kunev gives us a hope that S can be embedded in $\mathbb{P}^4(1, 2, 2, 3, 3)$, i.e. in $\text{Proj } \mathbb{C}(s_0, s_1, s_2, s_3, s_4)$, where $\deg s_0 = 1, \deg s_1 = 2, \deg s_2 = 2, \deg s_3 = 3$ and $\deg s_4 = 3$.

Remark. — From now on all curves C_i will be fixed, where C_i is the divisor of s_i for all $i > 0$ and C_i are nonsingular and have the properties described above.

In order to prove Theorem 1, we need the following construction:

THE CONSTRUCTION OF X_4 . — From the fact $C_1 \in |2K_S|$ and the results of Wawrik [W] we can construct a $\mathbb{Z}_2$ cyclic covering $p_1 : X_1 \rightarrow S$ ramified over C_1 . Let me denote by $|H_1|$ the complete linear system $|p_1^* K_S|$. Let $p_2 : X_2 \rightarrow X_1$ be a $\mathbb{Z}_2$ covering of X_1 ramified over $p_1^* C_2$. Let me denote by $H_2 = p_2^* H_1$. It is clear that $(p_1 p_2)^* C_3$ belongs to $|3H_2|$ and we can construct a cyclic $\mathbb{Z}_3$ covering $p_3 : X_3 \rightarrow X_2$ ramified over $(p_2 p_1)^* C_3$. Again I will denote by $H_3 = p_3^* H_2$. We see immediately that $(p_3 p_2 p_1)^* C_4$ belongs to $|3H_3|$ so that we can construct a cyclic $\mathbb{Z}_3$ covering $p_4 : X_4 \rightarrow X_3$ ramified over $(p_3 p_2 p_1)^* C_4$. From the fact that all C_i are nonsingular and transect each other transversally, we conclude that all X_i are nonsingular surfaces, $i = 1, 2, 3, 4$. If we can prove that X_4 can be embedded as a

complete intersection of type (6, 6) in $\mathbb{P}^4$, Theorem 1 will be proved, because $\mathbb{P}^4(1, 2, 2, 3, 3) = \mathbb{P}^4/G$, where G is a group which acts in the following way

$$(g, (x_0 : x_1 : x_2 : x_3 : x_4)) = (x_0 g_0 : x_1 g_1 : x_2 g_2 : x_3 g_3 : x_4 g_4)$$

$$g_i = \exp(2\pi b_i/w_i), \quad 0 \leq b_i < w_i.$$

The equivalence of these two definitions is proved in [D]. Thus our aim is to prove that X_4 is a complete intersection of type (6, 6) in $\mathbb{P}^4$.

LEMMA 1. — (a) $\dim H^0(X_1, \mathcal{O}(H_1)) = 2$, (b) $|H_1|$ does not have fixed components, (c) $(H_1, H_1) = 2$.

Proof. — The proof is based on the following remark: $\mathbb{Z}_2 = (1, s)$ acts on $H^0(X_1, \mathcal{O}(H_1))$ and so $H^0(X_1, \mathcal{O}(H_1)) = H^0(\mathcal{O}(H_1))^+ \oplus H^0(\mathcal{O}(H_1))^-$, where $H^0(\mathcal{O}(H_1))^+$ is the invariant and $H^0(\mathcal{O}(H_1))^-$ is the anti-invariant subspace. It is a well-known fact that $H^0(\mathcal{O}(H_1))^+ = p_1^* H^0(\mathcal{O}(K_S))$ and thus $\dim H^0(\mathcal{O}(H_1))^+ = 1$. Now we must compute $\dim H^0(\mathcal{O}(H_1))^-$. Notice that $\mathcal{O}(H_1) = p_1^* \mathcal{O}(K_S)$ and it follows that the cocycle defining $\mathcal{O}(H_1)$ is of the form $f_{ij} = p_1^*(g_{ij})$. Let U_i be a covering of X_1 by polycylinders. If $f \in H^0(\mathcal{O}(H_1))^-$ then it follows that $f^s = -f$ and $f_i^s = -f_i$, where $f_i = f|_{U_i}$. Indeed, from the definition of f it follows that $f_i = f_{ij} f_j$ and so from $f_{ij}^s = f_{ij}$ it follows that $f_i^s = -f_i$. Now let U_i contains the branch locus of p_1 , C'_1 . It is a well-known fact that we can choose the local coordinate system (x_i, y_i) in U_i in such a manner that $x_i^s = x_i$ and $y_i^s = -y_i$, where y_i is the local equation of C'_1 in U_i . Now let

$$(1.1) \quad f_i(x_i, y_i) = \sum a_{mn} x_i^m y_i^n \quad \text{and} \quad f_i^s = \sum (-1)^n a_{mn} x_i^m y_i^n,$$

$$(1.2) \quad f_i^s = -f_i \quad \text{iff} \quad f_i = \sum a_{mn} x_i^m y_i^{2n+1}, \quad \text{where } m \text{ and } n > 0.$$

So

$$(1.3) \quad f_i = -f_i \quad \text{iff} \quad f_i = y_i g_i(x_i, y_i^2).$$

From (1.3) it follows that if $f^s = -f$ then $(f) = C'_1 + D$, where D is an effective divisor on X_1 . If we can prove that C'_1 is rationally equivalent to H_1 , then from (1, 3) it will follow that $\dim H^0(\mathcal{O}(H_1))^- = 1$.

PROPOSITION 1.1. — The branch locus of C'_1 is rationally equivalent to H_1 .

Proof. — See [W].

Q.E.D.

Proposition 1.1 proves (a) of Lemma 1.

Q.E.D.

$|H_1|$ does not have fixed components because $C'_1 \in |H_1|$ and it is a nonsingular curve. Thus (b) is proved.

On S we have $(K_S, C_1) = (K_S, 2K_S) = 2$ and on X_1 we have

$$(p_1^* K_S, p_1^* C_1) = \deg(p_1) \times (K_S, C_1) = 4 = (H_1, 2C'_1) = (H_1, 2H_1).$$

So we obtain that $(H_1, H_1) = 2$.

Q.E.D.

LEMMA 2. — $\dim H^0(O(H_2))=3$. We can choose C_2 in such a manner that the linear system $|H_2|$ gives a holomorphic map $X_2 \rightarrow \mathbb{P}^2$, $(H_2, H_2)=4$.

Proof. — We know that $\mathbb{Z}_2=(1, s)$ acts on X_2 and $X_2/s=X_1$ and so we can repeat the arguments of Lemma 1 and conclude that $H^0(O(H_2))=H^0(O(H_2))^+ + H^0(O(H_2))^-$, where $H^0(O(H_2))^+ = p_2^*(H^0(O(H_1)))$ and $H^0(O(H_2))^-$ is generated by f , where $(f)=C'_2$ is the branch locus of p_2 . From all these facts and Lemma 1 we get that $\dim H^0(O(H_2))=3$.

(b) If $|H_1|$ has base points, these points can be at most two because of $(H_1, H_1)=2$. Let these two points be P_1 and P_2 . From Kunev's theorem it follows that we can choose C_2 in a such a manner that C_2 does not contain the images of P_1 and P_2 on S . Now our result follows from the decomposition

$$H^0(O(H_2))=H^0(O(H_2))^+ \oplus H^0(O(H_2))^- = p_2^* H^0(O(H_1)) + \mathbb{C} f,$$

where $(f)=C'_2$ the branch locus of p_2 and $(p_2 p_1)^* C_2 = 2 C'_2$.

(c) The proof of $(H_2, H_2)=4$ is the same as the proof of $(H_1, H_1)=2$.

Q.E.D.

LEMMA 3. — (a) $\dim H^0(X_3, O(H_3))=4$, (b) the complete system $|H_3|$ gives a holomorphic map $g_3 : X_3 \rightarrow Y \subset \mathbb{P}^3$, Y is a hypersurface of degree 6, X_3 is a double covering of Y ramified over a curve rationally equivalent to $6H$, H is the hypersurface section on Y . (c) $(H_3, H_3)=12$.

Proof. — The proof is based on several steps.

STEP 1. — $\dim H^0(O(H_3))=4$.

Proof. — $\mathbb{Z}_3=(1, s, s^2)$ acts on X_3 and thus on $H^0(O(H_3))$. From here it follows that

$$H^0(O(H_3))=H^0(O(H_3))^+ \oplus H^0(O(H_3))^\varepsilon \oplus H^0(O(H_3))^{\varepsilon^2},$$

where $H^0(O(H_3))^+$ is the invariant subspace and $H^0(O(H_3))^\varepsilon$ and $H^0(O(H_3))^{\varepsilon^2}$ are eigen subspaces with eigen values ε and ε^2 , where $\varepsilon^3=1$ and $\varepsilon \neq 0$. From $H^0(O(H_3))^+ = p_3^* H^0(O(H_2))$ follows that $\dim H^0(O(H_3))^+ = 3$ (this is Lemma 2).

PROPOSITION 3.1. — $\dim H^0(O(H_3))^\varepsilon = 1$ and $\dim H^0(O(H_3))^{\varepsilon^2} = 0$.

Proof. — Let U_i be a covering of X_3 . Let f and g be elements of $H^0(O(H_3))^\varepsilon$ and $H^0(O(H_3))^{\varepsilon^2}$ respectively. Let me denote by f_i and g_i , $f|_{U_i}$ and $g|_{U_i}$. If $U_i \cap C'_3 \neq \emptyset$, where C'_3 is the branch locus of p_3 , then we can choose the coordinates in U_i in the following manner: $x_i^s = x_i$ and $y_i^s = \varepsilon y_i$, where y_i is the local equation of C'_3 in U_i . Repeating the same arguments as in Lemma 1 we get

$$(3.2) \quad f_i^s = \varepsilon f_i \quad \text{and} \quad g_i^s = \varepsilon^2 g_i \quad \text{if} \quad f^s = \varepsilon f \quad \text{and} \quad g^s = \varepsilon^2 g.$$

Let

$$(3.3) \quad f_i = \sum a_{mn} x_i^m y_i^n \quad \text{and} \quad g_i = \sum b_{mn} x_i^m y_i^n.$$

From (3.2) and (3.3) we obtain:

$$(3.4) \quad f_i^s = \varepsilon f_i \quad \text{iff} \quad f_i = y_i f'_i(x_i, y_i^3) \quad \text{and} \quad g_i = \varepsilon^2 g_i \quad \text{iff} \quad g_i = y_i^2 g'_i(x_i, y_i^3).$$

From (3.4) it follows that if f and g are elements of $H^0(O(H_3))^\varepsilon$ and $H^0(O(H_3))^{\varepsilon^2}$ respectively, then $(f)=C'_3+D$ and $(g)=2C'_3+D_1$. Proposition 3.1 follows from the fact that C'_3 is rationally equivalent to H_3 . For the proof of this fact, see [W].

Q.E.D.

Remark. — Notice that we have proved that $H^0(O(H_3))=p_3^*H^0(O(H_2))+\mathbb{C}y$, where $\mathbb{C}$ is the complex number field and $(y)=C'_3$.

STEP 2. — $(H_3, H_3)=12$.

Proof. — The proof is the same as the proof for $(H_1, H_1)=2$.

Q.E.D.

STEP 3. — $\deg g_3(X_3)$ is one of the following numbers: 2, 3, 4, 6 and 12.

Proof. — It follows from Lemma 2 and the remark after Step 1 that the complete linear system $|H_3|$ gives a holomorphic map $g_3: X_3 \rightarrow Y \hookrightarrow \mathbb{P}^3$. Now Step 3 follows from the following formula: $(H_3, H_3)=\deg g_3 \times (H, H)_Y$, where $(H, H)_Y$ is the selfintersection number of the hyperplane section on Y .

Q.E.D.

STEP 4. — Let x_1, x_2, x_3 and x_4 be sections of $H^0(O(H_3))$, which are linearly independent and generate $H^0(O(H_3))$. Then all monomials formed from x_1, x_2, x_3 and x_4 and having degree 4 are linearly independent in $H^0(O(4H_3))$. We suppose that $\deg x_i=1$ for all i .

Proof. — The proof is based on several propositions.

PROPOSITION 3.2 :

$$H^0(O(4H_3))=p_3^*H^0(O(H_2))+x_4p_3^*H^0(O(3H_2))+x_4^2p_3^*H^0(O(2H_2)),$$

where x_4 is such that $(x_4)=C'_3$, the branch locus of p_3 .

Proof. — From the way we constructed X_3 we know that $\mathbb{Z}_3$ acts on X_3 . From here it follows that $\mathbb{Z}_3$ acts on $H^0(O(H_3))$. From this action we get the following decomposition

$$H^0(O(4H_3))=H^0(O(4H_3))^+ + H^0(O(4H_3))^\varepsilon + H^0(O(4H_3))^{\varepsilon^2},$$

where $H^0(O(4H_3))^+$ is the invariant subspace, $H^0(O(4H_3))^\varepsilon$ and $H^0(O(4H_3))^{\varepsilon^2}$ are eigen subspaces with eigen values ε and ε^2 . Repeating the same arguments as in Step 1, we get

$$(3.5) \quad \begin{cases} f \in H^0(O(4H_3))^\varepsilon & \text{iff } f|_{U_i} = f_i(x_i, y_i) = y_i f'_i(x_i, y_i^3), \\ g \in H^0(O(4H_3))^{\varepsilon^2} & \text{iff } g|_{U_i} = g_i(x_i, y_i) = y_i^2 g'_i(x_i, y_i^3), \end{cases}$$

where (x_i, y_i) is a local coordinate system in U_i such that $x_i^3 = x_i y_i n dy_i^3 = y_i$, y_i is the local equation of C'_3 in U_i . From (3.5) and the fact that C'_3 is rationally equivalent to H_3 we obtain:

$$(3.6) \quad \begin{cases} f \in H^0(O(4H_3))^\varepsilon & \text{iff } f = x_4 f', \quad \text{where } (x_4)=C'_3 \text{ and } f' \in p_3^*H^0(O(3H_2)), \\ g \in H^0(O(4H_3))^{\varepsilon^2} & \text{iff } g = x_4^2 g', \quad \text{where } g' \in p_3^*H^0(O(2H_2)). \end{cases}$$

PROPOSITION 3.2. — *Follows from (3.6) and the fact $H^0(O(4H_3))^+ = p_3^* H^0(O(4H_2))$.*

Q.E.D.

PROPOSITION 3.3:

$$\begin{aligned} H^0(O(4H_2)) = & (p_2 p_1)^* H^0(O(4K_S)) + y_2 (p_2 p_1)^* H^0(O(3K_S)) \\ & + y_1 (p_2 p_1)^* H^0(O(3K_S)) + y_2 y_1 (p_2 p_1)^* H^0(O(2K_S)), \end{aligned}$$

where $(y_2) = C'_2$ (the branch locus of p_2) and $y_1 = (p_2)^* z_1$, $(z_1) = C'_1$, the branch locus of p_1 .

Proof. — $\mathbb{Z}_2$ acts on X_2 and so it acts on $H^0(O(4H_2))$. Thus we have

$$H^0(O(4H_2)) = H^0(O(4H_2))^+ + H^0(O(4H_2))^-.$$

We know that

$$H^0(O(4H_2))^+ = p_2^* H^0(O(4H_1)).$$

Repeating the same arguments as in Proposition 4.1 we will get that

$$H^0(O(4H_2))^- = y_2 p_2^* H^0(O(3H_1)).$$

where $(y_2) = C'_2$. So we get

$$(3.7) \quad H^0(O(4H_2)) = p_2^* H^0(O(4H_1)) + y_2 p_2^* H^0(O(3H_1)).$$

We know that X_1 is a double covering of S ramified over C_1 . $\mathbb{Z}_2$ acts on X_1 . From this action we get

$$(3.8) \quad H^0(O(4H_1)) = p_1^* H^0(O(4H_1))^+ + H^0(O(4H_1))^-.$$

Repeating the arguments of Remark 2 we obtain:

$$(3.9) \quad H^0(O(4H_1))^+ = p_1^* H^0(O(4K_S)),$$

$$(3.10) \quad H^0(O(4H_1))^- = z_1 p_1^* H^0(O(3K_S)),$$

where $(z_1) = C'_1$ the branch locus of p_1 and $z_1 \in H^0(O(H_1))$.

Repeating the same discussion for $H^0(O(3H_1))$ we get that

$$(3.11) \quad H^0(O(3H_1)) = p_1^* H^0(O(3K_S)) + z_1 p_1^* H^0(O(2K_S)).$$

Combining (3.8), (3.9) and (3.10) we get

$$(3.12) \quad H^0(O(4H_1)) = p_1^* H^0(O(4K_S)) + z_1 p_1^* H^0(O(3K_S)).$$

Putting (3.11) and (3.12) in (3.7) leads us to

$$\begin{aligned} (3.13) \quad H^0(O(4H_2)) = & (p_2 p_1)^* H^0(O(4K_S)) + p_2 (z_1) (p_2 p_1)^* H^0(O(3K_S)) \\ & + y_2 (p_2 p_1)^* H^0(O(2K_S)) + y_2 p_2^* (z_1) H^0(O(2K_S)). \end{aligned}$$

(3.13) Proves Proposition 3.3 if we take into account that $y_1 = p_2^*(z_1)$.

Q.E.D.

Remark. — We can choose x_1, x_2, x_3 and x_4 (a basis of $H^0(X_3, \mathcal{O}(H_3))$) in a such way that $x_1^2 = (p_3 p_2 p_1)^*(s_1)$, $x_2^2 = (p_3 p_2 p_1)^*(s_2)$, $x_3 = (p_3 p_2 p_1)^*(s_0)$ and $x_4^3 = (p_3 p_2 p_1)^*(s_3)$, where $(s_0) = K_S$, $(s_1) = C_1$, $(s_2) = C_2$ and $(s_3) = C_3$.

Proof. — In Lemma 1 we proved that $H^0(\mathcal{O}(H_1)) = p_1^* H^0(\mathcal{O}(K_S)) + \mathbb{C} z_1$, where $(z_1) = C'_1$, the branch locus of p_1 . From the fact that X_1 is a double covering of S ramified over C_1 , it follows that $p_1^*(s_1) = z_1^2$. In Lemma 2 we proved that

$$H^0(\mathcal{O}(H_2)) = p_2^* H^0(\mathcal{O}(H_1)) + \mathbb{C} y_2,$$

where $(y_2) = C'_2$, the branch locus of p_2 . From the fact that X_2 is a double covering of X_1 ramified over $p_1^*(C_2)$, it follows that $(p_2 p_1)^*(s_2) = y_2^2$. In Lemma 3 we proved that

$$H^0(\mathcal{O}(H_3)) = p_3^* H^0(\mathcal{O}(H_2)) + \mathbb{C} x_4,$$

where $(x_4) = C'_3$. From the fact that X_3 is a cyclic $\mathbb{Z}_3$ covering of X_2 ramified over $(p_2 p_1)^*(C_3)$ it follows that $(p_3 p_2 p_1)^*(s_3) = x_4^3$. Combining all these facts we conclude that

$$H^0(\mathcal{O}(H_3)) = (p_3 p_2 p_1)^* H^0(\mathcal{O}(K_S)) + \mathbb{C} (p_2 p_1)^*(z_1) + \mathbb{C} p_3^*(y_2) + \mathbb{C} x_4.$$

Now taking into account that $H^0(\mathcal{O}(K_S)) = \mathbb{C} s_0$ and denoting by $x_1 = (p_3 p_2)^*(z_1)$, $x_2 = p_3^*(y_2)$, $x_3 = (p_3 p_2 p_1)^*(s_0)$ we can state that $H^0(\mathcal{O}(H_3))$ is generated by x_1, x_2, x_3 and x_4 .

Q.E.D.

PROPOSITION 3.4:

$$(a) \quad H^0(\mathcal{O}(3H_2)) = (p_2 p_1)^* H^0(\mathcal{O}(3K_S)) + y_2 (p_2 p_1)^* H^0(\mathcal{O}(2K_S)) \\ + y_1 (p_2 p_1)^* H^0(\mathcal{O}(2K_S)) y_1 y_2 (p_2 p_1)^* H^0(\mathcal{O}(K_S)).$$

y_1 and y_2 have the same meaning as in Proposition 3.4.

$$(b) \quad H^0(\mathcal{O}(2H_2)) = (p_2 p_1)^* H^0(\mathcal{O}(2K_S)) + y_1 (p_2 p_1)^* H^0(\mathcal{O}(K_S)) \\ + y_2 (p_2 p_1)^* H^0(\mathcal{O}(K_S)) + \mathbb{C} y_1 y_2.$$

Proof. — Repeat the proof of Proposition 3.3.

Q.E.D.

PROPOSITION 3.5. — $H^0(\mathcal{O}(4K_S))$ is generated by $s_0^4, s_0^2 s_1, s_0^2 s_2, s_0 s_3, s_0 s_4, s_1^2, s_2^2$ and $s_1 s_2$. The s_i are chosen in the way pointed out on Paragraph 1.

Proof. — From the exact sequence

$$0 \rightarrow \mathcal{O}(3K_S) \xrightarrow{\otimes s_0} \mathcal{O}(4K_S) \rightarrow \mathcal{O}(4K_S)|_{K_S} \rightarrow 0$$

we get the following inclusion

$$0 \rightarrow H^0(\mathcal{O}(3K_S)) \xrightarrow{\otimes s_0} H^0(\mathcal{O}(4K_S)).$$

From this inclusion it follows that $s_0^4, s_0^2 s_1, s_0^2 s_2, s_0 s_3, s_0 s_4$ are linearly independent. Let me denote the vector space spanned by these linearly independent vectors by V_1 . The subspace V_1 has dimension 5. Let me denote the subspace spanned by s_1^2, s_2^2 and $s_1 s_2$ by V_2 . We will show that $\dim V_2 = 3$. If $\dim V_2 < 3$ then we will have $a_1 s_1^2 + a_2 s_2^2 + a_3 s_1 s_2 = 0$. From this equation we get $a_1 s_1^2 = s_2(a_2 s_2 + a_3 s_1)$. From the last equation it follows that C_2 is contained in C_1 . This is impossible. If $V_1 \cap V_2 = \emptyset$, then Proposition 3.5 will be proved. Suppose that $V_1 \cap V_2 \neq \emptyset$ and let $v \in V_1 \cap V_2$ and $v \neq 0$. Thus

$$v = b_1 s_1 + b_2 s_2 + b_3 s_1 s_2 = s_0(c_1 s_0^3 + c_2 s_0 s_1 + c_3 s_3 + c_4 s_0 s_2 + c_5 s_4).$$

From this formula we obtain:

$$(3.14) \quad b_1 s_1^2 + b_2 s_2^2 + b_3 s_1 s_2 \equiv 0 \quad \text{on } K_S.$$

Notice that it is impossible. Indeed, let U be a neighborhood of a point on K_S . Let $s_1|_U = f_1$ and $s_2|_U = f_2$. From the definition of s_1 and s_2 and Kunev's theorem it follows that we can find a point $P \in K_S \cap U$ such that $f_1(P) \neq 0$ and $f_2(P) \neq 0$. This fact contradicts (3.14). Proposition 3.5 is thus proved.

Q.E.D.

The end of the proof of Step 4. — Let me denote by $P_3 = p_3 p_2 p_1$. Combining Propositions 3.2, 3.3 and 3.4 and taking into account the remark after Proposition 3.3, we will obtain the following formula

$$(3.15) \quad \begin{aligned} H^0(O(4H_3)) &= P_3^* H^0(O(4K_S)) + x_1 P_3^* H^0(O(3K_S)) + x_2 P_3^* H^0(O(3K_S)) \\ &+ x_1 x_2 P_3^* H^0(O(2K_S)) + x_4 x_1 P_3^* H^0(O(2K_S)) + x_4 x_2 P_3^* H^0(O(2K_S)) \\ &+ x_4 P_3^* H^0(O(3K_S)) + x_4 x_1 x_2 P_3^* H^0(O(K_S)) + x_4^2 P_3^* H^0(O(K_S)) \\ &+ x_4^2 x_1 P_3^* H^0(O(K_S)) + x_4^2 x_2 P_3^* H^0(O(K_S)) + x_4^2 x_1 x_2 C. \end{aligned}$$

Note that this is a decomposition into a direct sum. From (3.15) we come to:

PROPOSITION 3.6. — *The basis of $H^0(X_3, O(3H_3))$ consists of all monomials of degree 4 formed of x_1, x_2, x_3 and x_4 plus $x_1 P_3(s_4), x_2 P_3(s_4), x_3 P(s_4)$ and $x_4 P_3(s_4)$.*

STEP 5. — $\deg g_3(X_3) = 6$, i.e. $g_3(X_3)$ is a hypersurface of degree 6 in $\mathbb{P}^3$.

Proof. — From Step 4 we see that $Y = g_3(X_3)$ cannot be a hypersurface of degree less or equal to 6. From Step 3 it follows that $\deg Y$ is either 6 or 12. Suppose that $\deg Y = 12$. From this fact it follows that all monomials of degree 6 formed of x_1, x_2, x_3 and x_4 are linearly independent in $H^0(X_3, O(6H_3))$. It is clear that we have the following inclusion; $P_3^* : H^0(S, O(6K_S)) \hookrightarrow H^0(X_3, O(6H_3))$. From this inclusion and the fact that all monomials of degree 6 formed of x_1, x_2, x_3 and x_4 are linearly independent it follows that $s_0^6, s_0 s_1 s_3, s_0 s_2 s_3, s_0^4 s_1, s_0^4 s_2, s_0^3 s_3, s_0^2 s_1^2, s_0^2 s_2^2, s_0^2 s_1 s_2, s_2^3, s_1^3, s_2^3, s_1^2 s_2$ and $s_1 s_2^2$ are linearly independent vectors in $H^0(S, O(6K_S))$ and spanned a vector subspace V of dimension 14. (Formally the proof of the fact that $\dim V = 14$ follows from the remark on Paragraph 7 and the above inclusion.) We have the following standart exact sequence

$$0 \rightarrow H^0(O(3K_S)) \xrightarrow{\otimes s_4} H^0(O(6K_S)) \xrightarrow{r} H^0(O(6K_S)|_{C_4}).$$

From this exact sequence it follows that if $v \neq 0$ and $v \in V$, then $r(v) \neq 0$, so $V \cap s_4 \otimes H^0(O(3K_S)) = \emptyset$. From this fact we obtain that

$$\dim H^0(O(6K_S)) \geq \dim V + \dim H^0(O(3K_S)) = 19.$$

From Kodaira vanishing theorem for surfaces of general type, i.e. $\dim H^i(S, O(nK_S)) = 0$ if i and n are greater than 0, and Riemann-Roch theorem, we get that $\dim H^0(O(6K_S)) = 17$. This contradiction proves Proposition 3.6.

Q.E.D.

STEP 6. — Suppose that K_S is an ample divisor. Then Y is a nonsingular variety.

Proof. — Mumford proved that $\text{Proj}(\oplus H^0(S, O(nK_S)))$ is a nonsingular model of a surface of general type S if K_S is an ample divisor. K_S is the canonical class of S . From this result it follows that:

PROPOSITION 3.7. — $\text{Proj}(\oplus H^0(X_3, O(nH_3)))$ is a nonsingular model of X_3 .

Proof. — From Lemma 3 it follows that $q(X_4) = Y$ is a surface of degree 6 in $\mathbb{P}^3$. If X_1 . Let me denote by R_1 the ring $\oplus H^0(X_1, O(nH_1))$ and by R the canonical ring of S , i.e. $R = H^0(S, O(nK_S))$. We must prove that for any maximal ideal m in R_1 , the local ring $R_{1(m)}$ is regular. Notice that $R_1 = R[X]/(X^2 - s_1)$, so $m' = m \cap R$ is a maximal ideal in R if m is maximal one in R_1 . For the proof of this fact look at Zariski and Samuel book *Commutative Algebra*. If the ideal m' does not contain the ideal (s_0) then $\hat{R}_{(m')} \cong \hat{R}_{1(m)}$. For the proof of this see Zariski and Samuel (the sign $\wedge$ means the completion in the m -adic topology). Now it is a standart fact from the local algebra that if the completion of a local ring is a regular one then the local ring is also regular. So in this case Proposition 3.7 is proved. Now suppose that $(s_1) \subset m'$. It is clear that we have the following isomorphism: $R_{1(m)} = R_{(m')}[X]/(X^2 - s_1)$. The ring $R_{(m')}[X]/(X^2 - s_1)$ is regular iff $s_1 \neq 0 \pmod{m'^2}$, i.e. s_1 is a local parameter in $R_{(m')}$. The last condition is fulfilled because the divisor of s_1 , C_1 , is a nonsingular curve. So from here Proposition 6.1 follows. The criterium we used is proved in Serre book *Local Algebra in Springer Lecture Notes*. If we repeat the same arguments for the rings $R_i = H^0(X_i, O(nH_i))$, $i=2, 3$ and 4 we will get that $\text{Proj}(R_i)$ is a nonsingular model of X_i .

Q.E.D.

PROPOSITION 3.8. — Y is a nonsingular hypersurface in $\mathbb{P}^3$.

Proof. — First we will prove that $Z_2 = (1, s)$ acts on X_3 and $X_3/s = Y$, i.e. $g_3 : X_3 \rightarrow Y$ is the natural map $X_3 \rightarrow X_3/s$. Let me denote by $K(X_3)$ the field of rational functions on X_3 and by $K(Y)$ the field of rational functions on Y . We have the natural inclusion: $K(Y) \subset K(X_3)$. From Step 5, i.e. $\deg g_3 = 2$, we get that $\deg(K(X_3) : K(Y)) = 2$. So $K(X_3)$ is a Galois extension of $K(Y)$ with a Galois group $G = Z_2$, i.e. $K(Y) = K(X_3)^G$. From the fact that $K(Y)$ is the quotient field of the subring $R' \subset R_3$ generated by x_1, x_2, x_3 and x_4 , it follows that $K(Y) \cap R_3 = R'$. From this fact we get immediately that $R' = R_3^G$. That G acts on R_3 follows from the following theorem: Every birational automorphism is a biregular one on the minimal model of a surface of general type. From the definition of R' it follows that $\text{Proj}(R') = Y = g_3(X_3)$. Now it is clear that

$Y = X_3/s$ and since Y is a factor of a nonsingular surface X_3 by the action of a group Z_2 , it follows that Y is a normal hypersurface in $\mathbb{P}^3$. This fact leads us to conclude that Y can have at most isolated singular points. These singular points can be ordinary double points because $Y = X_3/Z_2$ and their number is equal to the number of the fixed points by the action of Z_2 . Let me denote the fixed points by the action of Z_2 by p_i . To obtain a nonsingular model $\hat{Y}$ of Y , we first blow X_3 at all fixed points p_i and obtain a surface $\hat{X}_3$. It is easy to see that the involution s can be lifted to an involution $\hat{s}$ on $\hat{X}_3$. Let p be the canonical map $p : \hat{X}_3 \rightarrow X_3$. Let $E'_i = p^{-1}(p_i)$, then $s|_{E'_i} = \text{id}$. This implies that the quotient space $\hat{Y}$ of $\hat{X}_3$ by the involution s is nonsingular. Moreover, the morphism p induces a morphism $\hat{p} : \hat{X}_3 \rightarrow \hat{Y}$ which gives a resolution of singularities of Y . From this whole discussion it follows that we have a map $\hat{g}_3 : \hat{X}_3 \rightarrow \hat{Y}$, where $\hat{X}_3$ and $\hat{Y}$ are nonsingular varieties and $\hat{Y} = \hat{X}_3/Z_2$. These facts show us that the ramification divisor of $\hat{g}_3$ consists of the disjoint union of nonsingular curves. Now let me compute the canonical class of X_3 . We will use the following lemma proved in [M] on p. 110.

LEMMA. — *Let $f : X^r \rightarrow Y^r$ be a regular dominating map of smooth r -dimensional varieties with a branch locus B . Then for all rational r -forms w on Y :*

$$(3.16) \quad (f^* w) = B + f^{-1}((w)).$$

From this formula we immediately get

$$(3.17) \quad K_{X_3} = 5 K_3.$$

Note that $K_Y = 2H$. Let the branch locus of g_3 be $C + \sum E'_i$. It is a standard fact that $K_{X_3} = p^* K_Y + \sum E'_i$. From formula (3.16) we obtain:

$$(3.18) \quad 5H_3 + \sum E'_i = 2H_3 + C + \sum E'_i.$$

From (3.18) we deduce that C is rationally equivalent to $3H_3$. Let R be the ramification divisor of g_3 . From the fact that X_3 is a double covering of Y ramified over R , it follows that $g_3^*(R) = 2C \sim 6H_3$, where $\sim$ means rationally equivalent. Thus we get

$$(3.19) \quad R \sim 6H.$$

Next we will prove that Y is a nonsingular surface. If we prove that Z_2 acts without isolated fixed points, then Y will automatically be nonsingular. Let me denote by n the number of fixed points on X_3 . The proof of the fact that Y is a nonsingular surface is based on the following formula, connecting the topological Euler characteristics of X and Y , where X is a Z_n cyclic covering of Y ramified over R :

$$(3.20) \quad \chi(X) = n\chi(Y) - (n-1)\chi(R),$$

where are the topological Euler characteristics.

Using (3.20) it is very easy to compute $\chi(X_3)$ and we will get that

$$(3.21) \quad \chi(X_3) = 504.$$

Notice that

$$(3.21) \quad \chi(\tilde{X}_3) = \chi(X_3) + n,$$

where n is the number of fixed points of the action of $\mathbb{Z}_2$ on X_3 .

From (3.20) we obtain:

$$(3.23) \quad \chi(X_3) = 2\chi(Y) - \chi(R),$$

where R is the ramification divisor of g_3 .

Let us compute $\chi(Y)$ and $\chi(R)$. Because Y has only ordinary double points, then from the results of Briescorn it follows that the minimal nonsingular model $\tilde{Y}$ of Y is diffeomorphic to a nonsingular hypersurface of degree 6 in $\mathbb{P}^3$. Let Z be a hypersurface of degree 6 in $\mathbb{P}^3$. Then from the well known formula: $12(p_g - q + 1) = (K_Z, K_Z) + \chi(Z)$ we can conclude that

$$(3.24) \quad \chi(Z) = 108.$$

Notice that $R = C + \sum E_i$, where C is rationally equivalent to $6H$ and E_i is an exceptional curve of the second type and as all E_i are $\mathbb{P}^1$ we get that $\chi(E_i) = 2$. From the adjunctional formula on Y we get that $2p_g(C) - 2 = (C, C + K_Y) = (6H, 6H + 2H) = 6 \times 6 \times 8 = 288$. So $\chi(C) = -288$:

$$(3.25) \quad \chi(R) = \chi(C) + \sum \chi(E_i) = -288 + 2n.$$

From (3.22), (3.23), (3.24) and (3.25) we get

$$(3.26) \quad \chi(X_3) + n = 2 \times 108 + 288 - 2n.$$

Combining (3.21) and (3.26) we see that $n = 0$, so thus proving Step 6 and Lemma 3.

Q.E.D.

LEMMA 4. — Let X_4 be a $\mathbb{Z}_3$ cyclic covering of X_3 ramified over $(p_3 p_2 p_1)^*(C_4)$, where $C_4 = (s_4) \in |3K_S|$. Then: (a) $\dim H^0(H_4, \mathcal{O}(H_4)) = 5$ and $(H_4, H_4) = 36$, (b) the complete linear system $|H_4|$ gives a map $g_4: X_4 \rightarrow \mathbb{P}^4$, $\deg g_4 = 1$ and $g_4(X_4)$ is a nonsingular variety, which is a complete intersection of type $(6, 6)$.

Proof. — The proof of (a) is the same as proof of Lemma 3. Notice that we have $H^0(\mathcal{O}(H_4)) = p_4^* H^0(\mathcal{O}(H_3)) + \mathbb{C}x_5$, where $(x_5) = C'_4$, the branch locus of $p_4: X_4 \rightarrow X_3$.

The proof of (b).

PROPOSITION 4.1. — $\deg g_4 = 1$.

Proof. — Let me consider the composition of maps $X_4 \xrightarrow{p_4} X_3 \xrightarrow{g_3} Y$ and let me denote this composition by q , i.e. $q: X_4 \rightarrow Y$. Notice that q is given by the linear system $p_4^* H^0(\mathcal{O}(H_3)) \subset H^0(X_4, \mathcal{O}(H_4))$. Let x_1, x_2, x_3 and x_4 be a basis for $p_4^* H^0(\mathcal{O}(H_3))$. From condition (a) it follows that x_1, x_2, x_3, x_4 and x_5 is a basis of $H^0(\mathcal{O}(H_4))$, where $(x_5) = C'_4$, the

branch locus of p_4 . Suppose that x and $y \in X_4$ and $q(x) \neq q(y)$, then it follows that $g_4(x) \neq g_4(y)$. Now suppose that a point $P \in Y$, $P \notin R$ (the ramification divisor of g_3) and $P \notin g_3(D_4)$ (the image of the ramification divisor of p_4). From these two conditions, it follows that (a) $g_3^{-1}(P) = (Q_1, Q_2)$ and $Q_1 \neq Q_2$, (b) $p_4^{-1}(Q_1) = (P_{11}, P_{12}, P_{13})$, $p_4^{-1}(Q_2) = (P_{21}, P_{22}, P_{23})$, where $P_{1i} \neq P_{1j}$ for $1 \leq i, j \leq 3$ and $P_{2i} \neq P_{2j}$ for all $1 \leq i, j \leq 3$. Note that $q(P_{ki}) = P$ for all k and i . First we will prove that $s_4(Q_1) \neq s_4(Q_2)$. If for all $(Q_1, Q_2) = g_3^{-1}(P)$ $s_4(Q_1) = s_4(Q_2)$, then it will follow that s_4 is invariant under the action of $\mathbb{Z}_2$ ($\mathbb{Z}_2$ acts on X_3 and $X_3/\mathbb{Z}_2 = g_3(X_3) = q(X_4)$). On the other hand because $H_3 = g_3^*(H)$, it follows that $\mathbb{Z}_2$ acts on $H^0(X_3, \mathcal{O}(H_3))$ and thus

$$H^0(\mathcal{O}(H_3)) = H^0(\mathcal{O}(H_3))^+ \oplus H^0(\mathcal{O}(H_3))^-.$$

From the fact that

$$(a) H^0(\mathcal{O}(H_3))^+ = g_3^* H^0(Y, \mathcal{O}(H)) \quad \text{and} \quad (b) H^0(\mathcal{O}(3H_3))^+$$

is generated by x_1, x_2, x_3 and x_4 (this is Step 4 of Lemma 3) it follows that $H^0(\mathcal{O}(3H_3))^+$ is generated by all monomials of degree 3 formed by x_1, x_2, x_3 and x_4 . Now it is clear that $p_3^*(s_4) \in H^0(\mathcal{O}(3H_3))^+$. Here we use the notations of remark after proposition 3.3. Next we have to prove that $x_4(P_{1i}) \neq x_4(P_{1j})$ for all i and j . This follows from the formula $p_4^*(s_4) = x_4^3$ and the fact that the Galois group $\text{Gal}(X_4/X_3) = (1, s, s^2)$ acts on x_4 in the following manner: $x_4^s = \varepsilon x_4$. (This follows from the fact that $(x_4) = C'_4$ the branch locus of p_4 .) From this and the fact that, say $P_{12} = s(P_{11})$ and $P_{13} = s(P_{12}) = s^2(P_{11})$ we obtain $x_4(P_{12}) = \varepsilon x_4(P_{11}) = \varepsilon^2 x_4(P_{13})$, where $\varepsilon \neq 1$ and $\varepsilon^3 = 1$. Thus we get that $x_4(P_{1i}) \neq x_4(P_{1j})$ for all $i \neq j$. The same holds true for P_{2i} . Thus we have proved that for general points x and y such that $x \neq y$, $g_4(x) \neq g_4(y)$. Indeed we have two possibilities (a) $q(x) \neq q(y)$: then from the fact $g_4 = (q, x_4)$, we get that $g_4(x) \neq g_4(y)$, (b) $q(x) = q(y)$; then we have proved that for general points x and y , $x_4(x) \neq x_4(y)$.

Q.E.D.

PROPOSITION 4.2. — $g_4(X_4)$ is a nonsingular surface in $\mathbb{P}^4$.

Proof. — The proof is based on the following sublemma:

SUBLEMMA. — Let x be any point on X_4 and let U be a neighborhood of x , then we can find two sections s_1 and $s_2 \in H^0(X_4, \mathcal{O}(H_4))$ such that the curves (s_1) and (s_2) are nonsingular in U and $x \in (s_1) \cap (s_2)$.

Proof. — We will consider two different cases: (a) $x \notin (x_5) = C'_5$. Let me consider $q(x) \in Y$. For the definition of q , see Proposition 4.6, i.e. $q = g_3 \circ p_4$. From the Bertini theorem it follows that we can find two hyperplane sections H_1 and H_2 such that (1) H_1 and H_2 are nonsingular curves (2) $q(x) \in H_1 \cap H_2$, (3) H_1 and H_2 transect R , the ramification divisor, transversally. From condition (1) and (3) it follows that $g_3(H_1)$ and $g_3(H_2)$ are nonsingular curves on X_3 . From the fact that p_4 is a local isomorphism around y , [this is condition (a), i.e. $x \notin C'_4$ the branch locus of p_4], it follows that $p_4^*(g_3^*(H_1))$ and $p_4^*(g_3^*(H_2))$ are nonsingular curves in some neighborhood of x . For this case the sublemma is proved.

(b) $x \in C'_4$. Let me consider again $q(x) \in Y$ and $p_4(x)$. For $p_4(x)$ we have two possibilities: (1) $p_4(x) \notin R' \cap D_4$, where D_4 is the ramification divisor of p_4 . In a

neighborhood of $p_4(x)$, g_3 is a local isomorphism, so that $g_3(D_4)=q(C'_4)$ is a nonsingular curve in some neighborhood of $q(x)$. Now let H be a nonsingular hyperplane section of Y such that H intersects $q(C'_4)$ transversally in $q(x)$. From this review it follows that $q^*(H)$ is a nonsingular curve transecting C'_4 transversally. C'_4 and $q^*(H)$ are then nonsingular curves in some neighborhood of x containing x .

(2) $p_4(x) \in R' \cap D_4$. In this case we have two possibilities (a) R' and D_4 intersect each other transversally, then in a neighborhood of $q(x)$, $g_3(D_4)$ is a nonsingular curve. Now let H be a nonsingular hyperplane section intersecting R and $g_3(D_4)$ transversally. Then $q(H)$ is a nonsingular curve such that $q(H)$ intersects C'_4 transversally in x . (b) Let D_4 and R' be tangent at $p_4(x)$. Now let H be a nonsingular hyperplane section of Y transversal to R at $q(x)$. Then $q^*(H)$ is a nonsingular curve in a neighborhood of x intersecting transversally C'_4 . Thus $q^*(H)$ and C'_4 are the with the required properties.

Q.E.D.

Remark. — We have proved even more, namely, that through any point $x \in X_4$ we can find two sections s_1 and s_2 of $H^0(X_4, \mathcal{O}(H_4))$ such that (s_1) and (s_2) are nonsingular curves meeting in x transversally.

Now let me prove Proposition 4.2. The map g_4 is given by

$$x \rightarrow (q_0(x), \dots, q_4(x)).$$

Now we may suppose that in a neighborhood of x , $q_0 \neq 0$ and q_1 and q_2 have the properties stated in the remark after the sublemma. From this remark it follows that q_1/q_0 and q_2/q_0 are local coordinates in U . Let me denote these local coordinates by x and y . The map $g_{4U}: U \rightarrow \mathbb{C}^4 = (t_1, t_2, t_3, t_4)$, where $t_1 = x$ and $t_2 = y$, $t_3 = q_3/q_0$ and $t_4 = q_4/q_0$. From the fact that x and y are local coordinates in U it follows that $q_3/q_0 = F(x, y)$ and $q_4/q_0 = G(x, y)$, so the image of U in $\mathbb{C}^4$, i.e. $g_4(U)$ in $\mathbb{C}^4$ is given by the following equations: $t_3 = F(t_1, t_2)$ and $t_4 = G(t_1, t_2)$. From these two equations immediately come to the conclusion that $g_4(X_4)$ is a nonsingular variety.

Q.E.D.

PROPOSITION 4.3. — $g_4(X_4)$ is a complete intersection of type $(6, 6)$ in $\mathbb{P}^4$.

Proof. — From Lemma 3 it follows that $q(X_4) = Y$ is a surface of degree 6 in $\mathbb{P}^3$. If x_1, x_2, x_3 and x_4 is a basis of $p_4^* H^0(X_3, \mathcal{O}(H_3)) = q^* H^0(Y, \mathcal{O}(H))$, then there is a relation of degree 6 among x_1, x_2, x_3, x_4 , i.e. $h_6(x_1, x_2, x_3, x_4) = 0$ in $H^0(X_4, \mathcal{O}(H_4))$. From here it follows that $g_4^*(X_4)$ is contained in a hypersurface of degree 6 in $\mathbb{P}^4$. In Step 4 of lemma 3 we proved that $H^0(X_3, \mathcal{O}(3H_3))$ is generated by x_1, x_2, x_3, x_4 and s_4 , i.e. from all monomials of degree 3 formed from x_1, x_2, x_3, x_4 and $(p_3 p_2 p_1)^* s_4$. Notice that the branch locus of $g_3 R'$ is an element of $|3H_3|$. It follows Lemma 3. Let $(z) = R'$, where $z \in H^0(\mathcal{O}(3H_3))$, so that $z = g(x_1, x_2, x_3, x_4, s_4)$. On the band, we have $R \sim 6H$ and $g_3^*(R) = 2R'$, so from $R \sim 6H$ it follows that R is given by the equation $f(x_1, x_2, x_3, x_4)$. From $g_3^* R = 2R'$ we get

$$z^2 = g^2(x_1, x_2, x_3, x_4, s_4) = f(x_1, x_2, x_3, x_4).$$

From this equation we obtain a second relation of $\deg = 6$ in $H^0(X_4, \mathcal{O}(6H_4))$ among monomials of $\deg = 6$ formed from x_1, x_2, x_3, x_4 and x_5 . From here we conclude that $g_4(X_4)$

is contained in the intersection of two hypersurfaces of degree 6 in $\mathbb{P}^4$. From the fact that $(H_4, H_4) = 36$ we immediately understand that $g_4(X_4)$ is a complete intersection of type $(6, 6)$ in $\mathbb{P}^4$.

Q.E.D.

Theorem 1 is proved.

Q.E.D.

Remarks. — (1) From Theorem 1 it follows that the moduli space of all surfaces with $p_g = 1$ and $(K, K) = 1$ consists of one component. (2) All surfaces with $p_g = 1$ and $(K, K) = 1$ are simply connected. (3) The moduli space of surfaces with $p_g = 1$ $(K, K) = 1$ is a rational variety.

2. Deformation theory of surfaces with $p_g = 1$ and $(K, K) = 1$

THEOREM 2. — *Let S be a surface with $p_g = 1$ and $(K, K) = 1$ for which K_S is an ample divisor. Then $H^2(S, \Theta_S) = 0$ and $\dim H^1(S, \Theta_S) = 18$, where Θ_S is the tangent bundle sheaf.*

Proof. — From the Serre duality it follows that $H^2(S, \Theta_S)^* = H^0(S, \Omega_S^1(K_S))$. If we can prove that $H^0(S, \Omega_S^1(K_S)) = 0$, then we will get that $H^2(S, \Theta_S) = 0$. That $\dim H^1(S, \Theta_S) = 18$ follows directly from Riemann-Roch-Hirzebruch theorem and the fact that for surfaces of general type we have $H^0(S, \Theta_S) = 0$. Our theorem then will be proved. In Theorem 1 we have proved that a surface X can be constructed, which is a complete intersection of type $(6, 6)$ in $\mathbb{P}^4$ and on X there is a group $G = \mathbb{Z}_6 \oplus \mathbb{Z}_6$ in such a way that $X/G = S$. From this fact we can deduce that $H^0(S, \Omega_S^1(K_S)) = H^0(X, \Omega_X^1(H))^G$. Notice that we have proved that $p^*(K_S) = H$, the hyperplane section of X , where $p: X \rightarrow X/G = S$. So if we prove that $H^0(X, \Omega_X^1(H)) = 0$, then Theorem 2 will be proved.

LEMMA 2.1. — $H^0(X, \Omega_X^1(H)) = 0$.

Proof. — The proof will be given in several steps.

STEP 1. — $H^0(X, \Omega_X^1(H)) = H^0(X, \Omega_{\mathbb{P}^4}^1(H)|_X)$.

Proof. — We have the following exact sequence

$$(2.2) \quad 0 \rightarrow \Theta_X \rightarrow \Theta_{\mathbb{P}^4}|_X \rightarrow N_{\mathbb{P}^4/X} \rightarrow 0.$$

We will take the dual of (2.2), multiply it by $\mathcal{O}_X(H)$ and take into account that $N_{\mathbb{P}^4/X}^* = \mathcal{O}_X(-6H) \oplus \mathcal{O}_X(-6H)$, thus obtaining:

$$(2.3) \quad 0 \rightarrow \mathcal{O}_X(-6H) \oplus \mathcal{O}_X(-6H) \rightarrow \Omega_{\mathbb{P}^4}^1(H)|_X \rightarrow \Omega_X^1(H) \rightarrow 0.$$

From (2.3) we get

$$(2.4) \quad 0 \rightarrow H^0(X, \Omega_{\mathbb{P}^4}^1(H)|_X) \rightarrow H^0(X, \Omega_X^1(H)) \rightarrow H^1(X, \mathcal{O}_X(-5H)).$$

PROPOSITION 2.5. — $H^1(X, \mathcal{O}(-5H)) = 0$.

Proof. — This follows immediately from Mumford vanishing theorem. See [M].

Q.E.D.

From (2.4) and (2.5) we get Step 1.

Q.E.D.

STEP 2. — $H^0(X, \Omega_{\mathbb{P}^4}^1(H)|_X) = 0$.

Proof. — From the Serre duality we get that $H^0(X, \Omega_{\mathbb{P}^4}^1(H)|_X)^* = H^2(X, \Theta_{\mathbb{P}^4}(6H))$. We must prove then that $H^2(\Theta_{\mathbb{P}^4}(6H)|_X) = 0$. From the fact that X is a complete intersection in $\mathbb{P}^4$ we receive the following exact sequence

$$(2.9) \quad 0 \rightarrow J_X = \mathcal{O}_{\mathbb{P}^4}(-6H) \oplus \mathcal{O}_{\mathbb{P}^4}(-6H) \rightarrow \mathcal{O}_{\mathbb{P}^4} \rightarrow \mathcal{O}_X \rightarrow 0,$$

J_X the sheaf of ideals that define X in $\mathbb{P}^4$. Let us multiply (2.9) by $\Theta_{\mathbb{P}^4}(6H)$ then

$$(2.10) \quad 0 \rightarrow \Theta_{\mathbb{P}^4} \oplus \Theta_{\mathbb{P}^4} \rightarrow \Theta_{\mathbb{P}^4}(6) \rightarrow \Theta_{\mathbb{P}^4}(6)|_X \rightarrow 0.$$

From (2.10) we obtain:

$$(2.11) \quad H^2(\mathbb{P}^4, \Theta_{\mathbb{P}^4}(6)) \rightarrow H^2(X, \Theta_{\mathbb{P}^4}(6)|_X) \rightarrow \oplus H^3(\mathbb{P}^4, \Theta_{\mathbb{P}^4}).$$

From Bott's results we get that $H^2(\mathbb{P}^4, \Theta_{\mathbb{P}^4}(6)) = H^3(\mathbb{P}^4, \Theta_{\mathbb{P}^4}) = 0$, see [B]. From here it follows that $H^2(X, \Theta_{\mathbb{P}^4}(6H)|_X) = 0$.

Q.E.D.

From Step 1 and Step 2 we get that $H^0(X, \Omega_X^1(H)) = 0$ and, as we have seen, Theorem 2 follows from here.

Q.E.D.

3. Canonical Galois coverings of $\mathbb{P}^2$, that are surfaces with $p_g = 1$ and $(K, K) = 1$

The aim of this chapter is to describe all surfaces with $p_g = 1$ and $(K, K) = 1$ for which the map $f_{|2K_S|}: S \rightarrow \mathbb{P}^2$ is a Galois covering with the following additional properties: (1) K_S is an ample divisor (2) K_S is a nonsingular curve.

THEOREM 3. — *Let S be a surface with $p_g = 1$ and $(K, K) = 1$ with the properties described above, i.e. K_S is an ample divisor, K_S is a nonsingular curve and $f_{|2K_S|}: S \rightarrow \mathbb{P}^2$ is a Galois covering. Then:*

- (a) $\text{Gal}(S/\mathbb{P}^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$;
- (b) one of the involutions, say s_1 , restricted to K_S is the identity map.

Proof. — Proof of (a).

PROPOSITION 3.1. — $\deg f_{|2K_S|} = 4$.

Proof. — Let p be a point outside the ramification divisor of $f_{|2K_S|}$. Let L_1 and L_2 be two lines intersecting in p . By the definition of deg of a map we have

$$\deg f_{|2K_S|} = (f_{|2K_S|}^{-1}(L_1), f_{|2K_S|}^{-1}(L_2)) = (2K_S, 2K_S) = 4.$$

Q.E.D.

PROPOSITION 3.2. — $\text{Gal}(S/\mathbb{P}^2)$ is either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. — The order of $\text{Gal}(S/\mathbb{P}^2)$ must be 4 because $\deg f_{|2K_S|} = 4$. There are only two groups of order 4 and they are $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$.

Q.E.D.

PROPOSITION 3.3. — $f_{|2K_S|}$ restricted to the canonical divisor K_S is the canonical map $K_S \rightarrow \mathbb{P}^1$, i. e. $p_g(K_S) = 2$ and so $\deg f_{|2K_S|}|_{K_S} = 2$.

Proof. — Let me consider the exact sequence

$$(3.4) \quad 0 \rightarrow \Omega_S^2 \rightarrow \Omega_S^2(K_S) \xrightarrow{\text{res}} \Omega_{K_S}^1 \rightarrow 0.$$

Res is the Poincaré residue map. From (3.4) we have

$$(3.5) \quad 0 \rightarrow H^0(\Omega_S^2) \rightarrow H^0(\Omega_S^2(K_S)) \rightarrow H^0(\Omega_{K_S}^1) \rightarrow H^1(\Omega_S^2) = 0 \quad (q(S) = 0).$$

From (3.5) we get that the restriction of $f_{|2K_S|}$ on K_S is the canonical map. Note that K_S is a nonsingular curve of genus 2 and so it is a hyperelliptic curve and so the canonical map has degree 2.

Q.E.D.

PROPOSITION 3.4. — $\text{Gal}(S/\mathbb{P}^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. — Suppose that $\text{Gal}(S/\mathbb{P}^2) = \mathbb{Z}_4$.

SUBLEMMA. — Let s be the generator of $\mathbb{Z}_4$, then $s(K_S) = K_S$.

Proof. — Notice that $\mathbb{Z}_4$ acts on $H^0(S, \Omega_S^2)$. Which leads to the following possibilities: (a) $w^s = \pm w$, (b) $w^s = \pm iw$, where w is an element of $H^0(S, \Omega_S^2) = \mathbb{C}w$. From these two possibilities and the fact that K_S is the divisor of w , we get what is necessary.

Q.E.D.

From this sublemma it follows that we can find 6 different points on K_S such that $s(p_i) = p_i$; $i = 1, 2, \dots, 6$ and p_i are the Weierstrass points on the hyperelliptic curve of genus two K_S . From the fact that $s(p_i) = p_i$ we get a representation of $\mathbb{Z}_4$ to the tangent space at p_i . This means that we have a map $g: \mathbb{Z}_4 \rightarrow \text{Aut}(T_{p_i, S})$. Let M be the matrix equal to $g(s)$. Notice that $M^4 = E$, $M^2 \neq E$ and $M^3 \neq E$. If $M = E$ or $M^2 = E$, it will mean that in a neighborhood of p_i , $f_{|2K_S|}$ will have degree 1 or 2 and this contradicts proposition 3.1. Because $M^3 \neq E$ and $M^4 = E$ and from the Jordan decomposition of any linear operator it follows that we can find a basis in $T_{p_i, S}$ for which M will be diagonal. The matrix will be one of the following types

$$(1) \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad (2) \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \quad (3) \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

$$(4) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (5) \begin{pmatrix} -1 & 0 \\ 0 & -i \end{pmatrix}, \quad (6) \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix}, \quad (7) \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}.$$

All these matrixes correspond to the fact that we can choose the local coordinates (u, v) around p_i such that (1) $u^s = u, v^s = iv$, (2) $u^s = u, v^s = -iv$, (3) $u^s = iu, v^s = iv$, (4) $u^s = iu, v^s = -iv$, (5) $u^s = -u, v^s = -iv$, (6) $u^s = u, v^s = iv$ and (7) $u^s = iu, v^s = -iv$. It would be an easy exercise to find the invariants, to see that if $\mathbb{Z}_4$ acts as in cases (3), (4), (5), (6) and (7) then $S/\mathbb{Z}_4$ will have isolated singularities, but this is impossible because we know that $f_{|2K_S|}: S \rightarrow S/\mathbb{Z}_4 = \mathbb{P}^2$. If $\mathbb{Z}_4$ acts as in cases (1) and (2), then we will see that the action of $\mathbb{Z}_4$ restricted to a curve defined

by $v=0$ in a neighborhood of p_i is the identity. So from here we get that the map $f_{|2K_S|} : S \rightarrow S/\mathbb{Z}_4$ is either one to one or four to one set theoretically. This contradicts the fact that on K_S , which contains p_i , the map $S \rightarrow S/\mathbb{Z}_4$ is set theoretically two to one. From here we conclude that $\text{Gal}(S/\mathbb{P}^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Q.E.D.

PROPOSITION 3.5. — *Let s be an element of $\text{Gal}(S/\mathbb{P}^2)$, then either $s|_{K_S} = \text{id}$ or it has 6 fixed points.*

Proof. — From the Hurwitz formula (see [H]) it follows that either s has 6 fixed points or s has two fixed points or s is the identity, when we restrict s on K_S . Suppose that s has two fixed points on K_S . From the Hurwitz formula it follows that $K_S/s = C$ is an elliptic curve. Let $s_1 \in \text{Gal}(S/\mathbb{P}^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and such that $s \circ s_1 \neq \text{id}$. Notice that if $Y = S/s$, then $Y/s_1 = \mathbb{P}^2$, i. e. the composition map $S \xrightarrow{g_1} Y \xrightarrow{g_2} \mathbb{P}^2$ is $f_{|2K_S|}$. From the fact that $f_{|2K_S|}(K_S) = \mathbb{P}^1$ it follows that $g_2(C) = g_2(g_1(K_S)) = \mathbb{P}^1 = C/s_1$. The map $g_2 : C \rightarrow \mathbb{P}^1$ has degree 2 and from here it follows that set theoretically, the map $f_{|2K_S|} : K_S \rightarrow \mathbb{P}^1$ has degree 4. This contradicts proposition 3.3. So we have the following possibilities: either $s|_{K_S}$ is the identity map or $s|_{K_S}$ is the canonical involution of the hyperelliptic curve.

Q.E.D.

Let s_1 and s_2 be elements of $\text{Gal}(S/\mathbb{P}^2)$ such that $s_1 \circ s_2 \neq \text{id}$. Such elements exist because $\text{Gal}(S/\mathbb{P}^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. From proposition 3.7 it follows that $s_1|_{K_S}$ is either the identity or is the canonical involution.

If both s_1 and s_2 restricted to K_S are the canonical involutions, then $s_1 s_2$ restricted to K_S will be the identity map and $s_1 s_2 \neq \text{id}$ on S . So it follows that one of the involutions of $\text{Gal}(S/\mathbb{P}^2)$ restricted to K_S is the identity map.

Q.E.D.

THEOREM 4 (due to Kunev). — *Suppose that S is a surface with $p_g = 1$ and $(K, K) = 1$ with (a) an ample nonsingular canonical class, (b) S is a canonical Galois covering of $\mathbb{P}^2$. Then local Torelli theorem is not true for S .*

Proof. — Griffiths proved the following criterion in [G] for proving the period map to be local isomorphism: The period map is a local isomorphism iff the natural pairing

$$(4.1) \quad H^1(\Omega_S^1) \otimes H^0(\Omega_S^2) \rightarrow H^1(\Omega_S^1 \otimes \Omega_S^2)$$

is surjective. We have the following exact sequence

$$(4.2) \quad 0 \rightarrow \Omega_S^1 \xrightarrow{\otimes w} \Omega_S^1(K_S) \rightarrow \Omega_S^1(K_S)|_{K_S} \rightarrow 0$$

and

$$(4.3) \quad H^1(\Omega_S^1) \xrightarrow{\otimes w} H^1(\Omega_S^1(K_S)) \rightarrow H^1(\Omega_S^1(K_S)|_{K_S}) \rightarrow H^2(S, \Omega_S^1) = 0.$$

From (4.3) it follows that if $H^1(\Omega_S^1(K_S)|_{K_S}) \neq 0$, then the local Torelli is not true.

PROPOSITION 4.1. — $H^1(\Omega_S^1(K_S)|_{K_S}) \neq 0$.

Proof. — From Theorem 3 condition (b) it follows that there exists $s \in \text{Gal}(S/\mathbb{P}^2)$ such that $s|_{K_S} = \text{id}$. From this fact it follows that s acts on $\Omega_S^1(K_S)|_{K_S}$, so s acts on $\Omega_S^1|_{K_S}$ and $\Omega_S^1|_{K_S} = \Omega_S^1|_{K_S}^+ + \Omega_S^1|_{K_S}^-$, where $\Omega_S^1|_{K_S}^+ = \Omega_{K_S}^1$ and $\Omega_S^1|_{K_S}^- = N_{S/K_S}^*$ (the conormal bundle).

Thus

$$\begin{aligned} \Omega_S^1|_{K_S}(K_S) &= \Omega_S^1|_{K_S} \leftarrow N_{S/K_S} \quad [\text{because } \mathcal{O}_S(K_S)_{K_S} = N_{S/K_S}] \\ &= \Omega_{K_S}^1 \otimes N_{S/K_S} \oplus N_{S/K_S}^* \otimes N_{S/K_S} = \Omega_{K_S}^1 \otimes N_{S/K_S} \oplus \mathcal{O}_{K_S}. \end{aligned}$$

From this decomposition we get that $H^1(\Omega_S^1(K_S)|_{K_S}) = H^1(\mathcal{O}_{K_S}) = \mathbb{C}^2 \neq 0$. This is because K_S is a nonsingular curve of genus two.

Q.E.D.

THEOREM 5. — *Let S be a surface with the following properties: (a) $p_g(S) = 1$ and $(K_S, K_S) = 1$. (b) K_S is a nonsingular curve. (c) S is a canonical Galois covering of $\mathbb{P}^2$.*

Suppose that $s \in \text{Gal}(S/\mathbb{P}^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$, such that $s|_{K_S} = \text{id}$ and $s \neq \text{id}$ (such an automorphism exists according to Theorem 3).

Then $S/s = Y$ is a $K-3$ surface, which is a double covering of $\mathbb{P}^2$ and s has 9 fixed points outside K_S .

Proof:

PROPOSITION 5.1. — *s acts on $H^0(\Omega_S^2)$ as the identity.*

Proof. — Let $w \neq 0$ and $w \in H^0(\Omega_S^2)$. We must prove that $w^s = w$. Let U be a neighborhood of a point x on K_S . In U we can choose a local coordinate system (x, y) such that $x^s = x$ and $y^s = -y$. Notice that y is the local equation of K_S in U . From the fact that the divisor of w is K_S , we obtain

$$w_U = y dx \wedge dy, \text{ so } w_U^s = -y dx \wedge d(-y) = y dx \wedge dy.$$

Proposition 5.1 is thus proved.

Q.E.D.

PROPOSITION 5.2. — *s can have only isolated fixed points outside K_S .*

Proof. — Suppose that $s(p) = p$ and $p \notin K_S$. Let U_p be a neighborhood of p . From (5.1) it follows that s preserve w , i. e. $w^s = w$. From this fact it follows that the representation of $\mathbb{Z}_2$ in $T_{p,S}$ must preserve the skewsymmetric form w . This representation must be a $\text{SL}_2(\mathbb{C})$ representation. From this fact it follows that we can find a local coordinate system in $U_p(x, y)$ such that $x^s = -x$ and $y^s = -y$. So p must be an isolated fixed point.

Q.E.D.

Let me blow up all isolated fixed points of s . We will denote by S' the modified S . Let $p: S' \rightarrow S$ be the morphism that blows down all exceptional curves of the first kind. It is a well known fact that $K_{S'} = p(K_S) + \sum \mathbb{P}_i^1$. See [H]. We can continue the action of s on S' . An easy calculation shows that $s|_{\mathbb{P}_i^1} = \text{id}$ and $S'/s = Y$ is a nonsingular variety. These are standart facts. From Proposition 5.1 we get that $H^0(S, \Omega_S^2)^G = H^0(Y, \Omega_Y^2) \cong \mathbb{C}$, so that $p_g(Y) = 1$. We know that $s|_{K_S} = \text{id}$. Let x be a point on K_S . In a neighborhood of $x \in U$, $w|_U = u du \wedge dv$, where u is the local equation of K_S in U and $u^s = -u$ and $v^s = v$. Around $q(x)$ the local coordinates are u^2 and v , $q: S' \rightarrow S'/s = Y$. So $w_Y|_{q(U)} = du^2 \wedge dv$ is a globally

defined form on Y such that $q^*(w_Y) = w$. Notice that the divisor of w_Y is zero. From the fact that $q(S') = 0$ it follows that $q(Y) = 0$, so from the classification theory of algebraic surfaces it follows that Y is a $K-3$ surface. See [Š].

Now let me calculate the number of the fixed points of s . The following formula is true

$$(5.3) \quad \chi(S') = 2\chi(Y) - \chi(D),$$

where χ is the topological Euler characteristics and D is the ramification divisor of q .

It is a well known fact that the Euler characteristics of a $K-3$ surface $\chi(Y) = 24$, see [Š]. From the Noether formula: $12(p_g - q + 1) = (K_S, K_S) + \chi(S)$ we get that $\chi(S) = 23$. Now let me denote by n the number of blown up points on S . We get

$$(5.4) \quad \chi(S') = 23 + n, \quad \chi(Y) = 24.$$

Notice that $D = K_S + \sum_{i=1}^n P_i^1$, $\chi(K_S) = -2$ and $\chi(P_i^1) = 2$. So

$$(5.5) \quad \chi(D) = -2 + 2n.$$

Now from (5.3), (5.4) and (5.5) we get that $3n = 27$ so $n = 9$.

Q.E.D.

4. Examples and the description of surfaces with $p_g = 1$ and $(K, K) = 1$ that are canonical Galois coverings of P^2

THEOREM 6. — Let $S \subset \mathbb{P}^4(1, 2, 2, 3, 3)$ which is complete intersection of type $(6, 6)$ with the following properties: (a) the equations that define S contain s_0 in even degrees, (b) K_S is a nonsingular curve. ($\deg s_0 = 1$), where $\mathbb{P}^4(1, 2, 2, 3, 3) \subseteq \mathbb{P}^4(s_0, s_1, s_2, s_3, s_4)$. Then S is a canonical Galois covering of $\mathbb{P}^2$. The ramification divisor of $f_{|2K_S|}$ consists of two nonsingular curves of degree 3 in $\mathbb{P}^2$ meeting in 9 distinct points and a line.

Proof. — Theorem 1 shows that $p_g(S) = 1$ and $(K_S, K_S) = 1$. Let β is an automorphism of $\mathbb{P}^4(1, 2, 2, 3, 3)$ and $\beta(s_0, s_1, s_2, s_3, s_4) = (-s_0, s_1, s_2, s_3, s_4)$. From condition (a) it follows that β is an involution on S and $\beta|_{K_S} = \text{id}$. It is not difficult to prove that the fixed points of β outside K_S are the points $(s_0, s_1, s_2, 0, 0)$ on S . These points are exactly the intersection points of the curves $(s_3) = C_3$ and $(s_4) = C_4$. We are supposing that we have chosen s_1, s_2, s_3 and s_4 exactly in the same way as in theorem 1, i.e. $(s_1) = C_1, (s_2) = C_2, (s_3) = C_3$ and $(s_4) = C_4$ are nonsingular curves intersecting each other transversally. From the fact that C_i for $i = 3$ and 4 birationally equivalent to $3K_S$ we get that the number of the fixed points of β outside K_S is equal to 9. Now let me denote by $Y = S/\beta$. From Theorem 5 we know that Y is a $K-3$ surface.

PROPOSITION 6.1. — Y is a double covering of $\mathbb{P}^2$ ramified over two cubic curves meeting each other in 9 distinct points.

Proof. — It is a well-known fact that if C is a nonsingular curve on a $K-3$ surface, then the complete linear system $|C|$ gives a holomorphic map if $p_g(C) \geq 1$. See [Š], Chapter 10. Let me denote by C the image of K_S on Y . Because K_S is fixed by the

involution β , it follows that C is isomorphic to K_S , so that $p_g(C)=2$. By the theorem mentioned above the complete linear system $|C|$ gives a holomorphic map. From Riemann-Roch theorem it follows that we have a holomorphic map $f_{|C|}: Y \rightarrow \mathbb{P}^2$. Because Y is a $K-3$ surface and the map $f_{|C|}$ has degree 2, the ramification divisor of $f_{|C|}$ is rationally equivalent to $6L$, L is a line in $\mathbb{P}^2$. For the proof of this see [W]. Now I claim that the branch locus of $f_{|C|}$ consists of the images of C_3 and C_4 in Y . Let me denote these two images by D_3 and D_4 . From the definition of β it follows that β leaves C_3 and C_4 invariant. Let me compute the number of the fixed points of the action of β on D_3 and D_4 . We have 9 points that are the fixed points of β outside K_S and these 9 points are $C_3 \cap C_4$. On the other hand $\beta|_{K_S} = \text{id}$, so that $K_S \cap C_i$ ($i=3$ and 4) are fixed points on C_3 and C_4 . From $(C_i, K_S) = (3K_S, K_S) = 3$ for $i=3$ and 4 , we get that the number of fixed points of β on C_3 and C_4 is equal to $9+3=12$. Of course we have 12 fixed points on each of C_i , $i=3$ and 4 . From the adjunctional formula we get that $p_g(C_i)=7$ for $i=3$ and 4 . From the Hurwitz formula it follows that $p_g(D_i)=1$ for $i=3$ and 4 . We need to compute (C, C_i) on Y . Let me denote by p the natural map $p: S \rightarrow S/\beta = Y$. From the formula $(p^*D_3, p^*C) = (3K_S, 2K_S) = \deg p$. $(D_3, C) = 2$. $(D_3, C) = 6$ we get $(C, D_3) = (C, D_4) = 3$. From these calculations we get that the degree of the line bundle $O_Y(C)$ restricted to both elliptic curves D_3 and D_4 is 3. So $f_{|C|}$ restricted to D_3 and D_4 gives one to one map, i.e. $f_{|C|}: D_i \rightarrow \mathbb{P}^2$ for $i=3$ and 4 . This is a standard fact about elliptic curves. See [H]. From this discussion we conclude that the images of D_3 and D_4 are contained in the ramification divisor of $Y \rightarrow \mathbb{P}^2$. From the fact that the ramification divisor is rationally equivalent to $6L$, we get what is necessary.

Q.E.D.

Theorem 6 follows from Proposition 6.1 and Propositions (3.1) and (3.3).

Q.E.D.

Remark. — From Theorem 6 we get an explicit description of all Galois (canonical) coverings of $\mathbb{P}^2$ in terms of the equations of S in $\mathbb{P}^4$ $(s_0, s_1, s_2, s_3, s_4) = \mathbb{P}^4(1, 2, 2, 3, 3)$, i.e. these are all surfaces in $\mathbb{P}^4(1, 2, 2, 3, 3)$ that are complete intersections of type $(6, 6)$ and the equations that define S must contain s_0 in even degree.

This is one way of describing the canonical Galois coverings of $\mathbb{P}^2$. The other way is the following one and it is due to H. Clemens. Let Y be a double covering of $\mathbb{P}^2$, ramified over two elliptic curves meeting in 9 different points. Let Y' be the surface obtained by blowing up all 9 double points on Y . Let $E_1, \dots, E_9$ be the exceptional curves of the second type on Y' . Let C be the preimage of the line L that does not contain any of the intersection points of the ramification divisor. It is not very difficult to prove that $C + E_1 + \dots + E_9$ is divisible by two in $H_2(Y', \mathbb{Z})$. Indeed, it is not difficult to see that $3C \sim 2D_3 + E_1 + \dots + E_9$ or $C \sim 2D_3 - 2C + E_1 + \dots + E_9$ and so $C + E_1 + \dots + E_9 \sim 2D_3 - 2C + 2(E_1 + \dots + E_9)$. Now let me define S' as a double covering of Y' ramified over $C + E_1 + \dots + E_9$. It is not very difficult to prove that the minimal model of S' , S is a surface with $p_g=1$ and $(K_S, K_S)=1$. From here we can compute the number of moduli of all canonical Galois coverings of $\mathbb{P}^2$. First one can prove that if Y is a $K-3$ surface which contains 9 exceptional curves of the second type and a curve of genus two not intersecting these 9 projective lines, then the curve of genus two plus the 9 lines are divisible by two in the second

homology group. Thus repeating the second construction of Galois coverings of $\mathbb{P}^2$. Note that the number of moduli of $K-3$ surfaces with the above properties is equal to 10. Two more moduli are obtained from the choice of C . So the number of the moduli of all canonical Galois coverings of $\mathbb{P}^2$ is 12.

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