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COMPARISON THEOREMS FOR COMPACT SYMMETRIC SPACES (*)

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1. Introduction

In this paper we prove that a compact Riemannian manifold M whose curvature is similar to the curvature of a compact symmetric space $\overline{M}$ is diffeomorphic to a locally symmetric space. In case of the sphere as model space, similarity of curvature is measured in terms of a bound on sectional curvature. The same is true for complex projective space as model if we restrict our attention to Kähler manifolds. For symmetric spaces of rank one our result specializes to well-known theorems, compare [5], [6] and [8].

For a symmetric space of arbitrary rank as model, a description of the similarity of curvatures of the general manifold and the model in terms of sectional curvatures would be rather cumbersome. Instead, we measure this similarity with the norm of the curvature of the corresponding Cartan connection. E. Cartan first developed this connection to study affine and projective structures on manifolds. Since the curvature of the relevant Cartan connection vanishes in case the manifold is a locally symmetric space, its norm is a natural measure for the deviation of the local geometry of the general manifold from the local geometry of the model. We refer to [8] for an elaboration of this point of view.

In [2] Cheeger utilized other assumptions on the curvature. Our assumption has the advantage that it is independent of the injectivity radius, or equivalently of a lower bound for the volume of the general manifold M. For rank one models $\overline{M}$ this difference is of no importance because the curvature assumptions yield an estimate for the injectivity radius of M. For models of higher rank such estimates are not known *a priori*; they are a consequence of our theorem.

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In the proof, we reduce the situation to the special case where the model is a compact semi-simple Lie group. Here the local geometry is described by the Maurer-Cartan equation. It is natural therefore to expect that a simply connected manifold P with a form ω satisfying the appropriate Maurer-Cartan equation up to a small error should be diffeomorphic to a Lie group. We prove this result by solving the Maurer-Cartan equation $d\bar{\omega}+[\bar{\omega},\bar{\omega}]=0$ on P, where [,] is the bracket of the Lie algebra of the model. To prove the existence of a solution we follow the method of Newton-Kolmogorov-Moser and solve a linearized deformation equation $d'\alpha=-\Omega'$ approximately, where Ω', the error term, is interpreted as the curvature of a certain connection. The fact that $d'\circ d'\neq 0$ prevents us from solving the linearized equation exactly; however the Bianchi equation allows us to solve it well enough for the iteration to converge to a Maurer-Cartan form.

As explained in Chapter 3, we obtain an approximate solution of $d'\alpha=-\Omega'$ by solving $\Delta'\beta=-\Omega'$ and setting $\alpha=\delta'\beta$, where $\Delta'=d'\delta'+\delta' d'$, and δ' is the adjoint of d'. The main point of Chapter 4 is to prove that the solutions of $\Delta'\beta=-\Omega'$ satisfy a maximum principle. As a consequence, existence and uniqueness of β follows as well as an estimate for $\|\beta\|$. We were motivated to do the computations of this chapter by the well known vanishing theorem for the second cohomology group of a compact semi-simple Lie group. In Chapter 5 we obtain the estimates we need to prove the convergence of the iteration. The main tools are interior regularity estimates for solutions of elliptic partial differential equations in variational form and a generalized maximum principle for coercive differential operators. The maximum principle enables us to avoid assumptions on the injectivity radius of the exponential map of P.

The material of Chapter 5 could be simplified considerably if, instead of assuming a bound on the C^0-norm of the curvature, we would assume a bound on the C^1-norm, or even a Hölder norm, of the curvature. In this case, the standard theorems on elliptic differential equations would apply directly. We wish to thank L. Bérard Bergery for suggesting several improvements in our first version of this paper.

2. The results

Let G denote a compact semi-simple Lie group, $\mathfrak{g}$ its Lie algebra and $\bar{\omega}:TG\to\mathfrak{g}$ the Maurer-Cartan form. By definition, $\bar{\omega}$ is constant on left invariant vector fields and defines an isomorphism between the Lie algebra of left invariant vector fields on G and the tangent space $\mathfrak{g}=T_eG$ of G at the identity element. We wish to compare the local geometry of a general manifold P to that of G.

The following definitions are motivated by the Maurer-Cartan equation $d\bar{\omega}+[\bar{\omega},\bar{\omega}]=0$, where [,] is the Lie bracket in $\mathfrak{g}$. Let $\omega:TP\to\mathfrak{g}$ define a parallelization of P, i.e., $\omega:T_xP\to\mathfrak{g}$ is a vector space isomorphism of the tangent space T_xP with $\mathfrak{g}$ for all $x\in P$. We define the curvature of ω to be the $\mathfrak{g}$-valued 2-form $\Omega=d\omega+[\omega,\omega]$, where [,] is the Lie bracket of $\mathfrak{g}$. In case $\mathfrak{g}$ is compact and semi-simple, the Lie algebra carries a natural positive definite scalar product (minus the Cartan-Killing form) and the isomorphism ω induces a Riemannian metric on the manifold P. With respect to these metrics we define the maximum norm $\|\Omega\| = \max |\Omega(X_1, X_2)|$, where the maximum is taken over all unit vectors $X_1, X_2 \in TP$. Our first result is the following:

THEOREM 1. — *Let* $\mathfrak{g}$ *be a compact semi-simple Lie algebra,* $\omega : TP \to \mathfrak{g}$ *a parallelization of a compact manifold* P, *and* Ω *the curvature of* ω.

There exists a positive constant A *depending only on* $\mathfrak{g}$ *such that* $\|\Omega\| < A$ *implies that* P *is diffeomorphic to a quotient* $\Gamma \backslash G$, *where* G *is the simply connected Lie group with Lie algebra* $\mathfrak{g}$ *and* Γ *is a finite subgroup of* G.

The main application of this result is the following Theorem 2, where manifolds P satisfying the above assumption occur naturally as principal bundles. For this application we need a slightly more precise formulation of Theorem 1. We prove in fact the existence of a parallelization $\bar{\omega} : P \to \mathfrak{g}$ with vanishing curvature, i. e., which satisfies the Maurer-Cartan equation. Moreover, by choosing A small enough we can find such a $\bar{\omega}$ with $\|\omega - \bar{\omega}\|$ arbitrarily small. This implies that the diffeomorphism of the theorem is in fact a quasi-isometry with the dilatation controlled by the constant A of the theorem.

To motivate and clarify our assumption on the curvature of the general manifold M in Theorem 2 we first analyse the standard case. Let $\bar{M} = G/K$ be an irreducible riemannian symmetric space. The projection $G \to \bar{M}$ is a K-principal bundle representing the reduction of the bundle of orthonormal frames over $\bar{M}$ to the isotropy group K. The Maurer-Cartan form $\bar{\omega} : TG \to \mathfrak{g}$ is a Cartan connection of type (G, K) for $\bar{M}$ with vanishing curvature $\bar{\Omega} = d\bar{\omega} + [\bar{\omega}, \bar{\omega}]$. The flatness of $\bar{\omega}$ is topologically reflected in the fact that the G-principal bundle $G \times_K G \to \bar{M}$ obtained by extending the fibres of the bundle $G \to \bar{M}$ from K to G; and hence also the associated fibre bundle $G \times_K G/K \to \bar{M}$ with fibre $G/K = \bar{M}$ is canonically trivial. The trivialization is given by

$$G \times_K G \cong G/K \times G,$$
$$[(a, b)] \to (aK, ab),$$

where $[(a, b)]$ denotes the equivalence class $\{(ak, k^{-1} b) \mid k \in K\}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a Cartan decomposition of $\mathfrak{g}$ with respect to K. The $\mathfrak{k}$-valued part of $\bar{\omega}$ is the Levi-Civita connection of the symmetric space $\bar{M}$ and the $\mathfrak{m}$-valued part is the canonical soldering form given by the isomorphism $T\bar{M} \cong G \times_K \mathfrak{m}$, with K represented in $\mathfrak{m}$ *via* the adjoint action of G restricted to K.

The following assumptions on M serve to define the appropriate K-principal bundle P over M together with a parallelization $\omega : TP \to \mathfrak{g}$ satisfying the conditions of Theorem 1. Our first assumption on M is that there is a reduction $P \xrightarrow{\pi} M$ of the bundle of frames over M to the structure group K represented orthogonally in $\mathfrak{m} \cong \mathbb{R}^n$ as above. This is a purely topological assumption and is obviously a necessary condition for M to be diffeomorphic to a quotient of $\bar{M}$. On P we have the canonical $\mathfrak{m}$-valued soldering form θ given by the formula:

$$\theta : \quad T_u P \to \mathfrak{m},$$
$$X \mapsto u^{-1} \pi(X),$$

where $u \in P$ defines an isomorphism $u : \mathfrak{m} \cong T_{\pi(u)} M$. θ is an ad K-equivariant 1-form vanishing on vertical vectors.

Let η be a connection form on P. η is a Riemannian connection since the structure group K of P is compact, i. e., it preserves a Riemannian metric. We do not assume that η is a Levi-Civita connection. This would imply certain rigid integrability conditions for P, whereas our aim is to avoid assuming closed conditions.

We combine θ and η to define a $\mathfrak{g}$-valued 1-form $\omega = \eta + \theta : TP \to \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. This ω is a Cartan connection of type (G, K) with curvature $\Omega = d\omega + [\omega, \omega]$, where $[\ ,\ ]$ is the Lie bracket of $\mathfrak{g}$. In the following Theorem we define $\|\Omega\|$ as in Theorem 1. We shall not define the concept of Cartan connection because we do not use it except for the motivation of the term curvature for the expression $\Omega = d\omega + [\omega, \omega]$. For details we refer to [8].

THEOREM 2. — *Let $\overline{M} = G/K$ denote a simply connected compact irreducible symmetric space,* M *a compact Riemannian manifold, and* Ω *the curvature form of the Cartan connection* ω *on the principal bundle* P *over* M *defined above.*

There exists a positive constant A *depending only on* $\overline{M}$ *such that* $\|\Omega\| < A$ *implies that* M *is diffeomorphic to a quotient* $\Gamma \backslash \overline{M}$, *where* Γ *is a finite subgroup of* G.

In case the Riemannian connection η is invariant under a group H, it follows from the proof that H is isomorphic to a subgroup of G and the diffeomorphism $M \to \Gamma \backslash \overline{M}$ is equivariant with respect to the actions of H. Thus, in case $\overline{M} = S^n$, our result, except for the numerical constant, specializes to the main result of [5]. The proof also shows that the diffeomorphism of $M \to \Gamma \backslash \overline{M}$ is a quasi-isometry with the dilatation controlled by the constant A of the theorem.

We remark that we have made no explicit assumptions on the Riemannian curvature. In particular we do not assume that M has non-negative Riemannian sectional curvature. Our curvature assumptions imply only that the connection η has small torsion and has curvature very near to that of the model space $\overline{M}$.

Some of the previously known Comparison Theorems can be obtained as a consequence of Cheeger's finiteness theorem, compare [3]. This is not true of the theorems above because the curvature assumptions, *a priori*, do not yield an estimate for the injectivity radius or a lower bound for the volume of M.

3. The proof

The main work in the proof of Theorem 1 is to establish the existence of a Maurer-Cartan form $\bar{\omega}$ on P, i. e., a parallelization $\bar{\omega} : TP \to \mathfrak{g}$ which satisfies the Maurer-Cartan equation $d\bar{\omega} + [\bar{\omega}, \bar{\omega}] = 0$, where $[\ ,\ ]$ is the bracket in the Lie algebra $\mathfrak{g}$. In the first part of this chapter we set up a successive approximation scheme for the solution $\bar{\omega}$ and prove that the existence of $\bar{\omega}$ implies Theorem 1. In the second part we prove Theorem 2 as a Corollary of Theorem 1. We postpone the two main steps of the proof, the existence of a solution of the linearized equation, and the estimates necessary for the convergence of the iteration, to subsequent chapters.

Let ω be the $\mathfrak{g}$-valued 1-form of Theorem 1. To obtain a 1-form $\bar{\omega}=\omega+\alpha$ with vanishing curvature we must solve the equation:

$$d\alpha+[\omega, \alpha]+[\alpha, \omega]+[\alpha, \alpha]=-\Omega, \tag{3.1}$$

To prove that $\bar{\omega}$ is again a parallelization, and therefore a Maurer-Cartan form, we will establish the estimate $\|\alpha\|<c\|\Omega\|$ for some constant c depending only on $\mathfrak{g}$.

Since (3.1) is non-linear we first consider the linearized deformation equation:

$$d^{\omega}\alpha=d\alpha+[\omega, \alpha]+[\alpha, \omega]=-\Omega. \tag{3.2}$$

The operator d^{ω} is the exterior covariant derivative of the covariant derivative D on the trivial vector bundle $P\times\mathfrak{g}$ defined by the formula:

$$D_X s=X s+[\omega(X), s], \tag{3.3}$$

where $X\in TP$, $s : P\rightarrow\mathfrak{g}$ is a section of the bundle and Xs is the derivative of s in direction X. The curvature R^D of D is computed to be

$$R^D(X, Y)s=[\Omega(X, Y), s], \tag{3.4}$$

where $X, Y\in TP$, $s : P\rightarrow\mathfrak{g}$ and $\Omega=d\omega+[\omega, \omega]$.

The exterior covariant derivative on $\mathfrak{g}$-valued forms associated to D, denoted by d^{ω}, is given by the formula:

$$d^{\omega}\alpha(X_0, \ldots, X_p)=\sum_{i=0}^{p}(-1)^i D_{X_i}\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_p) + \sum_{j<k}(-1)^{j+k}\alpha([X_j, X_p], \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_p). \tag{3.5}$$

It is readily seen that d^{ω} on 1-forms is exactly the linear deformation operator of equation (3.2).

Using (3.4) we can write the Bianchi identity for the curvature simply as

$$d^{\omega}\Omega=0. \tag{3.6}$$

In general, it may not be possible to solve the linear equation (3.2) exactly. We will see that it can be solved well enough, in a sense to be made precise later, if the curvature Ω is sufficiently small. For technical reasons which will become apparent in the next chapter, we prefer to replace d^{ω} by a slightly modified operator d'. We obtain d' from d^{ω} by replacing the vector field bracket in (3.5) by the Lie algebra bracket. For the rest of this chapter the symbols X_i will denote parallel vector fields; i.e. $\omega(X_i)=\text{Const}$. We define

$$d'\alpha(X_0, \ldots, X_p)=\sum_{i=0}^{p}(-1)^i D_{X_i}\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_p) + \sum_{j<k}(-1)^{j+k}\alpha(\{X_j, X_k\}, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_p), \tag{3.7}$$

where $\{X_j, X_k\}=\omega^{-1}([\omega(X_j), \omega(X_k)])$.

In the following we will need the adjoint δ' of d'. Since $\mathfrak{g}$ is compact and semi-simple it carries a natural positive scalar product (minus the Cartan-Killing form), and the isomorphism ω induces a Riemannian metric on P. With respect to these metrics we obtain

$$(3.8)\quad \delta'\alpha(X_2, \ldots, X_p) = -\sum_{k=1}^{N} D_{e_k}\alpha(e_k, X_2, \ldots, X_p) + \frac{1}{2}\sum_{k=1}^{N}\sum_{l=2}^{p}\alpha(e_k, X_2, \ldots, X_{l-1}, \{e_k, X_l\}, X_{l+1}, \ldots, X_p),$$

where $\{e_k\}_{k=l,\ldots,N}$ is a parallel orthonormal base in TP, and the vector fields X_i are parallel as indicated above. To estimate the norm of d'-d^ω we observe that $-[X, Y]+\{X, Y\}=\omega^{-1}(\Omega(X, Y))$ holds for parallel vector fields X, Y, hence d'-d^ω is an operator of order zero. We list a bound for this operator, as well as a consequence *via* the Bianchi equation (3.6), for further use:

$$(3.9)\quad \begin{cases} \| d'\text{-}d^\omega \| < c\|\Omega\|, \\ \| d'\Omega \| < c\|\Omega\|^2, \end{cases}$$

where c depends only on $\mathfrak{g}$.

Unlike in the case of the usual exterior derivative, where $d\circ d=0$, $d'\circ d'$ may not vanish. The following formula holds for $d'\circ d'\alpha$ in terms of parallel vector fields X_i:

$$(3.10)\quad d'd'\alpha(X_0 \ldots X_{p+1}) = \sum_{i<j}(-1)^{i+j}\,\omega^{-1}(\Omega(X_i, X_j))\,\alpha(X_0 \ldots \hat{X}_i \ldots \hat{X}_j \ldots X_{p+1}).$$

Proof. — For forms with $\alpha(X_1, \ldots, X_p)$ constant, d' is just the exterior derivative for the Lie algebra cohomology of $\mathfrak{g}$, and $d'd'\alpha=0$. Because of this it is sufficient to prove the formula for 0-forms. Let X, Y be vector fields of the parallelization and s a section in $P\times\mathfrak{g}$:

$$d'sX = Xs + [\omega(X), s],$$

$$\begin{aligned} d'd's(X, Y) &= XYs + [\omega(X)[\omega(Y), s]] \\ &\quad - YXs - [\omega(Y)[\omega(X), s]] - \{X, Y\}s - [\omega\{X, Y\}, s] \\ &= ([X, Y]-\{X, Y\})s = -\omega^{-1}(\Omega(X, Y))s, \end{aligned}$$

where we have used the Jacobi identity for the Lie bracket. For further reference we list the formula for $\delta'\circ\delta'$ in terms of parallel vector fields $X_3, \ldots, X_p$ and a parallel orthonormal base $\{e_i\}_{i=1,\ldots,N}$ in TP:

$$(3.11)\quad \delta'\delta'\alpha(X_3, \ldots, X_p) = \sum_{i,j=1}^{N}\omega^{-1}(\Omega(e_i, e_j))\,\alpha(e_i, e_j, X_3, \ldots, X_p).$$

After these preparations we are in a position to define the iteration for a sequence of 1-forms $\{\omega_i\}$ converging to a Maurer-Cartan form $\bar{\omega}$ on P. In fact we will define a sequence $\{\omega_i\}$ of parallelizations $\omega_i : TP \to \mathfrak{g}$ whose curvatures $\Omega_i = d\omega_i + [\omega_i, \omega_i]$ converge to zero. Starting with $\omega_0 = \omega$, we define $\omega_{i+1} = \omega_i + \alpha_i$, $i = 0, 1, \ldots$, where α_i is an approximate solution of (3.2). The reason for this choice of α_i is the following expression for the curvature of ω_{i+1}:

$$\Omega_{i+1} = \Omega_i + d^{\omega_i} \alpha_i + [\alpha_i, \alpha_i], \tag{3.12}$$

which shows that Ω_{i+1} will be of the order of magnitude $\|\Omega_i\|^2$, since $\|\alpha_i\|$ will be shown to be bounded by $c \|\Omega_i\|$.

To facilitate notation in the precise definition of α_i we denote α_i, Ω_i and Ω_{i+1} by α, Ω' and Ω'' respectively. We introduce the Laplacian $\Delta' = d' \delta' + \delta' d'$, where in the definitions (3.7) and (3.8) ω_i and Ω_i are to be used instead of ω and Ω.

We define:

$$\alpha = \delta' \beta, \tag{3.13}$$

where β is the unique solution of the potential equation:

$$\Delta' \beta = -\Omega'. \tag{3.14}$$

Existence and uniqueness of the 2-form β satisfying (3.14) will be proved in the next chapter as a consequence of the fact that the elliptic operator $\Delta' = d' \delta' + \delta' d'$ is positive definite on 2-forms in case $\|\Omega'\|$ is small enough.

The main requirements on α necessary for the convergence of the iteration described above are formulated in the following.

MAIN LEMMA. — *Let* $\omega' : \mathrm{TP} \to \mathfrak{g}$ *be a parallelization of a compact manifold* P, $\Omega' = d\omega' + [\omega', \omega']$, *and* [,] *the Lie bracket in the compact semi-simple Lie algebra* $\mathfrak{g}$.

There exists a constant $A' > 0$ *depending only on* $\mathfrak{g}$ *such that* $\|\Omega'\| < A'$ *implies that the* 1*-form* $\alpha = \delta' \beta$ *satisfies:*

$$(3.15) \quad \begin{cases} \text{(i)} & \|d' \alpha + \Omega'\| < c \|\Omega'\|^2; \\ \text{(ii)} & \|\alpha\| < c \|\Omega'\|; \\ \text{(iii)} & \|\alpha\|_{1,q} < c' \|\Omega'\|, \end{cases}$$

where c *is a constant depending only on* $\mathfrak{g}$, c' *a constant depending on* $\mathfrak{g}$ *and on the diameter of* P, $\| \; \|$ *is the maximum norm with respect to the metrics in* $\mathfrak{g}$ *and* P *defined earlier and* $\| \; \|_{1,q}$ *is the Sobolev norm with* $q > \dim \mathrm{P}$.

The proof as well as the definition of Sobolev norms will be given in the last chapter.

The estimates (3.15) (i) and (ii) together with (3.9) and (3.12) imply $\|\Omega_{i+1}\| < c \|\Omega_i\|^2$ with c depending only on $\mathfrak{g}$ and hence the sequence of curvatures converges rapidly to zero, provided that the initial curvature $\Omega = \Omega_0$ is small enough. Therefore by (3.15) (iii) the series $\sum_{i=0}^{\infty} \alpha_i$ and hence the sequence $\{\omega_i\}$ of connections converges in the Sobolev space $\mathrm{W}_{1,q}(\mathrm{P})$ to a connection form $\bar{\omega}$ with zero curvature. Since c' in (iii) might depend on the diameter of P, the rate of convergence in $\mathrm{W}_{1,q}$ might also depend on it. To prove that $\bar{\omega}$ is also a parallelization of the tangent bundle of P it suffices to show $\|\omega - \bar{\omega}\| < 1$ since the initial form $\omega=\omega_0$ is an isometry on each tangent space by the definition of the metric on P. We remark here that the metrics on P induced by ω_i at each stage of the iteration change with the iteration steps. However, since the change effected by an iteration step is controlled by $\|\alpha_i\|$, we may, by choosing the constant A of the theorem small enough, assume that

$$\frac{1}{2}\|\quad\|_0 \leqq \|\quad\|_i \leqq 2\|\quad\|_0,$$

where $\|\quad\|_i$ stands for any norm used in the ith iteration step. Therefore we may write unambiguously $\|\quad\|_{m,q}$ in the estimates at all stages of the iteration.

Because the constant c in (3.15) (ii) is independent of the diameter of P, the maximum norm $\left\|\sum_{i=0}^{\infty}\alpha_i\right\|=\|\omega-\bar{\omega}\|$ can be made as small as we please by choosing the constant A of the theorem small enough.

Thus $\bar{\omega}$ is a Maurer-Cartan form in the Sobolev space $W_{1,q}$. Now, because $\bar{\Omega}=0$ and because the vector fields of the parallelization defined by $\bar{\omega}$ are volume preserving, $\bar{\omega}$ satisfies the elliptic system of differential equations:

$$d\bar{\omega}+[\bar{\omega},\bar{\omega}]=0, \qquad \delta\bar{\omega}=0.$$

Therefore $\bar{\omega}$ is smooth.

To finish the proof of Theorem 1, let $\tilde{P}$ be the universal covering space of P and $\tilde{\omega}$ the pullback of $\bar{\omega}$ *via* the covering map. The connection form $\tilde{\omega}: T\tilde{P}\to\mathfrak{g}$ defines a vector space isomorphism between $\mathfrak{g}$ and a finite dimensional subspace $\tilde{\mathfrak{g}}$ of the space of vector fields on $\tilde{P}$. Explicitly, $\tilde{\mathfrak{g}}$ is defined by the property that $\tilde{\omega}(X)$ is constant for all $X\in\tilde{\mathfrak{g}}$. Since $\tilde{\omega}$ has curvature zero, we have

$$0=\tilde{\Omega}(X, Y)=X\tilde{\omega}(Y)-Y\tilde{\omega}(X)-\tilde{\omega}[X, Y]+[\tilde{\omega}(X), \tilde{\omega}(Y)].$$

This implies that $\tilde{\omega}:\tilde{\mathfrak{g}}\to\mathfrak{g}$ is a Lie algebra isomorphism.

To define a diffeomorphism $\tilde{F}:\tilde{P}\to G$, we fix an arbitrary point $\tilde{e}\in\tilde{P}$ and consider the over-determined system of partial differential equations

$$d\tilde{F}=\tilde{\omega}, \qquad \text{with initial condition } \tilde{F}(\tilde{e})=e, \tag{3.16}$$

where $d\tilde{F}$ is the differential of $\tilde{F}$, $e\in G$ is the identity element and $\tilde{\omega}$ is identified with the map $\tilde{\omega}: T_x(\tilde{P})\to\mathfrak{g}\cong T_{\tilde{F}(x)}G$ *via* left-invariant vector fields on G. This differential equation is integrable because $\tilde{\omega}:\tilde{\mathfrak{g}}\cong\mathfrak{g}$ is a Lie algebra isomorphism.

The solution $\tilde{F}:\tilde{P}\to G$ is globally defined because $\tilde{P}$ is simply connected. $\tilde{F}$ is in fact a diffeomorphism because G is simply connected as well. Explicitly, one can define $\tilde{F}$, or rather its graph, as follows: Consider the $\mathfrak{g}$-valued 1-form $\hat{\omega}=\pi_1^*\tilde{\omega}-\pi_2^*\bar{\omega}$ on the product $\tilde{P}\times G$, where π_1, π_2 are projections onto the respective factors and $\bar{\omega}$ is the Maurer-Cartan form on G. $\hat{\omega}$ defines an integrable distribution on $\tilde{P}\times G$ because $d\hat{\omega}+[\hat{\omega},\hat{\omega}]=\pi_1^*\tilde{\Omega}-\pi_2^*\bar{\Omega}=0$. The leaf of this foliation through the point $(\tilde{e}, e)$ is the graph of the diffeomorphism

$$\tilde{F}:\ \tilde{P}\to G. \tag{3.17}$$

To define the diffeomorphism $F : P \to \Gamma \backslash G$, we first identify the fundamental group $\pi(P)$ with a subset $\Gamma \subset G$ as follows. Let $\tilde{L}_\gamma$ denote the action of $\gamma \in \pi(P)$ on $\tilde{P}$. $\tilde{L}_\gamma$ leaves $\tilde{g}$ invariant. We define $\pi(P) \to G$ by $\gamma \to \tilde{F}(\tilde{L}_\gamma \tilde{e})$. This map is obviously injective and we use the same symbol for $\gamma \in \pi(P)$ and its image in $\Gamma \subset G$.

The diffeomorphism $\tilde{F} : \tilde{P} \to G$ is equivariant with respect to the actions $\tilde{L}_\gamma$ and L_γ (left translation) on $\tilde{P}$ and G respectively because the maps $\tilde{F}$ and $L_\gamma^{-1} \circ \tilde{F} \circ \tilde{L}_\gamma$ satisfy the same partial differential equations with initial condition (3.16) and hence are identical. As a consequence, $\pi(P) \to \Gamma \subset G$ is an injective group homomorphism and $\tilde{F}$ defines the diffeomorphism $F : P \to \Gamma \backslash G$ asserted in Theorem 1.

To prove Theorem 2 we note that the assumptions imply the existence of a parallelization ω of the principal bundle P satisfying the assumptions of Theorem 1.' Hence there exists another parallelization $\bar{\omega}$ satisfying the Maurer-Cartan equation. Now let $\tilde{P}$ and $\tilde{\omega}$ denote the bundle and connection form induced from P and $\bar{\omega}$ by the universal covering map $\tilde{M} \to M$. Although G is no longer assumed to be simply connected, integration of equation (3.16) still yields a global diffeomorphism $\tilde{F} : \tilde{P} \to G$ since the base spaces $\tilde{M}$ and G/K are simply connected and because small neighborhoods of the fibres in $\tilde{P}$ are mapped diffeomorphically onto neighborhoods of the cosets of the subgroup $K \subset G$. This follows from the facts that the original connection $\omega : TP \to \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ maps vectors tangent to the fibres onto $\mathfrak{k}$ and that $\| \omega - \bar{\omega} \|$ can be made arbitrarily small by choosing the constant A small enough. We now identify $\pi(M)$ acting on $\tilde{P}$ as covering transformations with a subgroup $\Gamma \subset G$ such that $\tilde{F}$ becomes equivariant. We define

$$\tilde{f}: \quad \tilde{M} \to G/K \quad \text{by } x \mapsto C_x, \tag{3.18}$$

where C_x denotes the center of mass of the map $\bar{\pi} \circ \tilde{F}$ restricted to the fibre $\pi^{-1}(x)$ ($\bar{\pi} : G \to G/K$ and $\pi : \tilde{P} \to \tilde{M}$) and the center of mass is defined as in [4] and [7]. For a map $h : Q \to \mathbb{R}^n$ from a measure space Q to euclidean space $\mathbb{R}^n$, the center of mass is simply the average over the measure in Q of the image points under h. In [4] this concept is generalized to the case where the target space is a Riemannian manifold. In case the image of h is contained in a small ball, all the properties of the euclidean center of mass are essentially conserved. Here we are concerned with the map $h = \bar{\pi} \circ \tilde{F}$ restricted to the fibres $\pi^{-1}(x)$, $x \in \tilde{M}$, and, as explained above, the fact that $\| \omega - \bar{\omega} \|$ can be made arbitrarily small by choosing the constant A of the theorem sufficiently small, implies that the image of $\bar{\pi} \circ \tilde{F}$ restricted to $\pi^{-1}(x)$ is contained in a small ball.

To obtain an estimate for the differential $d\tilde{f}$, we observe that, again because $\| \omega - \bar{\omega} \|$ is arbitrarily small, $d\bar{\pi} \circ d\tilde{F}$ restricted to horizontal tangent vectors in $T\tilde{P}$, is arbitrarily close to an isometry. Since the center of mass construction defined by averaging over the fibres essentially averages the differentials restricted to horizontal vectors, we conclude that $d\tilde{f}$ is a quasi-isometry with the dilatation constant controlled by the constant A of the theorem. Moreover, the map $\tilde{f}$ is equivariant with respect to the identification of the fundamental group $\pi(M)$ with a subgroup $\Gamma \subset G$. This is true because the center of mass is invariant under isometries.

Finally, to justify the remark on the injectivity radius of $\tilde{M}$ made in the introduction, we observe that $\tilde{f}$ is close to an isometry. Therefore the injectivity radii of $\tilde{M}$ and $\overline{M}$ are essentially the same.

4. The linear equation

In the preceding chapter we have shown that the proof hinges on the existence and uniqueness of the solution of the equation (3.14). In case P already is a compact semi-simple Lie group G, the operator Δ' reduces to the usual Laplace operator on 2-forms with values in $\mathfrak{g}$, and the result is well-known because the second cohomology vanishes. We have defined the operators d' and Δ' such that, even in the general case, Δ' is a positive operator. Since we need more information on Δ', we prove that the solutions of $\Delta' \beta = -\Omega'$ satisfy a maximum principle.

To compute the operator $\Delta' = d' \delta' + \delta' d'$ on 2-forms explicitly we introduce the following notation. Let $e_1, \ldots, e_N$ denote an orthonormal basis of vector fields on TP such that $\{ \omega(e_i) \}$ is a constant orthonormal basis in $\mathfrak{g}$. We define

$$c_{ij}^k = \langle \{ e_i, e_j \}, e_k \rangle. \tag{4.1}$$

Because of $\{ e_i, e_j \} = \omega^{-1}([\omega(e_i), \omega(e_j)])$, the expressions c_{ij}^k are constant on P. They are simply the structure constants of the Lie algebra $\mathfrak{g}$ with respect to the orthonormal basis $\{ \omega(e_i) \}$. The structure constants satisfy the identities

$$c_{ij}^k = -c_{ji}^k = -c_{ik}^j \quad \text{and} \quad c_{kl}^i \, c_{kl}^j = \delta^{ij}, \tag{4.2}$$

where the Einstein summation convention is used and δ^{ij} is the Kronecker symbol. The equations hold because the scalar product $\langle \ , \ \rangle$ is by definition the negative of the Killing form of $\mathfrak{g}$. Next we define

$$R_{ijk}^l = \frac{1}{4} c_{ij}^m \, c_{km}^l, \tag{4.3}$$

where R is the curvature tensor of the Lie group G with Lie algebra $\mathfrak{g}$ with respect to the metric $\langle \ \ \rangle$ and therefore satisfies the usual curvature identities.

For the computation of Δ' we split $d' = d_1 + d_2$, where

$$d_1 \alpha(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i D_{X_i} \alpha(X_0, \ldots, \hat{X}_i, \ldots, X_p) \tag{4.4}$$

and

$$d_2 \alpha(X_0, \ldots, X_p) = \sum_{j<k} (-1)^{j+k} \alpha(\{ X_j, X_k \}, X_0, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_p). \tag{4.5}$$

Here d_2 is an algebraic operator, just the usual coboundary operator for the cohomology of the Lie algebra $\mathfrak{g}$ with trivial coefficients. The adjoint δ' of d' splits accordingly as $\delta' = \delta_1 + \delta_2$, where

$$\delta_1 \alpha(X_2, \ldots, X_p) = -\sum_{k=1}^{N} D_{e_k} \alpha(e_k, X_2, \ldots, X_p) \tag{4.6}$$

and

$$(4.7)\qquad \delta_2\alpha(X_2, \ldots, X_p)=\frac{1}{2}\sum_{k=1}^{N}\sum_{l=2}^{p}\alpha(e_k, \ldots, X_{l-1}, \{e_k, X_l\}, X_{l+1}, \ldots, X_p).$$

The scalar product on p-forms used in the definitions of δ_1 and δ_2 is

$$\langle \alpha, \beta \rangle = \Sigma \langle \alpha(e_{i_1}, \ldots, e_{i_p}), \beta(e_{i_1}, \ldots, e_{i_p}) \rangle,$$

where the summation is over $i_1 < i_2 < \ldots < i_p$.

The Laplacian Δ' now has the following expression

$$(4.8)\qquad \Delta' = \Delta_1 + \Delta_2 + (d_1\delta_2 + \delta_1 d_2 + d_2\delta_1 + \delta_2 d_1),$$

where

$$\Delta_1 = d_1\delta_1 + \delta_1 d_1, \qquad \Delta_2 = d_2\delta_2 + \delta_2 d_2.$$

For the computation of the various terms in the above formula for Δ' on 2-forms we set $X_i = e_i$ and use the following abbreviation $\alpha_{i_1, \ldots, i_p} = \alpha(e_{i_1}, \ldots, e_{i_p})$:

$$(d_2\delta_2\alpha)_{ij} = -c_{ij}^l(\delta_2\alpha)_l = -\frac{1}{2}c_{ij}^l c_{kl}^m \alpha_{km} = 2 R_{ijk}^m \alpha_{mk},$$

$$(\delta_2 d_2\alpha)_{ij} = \frac{1}{2}c_{ki}^l (d_2\alpha)_{klj} - \frac{1}{2}c_{kj}^l (d_2\alpha)_{kli}$$

$$= \frac{1}{2}c_{kj}^l c_{kl}^m \alpha_{mi} + \frac{1}{2}c_{kj}^l c_{li}^m \alpha_{mk} + \frac{1}{2}c_{kj}^l c_{ik}^m \alpha_{ml} - \frac{1}{2}c_{ki}^l c_{kl}^m \alpha_{mj} - \frac{1}{2}c_{ki}^l c_{lj}^m \alpha_{mk} - \frac{1}{2}c_{ki}^l c_{jk}^m \alpha_{ml}$$

$$= \frac{1}{2}\alpha_{ij} + 2 R_{jki}^m \alpha_{mk} + 2 R_{jli}^m \alpha_{ml} + \frac{1}{2}\alpha_{ij} - 2 R_{ikj}^m \alpha_{mk} - 2 R_{ilj}^m \alpha_{ml} \qquad \text{using (4.2)}$$

$$= \alpha_{ij} + 4(R_{kij}^m + R_{jki}^m)\alpha_{mk} = \alpha_{ij} - 4 R_{ijk}^m \alpha_{mk},$$

where the Bianchi identity is used in the last equation.

Adding $d_2\delta_2\alpha + \delta_2 d_2\alpha$ we obtain:

$$(4.9)\qquad (\Delta_2\alpha)_{ij} = \alpha_{ij} - 2 R_{ijk}^m \alpha_{mk}$$

and hence $\Delta_2\alpha = (1/2)\alpha + (1/2)\delta_2 d_2\alpha$, which implies

$$(4.10)\qquad \langle \Delta_2\alpha, \alpha \rangle \geqq \frac{1}{2}|\alpha|^2.$$

In the following expressions the indices after the semi-colon indicate covariant differentiation (example: $\alpha_{;ij} = D_{e_j} D_{e_i}\alpha$). We continue with the computation of the mixed terms

$$(d_1\delta_2\alpha)_{ij} = \frac{1}{2}c_{kj}^l \alpha_{kl;i} - \frac{1}{2}c_{ki}^l \alpha_{kl;j},$$

$$(d_2\,\delta_1\,\alpha)_{ij}=c_{ij}^l\,\alpha_{kl;k},$$

$$(\delta_1\,d_2\,\alpha)_{ij}=c_{ij}^l\,\alpha_{lk;k}+c_{jk}^l\,\alpha_{li;k}+c_{ki}^l\,\alpha_{lj;k},$$

$$(\delta_2\,d_1\,\alpha)_{ij}=\frac{1}{2}c_{ki}^l\,\alpha_{kl;j}-\frac{1}{2}c_{kj}^l\,\alpha_{kl;i}+c_{jk}^l\,\alpha_{li;k}-c_{ik}^l\,\alpha_{lj;k}.$$

Summing up, we have

$$(4.11)\qquad (d_1\,\delta_2\,\alpha+\delta_1\,d_2\,\alpha+d_2\,\delta_1\,\alpha+\delta_2\,d_1\,\alpha)_{ij}=2\,(c_{ki}^l\,\alpha_{lj;k}-c_{kj}^l\,\alpha_{li;k}).$$

We continue with the computation of Δ_1. In the following we use the definition of curvature

$$R^D(X,\,Y)=D_X\,D_Y-D_Y\,D_X-D_{[X,\,Y]},$$

equations (3.4) and (4.2) as well as the definition of Ω', to replace the Lie bracket [X, Y] by $\{X,\,Y\}$:

$$(\Delta_1\,\alpha)_{ij}=-\alpha_{ij;kk}-\alpha_{jk;ik}+\alpha_{jk;ki}-\alpha_{ki;jk}+\alpha_{ki;kj}$$

$$=-\alpha_{ij;kk}+R_{ik}^D\,\alpha_{jk}-R_{jk}^D\,\alpha_{ik}+(c_{ik}^l-\Omega_{ik}^{\prime l})\,\alpha_{jk;l}-(c_{jk}^l-\Omega_{jk}^{\prime l})\,\alpha_{ik;l}$$

and

$$(4.12)\qquad (\Delta_1\,\alpha)_{ij}=-\alpha_{ij;kk}+[\Omega_{ik}',\,\alpha_{jk}]-[\Omega_{jk}',\,\alpha_{ik}]\ +(c_{ki}^l-\Omega_{il}^{\prime k})\,\alpha_{jl;k}-(c_{kj}^l-\Omega_{jl}^{\prime k})\,\alpha_{il;k}.$$

Adding (4.9), (4.11) and (4.12) we obtain:

$$(4.13)\quad (\Delta'\,\alpha)_{ij}=-\alpha_{ij;kk}+(c_{jl}^k+\Omega_{jl}^{\prime k})\,\alpha_{il;k}-(c_{il}^k+\Omega_{il}^{\prime k})\,\alpha_{jl;k}$$

$$+\alpha_{ij}-2\,R_{ijk}^m\,\alpha_{mk}+[\Omega_{ik}',\alpha_{jk}]-[\Omega_{jk}',\alpha_{ik}].$$

To prove that Δ' is positive, we compute the pointwise scalar product $\langle\Delta'\alpha,\,\alpha\rangle$. We obtain:

$$(4.14)\quad \langle\Delta'\,\alpha,\alpha\rangle-\frac{1}{2}\Delta|\,\alpha\,|^2-\langle\Delta_2\,\alpha,\alpha\rangle-\langle[\Omega_{ik}',\alpha_{jk}],\alpha_{ij}\rangle$$

$$=\frac{1}{2}\langle\,\alpha_{ij;k},\ \alpha_{ij;k}\rangle+\frac{1}{2}(c_{jl}^k+\Omega_{jl}^{\prime k})\langle\,\alpha_{il;k},\alpha_{ij}\rangle\ -\frac{1}{2}(c_{il}^k+\Omega_{il}^{\prime k})\langle\,\alpha_{jl;k},\alpha_{ij}\rangle$$

$$=\frac{1}{2}\langle\,\alpha_{ij;k},\alpha_{ij;k}\rangle+\frac{1}{2}(c_{lj}^k+\Omega_{lj}^{\prime k})\langle\,\alpha_{ij;k},\alpha_{il}\rangle\ +\frac{1}{2}(c_{li}^k+\Omega_{li}^{\prime k})\langle\,\alpha_{ij;k},\alpha_{lj}\rangle$$

$$=\frac{1}{2}\langle(\tilde{D}\,\alpha)_{ijk},\,(\tilde{D}\,\alpha)_{ijk}\rangle-\frac{1}{2}\langle(C\,\alpha+\Omega'\,\alpha)_{ijk},\,(C\,\alpha+\Omega'\,\alpha)_{ijk}\rangle,$$

where

$$(C\,\alpha)_{ijk}=\frac{1}{2}c_{lj}^k\,\alpha_{il}-\frac{1}{2}c_{li}^k\,\alpha_{jl},$$

$$(\Omega'\,\alpha)_{ijk}=\frac{1}{2}\Omega_{lj}^{\prime k}\,\alpha_{ik}-\frac{1}{2}\Omega_{li}^{\prime k}\,\alpha_{jl},$$

$$(\tilde{D}\,\alpha)_{ijk}=\alpha_{ij;k}+(C\,\alpha)_{ijk}+(\Omega'\,\alpha)_{ijk}.$$

To obtain a more manageable formula we compute $|C\alpha|$ and prove:

$$(4.15) \qquad \langle (C\alpha)_{ijk}, (C\alpha)_{ijk} \rangle = \langle \Delta_2 \alpha, \alpha \rangle.$$

$$\langle (C\alpha)_{ijk}, (C\alpha)_{ijk} \rangle = \frac{1}{2} c_{lj}^k \langle \alpha_{il}, (C\alpha)_{ijk} \rangle - \frac{1}{2} c_{li}^k \langle \alpha_{jl}, (C\alpha)_{ijk} \rangle$$

$$= c_{lj}^k \langle \alpha_{il}, (C\alpha)_{ijk} \rangle$$

since $(C\alpha_{ijk})$ is anti-symmetric in i and j

$$= \frac{1}{2} c_{lj}^k c_{mj}^k \langle \alpha_{il}, \alpha_{im} \rangle - \frac{1}{2} c_{lj}^k c_{mi}^k \langle \alpha_{il}, \alpha_{jm} \rangle$$

$$= \frac{1}{2} \langle \alpha_{im}, \alpha_{im} \rangle + 2\, R_{ljm}^i \langle \alpha_{il}, \alpha_{jm} \rangle$$

$$= \frac{1}{2} \langle \alpha_{im}, \alpha_{im} \rangle - R_{jml}^i \langle \alpha_{il}, \alpha_{jm} \rangle,$$

where we used the Bianchi equation and the anti-symmetry of j and m in α_{jm} in the second summand. Comparison of the last line in the equation above with (4.9) proves (4.15).

Finally we substitute (4.15) in the last line of equality (4.14) and obtain the following Weitzenböck formula for the Laplacian Δ':

$$(4.16) \quad \langle \Delta' \alpha, \alpha \rangle - \frac{1}{2} \Delta |\alpha|^2 = |\tilde{D}\alpha|^2 + \frac{1}{2} \langle \Delta_2 \alpha, \alpha \rangle$$

$$- \langle C\alpha, \Omega' \alpha \rangle - \frac{1}{2} |\Omega' \alpha|^2 + \langle \Omega'(\alpha), \alpha \rangle,$$

where $\Omega'(\alpha)_{ij} = [\Omega'_{ik}, \alpha_{jk}]$.

Substituting the inequality (4.10) in the above equation we have

$$(4.17) \qquad \langle \Delta' \alpha, \alpha \rangle - \frac{1}{2} \Delta |\alpha|^2 - |\tilde{D}\alpha|^2 \geqq \frac{1}{4} |\alpha|^2 - c(\|\Omega'\| + \|\Omega'\|^2) |\alpha|^2,$$

where c is a numerical constant depending only on $\mathfrak{g}$. Thus there exists $A' > 0$ depending only on $\mathfrak{g}$ such that $\|\Omega'\| < A'$ implies

$$(4.18) \qquad \langle \Delta' \alpha, \alpha \rangle - \frac{1}{2} \Delta |\alpha|^2 - |\tilde{D}\alpha|^2 > \frac{1}{5} |\alpha|^2.$$

This inequality implies in particular, by integration over P, that Δ' is positive definite on 2-forms. Therefore the equation $\Delta' \beta = -\Omega'$ has a unique solution. Moreover, (4.18) yields an estimate of the maximum norm of the solution β in terms of $\|\Omega'\|$ as follows: At a point where $|\beta|$ achieves its maximum we have $\Delta |\beta|^2 > 0$, and inequality (4.18) implies

$$\frac{1}{5} \|\beta\|^2 \leqq \langle \Delta' \beta, \beta \rangle \leqq c \|\Omega'\| . \|\beta\|,$$

i. e.,

$$\|\beta\| \leqq 5c\|\Omega'\|. \tag{4.19}$$

where c is a constant depending only on the dimension of $\mathfrak{g}$.

5. The estimates

Let β be the unique solution of $\Delta'\beta = -\Omega'$ and set $\alpha = \delta'\beta$. In this chapter we show that α is an approximate solution of $d^{\omega}\alpha = -\Omega'$ in the sense of the estimates 3.15 of the Main Lemma.

In order to avoid dependence of our constants on the injectivity radius of the metric on P, we lift our forms to the tangent space and make our estimates in terms of Sobolev norms defined there. Moreover we do not use the exponential map of the Levi-Civita connection to lift our forms since this would involve the metric's riemannian curvature, on which we do not impose any bounds. Instead, we introduce the connection ∇ on P defined by $\nabla\omega' = 0$, or equivalently, $\nabla_{e_i} e_j = 0$ for the vector fields e_j of the parallelization of TP, and use the exponential map of this connection. We show that the first derivatives of this exponential map can be controlled by constants depending only on $\mathfrak{g}$ and on $\|\Omega'\|$. To this effect let us fix a point $0 \in \mathrm{P}$ and compare $\exp_0 : \mathrm{T}_0\mathrm{P} \to \mathrm{P}$ with the exponential map $\mathrm{Exp} : \mathfrak{g} \to \mathrm{G}$ of the Lie algebra onto its simply connected Lie group.

In the following we define geodesics and Jacobi fields with respect to the metric connection ∇ and let Y be a Jacobi field along a geodesic $c(t)$ in P with initial conditions $c(0) = 0$, $c'(0) = \omega_0^{-1}(\mathrm{U})$, $\mathrm{Y}(0) = 0$, $\mathrm{Y}'(0) = \omega_0^{-1}(\mathrm{V})$, and let $\overline{\mathrm{Y}}$ be the corresponding Jacobi field in G along $\bar{c}(t)$ with the same initial conditions.

Y satisfies the equation:

$$\mathrm{Y}'' + (\mathrm{T}^{\nabla}(\mathrm{Y}, \dot{c}))' + \mathrm{R}^{\nabla}(\mathrm{Y}, \dot{c})\dot{c} = 0. \tag{5.1}$$

The curvature R^{∇} of ∇ is zero and the torsion T^{∇} is given by

$$\mathrm{T}^{\nabla}(\mathrm{X}, \mathrm{Y}) = -\{\mathrm{X}, \mathrm{Y}\} + \omega^{-1}(\Omega'(\mathrm{X}, \mathrm{Y})).$$

Let $\mathrm{X}(t) = \omega(\mathrm{Y}(t)) \in \mathfrak{g}$. Since $\nabla\omega = 0$, X satisfies

$$\mathrm{X}'' + ([\mathrm{U}, \mathrm{X}] + \Omega'(\mathrm{Y}, \dot{c}))' = 0, \tag{5.2}$$

and hence

$$\mathrm{X}' + [\mathrm{U}, \mathrm{X}] + \Omega'(\mathrm{Y}, \dot{c}) = \mathrm{Const.} = \mathrm{X}'(0) = \mathrm{V}. \tag{5.3}$$

Similarly, $\overline{\mathrm{X}}(t) = \bar{\omega}(\overline{\mathrm{Y}}(t))$ satisfies

$$\overline{\mathrm{X}}' + [\mathrm{U}, \overline{\mathrm{X}}] = \mathrm{V}. \tag{5.4}$$

Comparing (5.3) and (5.4) we note that if $\|\Omega'\|$ is sufficiently small then

$$|(X-\overline{X})(t)| \leqq c_1 (e^{c_2\|\Omega'\| t^2}-1), \tag{5.5}$$

where the constants depend on $\mathfrak{g}$ only.

This implies first of all that the exponential map of ∇ has maximal rank on a ball of radius r in the tangent space, with r depending only on $\mathfrak{g}$ and on an upper bound for $\|\Omega'\|$.

Let g_{kl} be the coordinate expression of the metric $\langle\ ,\ \rangle$ lifted to the ball B_r *via* the exponential map. Since g_{kl} is given by the scalar product of two Jacobi fields, the estimate (5.5) also gives us bounds (depending only on $\mathfrak{g}$ and $\|\Omega'\|$) for the maximum norms of the g_{kl} in B_r. In fact, since we can also bound the derivatives of the Jacobi fields in radial directions by (5.3) we even have bounds on the first derivatives of the g_{kl} in radial directions. This gives in particular a bound in terms of $\mathfrak{g}$ and $\|\Omega'\|$ for the modulus of continuity of the g_{kl} at the origin.

We use the exponential map to lift the equation $\Delta'\beta=-\Omega'$ to the ball $B_r \subset T_0 P$ and we denote the lifts by the same symbols. We now use the formula (4.13) to write this equation in the form

$$\Delta\beta_{ij}+(L'(\beta))_{ij}=-\Omega'_{ij}, \tag{5.6}$$

where the leading term is the Laplacian (acting on functions) of g_{kl} in B_r and L' is a first-order differential operator. By examining (4.13) it is easily seen that we have a bound for the maximum norm of the coefficients of L' in terms of $\mathfrak{g}$, $\|\Omega'\|$ and $\|g_{kl}\|$. The Laplacian Δ has the coordinate expression

$$-(\sqrt{\det g})^{-1}\frac{\partial}{\partial x^k}\left(\sqrt{\det g}\; g^{kl}\frac{\partial}{\partial x^l}\right)$$

and hence is in divergence form.

We now invoke interior regularity estimates for elliptic equations of this form, as in [9], Th. 5.5.5′, to obtain the following preliminary estimate:

$$\|\beta\|_{1,q} \leqq c(\|\Omega'\|_{0,q}+\|\beta\|_{0,q}), \tag{5.7}$$

where $\|\ \|_{m,q}$ denotes the Sobolev norm in B_r given by

$$\|u\|^q_{m,q}=\sum_{|\mu|\leqq m}\int_{B_r}|D^\mu u|^q \qquad \text{for} \quad m, q\in\mathbb{N}$$

and where the constant c depends on r, $N=\dim P$, the ellipticity constant for Δ, C^0-bounds for the coefficients of L', C^0-bounds for g_{kl}, and on the modulus of continuity of the g_{kl} at the origin, and hence only on $\mathfrak{g}$ and an upper bound for $\|\Omega'\|$.

Using the C^0-estimate (4.19) for β we have

$$\|\beta\|_{1,q} \leqq c\|\Omega'\|. \tag{5.8}$$

We remark that since Δ' is an operator of order 2 it is well known that an estimate for $\|\beta\|_{2,q}$ in terms of $\|\Omega'\|_{0,q}$ and $\|\beta\|_{0,q}$ also holds. But then the constant, see for example [1], would in general also depend on bounds for the $W_{1,q}$-norm of g_{kl}, which we are not willing to impose. However, for $\alpha=\delta'\beta$ and $d'\beta$ we shall show that we are still able to bound the $W_{1,q}$-norms with the constants depending only on $\mathfrak{g}$ and on an upper bound for $\|\Omega'\|$.

By definition, α satisfies the elliptic system of equations

$$d'\alpha=-\Omega'-\gamma \quad (\gamma=\delta' d'\beta), \qquad \delta'\alpha=\delta'\delta'\beta.$$

Therefore in variational form we have

$$(5.9) \qquad \int_{B_r} \langle d'\alpha+\Omega'+\gamma, d'\xi\rangle+\langle\delta'\alpha-\delta'\delta'\beta, \delta'\xi\rangle=0,$$

for all $\mathfrak{g}$-valued 1-forms ξ with compact support in B_r, where $\langle\ ,\ \rangle$ is the pointwise scalar product.

To obtain an estimate for $\|\alpha\|_{1,q}$ we observe that the leading term of Δ' on 1-forms is just the Laplacian on the component functions and that the rest is a system of first order differential operators whose coefficients are bounded in terms of $\mathfrak{g}$ and $\|\Omega'\|$. We computed the coefficients of Δ' on 2-forms in Chapter 4 where the main point was to prove that Δ' is positive. Here we do not need this information because we already have a bound for $\|\alpha\|_{0,q}$ and we omit the computations. The leading term of (5.9) being diagonal at the origin we can again use the results of [9], Th. 5.5.5', to obtain the following estimate:

$$(5.10) \qquad \|\alpha\|_{1,q} \leqq c(\|\Omega'\|_{0,q}+\|\gamma\|_{0,q}+\|\alpha\|_{0,q}+\|\delta'\delta'\beta\|_{0,q}),$$

where $q > N=\dim P$, and c depends only on q, $\mathfrak{g}$ and $\|\Omega'\|$. Now by (3.11) we have the estimate

$$\|\delta'\delta'\beta\|_{0,q} \leqq c\|\Omega'\|.\|\beta\|_{1,q} \leqq c\|\Omega'\|^2 \quad \text{by (5.8)}$$

and since $\|\alpha\|_{0,q} \leqq \|\beta\|_{1,q} \leqq c\|\Omega'\|$, again by (5.8) we obtain:

$$(5.11) \qquad \|\alpha\|_{1,q} < c(\|\Omega'\|+\|\Omega'\|^2+\|\gamma\|_{0,q}).$$

Similarly by considering the elliptic system of equations

$$d'(d'\beta)=(d'\circ d')\beta, \qquad \delta'(d'\beta)=\gamma,$$

we obtain the following estimate for $d'\beta$:

$$(5.12) \qquad \|d'\beta\|_{1,q} < c(\|\Omega'\|+\|\Omega'\|^2+\|\gamma\|_{0,q}).$$

We now estimate $\|\gamma\|$. This will be done with the help of a generalized maximum principle for elliptic operators in divergence form developed by Stampacchia in [10].

γ satisfies the elliptic system

$$(5.13) \qquad \begin{cases} d'\gamma = -d'(d'\alpha+\Omega'), \\ \delta'\gamma = \delta'\delta'(d'\beta). \end{cases}$$

To estimate $\|\gamma\|$ we first estimate the right hand sides of the above equations abbreviated by φ and ψ respectively. The formulae (3.9) and (3.10) imply

$$(5.14) \qquad \|\varphi\|_{0,q} \leqq c(\|\Omega'\|^2 + \|\Omega'\|.\|\alpha\|_{1,q}).$$

Similarly by (3.11) ψ can be estimated as

$$(5.15) \qquad \|\psi\|_{0,q} \leqq c\|\Omega'\|.\|d'\beta\|_{1,q}.$$

Here we may take the Sobolev norms in a ball of radius r in any tangent space $T_x P$, in particular at the point x_0 where $|\gamma|$ takes its maximum value.

Instead of (5.13) we may also consider

$$\Delta'\gamma = \delta'\varphi + d'\psi$$

and

$$\langle \Delta'\gamma, \gamma\rangle = \langle \delta'\varphi, \gamma\rangle + \langle d'\psi, \gamma\rangle.$$

Setting $u=|\gamma|^2$ we have by (4.18) the following differential inequality:

$$(5.16) \qquad \Delta u + \frac{2}{5}u \leqq 2\langle \delta'\varphi, \gamma\rangle + 2\langle d'\psi, \gamma\rangle.$$

To be able to apply [10], Th. 4.2 we rewrite the right hand side of the above equation as follows:

$$(5.17) \qquad 2\langle \delta'\varphi, \gamma\rangle + 2\langle d'\psi, \gamma\rangle = 2|\varphi|^2 + 2|\psi|^2 + \sum_{i=1}^{N} D_i f_i,$$

where the last summand is the coordinate expression of the divergence of the vector field $f_i = 2\varphi_{ikj}\gamma_{kj} + 4\psi_j\gamma_{ij}$. The estimates (5.14) and (5.15) imply

$$(5.18) \qquad \|f_i\|_{0,q} \leqq c(\|\Omega'\|^2 + \|\Omega'\|(\|\alpha\|_{1,q} + \|d'\beta\|_{1,q}))\|\gamma\|.$$

The operator $Lv = \Delta v + (2/5)v$ is coercive and therefore the Dirichlet problem:

$$Lv = 2|\varphi|^2 + 2|\psi|^2 + \sum D_i f_i \quad \text{in } B_r,$$

with boundary value $v|_{\partial B_r} = 0$ has a unique solution. The estimate of [10], Th. 4.2 applied to this solution now gives

$$(5.19) \qquad |v| < c(\|\Omega'\|^2(\|\Omega'\| + B)^2 + \|\gamma\|(\|\Omega'\|^2 + B\|\Omega'\|))$$

where $B = \|\alpha\|_{1,q} + \|d'\beta\|_{1,q}$.

Next, we apply the usual maximum principle to estimate the difference $w = u - v$. We first define a comparison function g on B_r to be the solution of $Lg = 0$ with boundary values $g|_{\partial B_r} = \text{Const.} = \| u \| = u(0)$. g satisfies the inequality $g(0) < (1-k) g_{\max} = (1-k) u(0)$ for some constant $k > 0$ depending only on $\mathfrak{g}$ and $\| \Omega' \|$. By the definition of v, $Lw \leqq 0$ holds and hence $w - g$ satisfies the maximum principle, i.e., has no positive maximum in the interior. The boundary values of $w - g$ are non-positive and we obtain:

$$w(0) = u(0) - v(0) \leqq g(0) < (1-k) \| u \|$$

and therefore

$$k \| \gamma \|^2 = k \| u \| < v(0) \leqq c (\| \Omega' \|^2 (\| \Omega' \| + B)^2 + \| \gamma \| (\| \Omega' \|^2 + B \| \Omega' \|)).$$

From this we conclude that

$$\| \gamma \| \leqq c \| \Omega' \| (\| \Omega' \| + B). \tag{5.20}$$

Now by (5.11) and (5.12) we have

$$B \leqq c (\| \Omega' \| + \| \Omega' \|^2 + \| \gamma \|_{0,q}) \leqq c (\| \Omega' \| + \| \Omega' \|^2 + \| \gamma \|).$$

Substituting this bound in (5.20) and choosing $\| \Omega' \|$ small enough we finally obtain the estimate (3.15) (i) of the Main Lemma:

$$\| \gamma \| = \| d' \alpha + \Omega' \| \leqq c \| \Omega' \|^2.$$

This in turn implies by (5.11):

$$\| \alpha \|_{1,q} \leqq c (\| \Omega' \| + \| \Omega' \|^2). \tag{5.21}$$

And the Sobolev Inequality for $q > N$ yields the estimate (3.15) (ii) of the Main Lemma.

For the final estimate (3.15) (iii) we note that the Sobolev norms $\| \quad \|_{m,q}$ can be defined for the whole manifold P by covering it with a finite number of patches which are not necessarily balls on which the exponential map of ∇ has maximal rank. Because the number of patches depends on the diameter of P, the constant c' in the estimate $\| \alpha \|_{1,q} < c' \| \Omega' \|$ obtained from summing up the estimates (5.21) depends also on this diameter. Actually, a careful analysis of the Jacobi fields with respect to the flat connection induced by ω' would probably yield an estimate of the diameter of P. Since we do not need such an estimate *a priori* and since the bound follows from Theorem 1 we do not elaborate on this point.

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Added in Proof. — A. Borel suggested that a reference to the work of H. E. Rauch would be appropriate. Indeed, in his pioneering work, *Geodesics, Symmetric Spaces, and Differential Geometry in the Large* (*Comment. Math. Helv.*, Vol. 27, 1953, pp. 294-320), H. E. Rauch initiated the study of comparison theorems for compact symmetric spaces. In view of the results of M. Berger, *Sur les groupes d'holonomie des variétés à connexion affine et des variétés riemanniennes* (*Bull. Soc. math. Fr.*, Vol. 83, 1955, pp. 279-330), H. E. Rauch reexamined his earlier work in this paper, *The Global Study of Geodesics in Symmetric and Nearly Symmetric Riemannian Manifolds* (*Comment. Math. Helv.*, Vol. 35, 1961, pp. 111-125). It turns out that the irreducible symmetric spaces of rank one are the only admissible models for his comparison theorem. For higher rank Rauch's holonomy assumption on the Levi-Civita connection implies by Berger's result that the general manifold is locally isometric to the model. This explains the choice of assumptions in our theorems. At least for small perturbations of the local geometry our results complete the program of H. E. Rauch.