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GENERALIZED VERMA MODULES, LOOP SPACE COHOMOLOGY AND MACDONALD-TYPE IDENTITIES

BY J. LEPOWSKY (1)

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1. Introduction

In this paper, we extend certain aspects of semisimple Lie theory to Kac-Moody Lie algebras, obtaining connections with topology and η-function identities. These possibly infinite-dimensional Lie algebras, also called GCM (generalized Cartan matrix) Lie algebras, were introduced and studied by Kac [15], (*a*)-(*d*) and Moody [22], (*a*)-(*c*).

Let $\mathfrak{g}$ be a complex simple Lie algebra. One of the most important Kac-Moody Lie algebras is the infinite-dimensional complex Lie algebra $\tilde{\mathfrak{g}}=\mathfrak{g}\otimes_{\mathbf{C}}\mathbf{C}[t, t^{-1}]$, where $\mathbf{C}[t, t^{-1}]$ is the commutative algebra of Laurent polynomials in the indeterminate t. (Strictly speaking, $\tilde{\mathfrak{g}}$ is the quotient of a certain Kac-Moody Lie algebra by a one-dimensional center.) Let $\mathfrak{u}=\mathfrak{g}\otimes\mathbf{C}[t]^{+}$, where $\mathbf{C}[t]^{+}$ denotes the space of polynomials in t without constant term. The cohomology of the Lie algebra $\mathfrak{u}$ is determined by Garland in [10], and is used to "explain" certain MacDonald identities [21], *cf.* also [11], (*a*). (Fundamental to the present paper and to [11], (*a*) are Kac's [15], (*c*) and Moody's [22], (*c*) interpretation of MacDonald's identities by means of Euclidean Kac-Moody Lie algebras.)

Now the subalgebra $\mathfrak{g}=\mathfrak{g}\otimes 1$ of $\tilde{\mathfrak{g}}$ acts naturally on $\mathrm{H}^*(\mathfrak{u})$. For all $j\geqq 0$, let M_j be the number of irreducible $\mathfrak{g}$-module components in $\mathrm{H}^j(\mathfrak{u})$. Garland observes "empirically" in [10] that for all $j\geqq 0$,

$$\mathrm{M}_j=\dim \mathrm{H}^{2j}(\Omega(\mathrm{G}), \mathbf{C}), \tag{1}$$

where $\Omega(\mathrm{G})$ is the loop space of the compact simply connected Lie group G with Lie algebra a compact real form of $\mathfrak{g}$. (Bott [3], (*a*) had used Morse theory to compute $\mathrm{H}^*(\Omega(\mathrm{G}), \mathbf{C})$ and to show that $\Omega(\mathrm{G})$ has the homotopy type of a countable CW-complex with only even-dimensional cells.)

The present paper was motivated by a desire to "understand" Garland's equality (1). We do this by determining the relative Lie algebra cohomology $\mathrm{H}^*(\tilde{\mathfrak{g}}, \mathfrak{g})$ as a graded vector space, and by showing that

$$\mathrm{M}_j=\dim \mathrm{H}^{2j}(\tilde{\mathfrak{g}}, \mathfrak{g}) \tag{2}$$

(Theorem 9.1). In particular, we find that

$$\mathrm{H}^*(\tilde{\mathfrak{g}}, \mathfrak{g})\simeq \mathrm{H}^*(\Omega(\mathrm{G}), \mathbf{C}) \tag{3}$$

(Theorem 10.4). The plausibility of (3) is explained in paragraph 10.

Formula (2) is a special case of Corollary 6.16, which at the same time generalizes work of Kostant [16], (*c*). Specifically, let R be the centralizer of a torus in G, so that G/R is a typical generalized flag manifold. Let $\mathfrak{r}$ be the complexified Lie algebra of R, and $\mathfrak{n}$ a nilpotent subalgebra of $\mathfrak{g}$ such that $\mathfrak{r}\oplus\mathfrak{n}$ is a parabolic subalgebra of $\mathfrak{g}$. For a finite-dimensional irreducible $\mathfrak{g}$-module V and $j\geqq 0$, let N_j be the number of irreducible $\mathfrak{r}$-module components in $\mathrm{H}^j(\mathfrak{n}, \mathrm{V})$. Kostant determines the $\mathfrak{r}$-module structure of $\mathrm{H}^j(\mathfrak{n}, \mathrm{V})$ [16], (*b*) and proves algebraically [16], (*c*) that

$$\mathrm{N}_j=\dim \mathrm{H}^{2j}(\mathfrak{g}, \mathfrak{r}). \tag{4}$$

Of course, there is a classical isomorphism

$$H^*(\mathfrak{g}, \mathfrak{r}) \simeq H^*(G/R, \mathbf{C}).$$

For the case in which R is a Cartan subgroup of G, (4) is Bott's "strange equality" [3], (*b*), p. 247, which motivated Kostant's papers [16], (*b*), (*c*).

Corollary 6.16 is a common generalization of (2) and (4) to all Kac-Moody Lie algebras. (The GCM of a Kac-Moody Lie algebra is assumed symmetrizable.) In particular, Garland's equality (1) and Bott's "strange equality" become special cases of the same phenomenon. The present paper may be viewed as a sequel to [11], (*a*) in which Garland's result on the $\mathfrak{g}$-module structure of $H^*(\mathfrak{u})$ and Kostant's result on the $\mathfrak{r}$-module structure of $H^*(\mathfrak{n}, V)$ are simultaneously generalized to all Kac-Moody Lie algebras. This result from [11], (*a*) is stated as Theorem 5.5 below. The modules generalizing V are the standard modules, introduced by Kac.

In [11], (*a*), a resolution of a standard module in terms of generalized Verma modules is established (*see* Theorem 5.1 below), generalizing a weak form of the Bernstein-Gelfand-Gelfand resolution [2] for finite-dimensional semisimple Lie algebras. This generalized Verma module resolution is central to the present paper. In Part I below, we set up some relative homological algebra (*cf.* also [14]), which enables us to derive from our resolution both Theorem 5.5 and our generalizations of (2) and (4) (*see* Corollary 6.16 and the other results in paragraph 6). For finite-dimensional semisimple Lie algebras, our method recovers Kostant's result (4), and Bott's "strange equality" in particular, in a new natural way.

A surprising by-product of our relative-homological approach is a new proof of the Theorem in [23] on "minimal K-types" for finite-dimensional irreducible representations of complex semisimple Lie groups regarded as real (*see* § 7). This homological idea is further used in [17], (*a*), (*b*) to obtain new results and to illuminate some known results in the representation theory of real semisimple Lie algebras.

Thanks to Kac's classification of the automorphisms of finite order of complex semisimple Lie algebras [15], (*b*), we can place formulas (2) and (3) in a context involving much more general path spaces than $\Omega(G)$. Part III is devoted to this. Kac shows that the automorphisms of finite order are described in a certain way by means of the Euclidean Lie algebras (introduced and studied by Kac [15], (*a*)-(*d*) and Moody [22], (*a*)-(*c*), which are a little more general than the Lie algebras $\tilde{\mathfrak{g}}$; *see* paragraph 8 for an exposition of these results of Kac. Each such automorphism θ gives rise to a path space in G which turns out to be homeomorphic to a quotient $E_\theta(G)$ of a certain equivariant loop space defined by θ (§ 10). The loop space $\Omega(G)$ is just $E_\theta(G)$ for the special case $\theta = 1$. Around 1960, Bott [unpublished] studied these path spaces using Morse theory, showing in particular that they have only even-dimensional cells. In paragraphs 9-11, we specialize Corollary 6.16 to Euclidean Lie algebras and note connections, partly conjectural, between relative Lie algebra cohomology and the spaces $E_\theta(G)$. In particular, we point out that these spaces are good analogues for Euclidean Lie algebras of generalized flag manifolds. The fact that $\Omega(G)$ behaves like a generalized flag manifold has already been shown by Garland-Raghunathan [12].

Before I became aware of Bott's unpublished result, H. Samelson and M. Shahshahani used Morse theory to study intensively the spaces $E_\theta(G)$ in the situation in which θ is an involution. In this case, $E_\theta(G)$ is a path space attached canonically to the most general compact simply connected symmetric space. Knowledge of Bott's result then provided the stimulus to allow θ to be any automorphism of finite order and to use Kac's paper [15], (*b*).

Kac's classification of these automorphisms in [15], (*b*), fundamental results of MacDonald in [21], and Kac's [15], (*c*) and Moody's [22], (*c*) Lie algebraic interpretation of MacDonald's identities lead in another interesting direction: To every automorphism θ of finite order of $\mathfrak{g}$ is associated naturally a one-variable specialization of a multivariable MacDonald identity (§ 13), in such a way that the formulas of Dyson [27] and MacDonald for $\eta(q)^{\dim \mathfrak{g}}$ [$\eta(q)$ being Dedekind's eta-function] come from $\theta = 1$ (§ 14). Many interesting new identities are produced. Conjectural connections between $E_\theta(G)$ and the identity corresponding to θ are made in Conjecture 10.6 and the subsequent Remark. These connections are suggested partly by the work of Garland-Raghunathan [12] and Kostant [16], (*c*).

By choosing θ to be Kostant's "principal" automorphism [16], (*a*), (*d*) of order equal to the Coxeter number h of $\mathfrak{g}$, we get new identities for arbitrary positive powers of $\eta(q)$ (§ 17). We call the corresponding specialization *principal specialization*. For example, principal specialization for $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C})$ gives a formula for $\eta(q)^n/\eta(q^n)$, and principal specialization for $\mathfrak{g} = \mathfrak{so}(2n-1, \mathbf{C})$ gives a formula for $\eta(q)^n/\eta(q^2)$. By choosing θ to be a certain interesting automorphism of order $h+1$, we get a formula for $\eta(q)^{\operatorname{rank} \mathfrak{g}}$ (§ 16). Outside the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$, the identities in paragraphs 16 and 17 all seem to be new. In particular, we get several new formulas, different from Dyson's [27], for the generating function $\eta(q)^{24}$ of Ramanujan's τ-function. Incidentally, an amusing new pattern involving what we call "polygonal numbers" emerges when we apply our systematic specialization procedure to MacDonald's identity for $\mathfrak{sl}(2, \mathbf{C})^{\sim}$ (which is Jacobi's classical "θ-function identity"). This is explained in paragraph 15, which motivates paragraphs 16 and 17. While the Dyson-MacDonald identities for $\eta(q)^{\dim \mathfrak{g}}$ are the natural generalizations of Jacobi's identity for $\eta(q)^3$, the identities in paragraphs 16 and 17 are the natural generalizations of Euler's formula for $\eta(q)$ and Gauss' formula for $\eta(q)^2/\eta(q^2)$, respectively.

In obtaining their formulas for $\eta(q)^{\dim \mathfrak{g}}$, Dyson and MacDonald invoke the Freudenthal-de Vries "strange formula" $(\rho, \rho) = (\dim \mathfrak{g})/24$, where ρ is half the sum of the positive roots of $\mathfrak{g}$ and $(.,.)$ is the canonical inner product (*see* [21], p. 95). Analogously, in paragraphs 16 and 17, we are led to conjecture and prove (by case-checking, unfortunately) two new, even stranger, formulas in the same spirit as the Freudenthal-de Vries formula. One of them involves $(\operatorname{rank} \mathfrak{g})/24$ and the other involves the exponents of $\mathfrak{g}$; *see* Th. 16.6 and 17.5.

At several stages in the preparation of this paper, I have profited considerably from enlightening and stimulating discussions with many people. I would particularly like to thank R. Bott, W. Dwyer, H. Garland, J. Millson, J. Milnor, H. Samelson, M. Shahshahani, J. Tits and P. Trauber for their time and interest.

After this work was completed, it came to my attention that the specialization that I use to prove Theorem 16.5 (which together with the new "strange formula" Theorem 16.6 yields the formula for $\eta(q)^{\operatorname{rank} \mathfrak{g}}$) had already been used for a different purpose by MacDonald in [21], p. 125.

This work was announced in [17], (*c*). Since the time the preprint of this paper was circulated in 1977, several uses have been made of the ideas introduced here: The idea of principal specialization (§ 17) proved to be unexpectedly effective in "unlocking" the standard modules and in relating them to combinatorics ([19], [9], [17], (*d*)). These ideas were used in [15], (*d*). Principal specialization was further exploited in [20] and its generalization [28], where the (2, 1, ..., 1)-specialization (§ 16) also turned out to play a key role. The idea that one can obtain new η-function identities from Dyson's and MacDonald's multivariable identities by specializing the exponentials of minus the simple roots of the corresponding Euclidean Lie algebra to powers of q (Part IV) was also used in [15], (*d*). The two new "strange formulas" (Th. 16.6 and 17.5) suggested to me that a similar "strange formula" might hold more generally whenever the automorphism of finite order of $\mathfrak{g}$ is such that the product side of the associated specialized identity can be written in the form $\Pi_i \varphi(q^{a_i}) \Pi_j \varphi(q^{b_j})^{-1}$, where $\varphi(q) = \Pi_{n \geqq 1}(1-q^n)$. Such "strange formulas" generalizing 16.6 and 17.5 were in fact obtained by Kac, using ideas of Deligne and Kazhdan [15], (*d*), and independently by MacDonald. Using these formulas, Kac [15], (*d*) and MacDonald independently generalized Theorems 16.7 and 17.6. The choice of the exponentials of minus the simple roots of the Euclidean Lie algebra as power series variables in Dyson's and MacDonald's identities (Part IV), and material in Part IV on non-principal specializations, were used in [19], pp. 27, 40, 41, 48, 49, to formulate multi-variable vector partition theorems. The reader is also referred to [17], (*e*).

PART I
SOME RELATIVE HOMOLOGICAL ALGEBRA

2. Relative homology and cohomology

Part I, which is largely expository (*cf.* [14]), consists of general material needed in Part II, as noted in the Introduction.

We shall begin by recalling the resolutions $V(\mathfrak{b}, \mathfrak{a}, N)$ discussed in [11], (*a*), § 1.

Let $\mathfrak{b}$ be a Lie algebra over a field k, and let $\mathfrak{a}$ be a subalgebra of $\mathfrak{b}$. Let $\mathscr{B}$ and $\mathscr{A}$ be the universal enveloping algebras of $\mathfrak{b}$ and $\mathfrak{a}$, respectively, and regard $\mathscr{A}$ as a subalgebra of $\mathscr{B}$. We shall identify Lie algebra modules with the corresponding universal enveloping algebra modules.

For each $j \in \mathbf{Z}_+$ (the set of nonnegative integers), the j-th exterior power $\Lambda^j(\mathfrak{b}/\mathfrak{a})$ is an $\mathfrak{a}$-module in a natural way, and we may form the corresponding induced $\mathfrak{b}$-module $D_j = \mathscr{B} \otimes_{\mathscr{A}} \Lambda^j(\mathfrak{b}/\mathfrak{a})$. Let $V(\mathfrak{b}, \mathfrak{a})$ be the sequence of $\mathfrak{b}$-modules and $\mathfrak{b}$-module maps

$$\ldots \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \xrightarrow{\varepsilon_0} k \to 0$$

constructed in [2], §9. To define $d_j\colon D_j \to D_{j-1}$ $(j>0)$, let $x_1, \ldots, x_j \in \mathfrak{b}/\mathfrak{a}$, and choose representatives $y_1, \ldots, y_j \in \mathfrak{b}$. Also, let $x \in \mathscr{B}$. Then

$$d_j(x \otimes x_1 \wedge \ldots \wedge x_j) = \sum_{i=1}^{j} (-1)^{i+1} (xy_i) \otimes x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_j \\ + \sum_{1 \leqq r < s \leqq j} (-1)^{r+s} x \otimes \pi [y_r, y_s] \wedge x_1 \wedge \ldots \wedge \hat{x}_r \wedge \ldots \wedge \hat{x}_s \wedge \ldots \wedge x_j,$$

where $\pi\colon \mathfrak{b} \to \mathfrak{b}/\mathfrak{a}$ is the natural map, and $\hat{}$ signifies the omission of a symbol. It is easily checked that d_j is independent of the choice of representatives $y_1, \ldots, y_j$, and that d_j is a $\mathfrak{b}$-module map. The map $\varepsilon_0\colon D_0 \to k$ is defined by the condition that $\varepsilon_0(b \otimes 1)$ $(b \in \mathscr{B})$ be the constant term of b. Theorem 9.1 of [2] states:

PROPOSITION 2.1. — $V(\mathfrak{b}, \mathfrak{a})$ *is an exact sequence.*

Remarks. — (1) The complex $V(\mathfrak{b}, 0)$ is the standard $\mathscr{B}$-free resolution of the trivial $\mathfrak{b}$-module k.

(2) If $\mathfrak{z}$ is a subalgebra of $\mathfrak{a}$ and of the center of $\mathfrak{b}$, then clearly $V(\mathfrak{b}/\mathfrak{z}, \mathfrak{a}/\mathfrak{z})$ may be regarded as a $\mathfrak{b}$-module complex which is naturally isomorphic to $V(\mathfrak{b}, \mathfrak{a})$.

Let N be a $\mathfrak{b}$-module. Denote by $V(\mathfrak{b}, \mathfrak{a}, N)$ the sequence (exact by Proposition 2.1) of tensor product $\mathfrak{b}$-modules and $\mathfrak{b}$-module maps

$$\ldots \xrightarrow{d_2 \otimes 1} D_1 \otimes N \xrightarrow{d_1 \otimes 1} D_0 \otimes N \xrightarrow{\varepsilon_0 \otimes 1} N \to 0.$$

Let $V'(\mathfrak{b}, \mathfrak{a}, N)$ be the $\mathfrak{b}$-module complex obtained by deleting the segment $\xrightarrow{\varepsilon_0 \otimes 1} N$ from $V(\mathfrak{b}, \mathfrak{a}, N)$.

Let $U_*(\mathfrak{b}, \mathfrak{a}, N)$ be the complex obtained from $V'(\mathfrak{b}, \mathfrak{a}, N)$ by applying the functor $k \otimes_{\mathscr{B}} \cdot$, where k is regarded as the trivial right $\mathfrak{b}$-module. That is, $U_*(\mathfrak{b}, \mathfrak{a}, N)$ is the complex

$$\ldots \xrightarrow{1 \otimes (d_2 \otimes 1)} k \otimes_{\mathscr{B}} (D_1 \otimes N) \xrightarrow{1 \otimes (d_1 \otimes 1)} k \otimes_{\mathscr{B}} (D_0 \otimes N) \to 0.$$

Let $U^*(\mathfrak{b}, \mathfrak{a}, N)$ be the complex dual to $U_*(\mathfrak{b}, \mathfrak{a}, N)$, i.e., the complex

$$\ldots \xleftarrow{\mathrm{Hom}(d_2 \otimes 1, 1)} \mathrm{Hom}_{\mathfrak{b}}(D_1 \otimes N, k) \xleftarrow{\mathrm{Hom}(d_1 \otimes 1, 1)} \mathrm{Hom}_{\mathfrak{b}}(D_0 \otimes N, k) \leftarrow 0.$$

Let $T\colon \mathscr{B} \to \mathscr{B}$ be the transpose map of $\mathscr{B}$, i.e., the unique anti-automorphism which is -1 on $\mathfrak{b}$. Denote by N^t the right $\mathfrak{b}$-module whose space is N and on which $\mathscr{B}$ acts by the formula $n.b = T(b).n$ for all $n \in N$ and $b \in \mathscr{B}$. The following is clear:

PROPOSITION 2.2. — $U_*(\mathfrak{b}, \mathfrak{a}, N)$ *is naturally isomorphic to the complex*

$$\ldots \xrightarrow{1 \otimes d_2} N^t \otimes_{\mathscr{B}} D_1 \xrightarrow{1 \otimes d_1} N^t \otimes_{\mathscr{B}} D_0 \to 0$$

and $U^*(\mathfrak{b}, \mathfrak{a}, N)$ *is naturally isomorphic to the dual complex*

$$\ldots \xleftarrow{\mathrm{Hom}(d_2,\, 1)} \mathrm{Hom}_{\mathfrak{b}}(D_1, N^*) \xleftarrow{\mathrm{Hom}(d_1,\, 1)} \mathrm{Hom}_{\mathfrak{b}}(D_0, N^*) \leftarrow 0,$$

where N^* *is the* $\mathfrak{b}$-*module contragredient to* N.

DEFINITIONS. — Call the homology of $U_*(\mathfrak{b}, \mathfrak{a}, N)$ the *relative homology of* $\mathfrak{b}$ *with respect to* $\mathfrak{a}$ *in* N^t, and denote it by $H_*(\mathfrak{b}, \mathfrak{a}, N^t)$. Write $H_*(\mathfrak{b}, \mathfrak{a}, k)$ as $H_*(\mathfrak{b}, \mathfrak{a})$ (where k is regarded as the trivial module) and call it the *relative homology of* $\mathfrak{b}$ *with respect to* $\mathfrak{a}$. Call the homology of $U^*(\mathfrak{b}, \mathfrak{a}, N)$ the *relative cohomology of* $\mathfrak{b}$ *with respect to* $\mathfrak{a}$ *in* N^*, and denote it by $H^*(\mathfrak{b}, \mathfrak{a}, N^*)$. More generally, if we replace N^* by an arbitrary $\mathfrak{b}$-module M in the second complex in Proposition 2.2, the homology of the resulting complex, which we denote by $S^*(\mathfrak{b}, \mathfrak{a}, M)$, is called the *relative cohomology of* $\mathfrak{b}$ *with respect to* $\mathfrak{a}$ *in* M and is denoted $H^*(\mathfrak{b}, \mathfrak{a}, M)$. Note that $S^*(\mathfrak{b}, \mathfrak{a}, N^*) \simeq U^*(\mathfrak{b}, \mathfrak{a}, N)$. Write $H^*(\mathfrak{b}, \mathfrak{a}, k)$ as $H^*(\mathfrak{b}, \mathfrak{a})$, and call it the *relative cohomology of* $\mathfrak{b}$ *with respect to* $\mathfrak{a}$.

Remarks. — (1) For each $j \in \mathbf{Z}_+$, $H^j(\mathfrak{b}, \mathfrak{a}, N^*)$ is naturally isomorphic to the dual vector space $H_j(\mathfrak{b}, \mathfrak{a}, N^t)^*$. In particular, $H^j(\mathfrak{b}, \mathfrak{a}) \simeq H_j(\mathfrak{b}, \mathfrak{a})^*$.

(2) From Remark (2) after Proposition 2.1, it follows easily that if $\mathfrak{z}$ is a subalgebra of $\mathfrak{a}$ and of the center of $\mathfrak{b}$, then for all $j \in \mathbf{Z}_+$, $H_j(\mathfrak{b}, \mathfrak{a})$ is naturally isomorphic to $H_j(\mathfrak{b}/\mathfrak{z}, \mathfrak{a}/\mathfrak{z})$, and hence $H^j(\mathfrak{b}, \mathfrak{a})$ is naturally isomorphic to $H^j(\mathfrak{b}/\mathfrak{z}, \mathfrak{a}/\mathfrak{z})$.

(3) $U_*(\mathfrak{b}, 0, N)$ is naturally isomorphic to the standard homology complex

$$\ldots \xrightarrow{\partial_2} N^t \otimes_k \Lambda^1(\mathfrak{b}) \xrightarrow{\partial_1} N^t \otimes_k \Lambda^0(\mathfrak{b}) \to 0,$$

where for all $j > 0$, $n \in N^t$ and $b_1, \ldots, b_j \in \mathfrak{b}$,

$$\begin{aligned}\partial_j(n \otimes b_1 \wedge \ldots \wedge b_j) = &\sum_{i=1}^{j} (-1)^{j+1} (n.b_i) \otimes b_1 \wedge \ldots \wedge \hat{b}_i \wedge \ldots \wedge b_j \\ &+ \sum_{1 \leqq r < s \leqq j} (-1)^{r+s} n \otimes [b_r, b_s] \wedge b_1 \wedge \ldots \wedge \hat{b}_r \wedge \ldots \wedge \hat{b}_s \wedge \ldots \wedge b_j,\end{aligned}$$

and its homology $H_*(\mathfrak{b}, 0, N^t)$ is the homology $H_*(\mathfrak{b}, N^t)$ of $\mathfrak{b}$ in the right $\mathfrak{b}$-module N^t (*cf.* [5], p. 282). Analogously, for a $\mathfrak{b}$-module M, $S^*(\mathfrak{b}, 0, M)$ is naturally isomorphic to the standard cohomology complex

$$\ldots \xleftarrow{\delta_2} \mathrm{Hom}_k(\Lambda^1(\mathfrak{b}), M) \xleftarrow{\delta_1} \mathrm{Hom}_k(\Lambda^0(\mathfrak{b}), M) \leftarrow 0,$$

where for all $j \in \mathbf{Z}_+$, $f \in \mathrm{Hom}_k(\Lambda^j(\mathfrak{b}), M)$ and $b_1, \ldots, b_{j+1} \in \mathfrak{b}$,

$$\begin{aligned}(\delta_{j+1} f)(b_1 \wedge \ldots \wedge b_{j+1}) = &\sum_{i=1}^{j+1} (-1)^{i+1} b_i . f(b_1 \wedge \ldots \wedge \hat{b}_i \wedge \ldots \wedge b_{j+1}) \\ &+ \sum_{1 \leqq r < s \leqq j+1} (-1)^{r+s} f([b_r, b_s] \wedge b_1 \wedge \ldots \wedge \hat{b}_r \wedge \ldots \wedge \hat{b}_s \wedge \ldots \wedge b_{j+1}),\end{aligned}$$

and its homology $H^*(\mathfrak{b}, 0, M)$ is the cohomology $H^*(\mathfrak{b}, M)$ of $\mathfrak{b}$ in M (again *cf.* [5], p. 282).

In view of the fact that $\mathscr{B} \otimes_{\mathscr{A}} \Lambda^j(\mathfrak{b}/\mathfrak{a})$ is naturally a $\mathfrak{b}$-module quotient of $\mathscr{B} \otimes_k \Lambda^j(\mathfrak{b})$ for each $j \in \mathbf{Z}_+$, we see that $\mathrm{Hom}_{\mathfrak{b}}(\mathscr{B} \otimes_{\mathscr{A}} \Lambda^j(\mathfrak{b}/\mathfrak{a}), M)$ may be naturally identified with a certain subspace of $\mathrm{Hom}_{\mathfrak{b}}(\mathscr{B} \otimes_k \Lambda^j(\mathfrak{b}), M)$, and it is also clear that this identification gives a natural injection of the complex $S^*(\mathfrak{b}, \mathfrak{a}, M)$ into the complex $S^*(\mathfrak{b}, 0, M)$. In terms of the identification of $S^*(\mathfrak{b}, 0, M)$ with the complex indicated in Remark (3) above, $S^*(\mathfrak{b}, \mathfrak{a}, M)$ identifies with the subcomplex whose j-th term is the subspace $\mathrm{Hom}_{\mathfrak{a}}(\Lambda^j(\mathfrak{b}/\mathfrak{a}), M)$

of $\mathrm{Hom}_k(\Lambda^j(\mathfrak{b}), \mathrm{M})$. [In particular, the maps δ_j in Remark (3) preserve this subcomplex.] This shows:

PROPOSITION 2.3. — *For a* $\mathfrak{b}$-*module* M, *the cohomology* $\mathrm{H}^*(\mathfrak{b}, \mathfrak{a}, \mathrm{M})$ *defined above is naturally isomorphic to the classical relative cohomology of* $\mathfrak{b}$ *with respect to* $\mathfrak{a}$ *in* M *defined by Chevalley and Eilenberg in* [6], § 28 (cf. also [14], p. 266). *In particular,* $\mathrm{H}^*(\mathfrak{b}, \mathfrak{a})$ *as defined above is naturally isomorphic to the classical relative cohomology of* $\mathfrak{b}$ *with respect to* $\mathfrak{a}$ (see [6], § 22).

Remark. — It is clear that $\mathrm{H}^0(\mathfrak{b}, \mathfrak{a}, \mathrm{M})$ is naturally isomorphic to the space of $\mathfrak{b}$-invariants in M.

Because it will be useful later (*see* Prop. 3.12, 4.3 and 4.7), we shall recall the following general (Hopf algebra) principle [11], (*a*), Prop. 1.7:

PROPOSITION 2.4. — *Let* M *be an* $\mathfrak{a}$-*module and* N *a* $\mathfrak{b}$-*module. Then there is a natural isomorphism of* $\mathfrak{b}$-*modules*

$$(\mathscr{B}\otimes_{\mathscr{A}}\mathrm{M})\otimes_k \mathrm{N} \simeq \mathscr{B}\otimes_{\mathscr{A}}(\mathrm{M}\otimes_k \mathrm{N});$$

here the left-hand side is the tensor product of $\mathfrak{b}$-*modules, and* $\mathrm{M}\otimes_k \mathrm{N}$ *on the right is the tensor product of* $\mathfrak{a}$-*modules, with* N *regarded as an* $\mathfrak{a}$-*module by restriction.*

3. $(\mathfrak{b}, \mathfrak{a})$-projective resolutions

Assume that the field k has characteristic zero, and assume that $\mathfrak{b}$ is a finitely semisimple $\mathfrak{a}$-module (under the adjoint action), i. e., that $\mathfrak{b}$ is a direct sum of finite-dimensional irreducible $\mathfrak{a}$-modules. Define $\mathrm{C}(\mathfrak{b}, \mathfrak{a})$ to be the full subcategory of the category of $\mathfrak{b}$-modules consisting of those $\mathfrak{b}$-modules which are finitely semisimple under $\mathfrak{a}$.

LEMMA 3.1. — *The tensor product of two finitely semisimple* $\mathfrak{a}$-*modules is finitely semisimple. In particular,* $\mathrm{C}(\mathfrak{b}, \mathfrak{a})$ *is closed under the formation of tensor products.*

Proof. — This well-known fact for finite-dimensional Lie algebras $\mathfrak{a}$ of characteristic zero is easily extended to infinite-dimensional $\mathfrak{a}$ (*cf.* for example [18], Lemma 2.1).

Q.E.D.

COROLLARY 3.2. — *As an* $\mathfrak{a}$-*module under the natural action,* $\mathscr{B}$ *is finitely semisimple.*

Proof. — $\mathscr{B}$ is an $\mathfrak{a}$-module quotient of the tensor algebra over $\mathfrak{b}$, and this algebra is finitely semisimple under $\mathfrak{a}$ by Lemma 3.1.

Q.E.D.

COROLLARY 3.3. — *Let* Q *be a finitely semisimple* $\mathfrak{a}$-*module. Then the induced* $\mathfrak{b}$-*module* $\mathscr{B}\otimes_{\mathscr{A}}\mathrm{Q}$ *is finitely semisimple under* $\mathfrak{a}$, i. e., *it lies in* $\mathrm{C}(\mathfrak{b}, \mathfrak{a})$.

Proof. — The tensor product $\mathfrak{a}$-module $\mathscr{B}\otimes_k \mathrm{Q}$ is finitely semisimple by Lemma 3.1 and Corollary 3.2, and as an $\mathfrak{a}$-module, $\mathscr{B}\otimes_{\mathscr{A}}\mathrm{Q}$ is a quotient of $\mathscr{B}\otimes_k \mathrm{Q}$.

Q.E.D.

DEFINITION. — A module $P \in C(\mathfrak{b}, \mathfrak{a})$ is called $(\mathfrak{b}, \mathfrak{a})$-*projective* if for every morphism f: $P \to N$ in $C(\mathfrak{b}, \mathfrak{a})$ and every surjection g: $M \to N$ in $C(\mathfrak{b}, \mathfrak{a})$, there is a morphism h: $P \to M$ in $C(\mathfrak{b}, \mathfrak{a})$ such that $g \circ h = f$.

DEFINITION. — A module $F \in C(\mathfrak{b}, \mathfrak{a})$ is called $(\mathfrak{b}, \mathfrak{a})$-*free* if there is a finitely semisimple $\mathfrak{a}$-module Q and an $\mathfrak{a}$-module map ι: $Q \to F$ such that for every $M \in C(\mathfrak{b}, \mathfrak{a})$ and every $\mathfrak{a}$-module map f: $Q \to M$, there is a unique morphism g: $F \to M$ in $C(\mathfrak{b}, \mathfrak{a})$ such that $g \circ \iota = f$. In this case, F is called a $(\mathfrak{b}, \mathfrak{a})$-*free module generated by* Q. If a $(\mathfrak{b}, \mathfrak{a})$-free module generated by Q exists, it is clearly uniquely determined up to natural isomorphism.

Corollary 3.3 and the standard properties of induced modules (*see* [8], § 5.1) imply:

PROPOSITION 3.4. — *For every finitely semisimple* $\mathfrak{a}$-*module* Q, *the* $(\mathfrak{b}, \mathfrak{a})$-*free module generated by* Q *exists, and it may be realized as the induced* $\mathfrak{b}$-*module* $\mathscr{B} \otimes_{\mathscr{A}} Q$ *together with the natural* $\mathfrak{a}$-*module injection* ι: $Q \to \mathscr{B} \otimes_{\mathscr{A}} Q$ *taking* $q \in Q$ *to* $1 \otimes q$.

Remark. — The terminologies "$(\mathfrak{b}, \mathfrak{a})$-projective" and "$(\mathfrak{b}, \mathfrak{a})$-free" are partly justified by the fact that when we take $\mathfrak{a} = 0$, $C(\mathfrak{b}, \mathfrak{a})$ becomes the category of $\mathscr{B}$-modules, the $(\mathfrak{b}, \mathfrak{a})$-projective modules are the projective $\mathscr{B}$-modules (*see* [5], p. 6), and the $(\mathfrak{b}, \mathfrak{a})$-free modules are the free $\mathscr{B}$-modules; the $(\mathfrak{b}, \mathfrak{a})$-free module generated by the vector space (i.e., finitely semisimple $\mathfrak{a}$-module) Q is the free $\mathscr{B}$-module generated by any basis of Q. But in addition to being generalizations of the classical concepts "projective" and "free", the present concepts are analogues of the classical ones, as we shall see presently, by imitating results on projective and free modules in [5], § I.2 and V.1.

PROPOSITION 3.5. — A $(\mathfrak{b}, \mathfrak{a})$-*free module is* $(\mathfrak{b}, \mathfrak{a})$-*projective*.

Proof. — Let F be the $(\mathfrak{b}, \mathfrak{a})$-free module generated by the finitely semisimple $\mathfrak{a}$-module Q and let f: $F \to N$ and g: $M \to N$ be morphisms in $C(\mathfrak{b}, \mathfrak{a})$, with g a surjection. Since M is finitely semisimple under $\mathfrak{a}$, $\operatorname{Ker} g$ has an $\mathfrak{a}$-module complement $L \subset M$, and $g \mid L$: $L \to M$ is an $\mathfrak{a}$-module isomorphism. By Proposition 3.4, we may regard Q as an $\mathfrak{a}$-submodule of F. There is clearly an $\mathfrak{a}$-module map h': $Q \to L$ such that $g \circ h' = f$: $Q \to N$. By the defining property of F, h' extends to a $\mathfrak{b}$-module map h: $F \to M$, and since $g \circ h \mid Q = f \mid Q$, we must have $g \circ h = f$.

Q.E.D.

PROPOSITION 3.6. — *Every module* $M \in C(\mathfrak{b}, \mathfrak{a})$ *can be embedded in an exact sequence*

$$0 \to N \to F \to M \to 0$$

in $C(\mathfrak{b}, \mathfrak{a})$, *where* F *is* $(\mathfrak{b}, \mathfrak{a})$-*free*.

Proof. — Let F be the $(\mathfrak{b}, \mathfrak{a})$-free module generated by M, regarded now as a finitely semisimple $\mathfrak{a}$-module. (F exists by Proposition 3.4) We clearly have a $\mathfrak{b}$-module surjection $F \to M$, and since F is finitely semisimple under $\mathfrak{a}$, the kernel of this surjection is a $\mathfrak{b}$-module in $C(\mathfrak{b}, \mathfrak{a})$.

Q.E.D.

The category $C(\mathfrak{b}, \mathfrak{a})$ is clearly closed under the formation of (not necessarily finite) direct sums; and by Proposition 3.4, the direct sum of $(\mathfrak{b}, \mathfrak{a})$-free modules is clearly $(\mathfrak{b}, \mathfrak{a})$-free. The following result for $(\mathfrak{b}, \mathfrak{a})$-projective modules is straightforward from the definitions, and we omit the proof (*cf.* [5], Prop. I.2.1, p. 6):

PROPOSITION 3.7. – *A direct sum of modules in* $C(\mathfrak{b}, \mathfrak{a})$ *is* $(\mathfrak{b}, \mathfrak{a})$-*projective if and only if each summand is* $(\mathfrak{b}, \mathfrak{a})$-*projective.*

The last three propositions immediately yield the following two characterizations of $(\mathfrak{b}, \mathfrak{a})$-projective modules:

PROPOSITION 3.8. – *A module in* $C(\mathfrak{b}, \mathfrak{a})$ *is* $(\mathfrak{b}, \mathfrak{a})$-*projective if and only if it is a direct summand in* $C(\mathfrak{b}, \mathfrak{a})$ *of a* $(\mathfrak{b}, \mathfrak{a})$-*free module.*

PROPOSITION 3.9. – *A module* $P \in C(\mathfrak{b}, \mathfrak{a})$ *is* $(\mathfrak{b}, \mathfrak{a})$-*projective if and only if every exact sequence*

$$0 \to N \to M \to P \to 0$$

in $C(\mathfrak{b}, \mathfrak{a})$ *splits.*

Let $M \in C(\mathfrak{b}, \mathfrak{a})$. *A complex over* M is a complex

$$\ldots \to X_n \xrightarrow{d_n} X_{n-1} \to \ldots \xrightarrow{d_1} X_0 \to 0,$$

denoted X, in $C(\mathfrak{b}, \mathfrak{a})$, together with a map $X_0 \xrightarrow{\varepsilon} M$ in $C(\mathfrak{b}, \mathfrak{a})$, called the *augmentation*, such that the composition $X_1 \to X_0 \to M$ is zero. X is called $(\mathfrak{b}, \mathfrak{a})$-*projective* [respectively, $(\mathfrak{b}, \mathfrak{a})$-*free*] if each X_i is $(\mathfrak{b}, \mathfrak{a})$-projective [respectively, $(\mathfrak{b}, \mathfrak{a})$-free]; and X is said to be a $(\mathfrak{b}, \mathfrak{a})$-*projective* [respectively, $(\mathfrak{b}, \mathfrak{a})$-*free*] *resolution of* M if X is $(\mathfrak{b}, \mathfrak{a})$-projective [respectively, $(\mathfrak{b}, \mathfrak{a})$-free] and the augmented complex $X \xrightarrow{\varepsilon} M \to 0$ is exact.

Let $M' \in C(\mathfrak{b}, \mathfrak{a})$ and let X' be the complex

$$\ldots \to X'_n \xrightarrow{d'_n} X'_{n-1} \to \ldots \xrightarrow{d'_1} X'_0 \to 0$$

over M', with augmentation $X'_0 \xrightarrow{\varepsilon'} M'$. Let $f\colon M \to M'$ be a map in $C(\mathfrak{b}, \mathfrak{a})$. A map $F\colon X \to X'$ of complexes [i.e., a family $F_0\colon X_0 \to X'_0$, $F_1\colon X_1 \to X'_1$, $\ldots$ of maps in $C(\mathfrak{b}, \mathfrak{a})$ such that the usual diagrams commute] is called a *map over* f if the diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & X' \\ \varepsilon \downarrow & & \downarrow \varepsilon' \\ M & \xrightarrow{f} & M' \end{array}$$

commutes.

Let $F, G\colon X \to X'$ be two maps [of complexes in $C(\mathfrak{b}, \mathfrak{a})$]. A *homotopy* H from F to G is a family $H_0\colon X_0 \to X'_1$, $H_1\colon X_1 \to X'_2$, $\ldots$ of maps in $C(\mathfrak{b}, \mathfrak{a})$ such that $d'_1 \circ H_0 = G_0 - F_0\colon X_0 \to X'_0$ and for all $i \geqq 1$,

$$d'_{i+1} \circ H_i + H_{i-1} \circ d_i = G_i - F_i\colon X_i \to X'_i.$$

Note that $-H$ is a homotopy from G to F. If X and X′ are complexes over the same module $M \in C(\mathfrak{b}, \mathfrak{a})$, then X and X′ are said to have the same *homotopy type* if there are maps $F: X \to X'$ and $G: X' \to X$ over the identity map of M such that $G \circ F$ and $F \circ G$ are homotopic to the identity maps of X and X′, respectively.

Straightforward imitation of the proofs of Propositions V.1.1 and V.1.2 on pp. 76-77 of [5] yields the following two results:

PROPOSITION 3.10. — *Let* X, X′ *be* $(\mathfrak{b}, \mathfrak{a})$*-projective resolutions of* $M, M' \in C(\mathfrak{b}, \mathfrak{a})$, *respectively.* [*More generally, we may suppose that* X *is a* $(\mathfrak{b}, \mathfrak{a})$*-projective complex over* M, *and that* X′ *is a complex over* M′ *such that the corresponding augmented complex* $X' \to M \to 0$ *is exact.*] *Let* $f: M \to M'$ *be a map in* $C(\mathfrak{b}, \mathfrak{a})$. *Then there is a map* $F: X \to X'$ *over* f, *and any two such maps are homotopic.*

PROPOSITION 3.11. — *Every module* $M \in C(\mathfrak{b}, \mathfrak{a})$ *has a* $(\mathfrak{b}, \mathfrak{a})$*-free resolution, and in particular, a* $(\mathfrak{b}, \mathfrak{a})$*-projective resolution. Any two* $(\mathfrak{b}, \mathfrak{a})$*-projective resolutions of* M *have the same homotopy type.*

The general Hopf algebra principle, Proposition 2.4, provides us with useful information about $(\mathfrak{b}, \mathfrak{a})$-projective resolutions and so on:

PROPOSITION 3.12. — *Let* $M, N \in C(\mathfrak{b}, \mathfrak{a})$, *with* M $(\mathfrak{b}, \mathfrak{a})$*-free* [*respectively,* $(\mathfrak{b}, \mathfrak{a})$*-projective*]. *Then* $M \otimes N$ [$\in C(\mathfrak{b}, \mathfrak{a})$ *by Lemma* 3.1] *is* $(\mathfrak{b}, \mathfrak{a})$*-free* [*respectively,* $(\mathfrak{b}, \mathfrak{a})$*-projective*]. *In particular, if* X *is a* $(\mathfrak{b}, \mathfrak{a})$*-free* [*respectively,* $(\mathfrak{b}, \mathfrak{a})$*-projective*] *resolution of* M, *then the complex* $X \otimes N$ (*defined in the obvious way*) *is a* $(\mathfrak{b}, \mathfrak{a})$*-free* [*respectively,* $(\mathfrak{b}, \mathfrak{a})$*-projective*] *resolution of* $M \otimes N$.

Proof. — Propositions 2.4 and 3.4 immediately imply the assertions about $(\mathfrak{b}, \mathfrak{a})$-free modules, and Proposition 3.8 now implies the assertions about $(\mathfrak{b}, \mathfrak{a})$-projective modules.

Q.E.D.

Suppose now that $\mathfrak{b}^1$ and $\mathfrak{b}^2$ are Lie algebras over k with subalgebras $\mathfrak{a}^1 \subset \mathfrak{b}^1$ and $\mathfrak{a}^2 \subset \mathfrak{b}^2$ acting finitely semisimply on $\mathfrak{b}^1$ and $\mathfrak{b}^2$. Take $\mathfrak{b}$ to be the direct product Lie algebra $\mathfrak{b}^1 \times \mathfrak{b}^2$ and $\mathfrak{a}$ to be the subalgebra $\mathfrak{a}^1 \times \mathfrak{a}^2$. Then $\mathfrak{b}$ is finitely semisimple under $\mathfrak{a}$, and so all the above considerations apply to the pair $(\mathfrak{b}, \mathfrak{a})$. Write $\mathscr{A}^1$, $\mathscr{A}^2$, $\mathscr{B}^1$ and $\mathscr{B}^2$ for the universal enveloping algebras of $\mathfrak{a}^1$, $\mathfrak{a}^2$, $\mathfrak{b}^1$ and $\mathfrak{b}^2$, respectively.

If M^i is a $\mathfrak{b}^i$-module $(i=1, 2)$, then we provide $M^1 \otimes M^2$ with the natural $\mathfrak{b}$-module structure given by the rule

$$(b^1, b^2).m^1 \otimes m^2 = b^1.m^1 \otimes m^2 + m^1 \otimes b^2.m^2$$

for all $b^i \in \mathfrak{b}^i$ and $m^i \in M^i$. Note that $M^1 \otimes M^2$ may be regarded as the ordinary $\mathfrak{b}$-module tensor product of M^1 with $\mathfrak{b}$-module structure

$$(b^1, b^2).m^1 = b^1.m^1$$

and M^2 with $\mathfrak{b}$-module structure

$$(b^1, b^2).m^2 = b^2.m^2$$

(using obvious notation). Analogous comments hold for the tensor product $\mathfrak{a}$-module formed from an $\mathfrak{a}^1$-module and an $\mathfrak{a}^2$-module. In particular, Lemma 3.1 implies that if Q^i is a finitely semisimple $\mathfrak{a}^i$-module ($i=1, 2$), then $Q^1 \otimes Q^2$ is a finitely semisimple $\mathfrak{a}$-module. Hence if $M^i \in C(\mathfrak{b}^i, \mathfrak{a}^i)$ ($i=1, 2$), then $M^1 \otimes M^2 \in C(\mathfrak{b}, \mathfrak{a})$.

Let Q^i be an $\mathfrak{a}^i$-module ($i=1, 2$). Then we have a natural $\mathfrak{b}$-module isomorphism

$$\mathscr{B} \otimes_{\mathscr{A}} (Q^1 \otimes Q^2) \simeq (\mathscr{B}^1 \otimes_{\mathscr{A}^1} Q^1) \otimes (\mathscr{B}^2 \otimes_{\mathscr{A}^2} Q^2).$$

In particular, the tensor product of a $(\mathfrak{b}^1, \mathfrak{a}^1)$-free module with a $(\mathfrak{b}^2, \mathfrak{a}^2)$-free module is $(\mathfrak{b}, \mathfrak{a})$-free. Hence by Proposition 3.8, the tensor product of a $(\mathfrak{b}^1, \mathfrak{a}^1)$-projective module with a $(\mathfrak{b}^2, \mathfrak{a}^2)$-projective module is $(\mathfrak{b}, \mathfrak{a})$-projective.

Let X^1, X^2 be two complexes of vector spaces (over k), so that X^i ($i=1, 2$) is of the form

$$\ldots \to (X^i)_n \xrightarrow{(d^i)_n} (X^i)_{n-1} \to \ldots \xrightarrow{(d^i)_1} (X^i)_0 \to 0.$$

The *tensor product* $X^1 \otimes X^2$ is defined as usual to be the sequence

$$\ldots \to (X^1 \otimes X^2)_n \xrightarrow{d_n} (X^1 \otimes X^2)_{n-1} \to \ldots \xrightarrow{d_1} (X^1 \otimes X^2)_0 \to 0,$$

where for all $n \in \mathbf{Z}_+$,

$$(X^1 \otimes X^2)_n = \coprod_{r+s=n} (X^1)_r \otimes (X^2)_s$$

and for all $n>0$, $x^1 \in (X^1)_r$ and $x^2 \in (X^2)_s$ with $r+s=n$, we have

$$d_n(x^1 \otimes x^2) = (d^1)_r x^1 \otimes x^2 + (-1)^r x^1 \otimes (d^2)_s x^2.$$

$X^1 \otimes X^2$ is clearly a complex. Let

$$(Z^i)_r = \operatorname{Ker}(d^i)_r \subset (X^i)_r$$

($i=1, 2$; $r \in \mathbf{Z}_+$). If $z^1 \in (Z^1)_r$ and $z^2 \in (Z^2)_s$ ($r, s \in \mathbf{Z}_+$), then

$$z^1 \otimes z^2 \in \operatorname{Ker} d_{r+s} \subset (X^1 \otimes X^2)_{r+s},$$

and for all $n \in \mathbf{Z}_+$, we get a well-defined map

$$\mu_n : \coprod_{r+s=n} H_r(X^1) \otimes H_s(X^2) \to H_n(X^1 \otimes X^2),$$

where the symbols H_t denote the obvious homologies. It is a standard fact (*see* for example [25], p. 228) that μ_n is a linear isomorphism. Suppose that the complex X^i is a $\mathfrak{b}^i$-module complex ($i=1, 2$). Then it is clear from the definitions that $X^1 \otimes X^2$ is a $\mathfrak{b}$-module complex and that μ_n is a $\mathfrak{b}$-module map. Hence we have:

PROPOSITION 3.13. — *In the above notation, μ_n is a $\mathfrak{b}$-module isomorphism.*

Since μ_0 is a $\mathfrak{b}$-module isomorphism from $H_0(X^1) \otimes H_0(X^2)$ onto $H_0(X^1 \otimes X^2)$, we immediately obtain:

PROPOSITION 3.14. — *Let* $M^i \in C(\mathfrak{b}^i, \mathfrak{a}^i)$, *and let* X^i *be a* $(\mathfrak{b}^i, \mathfrak{a}^i)$*-free* [*respectively*, $(\mathfrak{b}^i, \mathfrak{a}^i)$*-projective*] *resolution of* M^i *in* $C(\mathfrak{b}^i, \mathfrak{a}^i)$ $(i=1, 2)$. *Then* $X^1 \otimes X^2$ *is a* $(\mathfrak{b}, \mathfrak{a})$*-free* [*respectively*, $(\mathfrak{b}, \mathfrak{a})$*-projective*] *resolution of* $M^1 \otimes M^2$ *in* $C(\mathfrak{b}, \mathfrak{a})$.

4. The functors $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}$ and $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}$

Retain the notation and assumptions of paragraph 3.

DEFINITION. — Let $M, N \in C(\mathfrak{b}, \mathfrak{a})$, and let X be a $(\mathfrak{b}, \mathfrak{a})$-projective resolution of N (*see* Prop. 3.11). Write M^t for the right $\mathfrak{b}$-module associated with M as in paragraph 2. Define $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(M^t, N)$ to be the homology of the complex $M^t \otimes_{\mathscr{B}} X$.

PROPOSITION 4.1. — $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(M^t, N)$ *is independent of the* $(\mathfrak{b}, \mathfrak{a})$*-projective resolution* X *used in its definition, and defines a covariant functor in* M *and* N, *with values in the category of graded vector spaces.*

Proof. — Let $M \in C(\mathfrak{b}, \mathfrak{a})$, let $f: N \to N'$ be a map in $C(\mathfrak{b}, \mathfrak{a})$, and let X and X′ be $(\mathfrak{b}, \mathfrak{a})$-projective resolutions of N and N′, respectively. Then by Proposition 3.10, there is a map $F: X \to X'$ over f, and for any other such map $G: X \to X'$, there is a homotopy H from F to G. Applying the functor $M^t \otimes_{\mathscr{B}}$ and using the notation of paragraph 3, we get

$$(1 \otimes d_1') \circ (1 \otimes H_0) = 1 \otimes G_0 - 1 \otimes F_0 : M^t \otimes_{\mathscr{B}} X_0 \to M^t \otimes_{\mathscr{B}} X_0'$$

and for all $i \geqq 1$,

$$(1 \otimes d_{i+1}') \circ (1 \otimes H_i) + (1 \otimes H_{i-1}) \circ (1 \otimes d_i) = 1 \otimes G_i - 1 \otimes F_i : M^t \otimes_{\mathscr{B}} X_i \to M^t \otimes_{\mathscr{B}} X_i'.$$

It is thus clear that $1 \otimes F$ and $1 \otimes G$ induce the same map from the homology of $M^t \otimes_{\mathscr{B}} X$ to the homology of $M^t \otimes_{\mathscr{B}} X'$. Thus there is a natural map, depending only on f, from the homology of $M^t \otimes_{\mathscr{B}} X$ to the homology of $M^t \otimes_{\mathscr{B}} X'$. If $N' = N$ and f is the identity, then we also get a natural map on homology in the reverse direction, and the two maps must be inverses of each other because the identity map of X and the identity map of X′ are maps over the identity map of N. Thus $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(M^t, N)$ is independent of the choice of X, up to natural isomorphism, and the rest of the proposition is straightforward.

Q.E.D.

Here is the relation between $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}$ and relative homology:

PROPOSITION 4.2. — *The relative homology* $H_*(\mathfrak{b}, \mathfrak{a})$ *of* $\mathfrak{b}$ *with respect to* $\mathfrak{a}$ *is naturally isomorphic to* $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(k, k)$, *where k is regarded as the trivial* (*right and left*) $\mathfrak{b}$*-module. Let* $M \in C(\mathfrak{b}, \mathfrak{a})$. *The relative homology* $H_*(\mathfrak{b}, \mathfrak{a}, M^t)$ *is naturally isomorphic to* $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(M^t, k)$.

Proof. — For each $j \in \mathbf{Z}_+$, $\Lambda^j(\mathfrak{b}/\mathfrak{a})$ is a finitely semisimple $\mathfrak{a}$-module, and so $\mathscr{B} \otimes_{\mathscr{A}} \Lambda^j(\mathfrak{b}/\mathfrak{a})$ is a $(\mathfrak{b}, \mathfrak{a})$-free module in $C(\mathfrak{b}, \mathfrak{a})$, by Proposition 3.4. Thus $V(\mathfrak{b}, \mathfrak{a})$, with the segment $\xrightarrow{\varepsilon_0} k$ deleted (*see* § 2), is a $(\mathfrak{b}, \mathfrak{a})$-free, and hence $(\mathfrak{b}, \mathfrak{a})$-projective, resolution of k, by Proposition 2.1. The rest follows from Propositions 2.2 and 4.1, and the definitions.

Q.E.D.

The "commutativity" of Tor is often proved using double complexes (*cf.* [5], p. 109). But because of the general principle Proposition 2.4 (*see* also Prop. 3.12) available to us, we can directly prove even more:

PROPOSITION 4.3. — *Let* M, $N \in C(\mathfrak{b}, \mathfrak{a})$. *Then there are natural isomorphisms*

$$\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(M^t, N) \simeq \mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(N^t, M) \simeq \mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(k, M \otimes N) \simeq \mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}((M \otimes N)^t, k),$$

where k is regarded as the trivial (right or left) $\mathfrak{b}$*-module.*

Proof. — Let X be a $(\mathfrak{b}, \mathfrak{a})$-projective resolution of k, so that by Proposition 3.12, $X \otimes M$ and $X \otimes M \otimes N$ are $(\mathfrak{b}, \mathfrak{a})$-projective resolutions of M and $M \otimes N$, respectively. Now the complex $k \otimes_{\mathscr{B}} (X \otimes M \otimes N)$ (using obvious notation) is naturally isomorphic to the complex $N^t \otimes_{\mathscr{B}} (X \otimes M)$ (*cf.* Prop. 2.2). Hence their homologies are naturally isomorphic, and so we have a natural isomorphism $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(k, M \otimes N) \simeq \mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(N^t, M)$, proving the middle isomorphism in the statement of the proposition. The last isomorphism now follows by taking $M = k$, and the first follows from the natural isomorphism $M \otimes N \simeq N \otimes M$.

Q.E.D.

The last two results give:

COROLLARY 4.4. — *For all* $N \in C(\mathfrak{b}, \mathfrak{a})$, $H_*(\mathfrak{b}, \mathfrak{a}, N^t)$ *is naturally isomorphic to* $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(k, N)$. *More generally, for all* M, $N \in C(\mathfrak{b}, \mathfrak{a})$, $H_*(\mathfrak{b}, \mathfrak{a}, (M \otimes N)^t)$ *is naturally isomorphic to* $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(M^t, N)$.

Let $\mathfrak{c}$ be a Lie subalgebra of $\mathfrak{b}$ such that $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{c}$ as a vector space, and write $\mathscr{C}$ for the corresponding universal enveloping algebra. Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{a}$ such that $[\mathfrak{s}, \mathfrak{c}] \subset \mathfrak{c}$. For a $\mathfrak{b}$-module M, $\mathfrak{s}$ acts in a natural way on each homology space $H_j(\mathfrak{c}, M^t)$ $(j \in \mathbf{Z}_+)$, giving rise to the *standard action* of $\mathfrak{s}$ on $H_*(\mathfrak{c}, M^t)$; *see* [11], (*a*), § 1. We shall now reconstruct this standard action *via* relative homological algebra.

It is easy to see from the property $[\mathfrak{s}, \mathfrak{c}] \subset \mathfrak{c}$ that if M and N are $\mathfrak{b}$-modules, then $M^t \otimes_{\mathscr{C}} N$ is a well-defined $\mathfrak{s}$-module in a natural way by the rule

$$s.(m \otimes n) = -m.s \otimes n + m \otimes s.n,$$

where $s \in \mathfrak{s}$, $m \in M^t$ and $n \in N$; recall that $\mathfrak{s}$ acts on the right on M^t.

DEFINITION. — Let M, $N \in C(\mathfrak{b}, \mathfrak{a})$, and let X be a $(\mathfrak{b}, \mathfrak{a})$-projective resolution of N (*see* Prop. 3.11). Define $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}(M^t, N)$ to be the homology (regarded as a graded $\mathfrak{s}$-module) of the $\mathfrak{s}$-module complex $M^t \otimes_{\mathscr{C}} X$.

Imitation of the proof of Proposition 4.1 yields:

PROPOSITION 4.5. — $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}(M^t, N)$ *is independent of the* $(\mathfrak{b}, \mathfrak{a})$*-projective resolution* X *used in its definition, and defines a covariant functor in* M *and* N, *with values in the category of graded* $\mathfrak{s}$*-modules.*

Proposition 1.4 of [11], (*a*) and the proof of Proposition 4.2 above now show:

PROPOSITION 4.6. — *The homology* $H_*(\mathfrak{c})$ *with the standard action of* $\mathfrak{s}$ *is naturally isomorphic to* $\operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}(k, k)$, *where* k *is regarded as the trivial (right and left)* $\mathfrak{b}$*-module. Let* $M \in C(\mathfrak{b}, \mathfrak{a})$. *The homology* $H_*(\mathfrak{c}, M^t)$ *with the standard action of* $\mathfrak{s}$ *is naturally isomorphic to* $\operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{s}, \mathfrak{c}, \mathfrak{s})}(M^t, k)$.

Using Proposition 3.12 and the proof of Proposition 4.3, we get the following analogues of Proposition 4.3 and Corollary 4.4:

PROPOSITION 4.7. — *Let* $M, N \in C(\mathfrak{b}, \mathfrak{a})$. *Then there are natural isomorphisms*

$$\operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}(M^t, N) \simeq \operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}(N^t, M) \simeq \operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}(k, M \otimes N) \simeq \operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}((M \otimes N)^t, k),$$

where k *is regarded as the trivial (right or left)* $\mathfrak{b}$*-module.*

COROLLARY 4.8. — *For all* $N \in C(\mathfrak{b}, \mathfrak{a})$, $H_*(\mathfrak{c}, N^t)$ *with the standard action of* $\mathfrak{s}$ *is naturally isomorphic to* $\operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}(k, N)$. *More generally, for all* $M, N \in C(\mathfrak{b}, \mathfrak{a})$, $H_*(\mathfrak{c}, (M \otimes N)^t)$ *with the standard action of* $\mathfrak{s}$ *is naturally isomorphic to* $\operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}(M^t, N)$.

Assume now that $\mathfrak{s} = \mathfrak{a}$, so that in particular, $[\mathfrak{a}, \mathfrak{c}] \subset \mathfrak{c}$. We shall set up a useful relationship between $\operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{a})}$ and $\operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}$. First we note the following two lemmas:

LEMMA 4.9. — *Let* M *and* N *be* $\mathfrak{b}$*-modules. Then the correspondence* $m \otimes n \mapsto 1 \otimes (m \otimes n)$ *defines a natural isomorphism from the vector space* $M^t \otimes_{\mathscr{B}} N$ *to* $k \otimes_{\mathscr{A}} (M^t \otimes_{\mathscr{C}} N)$, *where* k *is regarded as the trivial right* $\mathfrak{a}$*-module and* $M^t \otimes_{\mathscr{C}} N$ *is regarded as an* $\mathfrak{a}$*-module* (i. e., $\mathfrak{s}$*-module*) *as above.*

Proof. — The given map is clearly well-defined, and its inverse is given by the condition $1 \otimes (m \otimes n) \mapsto m \otimes n$.

Q.E.D.

LEMMA 4.10. — *If* $M, N \in C(\mathfrak{b}, \mathfrak{a})$, *then* $k \otimes_{\mathscr{A}} (M^t \otimes_{\mathscr{C}} N)$ *is naturally isomorphic to the space of* $\mathfrak{a}$*-invariants in* $M^t \otimes_{\mathscr{C}} N$.

Proof. — This follows immediatly from the fact that $M^t \otimes_C N$ is a finitely semisimple $\mathfrak{a}$-module.

Q.E.D.

For a graded $\mathfrak{a}$-module V, we denote by $V^{\mathfrak{a}}$ the graded vector space whose components are the spaces of $\mathfrak{a}$-invariants in the components of V.

PROPOSITION 4.11. — *For* $M, N \in C(\mathfrak{b}, \mathfrak{a})$, *there is a natural isomorphism*

$$\operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(M^t, N) \simeq \operatorname{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{a})}(M^t, N)^{\mathfrak{a}}.$$

In particular, we have natural isomorphisms

$$H_*(\mathfrak{b}, \mathfrak{a}) \simeq H_*(\mathfrak{c})^{\mathfrak{a}}$$

and

$$H_*(\mathfrak{b}, \mathfrak{a}, M^t) \simeq H_*(\mathfrak{c}, M^t)^{\mathfrak{a}},$$

where $\mathfrak{a}$ *acts* via *the standard action on* $H_*(\mathfrak{c})$ *and* $H_*(\mathfrak{c}, M^t)$.

Proof. — Let X be a $(\mathfrak{b}, \mathfrak{a})$-projective resolution of N. Then $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(\mathrm{M}^t, \mathrm{N})$ is the homology of the complex $\mathrm{M}^t \otimes_{\mathscr{B}} \mathrm{X}$, while $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{a})}(\mathrm{M}^t, \mathrm{N})^{\mathfrak{a}}$ is the homology of the complex $(\mathrm{M}^t \otimes_{\mathscr{C}} \mathrm{X})^{\mathfrak{a}}$, the components of $\mathrm{M}^t \otimes_{\mathscr{C}} \mathrm{X}$ being finitely semisimple under $\mathfrak{a}$. By Lemma 4.10, the complex $(\mathrm{M}^t \otimes_{\mathscr{C}} \mathrm{X})^{\mathfrak{a}}$ is naturally isomorphic to the complex $k \otimes_{\mathscr{A}} (\mathrm{M}^t \otimes_{\mathscr{C}} \mathrm{X})$. Lemma 4.9 now establishes the first assertion. The rest follows from Propositions 4.2 and 4.6.

Q.E.D.

Let $\mathfrak{b}^1$, $\mathfrak{b}^2$, $\mathfrak{a}^1$ and $\mathfrak{a}^2$ be as in paragraph 3, and take $\mathfrak{b} = \mathfrak{b}^1 \times \mathfrak{b}^2$ and $\mathfrak{a} = \mathfrak{a}^1 \times \mathfrak{a}^2$. (We temporarily ignore $\mathfrak{c}$ and $\mathfrak{s}$.) Recall that if M^i is a $\mathfrak{b}^i$-module $(i=1, 2)$, then $\mathrm{M}^1 \otimes \mathrm{M}^2$ is a $\mathfrak{b}$-module in a natural way.

PROPOSITION 4.12. — *Let* $\mathrm{M}^i, \mathrm{N}^i \in \mathrm{C}(\mathfrak{b}^i, \mathfrak{a}^i)$ $(i=1, 2)$. *Then for each* $n \in \mathbf{Z}_+$, *we have a natural vector space isomorphism*

$$\mathrm{Tor}_n^{(\mathfrak{b}, \mathfrak{a})}((\mathrm{M}^1 \otimes \mathrm{M}^2)^t, \mathrm{N}^1 \otimes \mathrm{N}^2) \simeq \coprod_{r+s=n} \mathrm{Tor}_r^{(\mathfrak{b}^1, \mathfrak{a}^1)}((\mathrm{M}^1)^t, \mathrm{N}^1) \otimes \mathrm{Tor}_s^{(\mathfrak{b}^2, \mathfrak{a}^2)}((\mathrm{M}^2)^t, \mathrm{N}^2).$$

In particular, we have natural isomorphisms

$$\mathrm{H}_n(\mathfrak{b}, \mathfrak{a}, (\mathrm{M}^1 \otimes \mathrm{M}^2)^t) \simeq \coprod_{r+s=n} \mathrm{H}_r(\mathfrak{b}^1, \mathfrak{a}^1, (\mathrm{M}^1)^t) \otimes \mathrm{H}_s(\mathfrak{b}^2, \mathfrak{a}^2, (\mathrm{M}^2)^t)$$

and

$$\mathrm{H}_n(\mathfrak{b}, \mathfrak{a}) \simeq \coprod_{r+s=n} \mathrm{H}_r(\mathfrak{b}^1, \mathfrak{a}^1) \otimes \mathrm{H}_s(\mathfrak{b}^2, \mathfrak{a}^2).$$

Proof. — Let X^i be a $(\mathfrak{b}^i, \mathfrak{a}^i)$-projective resolution of N^i in $\mathrm{C}(\mathfrak{b}^i, \mathfrak{a}^i)$ $(i=1, 2)$. By Proposition 3.14, $\mathrm{X}^1 \otimes \mathrm{X}^2$ is a $(\mathfrak{b}, \mathfrak{a})$-projective resolution of $\mathrm{N}^1 \otimes \mathrm{N}^2$ in $\mathrm{C}(\mathfrak{b}, \mathfrak{a})$. From the definition of the tensor product of complexes, there is a natural isomorphism

$$(\mathrm{M}^1 \otimes \mathrm{M}^2)^t \otimes_{\mathscr{B}} (\mathrm{X}^1 \otimes \mathrm{X}^2) \simeq ((\mathrm{M}^1)^t \otimes_{\mathscr{B}^1} \mathrm{X}^1) \otimes ((\mathrm{M}^2)^t \otimes_{\mathscr{B}^2} \mathrm{X}^2),$$

where $\mathscr{B}^i$ is the universal enveloping algebra of $\mathfrak{b}^i$. Taking homology and using the fact that μ_n is a vector space isomorphism in paragraph 3 gives us the first assertion of the proposition. The second assertion follows from Proposition 4.2.

Q.E.D.

Now suppose that $\mathfrak{c}^i$ is a subalgebra of $\mathfrak{b}^i$ such that $\mathfrak{b}^i = \mathfrak{a}^i \oplus \mathfrak{c}^i$ as a vector space, and let $\mathfrak{s}^i$ be a subalgebra of $\mathfrak{a}^i$ such that $[\mathfrak{s}^i, \mathfrak{c}^i] \subset \mathfrak{c}^i$ $(i=1, 2)$. Define $\mathfrak{c} = \mathfrak{c}^1 \times \mathfrak{c}^2$ and $\mathfrak{s} = \mathfrak{s}^1 \times \mathfrak{s}^2$. These two subalgebras of $\mathfrak{b}$ satisfy the above conditions on $\mathfrak{c}$ and $\mathfrak{s}$, and so the above considerations hold here.

PROPOSITION 4.13. — *Let* $\mathrm{M}^i, \mathrm{N}^i \in \mathrm{C}(\mathfrak{b}^i, \mathfrak{a}^i)$ $(i=1, 2)$. *Then for each* $n \in \mathbf{Z}_+$, *we have a natural* $\mathfrak{s}$-*module isomorphism*

$$\mathrm{Tor}_n^{(\mathfrak{b}, \mathfrak{a}, \mathfrak{c}, \mathfrak{s})}((\mathrm{M}^1 \otimes \mathrm{M}^2)^t, \mathrm{N}^1 \otimes \mathrm{N}^2) \simeq \coprod_{r+s=n} \mathrm{Tor}_r^{(\mathfrak{b}^1, \mathfrak{a}^1, \mathfrak{c}^1, \mathfrak{s}^1)}((\mathrm{M}^1)^t, \mathrm{N}^1) \otimes \mathrm{Tor}_s^{(\mathfrak{b}^2, \mathfrak{a}^2, \mathfrak{c}^2, \mathfrak{s}^2)}((\mathrm{M}^2)^t, \mathrm{N}^2).$$

In particular, we have natural $\mathfrak{s}$-*module isomorphisms* (*using the standard actions of* $\mathfrak{s}$, $\mathfrak{s}^1$ *and* $\mathfrak{s}^2$):

$$\mathrm{H}_n(\mathfrak{c}, (\mathrm{M}^1 \otimes \mathrm{M}^2)^t) \simeq \coprod_{r+s=n} \mathrm{H}_r(\mathfrak{c}^1, (\mathrm{M}^1)^t) \otimes \mathrm{H}_s(\mathfrak{c}^2, (\mathrm{M}^2)^t)$$

and

$$H_n(\mathfrak{c}) \simeq \coprod_{r+s=n} H_r(\mathfrak{c}^1) \otimes H_s(\mathfrak{c}^2).$$

Proof. — Let X^i $(i=1, 2)$ be as in the last proof, so that $X^1 \otimes X^2$ is a $(\mathfrak{b}, \mathfrak{a})$-projective resolution of $N^1 \otimes N^2$. We clearly have a natural $\mathfrak{s}$-module isomorphism of $\mathfrak{s}$-module complexes

$$(M^1 \otimes M^2)^t \otimes_{\mathscr{C}} (X^1 \otimes X^2) \simeq ((M^1)^t \otimes_{\mathscr{C}^1} X^1) \otimes ((M^2)^t \otimes_{\mathscr{C}^2} X^2),$$

where $\mathscr{C}^i$ is the universal enveloping algebra of $\mathfrak{c}^i$. Take homology and apply Proposition 3.13 to $\mathfrak{s}$ in place of $\mathfrak{b}$. This proves the first assertion of the proposition; apply Proposition 4.6 for the rest.

Q.E.D.

PART II
THE RESOLUTION AND ITS APPLICATION TO LIE ALGEBRA HOMOLOGY AND COHOMOLOGY

5. A complex for computing certain relative homology

Part II contains our general results on Kac-Moody Lie algebra homology and cohomology, and an application to finite-dimensional Lie algebras in paragraph 7. In [11], (*a*), Th. 8.7, a standard module for a Kac-Moody Lie algebra is resolved using generalized Verma modules. This result, restated as Theorem 5.1 below, is central to the present paper. The main idea in Part II is to combine the relative homological algebra of Part I with Theorem 5.1 applied to the product of a Kac-Moody Lie algebra with itself (*see* § 6). The basic references on Kac-Moody Lie algebras are [15], (*a*)-(*d*) and [22], (*a*)-(*c*). We use the notation of [11], (*a*).

Let $l \in \mathbf{Z}_+$, let $A=(A_{ij})_{i,j\in\{1,\ldots,l\}}$ be an $l \times l$ (generalized) Cartan matrix (="Cartan matrix" in [11], (*a*)) which is symmetrizable, and let k be a field of characteristic zero. Let $\mathfrak{g}$ be the corresponding (possibly infinite-dimensional) Kac-Moody Lie algebra $\mathfrak{g}(A)$ over k, with canonical generators h_i, e_i, f_i $(1 \leqq i \leqq l)$ (*see* [11], (*a*), p. 47). Let $\mathfrak{h}$ be the span of the h_i, and let η be the involution of $\mathfrak{g}$ which interchanges e_i and f_i and sends h_i to $-h_i$ for each i. Let D_i $(1 \leqq i \leqq l)$ be the *i-th* degree derivation of $\mathfrak{g}$ with respect to the natural $\mathbf{Z}^l$-grading, and let $\mathfrak{d}_0$ be the l-dimensional abelian Lie algebra of derivations of $\mathfrak{g}$ spanned by the D_i. For a subspace $\mathfrak{d}$ of $\mathfrak{d}_0$, form the semidirect product Lie algebra $\mathfrak{g}^e = \mathfrak{d} \times \mathfrak{g}$, and let $\mathfrak{h}^e$ be the abelian subalgebra $\mathfrak{d} \oplus \mathfrak{h}$. Define the *simple roots* $\alpha_1, \ldots, \alpha_l \in (\mathfrak{h}^e)^*$ by the conditions $[h, e_i] = \alpha_i(h) e_i$ for all $h \in \mathfrak{h}^e$ and $i \in \{1, \ldots, l\}$. Call $\mathfrak{d}$ an *admissible* subspace of $\mathfrak{d}_0$ if $\alpha_1, \ldots, \alpha_l$ are linearly independent. Admissible subspaces exist; fix one.

If A is classical of finite type, then $\mathfrak{g}$ is the finite-dimensional split semisimple Lie algebra with Cartan matrix A. In this case, we may choose $\mathfrak{d}=0$, so that $\mathfrak{g}^e = \mathfrak{g}$, and then the roots, Weyl group and other concepts discussed below simply reduce to the usual classical ones for $\mathfrak{g}$.

Let $\Delta \subset (\mathfrak{h}^e)^*$ be the set of roots of $\mathfrak{g}$, Δ_+ the set of positive roots and Δ_- the set of negative roots. Then we have the root space decomposition

$$\mathfrak{g}=\mathfrak{h}\oplus \coprod_{\varphi\in\Delta_+} \mathfrak{g}^\varphi \oplus \coprod_{\varphi\in\Delta_-} \mathfrak{g}^\varphi.$$

It is easy to show that the center of $\mathfrak{g}$ is the subspace of $\mathfrak{h}\subset\mathfrak{h}^e$ on which all the roots of $\mathfrak{g}$ vanish (*see* [15], (*a*), Chap. II, § 1, Lemma 1).

For each $i\in\{1, \ldots, l\}$, define the linear automorphism r_i of $(\mathfrak{h}^e)^*$ by the condition $r_i\varphi=\varphi-\varphi(h_i)\alpha_i$ for all $\varphi\in(\mathfrak{h}^e)^*$. Then $r_i\alpha_i=-\alpha_i$, and r_i acts as the identity on the codimension 1 subspace consisting of all $\varphi\in(\mathfrak{h}^e)^*$ such that $\varphi(h_i)=0$. Let W (the *Weyl group*) be the group of automorphisms of $(\mathfrak{h}^e)^*$ generated by $r_1, \ldots, r_l$. Then W is a Coxeter group with generators r_i and relations which can be given in terms of the Cartan matrix A; each element of W preserves Δ, and W is naturally isomorphic to the group of linear automorphisms it induces on the span of Δ.

Define the set Δ_R of *real* roots to be the set of Weyl group transforms of $\alpha_1, \ldots, \alpha_l$, and define the set Δ_I of *imaginary* roots to be $\Delta-\Delta_R$. Then $\dim\mathfrak{g}^\varphi=1$ for all $\varphi\in\Delta_R$, but this need not be true for $\varphi\in\Delta_I$. We have $W\Delta_R=\Delta_R$, $W\Delta_I=\Delta_I$, $\Delta_R=-\Delta_R$, $\Delta_I=-\Delta_I$ and $W(\Delta_I\cap\Delta_+)=\Delta_I\cap\Delta_+$.

For all $w\in W$, define

$$\Phi_w=\Delta_+\cap w\Delta_-=\{\varphi\in\Delta_+ \mid w^{-1}\varphi\in\Delta_-\},$$

so that $\Phi_w\subset\Delta_R\cap\Delta_+$. Let $n(w)$ be the number of elements in Φ_w. Let $l(w)$ be the *length* of w, that is, the smallest nonnegative integer j such that w can be written as $r_{i_1}r_{i_2}\ldots r_{i_j}$ $(1\leq i_m\leq l)$. Then $n(w)=l(w)$ (a finite number).

Define $\rho\in(\mathfrak{h}^e)^*$ to be any fixed element satisfying the conditions $\rho(h_i)=1$ for all $i\in\{1, \ldots, l\}$. For every finite subset Φ of Δ, define $\langle\Phi\rangle\in(\mathfrak{h}^e)^*$ to be the sum of the elements of Φ. Then $\langle\Phi_w\rangle=\rho-w\rho$ for all $w\in W$, and if this is zero, then $l(w)=n(w)=0$, and so $w=1$.

Fix a subset S of $\{1, \ldots, l\}$ such that the square submatrix B of A defined in the obvious way by S is a classical Cartan matrix of finite type. Then the Lie subalgebra $\mathfrak{g}_S$ of $\mathfrak{g}=\mathfrak{g}(A)$ generated by $\{h_i, e_i, f_i\}_{i\in S}$ is isomorphic to the finite-dimensional split semisimple Lie algebra $\mathfrak{g}(B)$ whose Cartan matrix is B. Let $\mathfrak{h}_S$ be the span of $\{h_i\}_{i\in S}$; $\Delta^S=\Delta\cap\coprod_{i\in S}\mathbf{Z}\alpha_i$; $\Delta^S_+=\Delta_+\cap\Delta^S$; $\Delta^S_-=\Delta_-\cap\Delta^S$; $\Delta_+(S)=\Delta_+-\Delta^S_+$ and $\Delta_-(S)=\Delta_--\Delta^S_-$. Then

$$\mathfrak{g}_S=\mathfrak{h}_S\oplus \coprod_{\varphi\in\Delta^S_+} \mathfrak{g}^\varphi \oplus \coprod_{\varphi\in\Delta^S_-} \mathfrak{g}^\varphi.$$

Define the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{n}=\coprod_{\varphi\in\Delta_+}\mathfrak{g}^\varphi;\qquad \mathfrak{n}^-=\coprod_{\varphi\in\Delta_-}\mathfrak{g}^\varphi;\qquad \mathfrak{n}_S=\coprod_{\varphi\in\Delta^S_+}\mathfrak{g}^\varphi;$$

$$\mathfrak{n}_S^-=\coprod_{\varphi\in\Delta^S_-}\mathfrak{g}^\varphi;\qquad \mathfrak{u}=\coprod_{\varphi\in\Delta_+^{(S)}}\mathfrak{g}^\varphi;\qquad \mathfrak{u}^-=\coprod_{\varphi\in\Delta_-^{(S)}}\mathfrak{g}^\varphi;\qquad \mathfrak{r}=\mathfrak{g}_S+\mathfrak{h};$$

and $\mathfrak{p}=\mathfrak{r}\oplus\mathfrak{u}$ (a subalgebra because $[\mathfrak{r}, \mathfrak{u}]\subset\mathfrak{u}$). Then

$$\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n};\qquad \mathfrak{g}_S=\mathfrak{n}_S^-\oplus\mathfrak{h}_S\oplus\mathfrak{n}_S;\qquad \mathfrak{n}=\mathfrak{n}_S\oplus\mathfrak{u};$$
$$\mathfrak{n}^-=\mathfrak{n}_S^-\oplus\mathfrak{u}^-;\qquad \mathfrak{r}=\mathfrak{n}_S^-\oplus\mathfrak{h}\oplus\mathfrak{n}_S\quad\text{and}\quad \mathfrak{g}=\mathfrak{u}^-\oplus\mathfrak{p}.$$

Also, $\mathfrak{r}$ is a (finite-dimensional) reductive Lie algebra with commutator subalgebra $\mathfrak{g}_S$ and center a subalgebra of $\mathfrak{h}$.

Let V be an $\mathfrak{h}^e$-module (for example, a $\mathfrak{g}^e$-module regarded as an $\mathfrak{h}^e$-module by restriction), and let $\mu\in(\mathfrak{h}^e)^*$. Define the *weight space* $V_\mu\subset V$ corresponding to μ to be $\{v\in V \mid h.v=\mu(h)v \text{ for all } h\in\mathfrak{h}^e\}$. Call μ a *weight* of V if $V_\mu\neq 0$, and call the nonzero elements of V_μ *weight vectors* with weight μ.

A $\mathfrak{g}^e$-module V is called a *highest weight module* if it is generated by an $\mathfrak{n}$-invariant weight vector v. In this case, the *highest weight vector* v is uniquely determined up to nonzero scalar multiple, its weight is called the *highest weight* of V, and its weight space is the *highest weight space* of V. The highest weight space is one-dimensional, V is the direct sum of its weight spaces, which are all finite-dimensional, and the weights of V are all of the form $\mu-\sum_{i=1}^{l} n_i\alpha_i$ $(n_i\in\mathbf{Z}_+)$, where $\mu\in(\mathfrak{h}^e)^*$ is the highest weight.

For every $\mathfrak{d}$-invariant subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, denote by $\mathfrak{t}^e$ the subalgebra $\mathfrak{d}\oplus\mathfrak{t}$ of $\mathfrak{g}^e$.

There is a natural bijection, denoted $\lambda\mapsto M(\lambda)$, between the set P_S of all $\lambda\in(\mathfrak{h}^e)^*$ such that $\lambda(h_i)\in\mathbf{Z}_+$ for all $i\in S$, and the set of (isomorphism classes of) finite-dimensional irreducible $\mathfrak{r}^e$-modules which are irreducible under $\mathfrak{g}_S$. The highest weight space (relative to $\mathfrak{h}_S$ and $\mathfrak{n}_S$) of the $\mathfrak{g}_S$-module $M(\lambda)$ is $\mathfrak{h}^e$-stable, and λ is the resulting weight.

For all $\lambda\in P_S$, we define the *generalized Verma module* $V^{M(\lambda)}$ to be the $\mathfrak{g}^e$-module induced by the irreducible $\mathfrak{p}^e$-module which is $M(\lambda)$ as an $\mathfrak{r}^e$-module and which is annihilated by $\mathfrak{u}$. Let $\mathcal{G}^e$ and $\mathcal{P}^e$ (regarded as a subalgebra of $\mathcal{G}^e$) be the universal enveloping algebras of $\mathfrak{g}^e$ and $\mathfrak{p}^e$, respectively. Then $V^{M(\lambda)}=\mathcal{G}^e\otimes_{\mathcal{P}^e} M(\lambda)$. $V^{M(\lambda)}$ is a highest weight module with highest weight λ. The highest weight space of $V^{M(\lambda)}$ coincides, under the natural identification of $M(\lambda)$ with the $\mathfrak{p}^e$-submodule $1\otimes M(\lambda)$ of $V^{M(\lambda)}$, with the highest weight space (relative to $\mathfrak{h}_S$ and $\mathfrak{n}_S$) of the $\mathfrak{g}_S$-module $M(\lambda)$.

Let $\Psi=(\lambda_1, \ldots, \lambda_n)$ be a finite sequence of elements of P_S. A $\mathfrak{g}^e$-module V is said to be *of type* Ψ if (1) V has a $\mathfrak{g}^e$-module filtration $0=V_0\subset V_1\subset\ldots\subset V_n=V$ such that the sequence of $\mathfrak{g}^e$-modules $V_1/V_0, V_2/V_1, \ldots, V_n/V_{n-1}$ coincides up to rearrangement with the sequence of generalized Verma modules $V^{M(\lambda_1)}, \ldots, V^{M(\lambda_n)}$, and (2) V is finitely semisimple as an $\mathfrak{r}^e$-module. (Condition (2) was not assumed in the corresponding definition in paragraph 7 of [11], (*a*).)

Define W_S^1 to be the subset of the Weyl group W consisting of those $w\in W$ such that $\Phi_w\subset\Delta_+(S)$.

A $\mathfrak{g}^e$-module R is called *standard* (as in [7]; *quasisimple* in [15], (*c*) and [11], (*a*)) if R is a highest weight module with a highest weight vector x such that there exists $n\in\mathbf{Z}_+$ with $f_i^n.x=0$ for all $i\in\{1, \ldots, l\}$. The trivial one-dimensional module is standard; its highest weight is 0. Let P be the set of *dominant integral* linear forms, i. e., the set of all $\lambda\in(\mathfrak{h}^e)^*$ such that $\lambda(h_i) \in \mathbf{Z}_+$ for all $i \in \{1, \ldots, l\}$. Then the highest weight of a standard $\mathfrak{g}^e$-module lies in P, and for all $\mu \in$ P, there exists a standard $\mathfrak{g}^e$-module with highest weight μ. If A is a classical Cartan matrix of finite type and $\mathfrak{d}=0$ (*see* above), then the standard $\mathfrak{g}^e$-modules are just the finite-dimensional irreducible $\mathfrak{g}$-modules.

Theorem 8.7 of [11], (*a*) gives the following resolution of a standard module in terms of generalized Verma modules:

THEOREM 5.1. — *Let* R *be a standard* $\mathfrak{g}^e$*-module with highest weight* $\mu \in$ P. *For all* $j \in \mathbf{Z}_+$, *let* Ψ_μ^j *be the (finite) family of (distinct) elements of* $(\mathfrak{h}^e)^*$ $\{w(\mu+\rho)-\rho\}$ *as* w *ranges through the set of elements of* W_S^1 *of length* j; *each element of* Ψ_μ^j *lies in* P_S. *Then there is an exact sequence of* $\mathfrak{g}^e$*-modules*

$$\ldots \to E_1 \to E_0 \to R \to 0$$

where E_j *is of type* Ψ_μ^j *for each* j.

Remark. — Note that this result includes the information that each E_j is finitely semisimple under $\mathfrak{r}^e$.

Remark. — Theorem 5.1 implies Kac's theorem that for $\mu \in$ P, there is exactly one (up to equivalence) standard $\mathfrak{g}^e$-module with highest weight μ, and it is irreducible; *see* [11], (*a*), § 9. Thus P bijectively indexes the set of equivalence classes of standard $\mathfrak{g}^e$-modules.

In order to place ourselves in the context of paragraphs 3 and 4, we note the following:

PROPOSITION 5.2. — *Under the adjoint action of* $\mathfrak{r}^e$, $\mathfrak{g}^e$ *is finitely semisimple. In particular,* $\mathfrak{r}^e$ *is a reductive Lie algebra. The standard* $\mathfrak{g}^e$*-module* R *is finitely semisimple as an* $\mathfrak{r}^e$*-module.*

Proof. — For each $\nu \in \Delta \cup \{0\}$, let $\Delta(\nu)$ be the set of all elements of $\Delta \cup \{0\}$ which can be written in the form $\nu + \sum_{i \in S} n_i \alpha_i$ with $n_i \in \mathbf{Z}$. It is clear that there exist $\nu_1, \nu_2, \ldots \in \Delta \cup \{0\}$ such that $\mathfrak{g}^e$ is the direct sum of the spaces $\coprod_{\varphi \in \Delta(\nu_1)} \mathfrak{g}^\varphi$, $\coprod_{\varphi \in \Delta(\nu_2)} \mathfrak{g}^\varphi$, ..., and that each of these spaces is an $\mathfrak{r}^e$-submodule of $\mathfrak{g}^e$. Proposition 5.1 and Lemma 5.2 of [11], (*a*) imply that each such space is finitely semisimple under $\mathfrak{r}^e$, and so the first two assertions of the proposition are proved. The last assertion follows from the case $j=0$ of Proposition 6.3 of [11], (*a*).

Q.E.D.

Now let $\mathfrak{b}$ be a subalgebra of $\mathfrak{g}^e$ such that $\mathfrak{g}^e = \mathfrak{b} + \mathfrak{p}^e$, and suppose that the subalgebra $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{p}^e$ is a subalgebra of $\mathfrak{r}^e$ which is reductive in $\mathfrak{r}^e$. (One example of this is the case $\mathfrak{b} = \mathfrak{r}^e \oplus \mathfrak{u}^-$, $\mathfrak{a} = \mathfrak{r}^e$.) Then the action of $\mathfrak{a}$ on any finitely semisimple $\mathfrak{r}^e$-module is finitely semisimple (*see* for example [8], Prop. 1.7.9 (ii)), and so in particular, $\mathfrak{a}$ acts finitely semisimply on $\mathfrak{g}^e$ and hence on $\mathfrak{b}$, and also on R, by the last proposition. Thus in the notation of paragraph 3, the category $C(\mathfrak{b}, \mathfrak{a})$ is defined, and R and the modules E_j in Theorem 5.1 lie in this category.

PROPOSITION 5.3. — *The complex in* $C(\mathfrak{b}, \mathfrak{a})$:

$$\ldots \to E_1 \to E_0 \to 0$$

obtained from the complex in Theorem 5.1 *by omitting the segment* $\to$ R *is a* $(\mathfrak{b}, \mathfrak{a})$*-free* *resolution of* R. *For each* $j \in \mathbf{Z}_+$, E_j *is isomorphic to the* $(\mathfrak{b}, \mathfrak{a})$-*free module* (cf. Prop. 3.4) $\mathscr{B} \otimes_{\mathscr{A}} \coprod M(w(\mu+\rho)-\rho)$, *where* w *ranges through the set of elements of* W_S^1 *of length* j.

Proof. — It is sufficient to prove the last statement. We know that E_j has a $\mathfrak{g}^e$-module filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_n = E_j$ such that the sequence of $\mathfrak{g}^e$-modules V_1/V_0, $V_2/V_1, \ldots, V_n/V_{n-1}$ coincides with the sequence $V^{M(\lambda_1)}, \ldots, V^{M(\lambda_n)}$, where $\lambda_1, \ldots, \lambda_n$ are the elements $w(\mu+\rho)-\rho$ indicated in the statement of the proposition. But for each $i = 1, \ldots, n$, there is a natural $\mathfrak{b}$-module isomorphism $V^{M(\lambda_i)} \simeq \mathscr{B} \otimes_{\mathscr{A}} M(\lambda_i)$, where on the right-hand side, $M(\lambda_i)$ is regarded as an $\mathfrak{a}$-module (*see* [8], Prop. 5.1.14). By Propositions 3.4 and 3.5, each $V^{M(\lambda_i)}$ is thus $(\mathfrak{b}, \mathfrak{a})$-projective. Repeated application of Proposition 3.9 now proves the proposition.

Q.E.D.

Remark. — This argument is essentially the same as the one used to prove Lemma 7.8 of [11], (*a*).

COROLLARY 5.4. — *Retain the notation of Theorem* 5.1, *and let* $T \in C(\mathfrak{b}, \mathfrak{a})$. *The relative homology* $H_*(\mathfrak{b}, \mathfrak{a}, (R \otimes T)^t)$ *of* $\mathfrak{b}$ *with respect to* $\mathfrak{a}$ *in the right* $\mathfrak{b}$-*module* $(R \otimes T)^t$ (see § 2) *is naturally isomorphic to the homology of a complex of vector spaces of the form*

$$\ldots \to V_1 \to V_0 \to 0,$$

where for each $j \in \mathbf{Z}_+$, V_j *is the space of* $\mathfrak{a}$-*invariants in the tensor product* $\mathfrak{a}$-*module* $T \otimes \coprod M(w(\mu+\rho)-\rho)$, *as* w *ranges through the set of elements of* W_S^1 *of length* j.

Proof. — By Corollary 4.4, $H_*(\mathfrak{b}, \mathfrak{a}, (R \otimes T)^t)$ is naturally isomorphic to $\mathrm{Tor}_*^{(\mathfrak{b}, \mathfrak{a})}(T^t, R)$, and by Proposition 5.3, this is naturally isomorphic to the homology of a complex of the form $\ldots \to V_1 \to V_0 \to 0$, where for each $j \in \mathbf{Z}_+$,

$$V_j = T^t \otimes_{\mathscr{B}} \mathscr{B} \otimes_{\mathscr{A}} \coprod M(w(\mu+\rho)-\rho),$$

where w ranges through the set of elements of W_S^1 of length j. Clearly, there are natural isomorphisms

$$V_j \simeq T^t \otimes_{\mathscr{A}} \coprod M(w(\mu+\rho)-\rho) \simeq k \otimes_{\mathscr{A}} (T \otimes \coprod M(w(\mu+\rho)-\rho)),$$

where $T \otimes \coprod M(w(\mu+\rho)-\rho)$ is the tensor product of $\mathfrak{a}$-modules. Since this tensor product is a finitely semisimple $\mathfrak{a}$-module, V_j is naturally isomorphic to the space of $\mathfrak{a}$-invariants in $T \otimes \coprod M(w(\mu+\rho)-\rho)$.

Q.E.D.

The relative homological algebra in paragraph 4 enables us to deduce easily the main homology result (Th. 8.6) in [11], (*a*) from the resolution (Th. 5.1 above):

THEOREM 5.5. — *In the notation of Theorem* 5.1, $H_j(\mathfrak{u}^-, R^t)$ *with the standard action of* $\mathfrak{r}^e$ *is naturally isomorphic to* $\coprod M(w(\mu+\rho)-\rho)$, *where* w *ranges through the set of elements of* W_S^1 *of length* j. *In particular* (see [11], (*a*), Prop. 1.6), $H^j(\mathfrak{u}^-, R^*)$ *with the standard action of* $\mathfrak{r}^e$ *is naturally isomorphic to* $\coprod M(w(\mu+\rho)-\rho)^*$ (*same* w). *Let* $C_j(R)$ *be the* $\mathfrak{r}^e$-*module*

$\Lambda^j(\mathfrak{u}^-)\otimes \mathrm{R}$, *so that the standard* $\mathfrak{r}^e$*-module complex for computing the standard action of* $\mathfrak{r}^e$ *on* $\mathrm{H}_*(\mathfrak{u}^-, \mathrm{R}^t)$ *has the form*

$$\ldots \xrightarrow{\partial_2} \mathrm{C}_1(\mathrm{R}) \xrightarrow{\partial_1} \mathrm{C}_0(\mathrm{R}) \to 0.$$

Let $\mathrm{B}_j(\mathrm{R})$ *be the unique* $\mathfrak{r}^e$*-submodule of* $\mathrm{C}_j(\mathrm{R})$ *isomorphic to* $\coprod \mathrm{M}(w(\mu+\rho)-\rho)$, *with* w *as above, and let* $\mathrm{B}'_j(\mathrm{R})$ *be the unique* $\mathfrak{r}^e$*-module complement of* $\mathrm{B}_j(\mathrm{R})$ *in* $\mathrm{C}_j(\mathrm{R})$, i. e., *the sum of all irreducible* $\mathfrak{r}^e$*-submodules* $\mathrm{M}(\lambda)$ *of* $\mathrm{C}_j(\mathrm{R})$ *where* $\lambda\in \mathrm{P}_\mathrm{S}$ *is not of the form* $w(\mu+\rho)-\rho$ *for any* $w\in \mathrm{W}^1_\mathrm{S}$ *of length* j. *Then the* $\mathrm{B}_j(\mathrm{R})$ *form a subcomplex* $\mathrm{B}_*(\mathrm{R})$ *of* $\mathrm{C}_*(\mathrm{R})$ *all of whose maps are zero, and the* $\mathrm{B}'_j(\mathrm{R})$ *form a subcomplex* $\mathrm{B}'_*(\mathrm{R})$ *of* $\mathrm{C}_*(\mathrm{R})$ *whose homology is zero.*

Proof. — Apply Proposition 5.3 to the pair $(\mathfrak{b}, \mathfrak{a})=(\mathfrak{r}^e\oplus\mathfrak{u}^-, \mathfrak{r}^e)$ and take $\mathfrak{c}=\mathfrak{u}^-$ and $\mathfrak{s}=\mathfrak{r}^e$ in the first assertion of Corollary 4.8. The fact that $\mathrm{H}_j(\mathfrak{u}^-, \mathrm{R}^t)$ is naturally isomorphic to $\coprod \mathrm{M}(w(\mu+\rho)-\rho)$ now follows since the irreducible $\mathfrak{r}^e$-modules $\mathrm{M}(w(\mu+\rho)-\rho)$ are inequivalent as w ranges through W^1_S (last assertion of [11], (*a*), Th. 8.5). Because each $\mathfrak{r}^e$-module $\mathrm{M}(w(\mu+\rho)-\rho)$ ($w\in \mathrm{W}^1_\mathrm{S}$ of length j) occurs with multiplicity one in $\mathrm{C}_j(\mathrm{R})$, which is a sum of $\mathfrak{r}^e$-modules of the form $\mathrm{M}(\lambda)$, $\lambda\in\mathrm{P}_\mathrm{S}$ (*see* [11], (*a*), Th. 8.5), we now see that ∂_j must map $\mathrm{B}_j(\mathrm{R})$ to zero and $\mathrm{B}'_{j+1}(\mathrm{R})$ into $\mathrm{B}'_j(\mathrm{R})$. The rest is clear.

Q.E.D.

It will be convenient to introduce the analogues of the standard $\mathfrak{g}^e$-modules with the roles of the positive and negative roots reversed. We define a *lowest weight vector* in a $\mathfrak{g}^e$-module to be an $\mathfrak{n}^-$-invariant weight vector, and a *lowest weight module* to be a $\mathfrak{g}^e$-module generated by a lowest weight vector. The corresponding *lowest weight* of the module is uniquely determined, and lowest weight modules have obvious properties analogous to those of highest weight modules. In fact, statements about highest weights, highest weight modules, etc., imply the corresponding statements about lowest weights, lowest weight modules, etc., by application of the involution η of $\mathfrak{g}$ defined above; η may be extended to $\mathfrak{g}^e$ by defining it to be -1 on $\mathfrak{d}$.

We shall say that a $\mathfrak{g}^e$-module R is Δ_--*standard* if R is a lowest weight module with a lowest weight vector x such that there exists $n\in\mathbf{Z}_+$ with $e_i^n.x=0$ for all $i\in\{1, \ldots, l\}$. Then every Δ_--standard module is irreducible, the lowest weight of a Δ_--standard module lies in the set $-\mathrm{P}$, and for all $\mu\in -\mathrm{P}$, there is a unique such module with lowest weight μ. In this way, $-\mathrm{P}$ bijectively indexes the set of equivalence classes of Δ_--standard $\mathfrak{g}^e$-modules.

Let $\mathfrak{p}_-$ be the subalgebra $\mathfrak{r}\oplus\mathfrak{u}^-$ of $\mathfrak{g}$. For all $\lambda\in\mathrm{P}_\mathrm{S}$, we define the $\mathfrak{g}^e$-module $\mathrm{V}_{\Delta_-}^{\mathrm{M}(\lambda)^*}$ to be the $\mathfrak{g}^e$-module induced by the irreducible $(\mathfrak{p}^-)^e$-module which as an $\mathfrak{r}^e$-module is the contragredient $\mathfrak{r}^e$-module $\mathrm{M}(\lambda)^*$ and which is annihilated by $\mathfrak{u}^-$. Let $\Psi=(\lambda_1, \ldots, \lambda_n)$ be a finite sequence of elements of P_S. A $\mathfrak{g}^e$-module V is said to be *of* Δ_-*-type* Ψ if (1) V has a $\mathfrak{g}^e$-module filtration $0=\mathrm{V}_0\subset\mathrm{V}_1\subset\ldots\subset\mathrm{V}_n=\mathrm{V}$ such that the sequence of $\mathfrak{g}^e$-modules $\mathrm{V}_1/\mathrm{V}_0$, $\mathrm{V}_2/\mathrm{V}_1$, $\ldots$, $\mathrm{V}_n/\mathrm{V}_{n-1}$ coincides up to rearrangement with the sequence $\mathrm{V}_{\Delta_-}^{\mathrm{M}(\lambda_1)^*}, \ldots, \mathrm{V}_{\Delta_-}^{\mathrm{M}(\lambda_n)^*}$, and (2) V is finitely semisimple as an $\mathfrak{r}^e$-module.

From Theorem 5.1, we get:

THEOREM 5.6. — *Let* R *be a* Δ_-*-standard* $\mathfrak{g}^e$*-module with lowest weight* $\mu\in -\mathrm{P}$. *For all* $j\in\mathbf{Z}_+$, let $\Psi^j_{-\mu}$ *be the (finite) family of (distinct) elements of* $(\mathfrak{h}^e)^*$ $\{w(-\mu+\rho)-\rho\}$ *as* w *ranges through the set of elements of* W^1_S *of length* j; *each element of* $\Psi^j_{-\mu}$ *lies in* P_S. *Then there is an* *exact sequence of* $\mathfrak{g}^e$*-modules*

$$\ldots \to E_1 \to E_0 \to R \to 0$$

where E_j *is of* Δ_-*-type* $\Psi^j_{-\mu}$ *for each* j.

Theorem 5.5 [or Proposition 5.3 applied to the pair $(\mathfrak{b}, \mathfrak{a})=(\mathfrak{r}^e \oplus \mathfrak{u}, \mathfrak{r}^e)$ together with Corollary 4.8 applied to $\mathfrak{c}=\mathfrak{u}$ and $\mathfrak{s}=\mathfrak{r}^e$, as in the proof of Theorem 5.5] yields:

THEOREM 5.7. — *In the notation of Theorem* 5.6, $H_j(\mathfrak{u}, R^t)$ *with the standard action of* $\mathfrak{r}^e$ *is naturally isomorphic to* $\coprod M(w(-\mu+\rho)-\rho)^*$, *where* w *ranges through the set of elements of* W^1_S *of length* j. *In particular* (see [11], (*a*), Prop. 1.6), *the cohomology* $H^j(\mathfrak{u}, R^*)$ *with the standard* $\mathfrak{r}^e$*-action is naturally isomorphic to* $\coprod M(w(-\mu+\rho)-\rho)$ (*same* w).

6. The relative homology $H_*(\mathfrak{g}^e, \mathfrak{r}^e)$ and related relative homologies and cohomologies

We shall continue to use the notation of paragraph 5. Our principal aim is to implement Corollary 5.4 in a certain special situation.

Let A' be the $2l \times 2l$ symmetrizable Cartan matrix diag (A, A). Recall from paragraph 5 the involution η of $\mathfrak{g}(=\mathfrak{g}(A))$ which acts on the canonical generators h_i, e_i, f_i by interchanging e_i and f_i and taking h_i to $-h_i$ for all $i \in \{1, \ldots, l\}$. The direct product Lie algebra $\mathfrak{g}'=\mathfrak{g}\times\mathfrak{g}$ may be identified with the Lie algebra $\mathfrak{g}(A')$, where the $6l$ canonical generators are

$$(h_1, 0), \ldots, (h_l, 0), (0, \eta(h_1)), \ldots, (0, \eta(h_l));$$

$$(e_1, 0), \ldots, (e_l, 0), (0, \eta(e_1)), \ldots, (0, \eta(e_l));$$

$$(f_1, 0), \ldots, (f_l, 0), (0, \eta(f_1)), \ldots, (0, \eta(f_l)).$$

Let $\mathfrak{d}'_0$ be the direct product Lie algebra $\mathfrak{d}_0 \times \mathfrak{d}_0$, where as in paragraph 5, $\mathfrak{d}_0$ is the l-dimensional abelian Lie algebra spanned by the l degree derivations of $\mathfrak{g}$. Then $\mathfrak{d}'_0$ may be naturally identified with the Lie algebra spanned by the $2l$ degree derivations of $\mathfrak{g}'$. Recall that $\mathfrak{d}$ is an admissible subspace of $\mathfrak{d}_0$ (*see* § 5). Then $\mathfrak{d}'=\mathfrak{d}\times\mathfrak{d}$ is clearly an admissible subspace of $\mathfrak{d}'_0$. Let $(\mathfrak{g}')^e$ be the natural semidirect product Lie algebra $\mathfrak{d}'\times\mathfrak{g}'$. Then $(\mathfrak{g}')^e$ is naturally isomorphic to the direct product $\mathfrak{g}^e \times \mathfrak{g}^e$. Set $\mathfrak{h}'=\mathfrak{h}\times\mathfrak{h}\subset\mathfrak{g}'$, and

$$(\mathfrak{h}')^e=\mathfrak{d}'\oplus\mathfrak{h}'\simeq\mathfrak{h}^e\times\mathfrak{h}^e\subset(\mathfrak{g}')^e.$$

Identify $((\mathfrak{h}')^e)^*$ with $(\mathfrak{h}^e)^*\oplus(\mathfrak{h}^e)^*$ in the obvious way. Then the analogues for $(\mathfrak{g}')^e$ of $\alpha_1, \ldots, \alpha_l$ are the $2l$ linear functionals $(\alpha_1, 0), \ldots, (\alpha_l, 0), (0, -\alpha_1), \ldots, (0, -\alpha_l)$ in $((\mathfrak{h}')^e)^*$.

Let $\Delta'\subset((\mathfrak{h}')^e)^*$ be the set of roots of $\mathfrak{g}'$, Δ'_+ the set of positive roots and Δ'_- the set of negative roots. Then

$$\Delta'=(\Delta, 0)\cup(0, \Delta), \qquad \Delta'_+=(\Delta_+, 0)\cup(0, \Delta_-) \qquad \text{and} \qquad \Delta'_-=(\Delta_-, 0)\cup(0, \Delta_+).$$

The Weyl group W' of $\mathfrak{g}'$ is naturally isomorphic to the direct product group $W\times W$ acting in the obvious way on $((\mathfrak{h}')^e)^*=(\mathfrak{h}^e)^*\oplus(\mathfrak{h}^e)^*$. We may write the elements of W' as the pairs

(w_1, w_2), where each $w_i \in W$. The length of this element is $l(w_1)+l(w_2)$. For all $w'=(w_1, w_2)\in W'$, define $\Phi'_{w'}$ to be $\Delta'_+ \cap w' \Delta'_-$. This set is just

$$(\Delta_+ \cap w_1 \Delta_-, 0) \cup (0, -(\Delta_+ \cap w_2 \Delta_-)) = (\Phi_{w_1}, 0) \cup (0, -\Phi_{w_2}).$$

Define $\rho' \in ((\mathfrak{h}')^e)^*$ to be the functional $(\rho, -\rho)$, with $\rho \in (\mathfrak{h}^e)^*$ as in paragraph 5. Then ρ' takes the value 1 on the $2l$ canonical generators $(h_1, 0), \ldots, (h_l, 0), (0, -h_1), \ldots, (0, -h_l)$ of $\mathfrak{g}'$. For each $(w_1, w_2)\in W'$ $(w_i \in W)$, $\rho' - (w_1, w_2)\rho' = (\rho - w_1 \rho, -(\rho - w_2 \rho))$.

Let S′ be the subset $S \cup (S+l)$ of $\{1, \ldots, 2l\}$, and let B′ be the square submatrix of the Cartan matrix A′ defined in the obvious way by S′. Then the Lie subalgebra $\mathfrak{g}'_{S'}$ of $\mathfrak{g}'$ generated by

$$\{(h_i, 0), (e_i, 0), (f_i, 0)\}_{i\in S} \cup \{(0, -h_i), (0, f_i), (0, e_i)\}_{i\in S}$$

is isomorphic to the finite-dimensional split semisimple Lie algebra whose Cartan matrix is B′, and this algebra is just $\mathfrak{g}_S \times \mathfrak{g}_S \subset \mathfrak{g}'$. The analogues for $(\mathfrak{g}')^e$ and S′ of the subalgebras $\mathfrak{h}_S, \mathfrak{n}, \mathfrak{n}^-, \mathfrak{n}_S, \mathfrak{n}_S^-, \mathfrak{u}, \mathfrak{u}^-, \mathfrak{r}, \mathfrak{p}, \mathfrak{r}^e$ and $\mathfrak{p}^e$ associated with $\mathfrak{g}^e$ and S are, respectively,

$$\mathfrak{h}'_{S'} = \mathfrak{h}_S \times \mathfrak{h}_S; \quad \mathfrak{n}' = \mathfrak{n} \times \mathfrak{n}^-; \quad (\mathfrak{n}')^- = \mathfrak{n}^- \times \mathfrak{n}; \quad \mathfrak{n}'_{S'} = \mathfrak{n}_S \times \mathfrak{n}_S^-; \quad (\mathfrak{n}'_{S'})^- = \mathfrak{n}_S^- \times \mathfrak{n}_S;$$
$$\mathfrak{u}' = \mathfrak{u} \times \mathfrak{u}^-; \quad (\mathfrak{u}')^- = \mathfrak{u}^- \times \mathfrak{u}; \quad \mathfrak{r}' = \mathfrak{r} \times \mathfrak{r}; \quad \mathfrak{p}' = \mathfrak{p} \times \mathfrak{p}^-$$

(where $\mathfrak{p}^- = \mathfrak{r} \oplus \mathfrak{u}^-$, as in § 5);

$$(\mathfrak{r}')^e = \mathfrak{r}^e \times \mathfrak{r}^e \quad \text{and} \quad (\mathfrak{p}')^e = \mathfrak{p}^e \times (\mathfrak{p}^-)^e.$$

The analogue of P_S is

$$P'_{S'} = \{(\lambda, \mu) \in ((\mathfrak{h}')^e)^* \mid \lambda \in P_S, \mu \in -P_S\}.$$

For all $(\lambda, \mu) \in P'_{S'}$, the associated $\mathfrak{g}'_{S'}$-irreducible $(\mathfrak{r}')^e$-module $M'(\lambda, \mu)$ is naturally isomorphic to the $\mathfrak{r}^e \times \mathfrak{r}^e$-irreducible module $M(\lambda) \otimes M(-\mu)^*$; here $M(\lambda)$ (*see* § 5) is regarded as a module for the first factor $\mathfrak{r}^e$, and $M(-\mu)^*$ is the contragredient of the module $M(-\mu)$ for the second factor $\mathfrak{r}^e$. [To see this, note that if $x \in M(\lambda)$ is a highest weight vector and $y \in M(-\mu)^*$ is a lowest weight vector relative to $\mathfrak{h}_S$ and $\mathfrak{n}_S$, then $x \otimes y$ is a highest weight vector relative to $\mathfrak{h}'_{S'}$ and $\mathfrak{n}'_{S'}$, for the irreducible action of $\mathfrak{g}'_{S'}$ on $M(\lambda) \otimes M(-\mu)^*$; moreover, $x \otimes y$ is a weight vector for the action of $(\mathfrak{h}')^e$, and its weight is (λ, μ).] The analogue, in the present situation, of W_S^1 is $(W')^1_{S'} = W_S^1 \times W_S^1$.

The analogue of P is

$$P' = \{(\lambda, \mu) \in ((h')^e)^* \mid \lambda \in P, \mu \in -P\}.$$

We may construct the unique (up to equivalence) standard (irreducible) $(\mathfrak{g}')^e$-module R with highest weight $(\lambda, \mu) \in P'$ as follows: Let R^1 be the standard module with highest weight λ for the first factor $\mathfrak{g}^e$ in $(\mathfrak{g}')^e = \mathfrak{g}^e \times \mathfrak{g}^e$. Let R^2 be the Δ_--standard module with lowest weight μ for the second factor $\mathfrak{g}^e$ in $(\mathfrak{g}')^e$ (*see* § 5). Then $R^1 \otimes R^2$, with the obvious action of $(\mathfrak{g}')^e$, is by definition a standard $(\mathfrak{g}')^e$-module with highest weight (λ, μ), so that $R = R^1 \otimes R^2$. Thus Proposition 4.13 immediately implies:

PROPOSITION 6.1. — *Let* $(\lambda, \mu) \in \mathrm{P}'$, *and let* R, R^1 *and* R^2 *be as above. For all* $n \in \mathbf{Z}_+$, *there is a natural* $(\mathfrak{r}')^e$-*module isomorphism*

$$\mathrm{H}_n((u')^-, \mathrm{R}^t) \simeq \coprod_{r+s=n} \mathrm{H}_r(\mathfrak{u}^-, (\mathrm{R}^1)^t) \otimes \mathrm{H}_s(\mathfrak{u}, (\mathrm{R}^2)^t),$$

where $(\mathfrak{r}')^e$ *acts on the left by the standard action, and on the right by the tensor product of the standard actions of the two factors* $\mathfrak{r}^e$ *of* $(\mathfrak{r}')^e$. *In particular, there is a natural* $(\mathfrak{r}')^e$-*module isomorphism*

$$\mathrm{H}_n((\mathfrak{u}')^-) \simeq \coprod_{r+s=n} \mathrm{H}_r(\mathfrak{u}^-) \otimes \mathrm{H}_s(\mathfrak{u}).$$

Combining this with Theorems 5.5 and 5.7, we have:

COROLLARY 6.2. — *In the same notation, there are natural* $(\mathfrak{r}')^e$-*module isomorphisms*

$$\mathrm{H}_n((\mathfrak{u}')^-, \mathrm{R}^t) \simeq \coprod \mathrm{M}(w_1(\lambda+\rho)-\rho) \otimes \mathrm{M}(w_2(-\mu+\rho)-\rho)^*$$

and

$$\mathrm{H}_n((\mathfrak{u}')^-) \simeq \coprod \mathrm{M}(w_1\rho-\rho) \otimes \mathrm{M}(w_2\rho-\rho)^*,$$

where the direct sums are over those $w_1, w_2 \in \mathrm{W}_{\mathrm{S}}^1$ *such that* $l(w_1)+l(w_2)=n$.

Remark. — Corollary 6.2 may be proved without referring to Proposition 4.13 by applying the first assertion of Theorem 5.5 to $(\mathfrak{g}')^e$, $(\mathfrak{u}')^-$ and R; we get a natural $(\mathfrak{r}')^e$-module isomorphism

$$\mathrm{H}_n((\mathfrak{u}')^-, \mathrm{R}^t) \simeq \coprod \mathrm{M}'(w_1(\lambda+\rho)-\rho,\ w_2(\mu-\rho)+\rho)$$

where (w_1, w_2) ranges through the set of elements of $(\mathrm{W}')_{\mathrm{S}'}^1 = \mathrm{W}_{\mathrm{S}}^1 \times \mathrm{W}_{\mathrm{S}}^1$ of length n, proving Corollary 6.2. Proposition 6.1 then follows from Corollary 6.2 by means of Thorems 5.5 and 5.7.

Let $\mathfrak{a}$ be the diagonal subalgebra $\{(x, x) \mid x \in \mathfrak{r}^e\}$ of $(\mathfrak{r}')^e = \mathfrak{r}^e \times \mathfrak{r}^e$. Then $\mathfrak{a}$ is reductive in $(\mathfrak{r}')^e$, since in fact the adjoint action of $\mathfrak{a}$ on $(\mathfrak{r}')^e$ is the direct sum of two copies of the adjoint action of the reductive Lie algebra $\mathfrak{r}^e$ on itself. Hence the action of $\mathfrak{a}$ on any finitely semisimple $(\mathfrak{r}')^e$-module is finitely semisimple.

LEMMA 6.3. — *Let* $\lambda \in \mathrm{P}$, $\mu \in -\mathrm{P}$, $w_1, w_2 \in \mathrm{W}_{\mathrm{S}}^1$. *Denoting by superscript the space of invariants, we have*

$$(\mathrm{M}(w_1(\lambda+\rho)-\rho) \otimes \mathrm{M}(w_2(-\mu+\rho)-\rho)^*)^{\mathfrak{a}} = 0$$

unless $\lambda = -\mu$ *and* $w_1 = w_2$, *in which case there is a natural isomorphism*

$$(\mathrm{M}(w_1(\lambda+\rho)-\rho) \otimes \mathrm{M}(w_1(\lambda+\rho)-\rho)^*)^{\mathfrak{a}} \simeq \mathrm{End}_{\mathfrak{r}^e}\mathrm{M}(w_1(\lambda+\rho)-\rho),$$

a one-dimensional space with a distinguised basis (*the identity operator*).

Proof. — Since $\mathrm{M}(w_1(\lambda+\rho)-\rho)$ and $\mathrm{M}(w_2(-\mu+\rho)-\rho)$ are absolutely irreducible $\mathfrak{r}^e$-modules, it is enough to show that the condition $w_1(\lambda+\rho) = w_2(-\mu+\rho)$ implies that

$\lambda=-\mu$ and $w_1=w_2$. For ν, $\xi\in(\mathfrak{h}^e)^*$, we shall say that $\nu\leqq\xi$ if $\xi-\nu$ is of the form $\sum_{i=1}^{l} n_i\alpha_i$ for some $n_i\in\mathbf{Z}_+$. Let Y be the standard $\mathfrak{g}^e$-module with highest weight λ By [11], (*a*), Prop. 6.1, $w_2^{-1}w_1\lambda$ is a weight of Y, and so $w_2^{-1}w_1\lambda\leqq\lambda$. Also,

$$w_2^{-1}w_1\rho-\rho=-\langle\Phi_{w_2^{-1}w_1}\rangle\leqq 0,$$

and this element is zero only if $w_1=w_2$ (*see* § 5). Thus

$$-\mu=w_2^{-1}w_1(\lambda+\rho)-\rho\leqq\lambda,$$

and equality would imply that $w_1=w_2$. The same argument with the roles of λ and $-\mu$ reversed shows that $\lambda\leqq-\mu$. Thus equality does hold, and we have $w_1=w_2$ and $\lambda=-\mu$.

Q.E.D.

Remark. — This proof demonstrates (and follows from) the known facts that two W-equivalent dominant integral elements coincide, and that if $\xi\in(\mathfrak{h}^e)^*$ is regular dominant integral (i.e., $\xi(h_i)\in\mathbf{Z}_+-\{0\}$ for all $i\in\{1,\ldots,l\}$), then the only Weyl group element fixing ξ is the identity.

Corollary 6.2 and Lemma 6.3 immediately imply:

COROLLARY 6.4. — *Let $\mathfrak{a}$ be the diagonal subalgebra of $(\mathfrak{r}')^e$. In the notation of Corollary 6.2, suppose that $\lambda\neq-\mu$. Then*

$$H_*((\mathfrak{u}')^-,R^{\mathfrak{r}})^{\mathfrak{a}}=0.$$

Suppose that $\lambda=-\mu$. Then

$$H_n((\mathfrak{u}')^-,R^{\mathfrak{r}})^{\mathfrak{a}}=0$$

if $n\in\mathbf{Z}_+$ is odd; and if $n=2j$ $(j\in\mathbf{Z}_+)$, then there are natural vector space isomorphisms

$$H_n((\mathfrak{u}')^-,R^{\mathfrak{r}})^{\mathfrak{a}}\simeq\coprod\mathrm{End}_{\mathfrak{r}^e}M(w(\lambda+\rho)-\rho)\simeq\mathrm{End}_{\mathfrak{r}^e}\coprod M(w(\lambda+\rho)-\rho),$$

where both direct sums range over the elements $w\in W_S^1$ of length j. In particular,

$$H_n((\mathfrak{u}')^-)^{\mathfrak{a}}=0$$

if n is odd; and if $n=2j$, then there are natural vector space isomorphisms

$$H_n((\mathfrak{u}')^-)^{\mathfrak{a}}\simeq\coprod\mathrm{End}_{\mathfrak{r}^e}M(w\rho-\rho)\simeq\mathrm{End}_{\mathfrak{r}^e}\coprod M(w\rho-\rho),$$

where the direct sums range over the same w as before.

Remark. — Note that $H_{2j}((\mathfrak{u}')^-,R^{\mathfrak{r}})^{\mathfrak{a}}$ and $H_{2j}((\mathfrak{u}')^-)^{\mathfrak{a}}$ have distinguished bases.

We now have:

THEOREM 6.5. — *Let $\mathfrak{a}$ be the diagonal subalgebra of $(\mathfrak{r}')^e$, let $\lambda\in P$ and $\mu\in-P$, and let R be the standard $(\mathfrak{g}')^e$-module with highest weight $(\lambda,\mu)\in P'$. If $\lambda\neq-\mu$, then*

$$H_*(\mathfrak{a}\oplus(\mathfrak{u}')^-,\mathfrak{a},R^{\mathfrak{r}})=0.$$

Suppose that $\lambda = -\mu$. *Then*

$$H_n(\mathfrak{a} \oplus (\mathfrak{u}')^-, \mathfrak{a}, R^t) = 0$$

if $n \in \mathbf{Z}_+$ *is odd. If* $n = 2j$ $(j \in \mathbf{Z}_+)$, *then* $\dim H_n(\mathfrak{a} \oplus (\mathfrak{u}')^-, \mathfrak{a}, R^t)$ *is the number of elements of* W_S^1 *of length* j, *and there are natural vector space isomorphisms*

$$H_n(\mathfrak{a} \oplus (\mathfrak{u}')^-, \mathfrak{a}, R^t) \simeq \coprod \operatorname{End}_{\mathfrak{r}^e} \mathbf{M}(w(\lambda+\rho)-\rho) \simeq \operatorname{End}_{\mathfrak{r}^e} H_j(\mathfrak{u}^-, (R^1)^t),$$

where the direct sum ranges over the elements $w \in W_S^1$ *of length* j, *and* R^1 *is the standard* $\mathfrak{g}^e$*-module with highest weight* λ. *In particular,* $H_{2j}(\mathfrak{a} \oplus (\mathfrak{u}')^-, \mathfrak{a}, R^t)$ *has a distinguished basis. For* $R = k$, *we have*

$$H_n(\mathfrak{a} \oplus (\mathfrak{u}')^-, \mathfrak{a}) = 0$$

if n *is odd; and if* $n = 2j$, *then* $\dim H_n(\mathfrak{a} \oplus (\mathfrak{u}')^-, \mathfrak{a})$ *is the number of elements of* W_S^1 *of length* j, *and there are natural vector space isomorphisms*

$$H_n(\mathfrak{a} \oplus (\mathfrak{u}')^-, \mathfrak{a}) \simeq \coprod \operatorname{End}_{\mathfrak{r}^e} M(w\rho-\rho) \simeq \operatorname{End}_{\mathfrak{r}^e} H_j(\mathfrak{u}^-),$$

where the direct sum ranges over the same w *as before.* $H_{2j}(\mathfrak{a} \oplus (\mathfrak{u}')^-, \mathfrak{a})$ *has a distinguished basis.*

Proof. — Just combine Corollary 6.4 with Proposition 4.11 [applied to $\mathfrak{b} = \mathfrak{a} \oplus (\mathfrak{u}')^-$] and the first assertion of Theorem 5.5.

Q.E.D.

We shall now take advantage of Corollary 5.4. With $\mathfrak{a}$ as in Theorem 6.5, let $\mathfrak{b}$ be an arbitrary subalgebra of $(\mathfrak{g}')^e$ such that $(\mathfrak{g}')^e = \mathfrak{b} + (\mathfrak{p}')^e$ and $\mathfrak{a} = \mathfrak{b} \cap (\mathfrak{p}')^e$. Then all the hypotheses of Corollary 5.4 hold in the present context, and so we can make use of it. One example of such a $\mathfrak{b}$ is the diagonal subalgebra of $(\mathfrak{g}')^e = \mathfrak{g}^e \times \mathfrak{g}^e$. Another is $\mathfrak{a} \oplus (\mathfrak{u}')^-$, which we considered above. The following result generalizes Theorem 6.5:

THEOREM 6.6. — *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be as above, and let* λ, μ *and* R *be as in Theorem* 6.5. *If* $\lambda \neq -\mu$, *then*

$$H_*(\mathfrak{b}, \mathfrak{a}, R^t) = 0.$$

Suppose that $\lambda = -\mu$. *Then there is a natural isomorphism of graded vector spaces*

$$H_*(\mathfrak{b}, \mathfrak{a}, R^t) \simeq H_*(\mathfrak{a} \oplus (\mathfrak{u}')^-, \mathfrak{a}, R^t),$$

which in turn is described in Theorem 6.5. *In particular, there is a natural isomorphism*

$$H_*(\mathfrak{b}, \mathfrak{a}) \simeq H_*(\mathfrak{a} \oplus (\mathfrak{u}')^-, \mathfrak{a});$$

again see Theorem 6.5.

Proof. — Applying Corollary 5.4 to the case $T=k$, we see that $H_*(\mathfrak{b}, \mathfrak{a}, R^t)$ is naturally isomorphic to the homology of a complex of vector spaces

$$\ldots \to V_1 \to V_0 \to 0,$$

where for each $n \in \mathbf{Z}_+$, V_n is the space of $\mathfrak{a}$-invariants in the $(\mathfrak{r}')^e$-module

$$\coprod M'(w'((\lambda, \mu)+\rho')-\rho'),$$

where w' ranges through the elements of $(W')^1_{S'}$ of length n; recall that the notations M' and ρ' have been defined above. Thus V_n is the space of $\mathfrak{a}$-invariants in

$$\coprod M(w_1(\lambda+\rho)-\rho) \otimes M(w_2(-\mu+\rho)-\rho)^*,$$

where the direct sum is over the pairs $w_1, w_2 \in W^1_S$ such that $l(w_1)+l(w_2)=n$. If $\lambda \neq -\mu$, then each $V_n=0$, by Lemma 6.3. If $\lambda=-\mu$, then Lemma 6.3 implies for odd n that $V_n=0$, and for $n=2j$ that there is a natural isomorphism

$$V_n \simeq \coprod \operatorname{End}_{\mathfrak{r}^e} M(w(\lambda+\rho)-\rho),$$

where w ranges through the elements of W^1_S of length j. Hence the homology of the complex of V_n's is naturally isomorphic to the complex itself, and the rest is clear.

Q.E.D.

We summarize this result in the important special case in which $\mathfrak{b}$ is the diagonal subalgebra of $(\mathfrak{g}')^e$ and $R=k$:

COROLLARY 6.7. — *For odd* $n \in \mathbf{Z}_+$,

$$H_n(\mathfrak{g}^e, \mathfrak{r}^e)=0.$$

For $n=2j\,(j \in \mathbf{Z}_+)$, *there are natural vector space isomorphisms*

$$H_n(\mathfrak{g}^e, \mathfrak{r}^e) \simeq \coprod \operatorname{End}_{\mathfrak{r}^e} M(w\rho-\rho) \simeq \operatorname{End}_{\mathfrak{r}^e} H_j(\mathfrak{u}^-),$$

where the direct sum ranges over the elements $w \in W^1_S$ *of length* j. *In particular,* $H_{2j}(\mathfrak{g}^e, \mathfrak{r}^e)$ *has a distinguished basis, and* dim $H_{2j}(\mathfrak{g}^e, \mathfrak{r}^e)$ *is the number of* $w \in W^1_S$ *of length* j, *and is also the number of irreducible* $\mathfrak{r}^e$-*module components in* $H_j(\mathfrak{u}^-)$ [*or in* $H_j(\mathfrak{u}^-, R^t)$ *for any standard* $\mathfrak{g}^e$-*module* R].

To obtain other consequences of Corollary 5.4, we define $\mathfrak{a}^\vee$ to be the diagonal subalgebra of $\mathfrak{r}'=\acute{\mathfrak{r}} \times \mathfrak{r}$, so that $\mathfrak{a}^\vee \simeq \mathfrak{r}$, and $\mathfrak{a}^\vee$ is reductive in $\mathfrak{a}$ (defined as in Theorem 6.5) and hence in $(\mathfrak{r}')^e$. We start with the following analogue of Lemma 6.3:

LEMMA 6.8. — *Let* $w_1, w_2 \in W^1_S$. *Then* (*superscript denoting the space of invariants*):

$$(M(w_1\rho-\rho) \otimes M(w_2\rho-\rho)^*)^{\mathfrak{a}^\vee}=0$$

unless $w_1=w_2$, *in which case there is a natural isomorphism*

$$(M(w_1\rho-\rho) \otimes M(w_1\rho-\rho)^*)^{\mathfrak{a}^\vee} \simeq \operatorname{End}_{\mathfrak{r}} M(w_1\rho-\rho),$$

a one-dimensional space with a distinguished basis (*the identity operator*).

Proof. — Note that the modules $M(w_1\rho-\rho)$ and $M(w_2\rho-\rho)$ are absolutely irreducible under $\mathfrak{r}$. Hence it is enough to show that if $w_1\rho$ and $w_2\rho$ have the same restriction to $\mathfrak{h}\subset\mathfrak{h}^e$, then $w_1=w_2$. But if $w_1\rho$ and $w_2\rho$ agree on $\mathfrak{h}$, then $w_1\rho-w_2\rho$ is a W-fixed element of $(\mathfrak{h}^e)^*$ (*see* § 5). Hence in $(\mathfrak{h}^e)^*$,

$$\langle\Phi_{w_1^{-1}w_2}\rangle=\rho-w_1^{-1}w_2\rho=w_1^{-1}(w_1\rho-w_2\rho)$$
$$=w_2^{-1}(w_1\rho-w_2\rho)=w_2^{-1}w_1\rho-\rho=-\langle\Phi_{w_2^{-1}w_1}\rangle,$$

and so

$$\langle\Phi_{w_1^{-1}w_2}\rangle=\langle\Phi_{w_2^{-1}w_1}\rangle=0,$$

which implies that $w_1=w_2$ (*see* § 5).

Q.E.D.

By Corollary 6.2 and Lemma 6.8, we have (*cf.* Cor. 6.4):

COROLLARY 6.9. — *For all odd $n\in\mathbf{Z}_+$,*

$$H_n((\mathfrak{u}')^-)^{\mathfrak{a}^\vee}=0.$$

For $n=2j$ $(j\in\mathbf{Z}_+)$, there are natural vector space isomorphisms

$$H_n((\mathfrak{u}')^-)^{\mathfrak{a}^\vee}\simeq\coprod \operatorname{End}_{\mathfrak{r}} M(w\rho-\rho)\simeq\operatorname{End}_{\mathfrak{r}}\coprod M(w\rho-\rho),$$

where the direct sums range over the elements $w\in W_S^1$ of length j. In particular, $H_{2j}((\mathfrak{u}')^-)^{\mathfrak{a}^\vee}$ has a distinguished basis.

As in the proof of Theorem 6.5, we now obtain from Proposition 4.11 [applied to $\mathfrak{b}=\mathfrak{a}^\vee\oplus(\mathfrak{u}')^-$] and Theorem 5.5:

THEOREM 6.10. — *Let $\mathfrak{a}^\vee$ be the diagonal subalgebra of $\mathfrak{r}'$; and let $n\in\mathbf{Z}_+$. If n is odd, then*

$$H_n(\mathfrak{a}^\vee\oplus(\mathfrak{u}')^-,\ \mathfrak{a}^\vee)=0.$$

If $n=2j\,(j\in\mathbf{Z}_+)$. then dim $H_n(\mathfrak{a}^\vee\oplus(\mathfrak{u}')^-,\ \mathfrak{a}^\vee)$ *is the number of elements of* W_S^1 *of length j, and there are natural vector space isomorphisms*

$$H_n(\mathfrak{a}^\vee\oplus(\mathfrak{u}')^-,\ \mathfrak{a}^\vee)\simeq\coprod\operatorname{End}_{\mathfrak{r}} M(w\rho-\rho)\simeq\operatorname{End}_{\mathfrak{r}} H_j(\mathfrak{u}^-),$$

where the direct sum ranges over the elements $w\in W_S^1$ of length j. In particular, $H_{2j}(\mathfrak{a}^\vee\oplus(\mathfrak{u}')^-,\ \mathfrak{a}^\vee)$ has a distinguished basis.

Now let $\mathfrak{b}^\vee$ be any subalgebra of $(\mathfrak{g}')^e$ such that $(\mathfrak{g}')^e=\mathfrak{b}^\vee+(\mathfrak{p}')^e$ and $\mathfrak{a}^\vee=\mathfrak{b}^\vee\cap(\mathfrak{p}')^e$. [One example of such a $\mathfrak{b}^\vee$ is the diagonal subalgebra of $\mathfrak{g}'=\mathfrak{g}\times\mathfrak{g}$; another is $\mathfrak{a}^\vee\oplus(\mathfrak{u}')^-$.] Then Corollary 5.4 applies in the present context to the pair $(\mathfrak{b}^\vee,\ \mathfrak{a}^\vee)$ in the role of the pair $(\mathfrak{b},\ \mathfrak{a})$. Applying it to the case $R=T=k$, as in the proof of Theorem 6.6, and using Lemma 6.8, we obtain the following generalization of Theorem 6.10:

THEOREM 6.11. — *Let $\mathfrak{a}^\vee$ and $\mathfrak{b}^\vee$ be as above. Then there is a natural isomorphism of graded vector spaces*

$$H_*(\mathfrak{b}^\vee,\ \mathfrak{a}^\vee)\simeq H_*(\mathfrak{a}^\vee\oplus(\mathfrak{u}')^-,\ \mathfrak{a}^\vee),$$

which is described in Theorem 6.10.

In the important special case in which $\mathfrak{b}^{\vee}$ is the diagonal subalgebra of $\mathfrak{g}'$, this becomes:

COROLLARY 6.12. — *For odd* $n \in \mathbf{Z}_+$,

$$H_n(\mathfrak{g}, \mathfrak{r}) = 0.$$

For $n = 2j$ $(j \in \mathbf{Z}_+)$, *the are natural vector space isomorphisms*

$$H_n(\mathfrak{g}, \mathfrak{r}) \simeq \coprod \operatorname{End}_{\mathfrak{r}} M(w\rho - \rho) \simeq \operatorname{End}_{\mathfrak{r}} H_j(\mathfrak{u}^-),$$

where the direct sum ranges over the elements $w \in W_S^1$ *of length* j. *In particular,* $H_{2j}(\mathfrak{g}, \mathfrak{r})$ *has a distinguished basis, and* $\dim H_{2j}(\mathfrak{g}, \mathfrak{r})$ *is the number of* $w \in W_S^1$ *of length* j, *and is also the number of irreducible* $\mathfrak{r}$-*module components in* $H_j(\mathfrak{u}^-)$.

Recall from paragraph 5 that the center of $\mathfrak{g}$ is the subspace of $\mathfrak{h} \subset \mathfrak{h}^e$ on which all the roots of $\mathfrak{g}$ vanish. By Remark (2) after Proposition 2.2, we thus have:

COROLLARY 6.13. — *Let* $\mathfrak{z}$ *be the center of* $\mathfrak{g}$. *Then* $\mathfrak{z} \subset \mathfrak{r}$ (*in fact*, $\mathfrak{z} \subset \mathfrak{h}$), *and there is a natural isomorphism of graded vector spaces*

$$H_*(\mathfrak{g}/\mathfrak{z}, \mathfrak{r}/\mathfrak{z}) \simeq H_*(\mathfrak{g}, \mathfrak{r}),$$

which is described in Corollary 6.12. *Moreover,* $\mathfrak{z}$ *acting trivially on* $M(w\rho - \rho)$ *for* $w \in W_S^1$ [*and hence on* $H_*(\mathfrak{u}^-)$], *we may replace the symbol* $\operatorname{End}_{\mathfrak{r}}$ *by* $\operatorname{End}_{\mathfrak{r}/\mathfrak{z}}$ *in* 6.8-6.12.

The situation for cohomology in place of homology is now very simple. We recall Remark (1) after Proposition 2.2 (relating relative cohomology to relative homology), and we use the distinguished bases of the above homology spaces. Then the results above immediately imply the following corollaries:

COROLLARY 6.14. — *Let* $\mathfrak{a}$, $\mathfrak{b}$, λ, μ *and* R *be as in Theorem* 6.6. *If* $\lambda \neq -\mu$, *then*

$$H^*(\mathfrak{b}, \mathfrak{a}, R^*) = 0.$$

If $\lambda = -\mu$, *then there is a natural isomorphism of graded vector spaces*

$$H^*(\mathfrak{b}, \mathfrak{a}, R^*) \simeq H_*(\mathfrak{b}, \mathfrak{a}, R^t).$$

There are also natural graded vector space isomorphisms

$$H^*(\mathfrak{g}^e, \mathfrak{r}^e) \simeq H^*(\mathfrak{b}, \mathfrak{a}) \simeq H_*(\mathfrak{b}, \mathfrak{a}).$$

(See *Theorems* 6.5 *and* 6.6 *and Corollary* 6.7 *for descriptions.*)

COROLLARY 6.15. — *Let* $\mathfrak{a}^{\vee}$ *and* $\mathfrak{b}^{\vee}$ *be as in Theorem* 6.11. *Then there are natural isomorphisms of graded vector spaces*

$$H^*(\mathfrak{g}, \mathfrak{r}) \simeq H^*(\mathfrak{b}^{\vee}, \mathfrak{a}^{\vee}) \simeq H_*(\mathfrak{b}^{\vee}, \mathfrak{a}^{\vee});$$

see *Theorems* 6.10 *and* 6.11 *and Corollary* 6.12 *for descriptions.*

COROLLARY 6.16. — *Let* $\mathfrak{z}$ *be the center of* $\mathfrak{g}$. *Then* $\mathfrak{z}\subset\mathfrak{h}\subset\mathfrak{r}$, *and there is a natural graded vector space isomorphism*

$$H^*(\mathfrak{g}/\mathfrak{z},\ \mathfrak{r}/\mathfrak{z})\simeq H^*(\mathfrak{g},\ \mathfrak{r}).$$

For odd $n\in\mathbf{Z}_+$,

$$H^n(\mathfrak{g}/\mathfrak{z},\ \mathfrak{r}/\mathfrak{z})=0.$$

For $n=2j\,(j\in\mathbf{Z}_+)$, *there are natural vector space isomorphisms*

$$H^n(\mathfrak{g}/\mathfrak{z},\ \mathfrak{r}/\mathfrak{z})\simeq\coprod \operatorname{End}_{\mathfrak{r}/\mathfrak{z}} M(w\rho-\rho)\simeq \operatorname{End}_{\mathfrak{r}/\mathfrak{z}} H^j(\mathfrak{u}^-),$$

where the direct sum ranges over the elements $w\in W_S^1$ *of length* j. *In particular*, $H^{2j}(\mathfrak{g}/\mathfrak{z},\ \mathfrak{r}/\mathfrak{z})$ *has a distinguished basis, and* $\dim H^{2j}(\mathfrak{g}/\mathfrak{z},\ \mathfrak{r}/\mathfrak{z})$ *is the number of* $w\in W_S^1$ *of length* j, *and is also the number of irreducible* $\mathfrak{r}/\mathfrak{z}$-*module components in* $H^j(\mathfrak{u}^-)$.

If we take $S=\emptyset$ in paragraphs 5 and 6, so that $\mathfrak{r}=\mathfrak{h}$, $\mathfrak{u}^-=\mathfrak{n}^-$, $W_S^1=W$, $P_S=(\mathfrak{h}^e)^*$ and $M(\lambda)$ is a one-dimensional weight space with weight λ for all $\lambda\in(\mathfrak{h}^e)^*$, then we obtain:

COROLLARY 6.17. — *Let* $\mathfrak{b}$ *be the diagonal subalgebra of* $(\mathfrak{g}')^e$, $\mathfrak{a}$ *the diagonal subalgebra of* $(\mathfrak{h}')^e$, *let* $\lambda\in P$, $\mu\in -P$ *and let* R *be the standard* $(\mathfrak{g}')^e$-*module with highest weight* $(\lambda,\ \mu)\in P'$. *If* $\lambda\neq-\mu$, *then*

$$H^*(\mathfrak{b},\ \mathfrak{a},\ R^*)=0.$$

Let $\lambda=-\mu$. *Then*

$$H^n(\mathfrak{g}^e,\ \mathfrak{h}^e)=H^n(\mathfrak{b},\ \mathfrak{a},\ R^*)=0$$

if $n\in\mathbf{Z}_+$ *is odd. If* $n=2j\ (j\in\mathbf{Z}_+)$, *then*

$$\dim H^n(\mathfrak{g}^e,\ \mathfrak{h}^e)=\dim H^n(\mathfrak{b},\ \mathfrak{a},\ R^*)=\dim H^j(\mathfrak{n}^-,\ (R^1)^*),$$

and this is the number of elements of W *of length* j; *moreover, there are natural vector space isomorphisms*

$$H^n(\mathfrak{b},\ \mathfrak{a},\ R^*)\simeq\coprod \operatorname{End}_{\mathfrak{h}^e} M(w(\lambda+\rho)-\rho)\simeq \operatorname{End}_{\mathfrak{h}^e} H^j(\mathfrak{n}^-,\ (R^1)^*),$$

where the direct sum ranges over the Weyl group elements of length j, *and* R^1 *is the standard* $\mathfrak{g}^e$-*module with highest weight* λ. *In particular*, $H^{2j}(\mathfrak{b},\ \mathfrak{a},\ R^*)$ *has a distinguished basis. Let* $\mathfrak{z}$ *be the center of* $\mathfrak{g}$, *so that* $\mathfrak{z}\subset\mathfrak{h}$. *Then*

$$H^n(\mathfrak{g}/\mathfrak{z},\ \mathfrak{h}/\mathfrak{z})=H^n(\mathfrak{g},\ \mathfrak{h})=0$$

if $n\in\mathbf{Z}_+$ *is odd. If* $n=2j\,(j\in\mathbf{Z}_+)$, *then*

$$\dim H^n(\mathfrak{g}/\mathfrak{z},\ \mathfrak{h}/\mathfrak{z})=\dim H^n(\mathfrak{g},\ \mathfrak{h})=\dim H^j(\mathfrak{n}^-)$$

(*the number of Weyl group elements of length* j), *and there are natural vector space isomorphisms*

$$H^n(\mathfrak{g}/\mathfrak{z},\ \mathfrak{h}/\mathfrak{z})\simeq H^n(\mathfrak{g},\ \mathfrak{h})\simeq\coprod \operatorname{End}_{\mathfrak{h}/\mathfrak{z}} M(w\rho-\rho)\simeq \operatorname{End}_{\mathfrak{h}/\mathfrak{z}} H^j(\mathfrak{n}^-),$$

where the direct sum ranges over the Weyl group elements of length j. In particular, $H^{2j}(\mathfrak{g}/\mathfrak{z}, \mathfrak{h}/\mathfrak{z})$ *has a distinguished basis.*

Remark. — The last corollaries generalize, and give a new proof of, the well-known special case in which $\mathfrak{g}^e = \mathfrak{g}$ is a finite-dimensional complex semisimple Lie algebra, so that $\mathfrak{r}^e = \mathfrak{r}$ is the reductive part of a parabolic subalgebra. The numbers $\dim H^j(\mathfrak{g}, \mathfrak{r})$ in this finite-dimensional case, computed algebraically by Kostant in [16] (*c*), are the Betti numbers of a generalized flag manifold U/U_1, where U is a compact connected semisimple Lie group and U_1 is the centralizer of a torus in U (*see* [3], (*a*), Introduction and [16] (*c*), Introduction and Remark 5.3, for discussions of the history of the study of these Betti numbers). In this finite-dimensional case, the equality

$$\dim H^{2j}(\mathfrak{g}, \mathfrak{h}) = \dim H^j(\mathfrak{n}^-, (R^1)^*)$$

in Corollary 6.17 amounts to Bott's "strange equality" [3], (*b*), p. 247, which motivated Kostant's papers [16] (*b*), (*c*) (*see* [16] (*b*), Introduction). The above results considerably generalize Bott's strange equality (in an algebraic sense) and explain it in a new way. Our results also to a certain extent explain the relationship, observed by Garland [10], between the Betti numbers of the loop space of a compact Lie group and the cohomology of certain subalgebras of "affine" Kac-Moody Lie algebras; *see* Part III below.

7. Minimal $\mathfrak{k}$-types for complex semisimple Lie algebras

Here we shall use Corollary 5.4 to recover the theorem, proved in [23], § 2.2, on minimal types for finite-dimensional irreducible representations of complex semisimple Lie algebras regarded as real.

Retain the notation of paragraph 6 preceding Proposition 6.1. Assume that the Cartan matrix A is of finite type, that $\mathfrak{d} = 0$, and that S is the null set. Then $\mathfrak{g}^e = \mathfrak{g}$ is a finite-dimensional split semisimple Lie algebra, $(\mathfrak{g}')^e = \mathfrak{g}' = \mathfrak{g} \times \mathfrak{g}$ is also, $\mathfrak{r}^e = \mathfrak{h}^e = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and $(\mathfrak{r}')^e = (\mathfrak{h}')^e = \mathfrak{h}' = \mathfrak{h} \times \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}'$. Let $\mathfrak{k}$ be the diagonal subalgebra of $\mathfrak{g}'$, so that $\mathfrak{k} \simeq \mathfrak{g}$, and let $\mathfrak{m}$ be the diagonal subalgebra of $\mathfrak{h}'$, so that $\mathfrak{m} \simeq \mathfrak{h}$ and $\mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{k}$.

THEOREM 7.1 ([23], p. 394, Cor. 1). — *Let* R *be a finite-dimensional irreducible* $\mathfrak{g}'$*-module, with highest weight* $\mu \in P' \subset (\mathfrak{h}')^*$. *Denote by* $\nu \in \mathfrak{m}^*$ *the restriction of* μ *to* $\mathfrak{m}$, *so that* ν *is an integral linear form on the Cartan subalgebra* $\mathfrak{m}$ *of* $\mathfrak{k}$. *Let* T *be the unique (up to equivalence) irreducible* $\mathfrak{k}$*-module which contains* ν *as an extremal weight. Then* T *is contained in* R *(regarded as a* $\mathfrak{k}$*-module) with multiplicity one.*

Remark. — Let $\mu = (\mu_1, \mu_2)$, with $\mu_1 \in P$ and $\mu_2 \in -P$, and let R^1 (respectively, R^2) be the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight μ_1 (respectively, lowest weight μ_2). Then $R \simeq R^1 \otimes R^2$ as $\mathfrak{g}' = \mathfrak{g} \times \mathfrak{g}$-modules, where the first factor $\mathfrak{g}$ acts on R^1 and the second factor $\mathfrak{g}$ on R^2. Let Y be the finite-dimensional irreducible $\mathfrak{g}$-module containing $\mu_1 + \mu_2$ as an extremal weight. Then Theorem 7.1 amounts to the assertion that the tensor product $\mathfrak{g}$-module $R^1 \otimes R^2$ contains Y with multiplicity one. Note that if $\mu_1 = -\mu_2$, then this assertion simply reduces to Schur's lemma.

Proof of Theorem 7.1. — The multiplicity of T in R is the dimension of the space of $\mathfrak{k}$-invariants in $R^* \otimes T$, which is $\dim H^0(\mathfrak{k}, \mathfrak{m}, R^* \otimes T)$, by the Remark following Proposition 2.3. Hence in view of Remark (1) after Proposition 2.2, it is sufficient to prove that

$$\dim H_0(\mathfrak{k}, \mathfrak{m}, (R \otimes T^*)^t) = 1.$$

Applying Corollary 5.4 to $\mathfrak{k}$, $\mathfrak{m}$, R and T^* in place of $\mathfrak{b}$, $\mathfrak{a}$, R and T, respectively, we see that it is sufficient to show that $\dim V_0 = 1$ and $V_1 = 0$ (using the notation of that corollary). The only Weyl group element (for $\mathfrak{g}'$) of length zero is the identity, and the Weyl group elements of length one are the simple reflections $(r_i, 1)$ and $(1, r_i)$ (r_i a simple reflection for $\mathfrak{g}$ with respect to $\mathfrak{h}$). Identifying $\mathfrak{k}$ with $\mathfrak{g}$, $\mathfrak{m}$ with $\mathfrak{h}$ and T with Y (*see* the above Remark), we see that it suffices to show that $\mu_1 + \mu_2$ is a multiplicity-one weight of Y, but that for every simple reflection r_i, neither $\gamma_i = r_i \mu_1 + \mu_2 - \alpha_1$ nor $\delta_i = \mu_1 + r_i \mu_2 + \alpha_i$ is a weight of Y.

By definition of Y, $\mu_1 + \mu_2$ is an extremal, and hence a multiplicity-one, weight of Y. Let $(.,.)$ be the natural bilinear form on $\mathfrak{h}^*$ induced by the Killing form of $\mathfrak{g}$. Then for each i,

$$\begin{aligned}(\gamma_i, \gamma_i) - (\mu_1 + \mu_2, \mu_1 + \mu_2) = {} & -((\mu_1(h_i)+1)\alpha_i, \mu_1 + \mu_2) \\ & -(\mu_1 + \mu_2 - (\mu_1(h_i)+1)\alpha_i, (\mu_1(h_i)+1)\alpha_i) \\ & \geqq -((\mu_1(h_i)+1)\alpha_i, \mu_1) - (\mu_1 - (\mu_1(h_i)+1)\alpha_i, (\mu_1(h_i)+1)\alpha_i)\end{aligned}$$

(since $\mu_1 \in P$ and $\mu_2 \in -P$)

$$= (\mu_1(h_i)+1)\{-2(\alpha_i, \mu_1) + (\mu_1(h_i)+1)(\alpha_i, \alpha_i)\}.$$

Since $\mu_1(h_i) = 2(\mu_1, \alpha_i)/(\alpha_i, \alpha_i)$, the expression in braces is just (α_i, α_i), and so

$$(\gamma_i, \gamma_i) > (\mu_1 + \mu_2, \mu_1 + \mu_2).$$

Hence we also have $(\delta_i, \delta_i) > (\mu_1 + \mu_2, \mu_1 + \mu_2)$, because $\delta_i = r_i \gamma_i$. Thus neither γ_i nor δ_i can be a weight of Y, and the theorem is proved.

Q.E.D.

PART III
EUCLIDEAN LIE ALGEBRAS AND EQUIVARIANT LOOP SPACES

8. Automorphisms of finite order of semisimple Lie algebras

In Part III, we formulate the relative cohomology results of Part II in the important special case of Euclidean Lie algebras (which were introduced and studied by Kac [15], (*a*)-(*d*) and Moody [22] (*a*)-(*c*)), and we relate this to work of Bott on loop space cohomology. Kac's description [15], (*b*) of the automorphisms of finite order of semisimple Lie algebras is basic here. Because the notation convenient for us frequently differs from Kac's notation in [15] (*b*), we find it necessary to summarize the contents of [15], (*b*). The present section is devoted to this summary.

Assume that our field k of characteristic zero is algebraically closed; and let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k. Given an automorphism θ of finite period m of $\mathfrak{g}$, we obtain a gradation mod m of $\mathfrak{g}$ as follows: Let ε be a primitive m-th root of unity in k. For all $i \in \mathbf{Z}/(m)$, denote by $\mathfrak{g}_i$ the ε^i-eigenspace of θ in $\mathfrak{g}$. Then $\mathfrak{g} = \coprod_{i \in \mathbf{Z}/(m)} \mathfrak{g}_i$, and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, $j \in \mathbf{Z}/(m)$. Conversely, given a Lie algebra gradation $\mathfrak{g} = \coprod_{i \in \mathbf{Z}/(m)} \mathfrak{g}_i$ of $\mathfrak{g}$, the endomorphism of $\mathfrak{g}$ which multiplies each element of $\mathfrak{g}_i$ by ε^i is an automorphism of $\mathfrak{g}$ of period m.

Continuing to use the above notation, define

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_k k[t, t^{-1}],$$

the infinite-dimensional k-Lie algebra obtained by tensoring $\mathfrak{g}$ with the commutative algebra of finite Laurent series in one indeterminate t. Also define

$$\mathrm{L}(\mathfrak{g}, \theta) = \coprod_{i \in \mathbf{Z}} \mathfrak{g}_{i (\mathrm{mod}\ m)} \otimes t^i.$$

Then $\mathrm{L}(\mathfrak{g}, \theta)$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$, and $\tilde{\mathfrak{g}} = \mathrm{L}(\mathfrak{g}, 1)$.

We shall assume now that $\mathfrak{g}$ is nonzero and semisimple, and that the pair $(\mathfrak{g}, \theta)$ is *indecomposable*, i.e., that $\mathfrak{g}$ cannot be decomposed into a direct product of nonzero θ-invariant ideals. The centralizer $\mathfrak{g}_0$ of θ is reductive in $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, and identify $\mathfrak{g}_0$ with the corresponding subalgebra of $\mathrm{L}(\mathfrak{g}, \theta)$. Let D be the degree derivation of $\mathrm{L}(\mathfrak{g}, \theta)$ with respect to its natural $\mathbf{Z}$-grading. That is, D acts as multiplication by i on $\mathfrak{g}_{i (\mathrm{mod}\ m)} \otimes t^i$ for all $i \in \mathbf{Z}$. Let $\mathfrak{h}_1 = k\mathrm{D} \oplus \mathfrak{h}_0$, so that $\mathrm{L}(\mathfrak{g}\ \theta)$ has a natural weight space decomposition with respect to $\mathfrak{h}_1$. If a weight is nonzero on $\mathfrak{h}_0$, then the corresponding weight space is one-dimensional. The roots of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_0$ are identified with certain linear functionals on $\mathfrak{h}_1$ which vanish on D. Choose a system of positive roots of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_0$. The union of this set with the set of weights of $\mathrm{L}(\mathfrak{g}, \theta)$ (with respect to $\mathfrak{h}_1$) which are positive on D is called the set of *positive weights* of $\mathrm{L}(\mathfrak{g}, \theta)$. Those positive weights which cannot be written as the sum of two positive weights are called the *simple weights* $\alpha_0, \ldots, \alpha_l$ $(l \geqq 1)$. Every positive weight is a nonnegative integral linear combination of the simple weights, which form a basis of $\mathfrak{h}_1^*$. The restrictions $\alpha_0 | \mathfrak{h}_0, \ldots, \alpha_l | \mathfrak{h}_0$ are nonzero elements of $\mathfrak{h}_0^*$ which are linearly dependent over $\mathbf{Z}$.

The Killing form of $\mathfrak{g}$ is nonsingular on $\mathfrak{h}_0$ and thus induces a nonsingular form $(.,.)$ on $\mathfrak{h}_0^*$. This form is rational-valued and positive definite on the rational span of the weights. The $(l+1) \times (l+1)$ matrix A given by

$$\mathrm{A}_{ij} = \frac{2(\alpha_i | \mathfrak{h}_0, \alpha_j | \mathfrak{h}_0)}{(\alpha_i | \mathfrak{h}_0, \alpha_i | \mathfrak{h}_0)}$$

is thus well defined. A is called the *Cartan matrix* of $\mathrm{L}(\mathfrak{g}, \theta)$, and gives rise in the usual way to a Dynkin diagram. Specifically, we draw a vertex for each simple weight, we connect the

i-th and j-th vertices by $A_{ij} A_{ji}$ lines, and we draw an arrow pointing from the i-th vertex toward the j-th if $|A_{ij}| < |A_{ji}|$. (If $i \neq j$, then $A_{ij} \in -\mathbf{Z}_+$.) Also, above (or near) the vertices, we place the smallest positive integers which give a linear dependence relationship among the respective columns of the Cartan matrix; these numbers exist and are unique. They are the same as the smallest positive integers which give a linear dependence relationship among the restrictions to $\mathfrak{h}_0$ of the simple weights.

The Dynkin diagram of $L(\mathfrak{g}, \theta)$ is in Tables 1-3 (where each diagram has $l+1$ vertices).

For each $i=0, \ldots, l$, there is a unique element $x_i \in \mathfrak{h}_0$ such that $(x_i, h) = \alpha_i(h)$ for all $h \in \mathfrak{h}_0$; here $(.,.)$ denotes the Killing form of $\mathfrak{g}$. There is a unique rational multiple h'_i of x_i such that $\alpha_i(h'_i) = 2$. We may choose elements $e_i, f_i \in L(\mathfrak{g}, \theta)$ such that e_i lies in the weight space for α_i, f_i lies in the weight space for $-\alpha_i$, and $[e_i, f_i] = h'_i$. Then

$$[h'_i, e_j] = A_{ij} e_j, \ [h'_i, f_j] = -A_{ij} f_j \qquad \text{and} \qquad [e_i, f_j] = \delta_{ij} h'_i$$

for all $i, j = 0, \ldots, l$. The elements h'_i, e_i and f_i generate $L(\mathfrak{g}, \theta)$ as i ranges from 0 to l. We shall call them *canonical generators* of $L(\mathfrak{g}, \theta)$.

The Lie algebra $L(\mathfrak{g}, \theta)$ is graded-simple in that it has no proper nonzero ideals homogeneous with respect to the weight space decomposition.

Two Lie algebras of the type $L(\mathfrak{g}, \theta)$ which have the same Dynkin diagram (or Cartan matrix, up to index permutation) are isomorphic (without regard to gradation) by an isomorphism which sends a set of canonical generators for one to a set of canonical generators for the other.

Moreover, if $\mathfrak{g}$ is a simple Lie algebra of rank n and θ an automorphism of order j induced by an automorphism of order j of its Dynkin diagram, then the Dynkin diagram of $L(\mathfrak{g}, \theta)$ is the diagram $X_n^{(j)}$ of Table j ($j=1, 2, 3$), where X_n is the type of $\mathfrak{g}$. Call these $L(\mathfrak{g}, \theta)$'s *standard*.

Thus every $L(\mathfrak{g}, \theta)$ is isomorphic to a standard one (without regard to gradation).

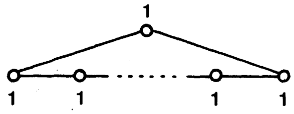
The Cartan matrix A of $L(\mathfrak{g}, \theta)$ is a symmetrizable (generalized) Cartan matrix. In the notation of paragraph 5, the Lie algebra $\mathfrak{g}(A)$ has a one-dimensional center, say $\mathfrak{c}$ (the subspace of $\mathfrak{h} = \text{span}\{h_i\}$ on which the roots vanish), and there is an exact sequence

$$0 \to \mathfrak{c} \to \mathfrak{g}(A) \xrightarrow{\pi} L(\mathfrak{g}, \theta) \to 0.$$

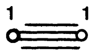
Here π is determined by the condition that it send the canonical generators h_i, e_i f_i for $\mathfrak{g}(A)$ (*see* § 5) to the respective canonical generators h'_i, e_i, f_i for $L(\mathfrak{g}, \theta)$ defined above. (Note however that the index set $\{0, \ldots, l\}$ now plays the role of the index set $\{1, \ldots, l\}$ in paragraph 5.) The Lie algebras $L(\mathfrak{g}, \theta)$ are called here the *Euclidean* Lie algebras. [Sometimes the central extensions $\mathfrak{g}(A)$ are called the Euclidean Lie algebras.] The corresponding Cartan matrices are called the *Euclidean* matrices. The Euclidean Lie algebras of the form $\tilde{\mathfrak{g}}$ ($\mathfrak{g}$ simple), or their central extensions, are called the *affine* Lie algebras, and the corresponding Cartan matrices are called the *affine* Cartan matrices.

TABLE 1

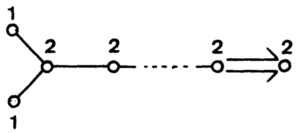
$A_l^{(1)}$ or $\tilde{A}_l$ $(l \geqq 2)$



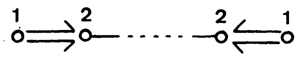
$A_1^{(1)}$ or $\tilde{A}_1$



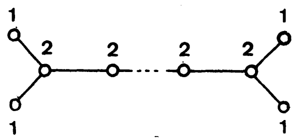
$B_l^{(1)}$ or $\tilde{B}_l$ $(l \geqq 3)$



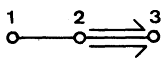
$C_l^{(1)}$ or $\tilde{C}_l$ $(l \geqq 2)$



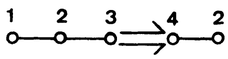
$D_l^{(1)}$ or $\tilde{D}_l$ $(l \geqq 4)$



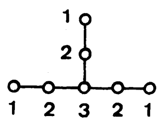
$G_2^{(1)}$ or $\tilde{G}_2$



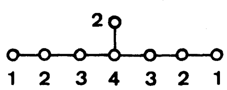
$F_4^{(1)}$ or $\tilde{F}_4$



$E_6^{(1)}$ or $\tilde{E}_6$



$E_7^{(1)}$ or $\tilde{E}_7$



$E_8^{(1)}$ or $\tilde{E}_8$

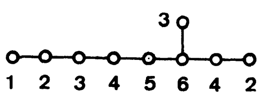
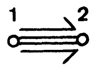


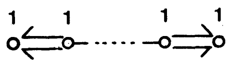
TABLE 2

$A_{2l}^{(2)}$ $(l \geqq 2)$

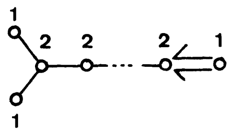
$A_2^{(2)}$



$D_{l+1}^{(2)}$ $(l \geqq 2)$



$A_{2l-1}^{(2)}$ $(l \geqq 3)$



$E_6^{(2)}$

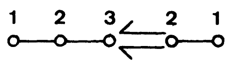
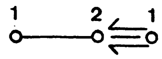


TABLE 3

$D_4^{(3)}$



Each Euclidean Lie algebra $L(\mathfrak{g}, \theta)$ is equipped with a natural **Z**-grading, as we observed above. This **Z**-grading is uniquely determined by asserting that $\mathfrak{h}_0$ have degree 0 and that the weight space corresponding to the weight $\pm\alpha_i$ have degree $\pm\alpha_i(D)$, for $i=0, \ldots, l$ (recall that D is the degree derivation). The numbers $\alpha_i(D)$ are nonnegative integers, not all zero. Conversely, given nonnegative integers $s_0, \ldots, s_l$, not all zero, we define the **Z**-grading of $L(\mathfrak{g}, \theta)$ of *type* $(s_0, \ldots, s_l)$ to be the unique **Z**-grading which assigns $\mathfrak{h}_0$ degree zero and the weight space for $\pm\alpha_i$ degree $\pm s_i$, for $i=0, \ldots, l$. Then every Euclidean Lie algebra, with its **Z**-grading, is graded-isomorphic to a standard Euclidean Lie algebra with a grading of type $(s_0, \ldots, s_l)$. Moreover, a standard Euclidean Lie algebra $L(\mathfrak{g}, \theta)$ with a grading of type $(s_0, \ldots, s_l)$ is graded-isomorphic to a Euclidean Lie algebra $L(\mathfrak{g}, \theta')$ (same $\mathfrak{g}$, but θ' possibly different from θ) with its natural **Z**-grading.

Suppose that $L(\mathfrak{g}, \theta)$ is standard, and that θ has order j, so that $j=1$, 2 or 3. For a positive integer r, define the ideal

$$I_r=(1-t^{jr})\,L(\mathfrak{g}, \theta)$$

of $L(\mathfrak{g}, \theta)$. Provide $L(\mathfrak{g}, \theta)$ with a grading of type $(s_0, \ldots, s_l)$, and let $m=jr\sum_{i=0}^{l} s_i b_i$, where the b_i are the integers above the vertices of the relevant Dynkin diagram in Table j. Then $L(\mathfrak{g}, \theta)/I_r$ is a (finite-dimensional) semisimple Lie algebra graded mod m and isomorphic (ignoring the grading) to a direct product of r copies of $\mathfrak{g}$. The corresponding automorphism of period m of $L(\mathfrak{g}, \theta)/I_r$ is inner if and only if $j=r=1$. Conversely, every automorphism θ' of finite order of a semisimple Lie algebra $\mathfrak{g}'$, such that $(\mathfrak{g}', \theta')$ is indecomposable, arises by the construction just given.

It is illuminating to know how to construct the diagrams in Tables 1-3 from the Dynkin diagrams for the (finite-dimensional) simple Lie algebras (*see* [15], (*a*)): To construct the diagram $X_l^{(1)}$ in Table 1, adjoin to the ordinary Dynkin diagram X_l one vertex corresponding to the lowest root of the associated simple Lie algebra (connecting this vertex to the original l vertices with the obvious lines and arrows). Place the integer 1 over this adjoined vertex, and place the expansion coefficients of the highest root in terms of the l simple roots over the original l vertices. To construct the diagrams $X_n^{(j)}$ in Table j ($j=2$, 3), let $\mathfrak{g}$ be a simple Lie algebra of type X_n; θ an automorphism of order j of $\mathfrak{g}$ induced by an automorphism of order j of the Dynkin diagram X_n; $\mathfrak{g}_0$, $\mathfrak{g}_1$ and possibly $\mathfrak{g}_2$ the components of $\mathfrak{g}$ for the mod j grading associated with θ; and Y_l the Dynkin diagram of $\mathfrak{g}_0$, which is a simple Lie algebra of rank l. To construct the diagram for $X_n^{(j)}$, adjoin to Y_l one vertex corresponding to the lowest weight of $\mathfrak{g}_0$ acting on the (irreducible) module $\mathfrak{g}_1$, place the integer 1 over this adjoined vertex, and place the expansion coefficients of the highest weight of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ (in terms of the l simple roots) over the original l vertices. If $X_n^{(j)}=D_{l+1}^{(2)}$, $A_{2l-1}^{(2)}$, $E_6^{(2)}$ or $D_4^{(3)}$, then $Y_l=B_l$, C_l, F_4 or G_2, respectively, and the highest weight of $\mathfrak{g}_1$ coincides with the highest short root of $\mathfrak{g}_0$. If $X_n^{(j)}=A_{2l}^{(2)}$ $(l\geqq 1)$, then $Y_l=B_l$, and the highest weight of $\mathfrak{g}_1$ coincides with twice the highest short root of $\mathfrak{g}_0$.

In each diagram in Tables 1-3, assign the adjoined vertex the index 0. The diagrams are all drawn so that the 0-th vertex occurs at the left.

Remarks. — (1) For each Dynkin diagram in Tables 1-3, the natural **Z**-grading of the corresponding standard Euclidean Lie algebra $L(\mathfrak{g}, \theta)$ is of type $(1, 0, \ldots, 0)$. Moreover, the corresponding mod m graded semisimple Lie algebra $L(\mathfrak{g}, \theta)/I_1$ is just $\mathfrak{g}$ provided with its mod j grading given by θ (where j is the table number). If $j=1$, then $L(\mathfrak{g}, \theta)=\tilde{\mathfrak{g}}$ and $L(\mathfrak{g}, \theta)/I_1=\mathfrak{g}$ with the trivial mod 1 grading.

(2) Suppose that we drop the hypothesis that the pair $(\mathfrak{g}, \theta)$ be indecomposable. Then with certain obvious modifications, everything in this section remains valid. Specifically, assume that $\mathfrak{g}=\mathfrak{g}_1 \times \ldots \times \mathfrak{g}_N$, where the $\mathfrak{g}_i$ are nonzero θ-invariant indecomposable semisimple Lie algebras. Then the centralizer $\mathfrak{g}_0$ of θ is the direct product of the centralizers $\mathfrak{g}_{0, i}$ of θ in the $\mathfrak{g}_i$, and the Cartan subalgebra $\mathfrak{h}_0$ is the direct product of Cartan subalgebras $\mathfrak{h}_{0, i}$ of the $\mathfrak{g}_{0, i}$. We have

$$L(\mathfrak{g}, \theta)=L(\mathfrak{g}_1, \theta) \times \ldots \times L(\mathfrak{g}_N, \theta),$$

and each factor $L(\mathfrak{g}_i, \theta)$ carries a natural degree derivation D_i which vanishes on all the other factors. Let $\mathfrak{h}_{1, i}=k D_i \oplus \mathfrak{h}_{0, i}$. Then $\mathfrak{h}_1=\mathfrak{h}_{1, 1} \times \ldots \times \mathfrak{h}_{1, N}$ is the abelian Lie algebra with respect to which we take the weight space decomposition of $L(\mathfrak{g}, \theta)$. The set of positive weights of $L(\mathfrak{g}, \theta)$ with respect to $\mathfrak{h}_1$ may be identified with the union of the sets of positive weights of the $L(\mathfrak{g}_i, \theta)$ with respect to the $\mathfrak{h}_{1, i}$, and similarly for the set of simple weights. The Dynkin diagram of $L(\mathfrak{g}, \theta)$ need not be connected, and is a disjoint union of diagrams from Tables 1-3. The Cartan matrix A of $L(\mathfrak{g}, \theta)$ is the direct sum, in the obvious sense, of the Cartan matrices of the Euclidean subalgebras $L(\mathfrak{g}_i, \theta)$; the center $\mathfrak{c}$ of $\mathfrak{g}(A)$ is N-dimensional; and there is an exact sequence

$$0 \to \mathfrak{c} \to \mathfrak{g}(A) \xrightarrow{\pi} L(\mathfrak{g}, \theta) \to 0$$

such that π takes canonical generators to canonical generators; this sequence is the direct sum of the N corresponding exact sequences for the Euclidean factors of $L(\mathfrak{g}, \theta)$.

9. The relative cohomology theorem for Euclidean Lie algebras

Continuing with the notation of paragraph 8, let θ be an automorphism of finite order of a semisimple Lie algebra $\mathfrak{g}$ such that $(\mathfrak{g}, \theta)$ is indecomposable, and consider the corresponding Euclidean Lie algebra $L(\mathfrak{g}, \theta)$, its $(l+1)\times(l+1)$ Cartan matrix A, and its central extension $\mathfrak{g}(A)$. Recall that π is the surjection $\mathfrak{g}(A) \to \mathfrak{g}(L, \theta)$. We are free to use the concepts and notation of paragraph 5 in discussing $\mathfrak{g}(A)$ (except that the index set $\{0, \ldots, l\}$ now plays the role of the index set $\{1, \ldots, l\}$ in paragraph 5). The centralizer $\mathfrak{g}_0$ of θ in $\mathfrak{g}$ embeds naturally in $L(\mathfrak{g}, \theta)$, and $\pi^{-1}(\mathfrak{g}_0)$ is a reductive subalgebra of $\mathfrak{g}(A)$. In fact, let $s_0, \ldots, s_l$ be the nonnegative integers such that the natural **Z**-grading of $L(\mathfrak{g}, \theta)$ is of type $(s_0, \ldots, s_l)$, and let $S \subset \{0, \ldots, l\}$ be the subset of those $i \in \{0, \ldots, l\}$ such that $s_i=0$. [These are just the $i \in \{0, \ldots, l\}$ for which the canonical generators e_i, f_i of $L(\mathfrak{g}, \theta)$ lie in $\mathfrak{g}_0$.] Then $\pi^{-1}(\mathfrak{g}_0)$ is exactly the subalgebra $\mathfrak{r}=\mathfrak{g}_S+\mathfrak{h}$ associated with S in paragraph 5. The center $\mathfrak{c}$ of $\mathfrak{g}(A)$ is the kernel of π, and is contained in $\mathfrak{h}$ and thus in $\mathfrak{r}$. We have natural isomorphisms $\mathfrak{g}(A)/\mathfrak{c} \simeq L(\mathfrak{g}, \theta)$ and $\mathfrak{r}/\mathfrak{c} \simeq \mathfrak{g}_0$.

Recall from paragraph 5 the Weyl group W, the subset W_S^1 of W, the length function defined on W, the subalgebra $\mathfrak{u}^-$ of $\mathfrak{g}(A)$, the element $\rho\in(\mathfrak{h}^e)^*$, and the finite-dimensional irreducible $\mathfrak{r}^e$-modules $M(\lambda)$ $(\lambda\in P_S)$. Recall also that $w\rho-\rho\in P_S$ for all $w\in W_S^1$ (*see* Th. 5.1), that $M(\lambda)$ is absolutely irreducible under $\mathfrak{r}$ for all $\lambda\in P_S$, and that for all $j\in\mathbf{Z}_+$, $H_j(\mathfrak{u}^-)$ with the standard action of $\mathfrak{r}$ is naturally isomorphic to the direct sum of inequivalent irreducible $\mathfrak{r}$-modules $\coprod M(w\rho-\rho)$, where w ranges through the set of elements of W_S^1 of length j (Th. 5.5 and Lemma 6.8). Corollaries 6.13 and 6.16 imply:

THEOREM 9.1. — *For odd* $n\in\mathbf{Z}_+$,

$$H^n(L(\mathfrak{g},\theta),\mathfrak{g}_0)=H_n(L(\mathfrak{g},\theta),\mathfrak{g}_0)=0.$$

For $n=2j$ $(j\in\mathbf{Z}_+)$, *there are natural vector space isomorphisms*

$$H^n(L(\mathfrak{g},\theta),\mathfrak{g}_0)\simeq H_n(L(\mathfrak{g},\theta),\mathfrak{g}_0)\simeq\coprod \operatorname{End}_{\mathfrak{g}_0} M(w\rho-\rho)\simeq \operatorname{End}_{\mathfrak{g}_0} H^j(\mathfrak{u}^-),$$

where the direct sum ranges over the elements $w\in W_S^1$ *of length* j. *In particular,* $H^{2j}(L(\mathfrak{g},\theta),\mathfrak{g}_0)$ *and* $H_{2j}(L(\mathfrak{g},\theta),\mathfrak{g}_0)$ *have distinguished bases, and* dim $H^{2j}(L(\mathfrak{g},\theta),\mathfrak{g}_0)$ *is the number of* $w\in W_S^1$ *of length* j, *and is also the number of irreducible* $\mathfrak{g}_0$*-module components in* $H^j(\mathfrak{u}^-)$.

Remarks. — (1) W_S^1 in the present context will be described concretely in Part IV.

(2) In Theorem 9.1, the case $\mathfrak{g}$ simple and $\theta=1$ (i.e., $L(\mathfrak{g},\theta)=\tilde{\mathfrak{g}}$ and $\mathfrak{g}_0=\mathfrak{g}$) gives $H^*(\tilde{\mathfrak{g}},\mathfrak{g})$ and $H_*(\tilde{\mathfrak{g}},\mathfrak{g})$.

(3) It is easy to see from Remark (2) at the end of paragaph 8 that Theorem 9.1 also holds when the assumption that $(\mathfrak{g},\theta)$ be indecomposable is dropped. *Cf.* also paragraph 6 and the last assertions of Propositions 4.12 and 4.13.

10. Equivariant loop spaces

Here we shall relate Euclidean Lie algebras with certain equivariant loop spaces, and we shall raise some questions.

Let X be a topological space, and let T^1 be the unit circle in the complex plane $\mathbf{C}$. Write $\Omega_u(X)$ for the space of (continuous) maps from T^1 to X, i.e., the space of unbased (or free) loops of X.

Now let G be a compact connected (real) Lie group. Then $\Omega_u(G)$ is a topological group under pointwise multiplication of functions.

Suppose that θ is an automorphism of finite period m of G, and let ε be a primitive m-th root of unity in T^1. The cyclic group $\mathbf{Z}/(m)$ of order m acts on G by sending $n+(m)$ $(n\in\mathbf{Z})$ to θ^n, and $\mathbf{Z}/(m)$ acts on T^1 by sending $n+(m)$ to multiplication by ε^n. Define $\Omega_u^\theta(G)$ to be the closed subgroup of $\Omega_u(G)$ consisting of the loops equivariant with respect to these two actions of $\mathbf{Z}/(m)$. Then

$$\Omega_u^\theta(G)=\{\omega\in\Omega_u(G)\mid \omega(\varepsilon z)=\theta\omega(z)\quad\text{for all } z\in T^1\}.$$

Note that if θ is the identity and $m=1$, we have $\Omega_u^\theta(G)=\Omega_u(G)$.

Let G_0 be the fixed set of θ in G, so that G_0 is a compact Lie subgroup of G. The set of constant maps from T^1 into G_0 forms a closed subgroup of $\Omega_u^\theta(G)$ naturally isomorphic to G_0, and we denote this closed subgroup also by G_0. Then we may form the homogeneous space $E_\theta(G)=\Omega_u^\theta(G)/G_0$ (E for "equivariant").

Examples. — (1) If $\theta=1$ and $m=1$, then $E_\theta(G)$ is clearly homeomorphic in a natural way to the ordinary (based) loop space $\Omega(G)$ of G consisting of the maps from T^1 into G sending $1\in T^1$ to the identity element of G.

(2) Let $G^{(2)}=G\times G$ and let θ be the involution of $G^{(2)}$ which takes (x, y) to (y, x) for all x, $y\in G$. The fixed set of θ is the diagonal subgroup of $G^{(2)}$ and is isomorphic to G, by projection of $G^{(2)}$ to either factor G. Then $E_\theta(G^{(2)})$ (with $m=2$) is naturally homeomorphic to the space $E_1(G)$ of Example (1), by projection of $G^{(2)}$ to either factor G. Hence by Example (1), $E_\theta(G^{(2)})$ is naturally homeomorphic to $\Omega(G)$.

(3) More generally, for $m\geqq 1$, let $G^{(m)}=G\times\ldots\times G$ (m times), and let θ be the automorphism of order m of G which takes $(x_1, \ldots, x_m)$ to $(x_m, x_1, x_2, \ldots, x_{m-1})$ for all $x_i\in G$. The fixed set of θ is the diagonal subgroup of $G^{(m)}$ and is naturally isomorphic to G, by projection of $G^{(m)}$ to any factor G. Then $E_\theta(G^{(m)})$ is naturally homeomorphic to $E_1(G)$ [*see* Ex. (1)] by projection of $G^{(m)}$ to any factor G, and so $E_\theta(G^{(m)})$ is naturally homeomorphic to $\Omega(G)$.

Let $\mathfrak{g}_{\mathbf{R}}$ be the (real) Lie algebra of G. Then the unbased loop space $\Omega_u(\mathfrak{g}_{\mathbf{R}})$ is an infinite-dimensional real Lie algebra under pointwise bracket of functions. Denote again by θ the Lie algebra automorphism of period m of $\mathfrak{g}_{\mathbf{R}}$ which is the differential of the automorphism θ of G. In a formal sense which we do not attempt to rigorize, the infinite-dimensional group $\Omega_u^\theta(G)$ has as its "Lie algebra" the infinite-dimensional real Lie algebra

$$\Omega_u^\theta(\mathfrak{g}_{\mathbf{R}})=\{\omega\in\Omega_u(\mathfrak{g}_{\mathbf{R}})\,|\,\omega(\varepsilon z)=\theta\omega(z)\quad\text{for all}\quad z\in T^1\}.$$

Note that $\Omega_u^\theta(\mathfrak{g}_{\mathbf{R}})$ consists of the maps from T^1 to $\mathfrak{g}_{\mathbf{R}}$ which are equivariant with respect to the above action of $\mathbf{Z}/(m)$ on T^1 and the action of $\mathbf{Z}/(m)$ on $\mathfrak{g}_{\mathbf{R}}$ which sends $n+(m)$ ($n\in\mathbf{Z}$) to θ^n. If $\theta=1$ and $m=1$, then $\Omega_u^\theta(\mathfrak{g}_{\mathbf{R}})=\Omega_u(\mathfrak{g}_{\mathbf{R}})$.

Write $\mathfrak{g}_{\mathbf{R},0}$ for the fixed set of θ in $\mathfrak{g}_{\mathbf{R}}$. Then the set of constant maps from T^1 into $\mathfrak{g}_{\mathbf{R},0}$ forms a Lie subalgebra of $\Omega_u^\theta(\mathfrak{g}_{\mathbf{R}})$ naturally isomorphic to $\mathfrak{g}_{\mathbf{R},0}$, and we denote this subalgebra also by $\mathfrak{g}_{\mathbf{R},0}$. Write $E_\theta(\mathfrak{g}_{\mathbf{R}})$ for the infinite-dimensional coset space $\Omega_u^\theta(\mathfrak{g}_{\mathbf{R}})/\mathfrak{g}_{\mathbf{R},0}$. In the same spirit that $\Omega_u^\theta(\mathfrak{g}_{\mathbf{R}})$ is formally the "Lie algebra" of $\Omega_u^\theta(G)$, we have that $E_\theta(\mathfrak{g}_{\mathbf{R}})$ is formally the "tangent space" at the origin $\{G_0\}$ of $E_\theta(G)=\Omega_u^\theta(G)/G_0$.

Denote by $\mathfrak{g}$, $\mathfrak{g}_0$ and θ the complexifications of $\mathfrak{g}_{\mathbf{R}}$, $\mathfrak{g}_{\mathbf{R},0}$ and θ, respectively, so that θ is an automorphism of period m of $\mathfrak{g}$ and $\mathfrak{g}_0$ is its fixed set. Then $\Omega_u(\mathfrak{g})$ is a complex Lie algebra under pointwise bracket of functions, and is the complexification of $\Omega_u(\mathfrak{g}_{\mathbf{R}})$. Moreover, the complex Lie algebra of equivariant loops

$$\Omega_u^\theta(\mathfrak{g})=\{\omega\in\Omega_u(\mathfrak{g})\,|\,\omega(\varepsilon z)=\theta\omega(z)\quad\text{for all}\quad z\in T^1\}$$

is the complexification of $\Omega_u^\theta(\mathfrak{g}_{\mathbf{R}})$, and $\mathfrak{g}_0$ may be identified with the subalgebra of constant loops in $\Omega_u^0(\mathfrak{g})$ with values in $\mathfrak{g}_0$. Write $E_\theta(\mathfrak{g})=\Omega_u^0(\mathfrak{g})/\mathfrak{g}_0$. Then $E_\theta(\mathfrak{g})$ is the complexification of $E_\theta(\mathfrak{g}_{\mathbf{R}})$, and is the "complexified tangent space" at the origin of $E_\theta(G)$. Note that $\Omega_u^\theta(\mathfrak{g})$ is the formal "complexified Lie algebra" of $\Omega_u^\theta(G)$.

Call a loop $\omega: T^1 \to \mathfrak{g}$ *algebraic* if it is the restriction of a (necessarily unique) everywhere defined algebraic morphism from the punctured plane $\mathbf{C}^*(=\mathbf{C}-\{0\})$ to $\mathfrak{g}$, and denote by $\Omega_a^\theta(\mathfrak{g})$ the set of algebraic loops in $\Omega_u^\theta(\mathfrak{g})$. Recall the notations $\tilde{\mathfrak{g}}=\mathfrak{g}\otimes_{\mathbf{C}}\mathbf{C}[t, t^{-1}]$ and $L(\mathfrak{g}, \theta)$ from paragraph 8, with $k=\mathbf{C}$. The following results are clear:

PROPOSITION 10.1. — *There is a natural Lie algebra isomorphism $\tilde{\mathfrak{g}} \to \Omega_a^1(\mathfrak{g})$ which takes $x\otimes t^i$ ($x\in\mathfrak{g}$, $i\in\mathbf{Z}$) to the map $z\mapsto z^i x$ from $\mathbf{C}^*$ (or T^1) into $\mathfrak{g}$. This isomorphism intertwines the action of $\mathbf{Z}/(m)$ on $\tilde{\mathfrak{g}}$ which takes $1+(m)\in\mathbf{Z}/(m)$ to the automorphism $x\otimes t^i\mapsto\theta(x)\otimes(\varepsilon^{-1}t)^i$ with the action of $\mathbf{Z}/(m)$ on $\Omega_a^1(\mathfrak{g})$ which takes $1+(m)$ to the automorphism $\omega(.)\mapsto\theta\omega(\varepsilon^{-1}.)$.*

PROPOSITION 10.2. — *The isomorphism in Proposition 10.1 restricts to a natural Lie algebra isomorphism $L(\mathfrak{g}, \theta) \to \Omega_a^\theta(\mathfrak{g})$ which maps $\mathfrak{g}_0\subset L(\mathfrak{g}, \theta)$ onto $\mathfrak{g}_0\subset\Omega_a^\theta(\mathfrak{g})$. $L(\mathfrak{g}, \theta)$ and $\Omega_a^\theta(\mathfrak{g})$ are exactly the centralizers of the actions of $\mathbf{Z}/(m)$ on $\tilde{\mathfrak{g}}$ and on $\Omega_a^1(\mathfrak{g})$, respectively.*

Thus $L(\mathfrak{g}, \theta)$ is the formal "complex algebraic Lie algebra" of $\Omega_u^\theta(G)$, and $L(\mathfrak{g}, \theta)/\mathfrak{g}_0$ is the formal "complex algebraic tangent space" at the origin of $E_\theta(G)$.

It is well known that if U is a compact connected Lie group and U_1 is a compact connected Lie subgroup, with complexified Lie algebras $\mathfrak{u}$ and $\mathfrak{u}_1\subset\mathfrak{u}$, respectively, then the topological cohomology $H^*(U/U_1, \mathbf{C})$ (with complex coefficients) is naturally isomorphic to the relative Lie algebra cohomology $H^*(\mathfrak{u}, \mathfrak{u}_1)$. The proof is based on the de Rham theorem and integration over a compact group.

Suppose that G is simply connected. We shall call a generalization or analogue of the above classical cohomology theory a *suitable de Rham theory* if it establishes that there is a natural isomorphism (at least of graded vector spaces):

$$H^*(E_\theta(G), \mathbf{C})\simeq H^*(L(\mathfrak{g}, \theta), \mathfrak{g}_0).$$

The cohomology on the right is the relative Lie algebra cohomology. The above heuristic discussion makes this isomorphism plausible. We shall give more evidence below.

PROBLEM 10.3. — Construct a suitable de Rham theory.

Remark. — Every automorphism θ of finite order of the complex semisimple Lie algebra $\mathfrak{g}$ (*see* paragraph 8 for the description of such θ) arises by the above process, in the following sense: It is known that θ preserves some compact real form $\mathfrak{g}_{\mathbf{R}}$ of $\mathfrak{g}$. (I thank J. Tits for this remark.) Let G be the compact simply connected Lie group corresponding to $\mathfrak{g}_{\mathbf{R}}$. Then θ exponentiates to an automorphism σ of finite order of G. Let us call the pair (G, σ) *indecomposable* if G is nontrivial and is not a direct product of proper nontrivial σ-invariant normal subgroups. Then it is clear that (G, σ) is indecomposable if and only if the corresponding pair $(\mathfrak{g}, \theta)$ is indecomposable. Moreover, G is in general a product of σ-invariant normal subgroups G_i such that the corresponding pairs (G_i, σ) are indecomposable. Note that the topological group $\Omega_u^\sigma(G)$ is the direct product of the corresponding subgroups $\Omega_u^\sigma(G_i)$, and that the space $E_\sigma(G)$ is the product of the corresponding spaces $E_\sigma(G_i)$. In view of the last assertion of Proposition 4.12, this is consistent with the solvability of Problem 10.3 [*cf.* Remark (3) at the end of paragraph 9].

Suppose that G is simple (in addition to being compact and simply connected) and that $\theta=1$ and $m=1$. Then $E_\theta(G)=\Omega(G)$ [Ex. (1) above], and $H^*(\Omega(G), \mathbf{C})$ is given by a well-known theorem of Bott. Specifically, Bott has proved [3], (*a*) using Morse theory that $\Omega(G)$ has the homotopy type of a countable CW-complex, with no odd-dimensional cells, and with known numbers of even-dimensional cells. In particular, $H^n(\Omega(G), \mathbf{C})=0$ for odd n, and $\dim H^{2j}(\Omega(G), \mathbf{C})$ is the number of cells of dimension $2j$ for each $j\in\mathbf{Z}_+$. Now the natural $\mathbf{Z}$-grading of $L(\mathfrak{g}, \theta)$ is of type $(1, 0, \ldots, 0)$ in this case [*see* Remark (1) at the end of paragraph 8], and the correspondingly determined subset S of $\{0, \ldots, l\}$ is $\{1, \ldots, l\}$ (*see* § 9). The subset W^1_S of W will be described in Part IV.

In this case, Garland has observed "empirically" in [10] that $\dim H^{2j}(\Omega(G), \mathbf{C})$, given by Bott, is exactly the number of elements of W^1_S of length j, or equivalently, the number of irreducible $\mathfrak{g}$-module components in $H^j(\mathfrak{u}^-)$ (*cf.* § 9). Combining Theorem 9.1 [and Remark (2) following it] with Bott's theorem and Garland's observation, we conclude:

THEOREM 10.4. — *There is an isomorphism of graded vector spaces*

$$H^*(\Omega(G), \mathbf{C}) \simeq H^*(\tilde{\mathfrak{g}}, \mathfrak{g}).$$

Suppose more generally that we are in the situation of Example (3) above with G simple; this includes as special cases the situations of Examples (1) and (2) with G simple. The associated Euclidean Lie algebra is $\tilde{\mathfrak{g}}$, and the associated $\mathfrak{g}_0$ is just $\mathfrak{g}$. That is, if we write $\mathfrak{g}^{(m)}$ for the complexified Lie algebra of $G^{(m)}$, then the pair $(L(\mathfrak{g}^{(m)}, \theta), \mathfrak{g}_0)$ is isomorphic to the pair $(\tilde{\mathfrak{g}}, \mathfrak{g})$. In view of Example (3) and Theorem 10.4, we thus have:

THEOREM 10.5. — *For $m\geq 1$, let $G^{(m)}=G\times\ldots\times G$ (m times), and let θ be the automorphism of order m of G which takes $(x_1, \ldots, x_m)$ to $(x_m, x_1, \ldots, x_{m-1})$ for all $x_i\in G$. Then there is an isomorphism of graded vector spaces*

$$H^*(E_\theta(G^{(m)}), \mathbf{C}) \simeq H^*(L(\mathfrak{g}^{(m)}, \theta), \mathfrak{g}_0).$$

Moreover, $E_\theta(G^{(m)})$ has the homotopy type of a countable CW-complex, with no odd-dimensional cells, and with $\dim H^{2j}(L(\mathfrak{g}^{(m)}, \theta), \mathfrak{g}_0)$ *$2j$-dimensional cells for all $j\in\mathbf{Z}_+$.*

Remarks. — (1) Granting that Problem 10.3 can be solved (and Theorems 10.4 and 10.5 give some evidence that this can be done), we have provided the promised "explanation" (*see* the last Remark in paragraph 6) of Garland's empirical observation, mentioned before Theorem 10.4, that $\dim H^{2j}(\Omega(G), \mathbf{C})$ is the number of irreducible $\mathfrak{g}$-module components in $H^j(\mathfrak{u}^-)$. This is because Theorem 9.1 and the theory behind it explain in a natural way why $\dim H^{2j}(\tilde{\mathfrak{g}}, \mathfrak{g})$ is the number of irreducible $\mathfrak{g}$-module components in $H^j(\mathfrak{u}^-)$. Our explanation is in the same spirit as Kostant's explanation [16], (*b*), (*c*) of Bott's "strange equality" (recall the last Remark in paragraph 6, where we have also pointed out that the present theory re-explains Bott's equality in the finite-dimensional special case dealt with by Kostant).

(2) Theorem 9.1, the discussion surrounding Problem 10.3, and Remark (1) suggest in striking ways that the spaces $\Omega(G)=\Omega_u(G)/G$ and more generally, the spaces $E_\theta(G)=\Omega^\theta_u(G)/G_0$, provide the correct extension to Euclidean Lie algebras of the concept of

generalized flag manifold U/U_1 (U a compact connected semisimple Lie group, U_1 the centralizer of a torus in U); *see* also the last Remark in paragraph 6. To see the analogy more clearly, note that the complexified "Lie algebras" of $\Omega_u^\theta(G)$ and G_0 are a Euclidean Lie algebra (or product of such) and the reductive part of a parabolic subalgebra, while the complexified Lie algebras of U and U_1 are a semisimple Lie algebra and the reductive part of a parabolic subalgebra. [Note that it is more illuminating to view the based loop space $\Omega(G)$ as a *quotient* $\Omega_u(G)/G$ of the unbased loop space $\Omega_u(G)$ than as a *subspace* of $\Omega_u(G)$.] More evidence for the analogy between the spaces $E_\theta(G)$ and generalized flag manifolds will be given later.

(3) Kostant's theorem on the cohomology of the nilradical of a parabolic subalgebra of a semisimple Lie algebra is well known to be equivalent to Bott's generalization of the Borel-Weil theorem on the realization of finite-dimensional irreducible modules; *see* [16], (*b*), paragraph 6. This of course suggests using the main cohomology theorem of [11], (*a*), Theorem 5.5 above, to obtain an analogue of the Borel-Weil-Bott theorem for standard irreducible modules for Euclidean (or more general) Kac-Moody Lie algebras. But note already that the present viewpoint ties together two quite different theorems of Bott in the same algebraic setting, in the following sense: The Borel-Weil-Bott-Kostant theorem is related to the cohomology theory of generalized flag manifolds (the case of semisimple Lie algebras) as Theorem 5.5 is related to Bott's cohomology theory of $\Omega(G)$ (the case of $\tilde{\mathfrak{g}}$). The transition in each case is *via* a "strange equality", explained by the present theory for all Kac-Moody Lie algebras. This analogy will be pushed further below. Note that Macdonald's identities, which are intimately related to Theorem 5.5 (*see* [11], (*a*)), form an essential part of this picture.

The following conjecture is motivated by the above discussion, the distinguished bases in Theorem 9.1, and Theorems 10.4 and 10.5 (*see* also the Remark below):

Conjecture 10.6. — Suppose that G is a compact simply connected Lie group and that θ is an automorphism of finite order of G. Then $E_\theta(G)=\Omega_u^\theta(G)/G_0$ has the homotopy type of a countable CW-complex with no odd-dimensional cells (this assertion follows from a theorem of Bott; *see* below), and with numbers of even-dimensional cells determined by a natural isomorphism

$$H^*(E_\theta(G), \mathbf{C}) \simeq H^*(L(\mathfrak{g}, \theta), \mathfrak{g}_0);$$

the structure of $H^*(L(\mathfrak{g}, \theta), \mathfrak{g}_0)$ is given in Theorem 9.1 and Remark (3) following it. The elements of length j ($j \in \mathbf{Z}_+$) in W_S^1 naturally index both the cells of dimension $2j$ in $E_\theta(G)$, as well as a natural basis of $H^{2j}(E_\theta(G), \mathbf{C})$ constructed from the natural basis of $\mathrm{End}_{\mathfrak{g}_0} H^j(\mathfrak{u}^-)$ (*see* Th. 9.1). Each such basis element of $H^{2j}(E_\theta(G), \mathbf{C})$ is realized by a "closed differential form" on $E_\theta(G)$ whose integral over the corresponding homology cell is nonzero and whose integral over every other cell is zero.

Remark. — This conjecture was first formulated for the case in which θ has order 2, so that G/G_0 is the most general compact simply connected symmetric space. The bulk of the Morse-theoretic work needed to construct the even-dimensional cell decomposition in this case was done by Samelson and Shahshahani. After this, Bott informed me of an unpublished Morse-theoretic result of his (Th. 11.1 below) which implies the even-dimensional cell structure in a situation even more general that of Conjecture 10.6 (*see* § 11). In fact, Bott's result was one of the stimuli to formulate the conjecture (part of which is Bott's Theorem) in the above form. It appears to require nontrivial work to translate the Morse-theoretic counting procedure for the Betti numbers into the present one. In addition, the conjectured naturality of the isomorphism

$$H^*(E_\theta(G), \mathbf{C}) \simeq H^*(L(\mathfrak{g}, \theta), \mathfrak{g}_0)$$

would involve more – presumably the solution of Problem 10.3. The conjecture on the construction of the "closed differential forms" from the natural basis of $\mathrm{End}_{\mathfrak{g}_0} H^j(\mathfrak{u}^-)$ is suggested partly by the paper of Garland-Raghunathan [12], (especially its last paragraph) and the work of Kostant [16], (*c*) on (finite-dimensional) generalized flag manifolds. The even-dimensional cells of $E_\theta(G)$ should undoubtedly be constructed using a generalized Bruhat decomposition as in Garland-Raghunathan [12] and Quillen [unpublished], as well as *via* Morse theory. This algebraic approach to the cell decomposition might also lead to a solution of Problem 10.3; that is, it might be possible to prove that $E_\theta(G)$ has the same homotopy type as a suitable "algebraic subspace" (*cf.* [12]) for which the desired de Rham theory can be carried out. Note that since the "closed differential forms" come from $H^j(\mathfrak{u}^-)$, they are closely related to Macdonald's identities; *cf.* [11], (*a*) and Part IV below.

11. Bott's theorem

In response to a preliminary version of Conjecture 10.6, Bott informed me of Theorem 11.1 below, which he had proved around 1960 [unpublished]. After stating the result, I shall relate it to the material in paragraph 10. I am grateful to W. Dwyer for helping me to formulate some of the arguments in this section.

As in paragraph 10, let G be a compact simply connected Lie group. Let θ be an automorphism of G, *not* necessarily of finite order, and consider the *θ-twisted adjoint action* of G on itself, given by $\mathrm{Ad}_\theta\, g(x) = \theta(g)\, x g^{-1}$ ($g, x \in G$). Denote by e the identity element of G, by I the interval [0, 1], and by PG the space of paths $\omega : I \to G$ such that $\omega(0) = e$. For every subset S of G, define the space

$$\Omega_S = \{ \omega \in PG \mid \omega(1) \in S \}.$$

THEOREM 11.1 (Bott). — *Let* S *be any θ-twisted adjoint* G*-orbit in* G. *Then Ω_S has the homotopy type of a countable* CW*-complex with no odd-dimensional cells.*

Remark. — There exists a procedure for determining the Betti numbers of Ω_S; *cf.* [26].

We now begin relating this result to the equivariant loop spaces of paragraph 10.

PROPOSITION 11.2. — *For all $x \in G$, let $\mathrm{Ad}\, x = \mathrm{Ad}_1\, x$ (the inner automorphism of* G *defined by x). Let* R *be the θ-twisted orbit $\mathrm{Ad}_\theta\, G(x)$, let $\tau = (\mathrm{Ad}\, x^{-1})\,\theta$, and let* S *be the τ-twisted orbit $\mathrm{Ad}_\tau\, G(e)$. Then Ω_R and Ω_S have the same homotopy type.*

Proof. — For all $g \in G$,

$$\mathrm{Ad}_\tau g(e) = \tau(g) g^{-1} = x^{-1} \theta(g) x g^{-1} = x^{-1} \mathrm{Ad}_\theta g(x).$$

Thus $x S = R$, and so Ω_S is homeomorphic to the space of paths from x to R. The rest is clear.

Q.E.D.

Note that Proposition 11.2 reduces Theorem 11.1 to the special case in which S is the θ-twisted adjoint G-orbit of e. Call this orbit S_θ, so that $S_\theta = \{ \theta(g) g^{-1} \mid g \in G \}$, and S_θ is a compact submanifold of G. Let G_0 be the fixed set of θ. Then the isotropy subgroup of G at e for the θ-twisted adjoint action is exactly G_0, and so we have a diffeomorphism

$$\iota : \ G/G_0 \to S_\theta$$
$$g G_0 \mapsto \theta(g) g^{-1}.$$

For a path $\omega : I \to G$ and $g \in G$, denote by $\omega g : I \to G$ the path $(\omega g)(t) = \omega(t) g$ for all $t \in I$. Let A_θ be the path space

$$A_\theta = \{ \omega : I \to G \mid \omega(1) = \theta(\omega(0)) \}.$$

Then G_0 acts on the right on A_θ by

$$\omega \mapsto \omega g_0.$$

PROPOSITION 11.3. — *The map*

$$h : \ A_\theta / G_0 \to \Omega_{S_\theta}$$
$$\omega G_0 \mapsto \omega(\omega(0)^{-1})$$

is well-defined and is a homeomorphism. The inverse of h is the (well-defined) map which takes $\omega \in \Omega_{S_\theta}$ *to* $(\omega g) G_0$, *where* $g G_0 = \iota^{-1}(\omega(1))$.

Proof. — Write $i : G/G_0 \to G$ for the injection defined by ι, write j for the fibration $PG \to G$ which takes $\omega \in PG$ to $\omega(1)$, and let P be the pullback defined by the diagram

$$\begin{array}{ccc} P & \rightarrow & PG \\ \downarrow & & \downarrow{\scriptstyle j} \\ G/G_0 & \xrightarrow{i} & G \end{array}$$

Then P is the space of pairs $(g G_0, \omega)$ such that $g \in G$, $\omega \in PG$ and $i(g G_0) = j(\omega)$, i. e., $\omega(1) = \theta(g) g^{-1}$. The projection map $P \to PG$ is clearly a homeomorphism from P onto Ω_{S_θ}.

Let B be the space of pairs

$$B = \{ (g, \omega) \mid g \in G, \ \omega \in PG, \ \omega(1) = \theta(g) g^{-1} \}.$$

Then G_0 acts on the right on B by the (well-defined) action

$$(g, \omega) g_0 = (g g_0, \omega),$$

and $B/G_0 = P$.

Consider the map

$$p: \quad A_\theta \to B,$$
$$\omega \mapsto (\omega(0), \omega(\omega(0)^{-1}))$$

and the map

$$q: \quad B \to A_\theta$$
$$(g, \omega) \mapsto \omega g.$$

Then p and q are well defined and are inverses of each other, and hence are homeomorphisms. Moreover, p and q each commute with the right actions of G_0 on A_θ and B. Thus p defines a homeomorphism from A_θ/G_0 onto P, and hence a homeomorphism

$$h: \quad A_\theta/G_0 \to \Omega_{S_\theta},$$
$$\omega G_0 \mapsto \omega(\omega(0)^{-1}).$$

The rest is clear.

Q.E.D.

Now A_θ is a topological group, under pointwise multiplication of functions from I to G. The constant paths with images in G_0 form a closed subgroup of A_θ isomorphic to G_0, and the above right action of G_0 on A_θ may be identified with the corresponding right multiplication in A_θ. Thus A_θ/G_0 may be regarded as a coset space.

Suppose now that θ has finite period m, and that ε is a primitive m-th root of unity in T^1, as in paragraph 10. Recall from paragraph 10 the closed subgroup $\Omega_u^\theta(G)$ of the topological group $\Omega_u(G)$, and also the homogeneous space $E_\theta(G) = \Omega_u^\theta(G)/G_0$. We clearly have:

PROPOSITION 11.4. — *The map*

$$d: \quad A_\theta \to \Omega_u^\theta(G),$$
$$\omega \mapsto \tau$$

where τ is the unique element of $\Omega_u^\theta(G)$ such that $\tau(\varepsilon^t) = \omega(t)$ for all $t \in I$, is well-defined and is an isomorphism of topological groups. The image of the subgroup G_0 of A_θ is the subgroup G_0 of $\Omega_u^\theta(G)$.

Propositions 11.3 and 11.4 imply:

COROLLARY 11.5. — *The spaces $E_\theta(G)$ and Ω_{S_θ} are naturally homeomorphic. Specifically, the map*

$$f: \quad \Omega_u^\theta(G)/G_0 \to \Omega_{S_\theta},$$
$$\tau G_0 \mapsto (d^{-1}\tau)\tau(1)^{-1}$$

is well-defined and is a homeomorphism. The inverse of f is the (well-defined) map which takes $\omega \in \Omega_{S_\theta}$ to $(d(\omega g)) G_0$, where $g G_0 = \iota^{-1}(\omega(1))$.

Remarks. — (1) Theorem 11.1 and Corollary 11.5 justify the assertion in Conjecture 10.6 about the cell structure of $E_\theta(G)$.

(2) For θ of finite order, Corollary 11.5 and the discussion in paragraph 10 show that the path spaces Ω_{S_θ} are the Euclidean Lie algebra analogues of generalized flag manifolds.

(3) It seems likely that for arbitrary θ not necessarily of finite order, there is a suitable automorphism σ of finite order of G such that Ω_{S_σ} and Ω_{S_θ} have the same homotopy type. If this is true, then Proposition 11.2, Corollary 11.5 and Remark (2) show that the most general space Ω_S considered in Theorem 11.1 is essentially a generalized flag manifold associated with a Euclidean Lie algebra.

(4) Theorem 11.1 appears to be closely related to Macdonald's identities, in view of the above Remarks and the comments preceding and following Conjecture 10.6.

PART IV
AUTOMORPHISMS OF FINITE ORDER AND SPECIALIZATIONS OF MACDONALD'S IDENTITIES

12. The affine root system T

Our next goal is to associate a power series identity to each automorphism of finite order of a simple Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic zero (*see* Th. 13.15). The identity is a specialization of a suitably rewritten Macdonald identity, and it leads in paragraphs 16 and 17 to new η-function identities. The material in Part IV is based on Macdonald's results [21], on Kac's and Moody's recognition ([15], (*c*) and [22], (*c*)) that Macdonald's identities amount to Weyl's denominator formula for the Euclidean Lie algebras, as well as on Kac's description [15], (*b*) of the automorphisms of finite order. We must quote frequently from [21] and [22], (*c*). The concrete description of W_S^1 promised in paragraph 9 is given in paragraph 13.

Throughout Part IV, our field k of characteristic zero is assumed to be algebraically closed. Let $L(\mathfrak{g}, \theta)$ be a standard Euclidean Lie algebra, defined as in paragraph 8. Then $\mathfrak{g}$ is a simple Lie algebra of type X_n over k, θ is an automorphism of $\mathfrak{g}$ induced by an automorphism of order j ($j=1, 2, 3$) of the Dynkin diagram X_n, and $L(\mathfrak{g}, \theta)$ has Dynkin diagram $X_n^{(j)}$ in Table j of paragraph 8. $L(\mathfrak{g}, \theta)$ is a certain subalgebra of $\tilde{\mathfrak{g}}$. As in paragraph 8, let $\mathfrak{g}_0$ be the centralizer of θ in $\mathfrak{g}$, so that $\mathfrak{g}_0$ is a simple Lie algebra of rank l which is naturally identified with a certain subalgebra of $L(\mathfrak{g}, \theta)$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$, and let $\alpha_0, \ldots, \alpha_l$ be the corresponding simple weights of $L(\mathfrak{g}, \theta)$; the α_i are linear functionals on $\mathfrak{h}_1 = k\,D \oplus \mathfrak{h}_0$, as in paragraph 8. Since $L(\mathfrak{g}, \theta)$ is standard, we have $\alpha_0(D)=1$ and $\alpha_i(D)=0$ for $i=1, \ldots, l$, by our convention in paragraph 8 which assigns the index 0 to the "adjoined" vertex in the Dynkin diagram $X_n^{(j)}$.

Let A be the Cartan matrix of $L(\mathfrak{g}, \theta)$, $\mathfrak{g}(A)$ the corresponding Kac-Moody Lie algebra defined in paragraph 5, $\mathfrak{c}$ the (one-dimensional) center of $\mathfrak{g}(A)$, and

$$0 \to \mathfrak{c} \to \mathfrak{g}(A) \xrightarrow{\pi} L(\mathfrak{g}, \theta) \to 0$$

the exact sequence defined by the condition that π take the canonical generators h_i, e_i, f_i of $\mathfrak{g}(A)$ (§ 5) to the corresponding canonical generators h'_i, e_i, f_i of $L(\mathfrak{g}, \theta)$ (*see* § 8). Then π maps the $(l+1)$-dimensional subalgebra $\mathfrak{h} = \text{span}\ \{h_i\}$ of $\mathfrak{g}(A)$ onto the l-dimensional subalgebra $\mathfrak{h}_0$ of $L(\mathfrak{g}, \theta)$, and π maps each root space of $\mathfrak{g}(A)$ isomorphically onto the corresponding weight space of $L(\mathfrak{g}, \theta)$. The set Δ of roots of $\mathfrak{g}(A)$ [defined as in paragraph 5 as elements of $(\mathfrak{h}^e)^*$] may, and will, be identified with the set of weights of $L(\mathfrak{g}, \theta)$ in $(\mathfrak{h}_1)^*$. Under this identification, the set Δ_+ of positive roots becomes the set of positive weights, and the roots α_i of paragraph 5 become the simple weights $\alpha_0, \ldots, \alpha_l$ of paragraph 8. We shall make use of the following objects defined in paragraph 5 : the set Δ_- of negative roots, the Weyl group W (which acts on the rational span of Δ), the simple reflections $r_0, \ldots, r_l$, which generate W, and the set Δ_R of real roots (the W-transforms of $\alpha_0, \ldots, \alpha_l$). For all $w \in W$, we have $\Phi_w = \Delta_+ \bigcap w\Delta_-$, which is a finite subset of $\Delta_R \bigcap \Delta_+$. The number of elements in Φ_w is the length of w. The sum of the elements of Φ_w is $\langle \Phi_w \rangle$.

Denote by $\mathfrak{h}_{0,Q}$ the rational span of the vectors $h'_0, \ldots, h'_l$ in $\mathfrak{h}_0$. The restriction $(.,.)$ to $\mathfrak{h}_0$ of the Killing form of $\mathfrak{g}$ is rational-valued and positive definite on $\mathfrak{h}_{0,Q}$ (*see* § 8). Let V be the real vector space $\mathfrak{h}_{0,Q} \otimes_Q \mathbf{R}$, with the positive definite bilinear form on V still denoted $(.,.)$. Define a norm on V by the condition $\|x\| = (x, x)^{1/2}$. Now regard the set V as an affine space E on which V acts faithfully and transitively by translations. Then E is a Euclidean space with metric given by the distance function $\|x-y\|$.

We recall some elementary notions about Euclidean affine spaces from [21]: An affine-linear functional on E is by definition a map $f: E \to \mathbf{R}$ for which there is a linear functional $\partial f \in V^*$, called the derivative of f, such that $f(x+v) = f(x) + (\partial f)(v)$ for all $x \in E$ and $v \in V$. We use the form $(.,.)$ on V to identify V and V*. Then we may regard $\partial f \in V$. Let F be the $(l+1)$-dimensional space of affine-linear functionals on E. Then F carries a bilinear form defined by $(f, g) = (\partial f, \partial g)$. This form is positive semidefinite, and the isotropic elements of F are the constant functions.

An affine-linear mapping from E to itself is a map $f : E \to E$ such that there is a linear map $\partial f \in \text{End } V$ (the derivative of f) such that $f(x+v) = f(x) + (\partial f)(v)$ for all $x \in E$ and $v \in V$. For each non-constant $f \in F$, we define

$$f^\vee = 2f/(f, f),$$

and we define the affine-linear isometry $w_f : E \to E$ by the condition

$$w_f(x) = x - f^\vee(x)\,\partial f$$

for all $x \in E$. Then w_f is the reflection through the affine hyperplane in E on which f vanishes. The reflection w_f also acts in a natural way on F by the rule

$w_f(g)=g\circ w_f^{-1}=g\circ w_f$ for all $g\in \mathrm{F}$. That is,

$$w_f(g)=g-(f^{\vee}, g)f=g-(f, g)f^{\vee}.$$

More generally, any affine-linear mapping $w : \mathrm{E}\to \mathrm{E}$ acts *via* transposition on F, by the condition $w(g)=g\circ w^{-1}$ for all $g\in \mathrm{F}$.

Following Macdonald [21], we define an *affine root system* on E to be a subset T (whose elements are the *affine roots*) of F satisfying the following conditions: T spans F; the elements of T are nonisotropic, i. e., nonconstant; $w_a \mathrm{T}=\mathrm{T}$ for all $a\in \mathrm{T}$; $(a, b^{\vee})\in \mathbf{Z}$ for all $a, b\in \mathrm{T}$; and the *Weyl group* W (T) of affine-linear isometries of E generated by the reflections w_a ($a\in \mathrm{T}$) acts properly on E, i. e., if K_1 and K_2 are compact subsets of E, then the set of elements $w\in \mathrm{W}(\mathrm{T})$ such that $w(\mathrm{K}_1)$ meets K_2 is finite.

We shall follow Moody [22], (*c*) in identifying the set Δ_{R} of real roots with an affine root system on E: Let Q be the $(l+1)$-dimensional real vector space $\mathbf{Q}\Delta\otimes_{\mathbf{Q}}\mathbf{R}$. Then $\alpha_0, \ldots, \alpha_l$ form a basis of Q. Since $\alpha_0(\mathrm{D})=1$ and $\alpha_i(\mathrm{D})=0$ for $i=1, \ldots, l$, evaluation at D is a well-defined linear functional on Q; this functional is just the 0-th dual basis element. Also, restriction to $\mathfrak{h}_0\subset\mathfrak{h}_1$ gives rise to a well-defined linear map δ from Q to the real vector space V*. We define a linear map $\zeta : \mathrm{Q}\to \mathrm{F}$ as follows: For all $\alpha\in \mathrm{Q}$, let $\zeta(\alpha)$ be the affine-linear functional on E which takes $x\in \mathrm{E}$ to $\alpha(\mathrm{D})+\delta(\alpha)(x)$. The derivative of $\zeta(\alpha)$ is $\delta(\alpha)\in \mathrm{V}^*$. The map ζ is clearly a linear isomorphism from Q to F.

Let $\mathrm{T}=\zeta(\Delta_{\mathrm{R}})$. It is shown in [22], (*c*) that T is an affine root system on E. Moreover, T is *irreducible* in the sense that it is not empty and not the direct sum of two or more nonempty affine root systems (*see* [21], § 3 for the precise definition) and *reduced* in that for every $a\in \mathrm{T}$, the only affine roots proportional to a are $\pm a$. Also, the isomorphism ζ sets up an isomorphism between W and W(T), and intertwines the actions of W on Q and of W(T) on F, in the following sense: For all $\alpha\in\Delta_{\mathrm{R}}$, let $r_\alpha\in \mathrm{W}$ be the Weyl reflection with respect to α, regarded as acting on Q. Then for all $\beta\in \mathrm{Q}$,

$$\zeta(r_\alpha \beta)=w_{\zeta(\alpha)}\zeta(\beta).$$

Thus the correspondence $r_\alpha\mapsto w_{\zeta(\alpha)}$ defines an isomorphism from W to W(T). We shall denote this isomorphism by ζ, so that in particular, $\zeta(r_\alpha)=w_{\zeta(\alpha)}$ for all $\alpha\in\Delta_{\mathrm{R}}$.

For all $i=0, \ldots, l$, let $a_i=\zeta(\alpha_i)$. Note that $2(a_i, a_j)/(a_i, a_i)=\mathrm{A}_{ij}$ (the Cartan matrix entry) for all i, j. Define the set $\mathrm{C}\subset \mathrm{E}$ as follows:

$$\mathrm{C}=\{x\in \mathrm{E}\mid a_i(x)>0 \text{ for all } i=0, \ldots, l\}.$$

Then C is an alcove (or chamber) in E (*see* [21], [22], (*c*)), and the walls of C are the affine hyperplanes on which $a_0, \ldots, a_l$ vanish. C is a nonempty open l-simplex. Let $x_0, \ldots, x_l$ be the vertices of C, with the vertex x_i opposite the wall on which a_i vanishes. Then $a_i(x_j)=0$ whenever $i\neq j$; $a_i(x_i)$ is a positive rational number for all i; $x_0=0\in \mathrm{E}$ ($=\mathrm{V}$ as a set); and the elements of C are the points $\sum_{i=0}^{l} c_i x_i$ with $c_i\in\mathbf{R}$, $\sum_{i=0}^{l} c_i=1$ and each $c_i>0$. The images under ζ of the simple reflections $r_0, \ldots, r_l\in \mathrm{W}$ are the simple reflections $w_{a_0}, \ldots, w_{a_l}$

in W(T) defined by the walls of C, and these $l+1$ reflections generate W(T). Denote by $l(.)$ the corresponding length function on W(T), so that for all $w \in W(T)$, $l(w)=l(\zeta^{-1} w)$. Let $T_+ = \zeta(\Delta_R \cap \Delta_+)$ and $T_- = \zeta(\Delta_R \cap \Delta_-)$, and call these the sets of *positive* and *negative* affine roots, respectively. Then every positive affine root is a nonnegative integral linear combination of $a_0, \ldots, a_l$ and is positive on C, and every negative affine root is a nonpositive integral linear combination of $a_0, \ldots, a_l$ and is negative on C. For all $w \in W(T)$, define

$$T(w) = T_+ \cap w T_-.$$

Then $T(w) = \zeta \Phi_{\zeta^{-1}w}$, and so $T(w)$ is a finite set with $l(w)$ elements. Let

$$s(w) = \sum_{a \in T(w)} a.$$

Then $s(w) = \zeta \langle \Phi_{\zeta^{-1}w} \rangle$. The derivatives $\partial a_1, \ldots, \partial a_l \in V$ are a basis of simple roots of the gradient root system $\partial T = \{ \partial a \mid a \in T \}$ of T.

By [21], Prop. 7.1, there is a unique element $r \in E$ on which the functions $\check{a}_0, \ldots, \check{a}_l$ all take the same value. The point r lies in C, and the common value is denoted g^{-1}, where g is a positive rational number. Proposition 7.5 of [21] asserts:

PROPOSITION 12.1. — *For all $w \in W(T)$ and $x \in E$, we have*

$$s(w)(x) = \frac{1}{2} g (\| wr - x \|^2 - \| r - x \|^2).$$

13. Specializations of Macdonald's identities

We continue in the setting of paragraph 12.

Macdonald's identities state:

THEOREM 13.1. — *We have*

$$\prod_{\varphi \in \Delta_+} (1 - e(-\varphi))^{\dim \mathfrak{g}(A)^{\varphi}} = \sum_{w \in W} (-1)^{l(w)} e(-\langle \Phi_w \rangle).$$

Here $\mathfrak{g}(A)^{\varphi}$ is the root space in $\mathfrak{g}(A)$ for the root φ, $l(w)$ is the length of w, and the symbol $e(.)$ is a formal exponential. The identity takes place in the formal power series ring $\mathbf{Z}[[e(-\alpha_0), \ldots, e(-\alpha_l)]]$ in the $l+1$ analytically independent variables $e(-\alpha_i)$.

The reader is referred to [22], (*c*) for a translation of Macdonald's original version of the identities into the above version, and to [15], (*c*) for the generalization to all Kac-Moody Lie algebras. In [11], (*a*), § 9, Th. 13.1 (for all Kac-Moody Lie algebras) is derived by the Euler-Poincaré principle from either the homology theorem (Th. 5.5 above) or the resolution (Th. 5.1 above).

Notation. — Write $u_i = e(-\alpha_i)$ for all $i = 0, \ldots, l$.

Each Macdonald identity may be written as an equality between two formal power series in $u_0, \ldots, u_l$. It is convenient to introduce the following:

Notation. — For every integral linear combination $\varphi=\sum_{i=0}^{l} c_i \alpha_i$ of the α_i, let $\xi_j(\varphi)=c_j (j=0, \ldots, l)$. For every integral linear combination $a=\sum_{i=0}^{l} c_i a_i$ of the a_i, also let $\xi_j(a)=c_j (j=0, \ldots, l)$. Note that under the identification ζ, the two maps ξ_j agree.

Macdonald's identity for $\mathfrak{g}(A)$ can clearly be rewritten in two ways as follows:

COROLLARY 13.2. — *In* $\mathbf{Z}[[u_0, \ldots, u_l]]$,

$$\prod_{\varphi \in \Delta_+} \left(1-\prod_{i=0}^{l} u_i^{\xi_i(\varphi)}\right)^{\dim \mathfrak{g}(A)^{\varphi}} = \sum_{w \in W} (-1)^{l(w)} \prod_{i=0}^{l} u_i^{\xi_i(\langle \Phi_w \rangle)} \tag{1}$$

and

$$\prod_{\varphi \in \Delta_+} \left(1-\prod_{i=0}^{l} u_i^{\xi_i(\varphi)}\right)^{\dim \mathfrak{g}(A)^{\varphi}} = \sum_{w \in W(T)} (-1)^{l(w)} \prod_{i=0}^{l} u_i^{\xi_i(s(w))}. \tag{2}$$

DEFINITION. — Let $(s_0, \ldots, s_l)$ be a sequence of nonnegative integers. Let q be an indeterminate. The homomorphism of power series rings

$$\mathbf{Z}[[u_0, \ldots, u_l]] \to \mathbf{Z}[[q]]$$

which sends u_i to q^{s_i} for all $i=0, \ldots, l$ is called the *q-specialization of type* $(s_0, \ldots, s_l)$. (It is not everywhere defined.)

We clearly have:

PROPOSITION 13.3. — *The q-specialization of type* $(s_0, \ldots, s_l)$ *of Corollary* 13.2 (2) *asserts:*

$$\prod_{\varphi \in \Delta_+} \left(1-q^{\sum_{i=0}^{l} s_i \xi_i(\varphi)}\right)^{\dim \mathfrak{g}(A)^{\varphi}} = \sum_{w \in W(T)} (-1)^{l(w)} q^{\sum_{i=0}^{l} s_i \xi_i(s(w))}.$$

Remark. — If some $s_i=0$, then this last equation merely says $0=0$. Indeed, the factor on the left corresponding to $\varphi=\alpha_i$ is $1-q^0=0$. We shall now remedy this defect. The following discussion (through Th. 13.4) holds in the generality of all Kac-Moody Lie algebras.

Let S be an arbitrary proper subset of $\{0, \ldots, l\}$. (For general Kac-Moody Lie algebras, we would choose S to be a subset of finite type, in the sense of [11], (*a*), § 3.) Recall from paragraph 5 the subsets Δ^S, Δ_+^S and $\Delta_+(S)$ of Δ_+, the subset W_S^1 of W, the reductive subalgebra $\mathfrak{r}^e$ of $\mathfrak{g}(A)^e$, the absolutely irreducible $\mathfrak{r}^e$-modules $M(\lambda)$ ($\lambda \in P_S$), and the fact that $-\langle \Phi_w \rangle = w\rho-\rho \in P_S$ for all $w \in W_S^1$. Note that for $w \in W_S^1$, $M(-\langle \Phi_w \rangle)$ may be regarded as an absolutely irreducible module for the reductive Lie algebra $\mathfrak{r}/\mathfrak{c}$. Also, let W_S be the subgroup of W generated by the simple reflections r_i for $i \in S$. Then W_S is the Weyl group of $\mathfrak{r}^e$ (and also of $\mathfrak{r}/\mathfrak{c}$), and is finite. Moreover, every element $w \in W$ can be written uniquely in the form $w=w_1 w^1$, where $w_1 \in W_S$ and $w^1 \in W_S^1$, and $l(w)=l(w_1)+l(w^1)$ (*see* [11], (*a*), p. 66). For every $\lambda \in P_S$, we denote by $\chi M(\lambda)$ the character of $M(\lambda)$, i.e., the finite formal exponential sum in the integral group ring of $(\mathfrak{h}^e)^*$:

$$\chi\,\mathrm{M}(\lambda)=\sum_{\mu\in(\mathfrak{h}^e)^*}\dim \mathrm{M}(\lambda)_\mu\, e(\mu),$$

where $\mathrm{M}(\lambda)_\mu$ is the μ-weight space of $\mathrm{M}(\lambda)$. We shall prove the following reformulation of Theorem 13.1:

THEOREM 13.4. — *We have*

$$\prod_{\varphi\in\Delta_+(\mathrm{S})}(1-e(-\varphi))^{\dim \mathfrak{g}(\mathrm{A})^\varphi}=\sum_{w\in \mathrm{W}_\mathrm{S}^1}(-1)^{l(w)}\chi\,\mathrm{M}(-\langle\Phi_w\rangle),$$

an identity in $\mathbf{Z}[[u_0,\ \ldots,\ u_l]]$.

Proof. — The left-hand side in Theorem 13.1 may be written

$$\prod_{\varphi\in\Delta_+^\mathrm{S}}(1-e(-\varphi))\prod_{\varphi\in\Delta_+(\mathrm{S})}(1-e(-\varphi))^{\dim \mathfrak{g}(\mathrm{A})^\varphi},$$

since $\dim \mathfrak{g}(\mathrm{A})^\varphi=1$ for all $\varphi\in\Delta_+^\mathrm{S}$, these φ being the positive roots of the finite-dimensional reductive Lie algebra $\mathfrak{r}^e$. In view of [11], (*a*), Prop. 2.5, the right-hand side equals

$$\sum_{w^1\in \mathrm{W}_\mathrm{S}^1}\sum_{w_1\in \mathrm{W}_\mathrm{S}}(-1)^{l(w^1)}(-1)^{l(w_1)}e(w_1 w^1\rho-\rho)$$

$$=\sum_{w^1\in \mathrm{W}_\mathrm{S}^1}(-1)^{l(w^1)}\sum_{w_1\in \mathrm{W}_\mathrm{S}}(-1)^{l(w_1)}e(w_1(w^1\rho-\rho)+(w_1\rho-\rho))$$

$$=\sum_{w^1\in \mathrm{W}_\mathrm{S}^1}(-1)^{l(w^1)}\sum_{w_1\in \mathrm{W}_\mathrm{S}}(-1)^{l(w_1)}e(w_1(-\langle\Phi_{w^1}\rangle)-\langle\Phi_{w_1}\rangle).$$

Now simply divide both sides by $\prod_{\varphi\in\Delta_+^\mathrm{S}}(1-e(-\varphi))$ and apply Weyl's character formula for the finite-dimensional irreducible $\mathfrak{r}^e$-modules with highest weights $-\langle\Phi_{w^1}\rangle$ $(w^1\in \mathrm{W}_\mathrm{S}^1)$.

Q.E.D.

We shall next determine the images under the correspondence ζ of W_S^1, W_S, etc.

Denote by F_S the subspace of F consisting of the affine-linear functionals on E which vanish at x_i for all $i\in\{0,\ \ldots,\ l\}$ *not* in S, and set $\mathrm{T}^\mathrm{S}=\mathrm{T}\cap \mathrm{F}_\mathrm{S}$. Let $\mathrm{W}(\mathrm{T})_\mathrm{S}$ be the subgroup of $\mathrm{W}(\mathrm{T})$ which fixes x_i for all $i\notin \mathrm{S}$. As in [21], p. 102, we have that the bilinear form $(.,\ .)$ is positive definite on F_S; T^S is a (finite, reduced) root system in F_S; $\{a_i\mid i\in \mathrm{S}\}$ is a basis of T^S; and $\mathrm{W}(\mathrm{T})_\mathrm{S}$ is the Weyl group of T^S.

The Weyl chambers of the root system T^S may be identified with the closures of the connected components in E of the complement of the union of the vanishing hyperplanes of the affine roots in T^S. The dominant chamber for T^S is the set

$$\mathrm{C}_\mathrm{S}=\{x\in \mathrm{E}\mid a_i(x)\geqq 0\quad \text{for all}\quad i\in \mathrm{S}\}.$$

It is clear that $\zeta(\Delta^\mathrm{S})=\mathrm{T}^\mathrm{S}$; $\zeta(\Delta_+^\mathrm{S})=\mathrm{T}_+^\mathrm{S}$ $(=\mathrm{T}_+\cap \mathrm{T}^\mathrm{S})$; $\zeta(\Delta_+(\mathrm{S})\cap\Delta_\mathrm{R})=\mathrm{T}_+-\mathrm{T}_+^\mathrm{S}$; and $\zeta(\mathrm{W}_\mathrm{S})=\mathrm{W}(\mathrm{T})_\mathrm{S}$. The following result is the concrete description of W_S^1 promised in Part III:

PROPOSITION 13.5. — *The image of* W_S^1 *under the isomorphism* $\zeta: W \to W(T)$ *is the set* $W(T)_S^1$ *of elements of* $W(T)$ *which take the (dominant) alcove* C *into the (dominant) chamber* C_S.

Proof. — Let $w \in W$. By [11], (a), Prop. 8.1, $w \in W_S^1$ if and only if $w^{-1} \Delta_+^S \subset \Delta_+$, i.e., if and only if $w^{-1} \alpha_i \in \Delta_+$ for all $i \in S$, i.e., if and only if $\zeta(w^{-1} \alpha_i) = \zeta(w)^{-1} a_i$ takes nonnegative values on C for all $i \in S$, i.e., if and only if a_i is nonnegative on $\zeta(w)(C)$ for all $i \in S$.

Q.E.D.

Remark. — It follows that every element of $W(T)$ can be written uniquely in the form $w_1 w^1$, where $w_1 \in W(T)_S$ and $w^1 \in W(T)_S^1$.

In order to understand $\chi M(-\langle \Phi_w \rangle)$ in Theorem 13.4, we prove the following:

PROPOSITION 13.6. — *Recall the meanings of* r *and* g *from Proposition* 12.1. *For all* $w \in W$ *and* $i = 0, \ldots, l$, *we have*

$$(w\rho)(h_i) = g \check{a_i}(\zeta(w)(r))$$

and

$$-\langle \Phi_w \rangle (h_i) = g(\partial \check{a_i})(\zeta(w) r - r) = g \check{a_i}(\zeta(w)(r)) - 1.$$

Proof. — It is sufficient to prove the first formula, for which we use induction on $l(w)$. The formula is certainly true if $l(w) = 0$. Given $w' \in W$ with $l(w') > 0$, write w' in the form $r_j w$, where $j = 0, \ldots, l$ (r_j being the j-th simple reflection) and $w \in W$ has length one less than that of w'. Suppose that the formula is true for w. Then

$$\begin{aligned}(w'\rho)(h_i) &= (r_j w\rho)(h_i) = (w\rho - (w\rho)(h_j)\alpha_j)(h_i) \\ &= (w\rho)(h_i) - \alpha_j(h_i)(w\rho)(h_j) = g\check{a_i}(\zeta(w)(r)) - A_{ij} g \check{a_j}(\zeta(w)(r)) \\ &= g(\check{a_i} - A_{ij}\check{a_j})(\zeta(w)(r)) = g(w_{a_j}(\check{a_i}))(\zeta(w)(r)) \\ &= g\check{a_i}(w_{a_j}\zeta(w)(r)) = g\check{a_i}(\zeta(r_j)\zeta(w)(r)) = g\check{a_i}(\zeta(w')(r)),\end{aligned}$$

and so the formula is true for w'.

Q.E.D.

COROLLARY 13.7. — *For all* $w \in W$ *and* $i = 0, \ldots, l$, *we have*

$$\langle \Phi_w \rangle (h_i) = (\partial \check{a_i}, \partial s(\zeta(w))).$$

Proof. — Combine the second formula in Proposition 13.6 with [21], Cor. 7.6; recall that V and V^* are identified by means of $(.,.)$.

Q.E.D.

The following is clear:

PROPOSITION 13.8. — *The set* ∂T^S *of derivatives of elements of* T^S *forms a root system in* V, *except perhaps that* ∂T^S *might not span* V. *The* ∂a_i *for* $i \in S$ *form a basis of* ∂T^S, *and* ∂T_+^S *is the corresponding positive system. The Weyl group is* $\partial W(T)_S$. *The dominant chamber in* V *is* $\{x \in V \mid (\partial a_i, x) \geqq 0$ *for all* $i \in S\}$. *The objects* ∂T^S, $\partial W(T)_S$, *etc., are naturally isomorphic to the root system, Weyl group, etc., of the reductive Lie algebra* $\mathfrak{r}/\mathfrak{c}$.

By Propositions 13.5 and 13.6 and the fact that $-\langle \Phi_w \rangle \in P_S$ for all $w \in W_S^1$, we get:

PROPOSITION 13.9. — *For all* $w \in W(T)_S^1$, $\mu = g(wr - r)$ *is a dominant integral weight in* V, i.e., $(\partial \check{a}_i, \mu) \in \mathbf{Z}_+$ *for all* $i \in S$.

We are now ready to give the improved version of Proposition 13.3. Combining Weyl's dimension formula with the above, we get:

PROPOSITION 13.10. — *Let* $(s_0, \ldots, s_l)$ *be a sequence of nonnegative integers, not all zero, and let* $S = \{ i \in \{0, \ldots, l\} \mid s_i = 0 \}$. Let $\rho_S = 1/2 \sum_{\beta \in \partial T_+^S} \beta \in V$ (see Prop. 13.8). *The q-specialization of type* $(s_0, \ldots, s_l)$ *of Theorem* 13.4 *states:*

$$\prod_{\varphi \in \Delta_+(S)} \left(1 - q^{\sum_{i=0}^{l} s_i \xi_i(\varphi)}\right)^{\dim \mathfrak{g}(A)^\varphi} = \sum_{w \in W(T)_S^1} (-1)^{l(w)} d_w \, q^{\sum_{i=0}^{l} s_i \xi_i(s(w))},$$

where

$$d_w = \prod_{\beta \in \partial T_+^S} \frac{(g(wr - r) + \rho_S, \beta)}{(\rho_S, \beta)}.$$

Moreover, d_w *is the dimension of the irreducible module with highest weight* $g(wr - r)$ *for the reductive Lie algebra* $\mathfrak{r}/\mathfrak{c}$, *with the identifications made in Proposition* 13.8. *Also,* d_w *is the dimension of the irreducible* $\mathfrak{r}^e$*-module* $M(-\langle \Phi_{\zeta^{-1}w} \rangle)$.

We now determine explicitly the vertices x_i of the alcove C and the positive rational numbers $a_i(x_i)$: Consider the basis $\partial a_1, \ldots, \partial a_l$ of simple roots of the gradient root system ∂T in V^*. The simple affine root a_0 is of the form

$$a_0 = 1 - \sum_{i=1}^{l} b_i \, \partial a_i,$$

where the b_i are positive integers. In fact, the $(l+1)$-tuple $(b_0 = 1, b_1, \ldots, b_l)$ is the list of integers written above the corresponding vertices in the Dynkin diagram of $\mathfrak{g}(A)$ in Table j ($j = 1, 2, 3$) of paragraph 8. Recall from paragraph 8 that the Dynkin diagram $X_n^{(j)}$ of $\mathfrak{g}(A)$ is obtained by adjoining a 0-th vertex to the Dynkin diagram X_l if $j = 1$ (in which case $n = l$) or to a Dynkin diagram Y_l if $j = 2, 3$. If $j = 1$, then $b_1, \ldots, b_l$ are the expansion coefficients of the highest root in terms of the simple roots of X_l; if $j = 2, 3$ and $X_n^{(j)} = D_{l+1}^{(2)}, A_{2l-1}^{(2)}, E_6^{(2)}$ or $D_4^{(3)}$, then $b_1, \ldots, b_l$ are the expansion coefficients of the highest short root in terms of the simple roots of Y_l ($= B_l, C_l, F_4$ or G_2, respectively); and if $j = 2$ and $X_n^{(j)} = A_{2l}^{(2)}$, then $Y_l = B_l$ and $b_1, \ldots, b_l$ are the expansion coefficients of twice the highest short root in terms of the simple roots of Y_l.

PROPOSITION 13.11. — *Let* $y_1, \ldots, y_l \in V$ *be the dual basis to the basis* $\partial a_1, \ldots, \partial a_l$ *of simple roots of the gradient root system* ∂T *in* V^*, *and let* $1 = b_0, b_1, \ldots, b_l$ *be the positive integers determined by the condition*

$$a_0 = 1 - \sum_{i=1}^{l} b_i \, \partial a_i$$

and described above. Then the vertices of the alcove C *are given as follows:*

$$x_0=0,$$
$$x_i=y_i/b_i \qquad \textit{for} \quad i=1, \ldots, l.$$

(*Here we identify* E *with* V.) *Moreover,*

$$a_i(x_i)=1/b_i \qquad \textit{for} \quad i=0, \ldots, l.$$

Proof. — The vertex x_i is determined by the conditions $a_j(x_i)=0$ if $i \neq j$. Hence $x_0=0$ and x_i is a multiple of y_i for $i>0$. The precise multiple comes from the condition $a_0(x_i)=0$. The formula for $a_i(x_i)$ is clear.

Q.E.D.

PROPOSITION 13.12. — *Given* $\mathbf{s}=(s_0, \ldots, s_l)$, *where the* s_i *are nonnegative integers not all* 0, *there exists a unique point* $t_{\mathbf{s}} \in \overline{\mathrm{C}}$ (*the closure of the alcove* C) *at which* $a_0, \ldots, a_l$ *take values proportional to* $s_0, \ldots, s_l$, *respectively. Let* $\mathrm{N}=\sum_{i=0}^{l} s_i b_i$ (see *Prop.* 13.11 *for the positive integers* b_i). *Then*

$$\mathrm{N} a_i(t_{\mathbf{s}})=s_i \qquad \textit{for} \quad i=0, \ldots, l,$$

and

$$t_{\mathbf{s}}=\sum_{i=0}^{l} \lambda_i x_i,$$

where

$$\lambda_i=s_i b_i/\mathrm{N} \qquad \textit{for} \quad i=0, \ldots, l;$$

we have $\Sigma \lambda_i=1$. *The point* $t_{\mathbf{s}} \in \mathrm{C}$ *if and only if each* $s_i>0$.

Proof. — Let $x \in \mathrm{E}$, so that $x=\sum_{i=0}^{l} \lambda_i x_i$ with $\lambda_i \in \mathbf{R}$ and $\sum \lambda_i=1$. Then for all $i=0, \ldots, l$,

$$a_i(x)=\lambda_i a_i(x_i)=\lambda_i/b_i,$$

by Proposition 13.11. By hypothesis, there is a constant c such that $s_i=c a_i(x)$ for each i. Then

$$1=\sum \lambda_i=\sum b_i a_i(x)=c^{-1} \sum s_i b_i=c^{-1} \mathrm{N},$$

so that $c=\mathrm{N}$. The rest is clear.

Q.E.D.

By combining Propositions 13.12 and 12.1, we immediately get a nice expression for the exponent of q on the right-hand sides of the formulas in Propositions 13.3 and 13.10:

PROPOSITION 13.13. — *Let* $\mathbf{s}$, $t_{\mathbf{s}}$ *and* N *be as in Proposition* 13.12, *and let* $w \in \mathrm{W}(\mathrm{T})$. *Then*

$$\sum_{i=0}^{l} s_i \xi_i(s(w))=\mathrm{N} s(w)(t_{\mathbf{s}})=\frac{\mathrm{N}g}{2}(\|wr-t_{\mathbf{s}}\|^2-\|r-t_{\mathbf{s}}\|^2).$$

The left-hand side of the formula in Proposition 13.10 can also be written in a nice way. Recall from paragraph 8 that our Lie algebra $L(\mathfrak{g}, \theta)$ provided with the grading of type $(s_0, \ldots, s_l)$ is graded-isomorphic to a Euclidean Lie algebra $L(\mathfrak{g}, \theta')$ (same $\mathfrak{g}$, but θ' possibly different from θ) with its natural $\mathbf{Z}$-grading. Moreover, θ' may be constructed from θ and $(s_0, \ldots, s_l)$ as follows: Provide $L(\mathfrak{g}, \theta)$ with its grading of type $(s_0, \ldots, s_l)$, and let I_1 be the ideal $(1-t^j) L(\mathfrak{g}, \theta)$ of $L(\mathfrak{g}, \theta)$. Then $L(\mathfrak{g}, \theta)/I_1$ is isomorphic to $\mathfrak{g}$ and is graded mod m, where $m=j \sum_{i=0}^{l} s_i b_i$, where the b_i are the integers above the relevant vertices of the Dynkin diagram in Table j. The automorphism of $\mathfrak{g}$ corresponding to this mod m grading is θ'. The graded level of each root $\varphi \in \Delta_+$ is precisely $\sum_{i=0}^{l} s_i \xi_i(\varphi)$. With S chosen as in Proposition 13.10, the level of φ is positive if and only if $\varphi \in \Delta_+(S)$. Thus the dimension of the d-th degree subspace $(d>0)$ of $L(\mathfrak{g}, \theta')$ is the number of roots $\varphi \in \Delta_+(S)$, with multiplicities counted, such that $\sum_{i=0}^{l} s_i \xi_i(\varphi)=d$. On the other hand, this dimension depends only on the congruence class of d mod m, and is just $\dim \mathfrak{g}_{d(\bmod m)}$, where $\mathfrak{g}$ is given its mod m grading. Hence we have:

PROPOSITION 13.14. — *For all* $d>0$, *let* $L(\mathfrak{g}, \theta')_d$ *be the d-th degree subspace of* $L(\mathfrak{g}, \theta')$, *and let* $\mathfrak{g}=\sum_{i \in \mathbf{Z}/(m)} \mathfrak{g}_i$ *be the* mod m *grading of* $\mathfrak{g}$ *associated with* θ' (see *above*). *Then the left-hand side of the formula in Proposition* 13.10 *may be written either*

$$\prod_{d>0} (1-q^d)^{\dim L(\mathfrak{g}, \theta')_d}$$

or

$$\prod_{d>0} (1-q^d)^{\dim \mathfrak{g}_{d(\bmod m)}}.$$

Combining Propositions 13.10, 13.13 and 13.14, we have the following conclusion:

THEOREM 13.15. — *Let* $\mathbf{s}=(s_0, \ldots, s_l)$ *be a sequence of nonnegative integers, not all zero;* $\mathfrak{g}=\sum_{i \in \mathbf{Z}/(m)} \mathfrak{g}_i$ *the corresponding* mod m *grading of* $\mathfrak{g}$ *described before Proposition* 13.14; $t_{\mathbf{s}}$ *and* N *as in Proposition* 13.12; r *and* g *as in Proposition* 12.1;

$$S=\{i \in \{0, \ldots, l\} \mid s_i=0\};$$

$W(T)_S^1$ *as in Proposition* 13.5; *and* d_w *for* $w \in W(T)_S^1$ *as in Proposition* 13.10. *Then the q-specialization of type* $\mathbf{s}$ *of Theorem* 13.4 *states:*

$$\prod_{d>0} (1-q^d)^{\dim \mathfrak{g}_{d(\bmod m)}} = \sum_{w \in W(T)_S^1} (-1)^{l(w)} d_w q^{(Ng/2)(\|wr-t_{\mathbf{s}}\|^2-\|r-t_{\mathbf{s}}\|^2)}.$$

If $s_i>0$ for all $i=0, \ldots, l$, then $W(T)_S^1=W(T)$, and $d_w=1$.

14. MacDonald's specialization of type $(1, 0, \ldots, 0)$

Most of the one-variable power series identities obtained by MacDonald in [21] are specializations of type $(1, 0, \ldots, 0)$ (*see* the definition following Corollary 13.2). That is, they correspond to the *natural* grading of the standard Euclidean Lie algebra $L(\mathfrak{g}, \theta)$ of paragraphs 12 and 13, and the associated automorphism of finite order of $\mathfrak{g}$ is just θ [*see* Remark (1) at the end of paragraph 8]. If the Dynkin diagram of $L(\mathfrak{g}, \theta)$ is in Table 1 of paragraph 8, then θ is just the identity automorphism. It is *via* this specialization (which from our viewpoint comes from $\theta = 1$) that MacDonald gets his famous identities for the $\dim \mathfrak{g}$ power of Dedekind's eta-function.

Here we apply Theorem 13.15 in the case $\mathbf{s} = (1, 0, \ldots, 0)$ to obtain another formulation of MacDonald's specialized identities. In our case, the identities involve the subset $W(T)^1_S$ of $W(T)$ instead of the translation lattice, which MacDonald uses. We set up some notation.

Let $\mathfrak{g}$ be a simple Lie algebra (over our algebraically closed field of characteristic zero); θ an automorphism of order j of $\mathfrak{g}$ induced by an automorphism of order j of its Dynkin diagram; $\mathfrak{g} = \sum_{i \in \mathbf{Z}/(j)} \mathfrak{g}_i$ the corresponding mod j grading of $\mathfrak{g}$; $\mathfrak{h}_0$ a Cartan subalgebra of the simple Lie algebra $\mathfrak{g}_0$; $\mathfrak{h}_{0,\mathbf{Q}}$ the rational subspace of $\mathfrak{h}_0$ on which the roots take rational values; $V = \mathfrak{h}_{0,\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$ the real Euclidean space with scalar product $(.,.)$ induced by the Killing form of $\mathfrak{g}$; $\|\,.\,\|$ the corresponding norm on V; Φ_+ a system of positive roots for $\mathfrak{g}_0$ with respect to $\mathfrak{h}_{0,\mathbf{Q}}$, with Φ_+ regarded as a subset of V by means of the identification provided by $(.,.)$; $\rho_0 = 1/2 \sum_{\beta \in \Phi_+} \beta \in V$; and $\beta_1, \ldots, \beta_l \in V$ the simple roots in Φ_+. Define $\beta_0 \in V$ as follows: If $j = 1$, then β_0 is the highest root in Φ_+. If $j = 2$ or 3, then β_0 is the highest weight of $\mathfrak{g}_0$ acting on the (irreducible) module $\mathfrak{g}_1$ (so that β_0 is either the highest short root or twice the highest short root in Φ_+; *see* § 8). Let C_0 be the open dominant Weyl chamber in V; $C = \{x \in C \mid (\beta_0, x) < 1\}$; $r \in C$ the unique point at which the numbers

$$2(\beta_1, r)/(\beta_1, \beta_1), \ldots, 2(\beta_l, r)/(\beta_l, \beta_l) \quad \text{and} \quad 2(1 - (\beta_0, r))/(\beta_0, \beta_0)$$

are all equal; g^{-1} their common value; W_a (the affine Weyl group) the group of affine isometries of V generated by the reflections through the $l+1$ walls of C; $l(w)$ the length of a minimal expression of $w \in W_a$ as a product of these $l+1$ reflections; and W_a^1 the subset of W_a consisting of the elements sending C into C_0. The following result is immediate from Theorem 13.15 and other relevant results from paragraph 13:

THEOREM 14.1. — *Let q be an indeterminate. In $\mathbf{Z}[[q]]$, we have:*

$$\prod_{d>0} (1 - q^d)^{\dim \mathfrak{g}_{d\,(\mathrm{mod}\, j)}} = \sum_{w \in W_a^1} (-1)^{l(w)} d_w q^{(g/2)(\|wr\|^2 - \|r\|^2)},$$

where

$$d_w = \prod_{\beta \in \Phi_+} \frac{(g(wr - r) + \rho_0, \beta)}{(\rho_0, \beta)}$$

Moreover, $g(wr\text{-}r)$ is a dominant integral element of V *with respect to Φ_+, and d_w is the dimension of the irreducible $\mathfrak{g}_0$-module with highest weight $g(wr\text{-}r)$.*

Remark. — If $j=1$, then the left-hand side of the power series identity is $\prod(1-q^d)^{\dim \mathfrak{g}}$. Also, $g=1/2$ and $r=2\rho_0$, since $(.,.)$ is the canonical bilinear form associated with the root system in V (*see* [21], Prop. 7.3 and 7.13).

15. The case $\tilde{A}_1$ and polygonal numbers

Partly to motivate the results in paragraphs 16 and 17, we point out an amusing sequence of specializations of MacDonald's identity for $\tilde{\mathfrak{g}}$ where $\mathfrak{g} \simeq \mathfrak{sl}(2, k)$. The identity, which is of course Jacobi's classical "θ-function identity" (*cf.* [21], p. 93) states, in the notation of Corollary 13.2:

PROPOSITION 15.1. — *In* $\mathbf{Z}[[u_0, u_1]]$, *we have*

$$\prod_{n \geq 1}(1-u_0^n u_1^n)(1-u_0^n u_1^{n-1})(1-u_0^{n-1} u_1^n)=\sum_{n \in \mathbf{Z}}(-1)^n u_0^{\binom{n}{2}} u_1^{\binom{n+1}{2}}.$$

This formula is written in various ways in the literature, and when one-variable specializations are deduced from it, the specialization procedures seem unmotivated and unsystematic.

First note that the version chosen above for the identity is natural from our point of view, because the variables used come from the simple roots of the underlying Kac-Moody Lie algebra [$u_0=e(-\alpha_0)$ and $u_1=e(-\alpha_1)$].

We specialize the identity by applying Theorem 13.15 to the sequence $\mathbf{s}_n=(1, n)$, $n \in \mathbf{Z}_+$. That is, we send u_0 to q and u_1 to q^n (but if $n=0$, we first divide both sides by $1-u_1$). The n-th such q-specialization corresponds to an automorphism of order $n+1$ of $\mathfrak{sl}(2, k)$.

Notation. — Write $\varphi(q)=\prod_{i>0}(1-q^i)$.

When $n=0$, we have $S=\{1\}$ in Theorem 13.15, and the result is

$$\varphi(q)^3=\sum_{i \in \mathbf{Z}_+}(-1)^i(2i+1) q^{(1/2) i(i+1)}. \tag{1}$$

When $n>0$, we have $S=\emptyset$, $W(T)_S^1=W(T)$ and $d_w=1$ in Theorem 13.15. For $n=1$, we get

$$\frac{\varphi(q)^2}{\varphi(q^2)}=\sum_{i \in \mathbf{Z}}(-1)^i q^{i^2} \tag{2}$$

and for $n=2$, we get

$$\varphi(q)=\sum_{i \in \mathbf{Z}}(-1)^i q^{(1/2) i(3i-1)}. \tag{3}$$

Formulas (1), (2) and (3) are due to Jacobi, Gauss and Euler, respectively (*see* [1], pp. 11, 23, 176).

From the exponents of $\varphi(q)$ on the left-hand sides of (3), (2) and (1), respectively, it might appear that these three identities, in this reverse order, form the beginning of a natural infinite sequence of identities. However, the "right" infinite sequence of identities begins with (1), (2) and (3) in forward order, and comes from the sequence $\mathbf{s}_n=(1, n)$ and the associated automorphisms of $\mathfrak{sl}(2, k)$ of order $n+1$. [In paragraphs 16 and 17, we *will* in fact produce identities for arbitrary powers of $\varphi(q)$, but from Kac-Moody Lie algebras other than $\tilde{A}_1$.]

It is easy to check that for $n>0$, the q-specialization of type $(1, n)$ of Proposition 15.1 states:

PROPOSITION 15.2. — *For all* $n>0$, *we have*

$$\sum_{i>0}(1-q^{(n+1)i})(1-q^{(n+1)i-1})(1-q^{(n+1)i-n})=\sum_{i\in\mathbf{Z}}(-1)^i q^{(1/2)i((n+1)i-(n-1))}.$$

Note that the exponents of q on the right-hand sides of formulas (1), (2) and (3) are the triangular numbers, the squares (i. e., the "quadrangular" numbers) and the pentagonal numbers, respectively. (*See* [24], p. 224 for a pictorial definition of the pentagonal numbers; this definition extends in a natural way to the obvious definition of what might be called the polygonal numbers, i. e., the r-agonal numbers for all $r\geqq 3$.) Here is a pleasant surprise — the completion of the above pattern in the q-exponents:

PROPOSITION 15.3. — *The exponenets of* q, *for* $i>0$, *on the right-hand side of the* n-*th identity in Proposition* 15.2, *are the* $(n+3)$-*agonal numbers.*

Proof. — The $(i+1)$ *st* $(n+3)$-agonal number is the i-th plus $(n+1)\,i+1$ (*cf.* the diagrams on p. 224 of [24]). The first $(n+3)$-agonal number is 1. By induction, the i-th is $(1/2)\,i\,((n+1)\,i-(n-1))$.

Q.E.D.

Recall from paragraph 8 that the *affine* Lie algebras are the Euclidean Lie algebras of the form $\tilde{\mathfrak{g}}$ for $\mathfrak{g}$ simple. In this section, we have focused on one affine Lie algebra $\tilde{\mathfrak{g}}$, and we have looked at a natural infinite sequence of specializations of the associated Macdonald identity, corresponding to different automorphisms of finite order of $\mathfrak{g}$. Note that among this infinite sequence, formulas (1), (2) and (3) are the "nicest". Formula (1) is the prototype of Macdonald's q-identities for all affine (or even Euclidean) Lie algebras, discussed in paragraph 14; when the Lie algebra is $\tilde{\mathfrak{g}}$, the corresponding automorphism of finite order is the identity automorphism of $\mathfrak{g}$. It turns out that (2) and (3) also have natural extensions to all affine Lie algebras $\tilde{\mathfrak{g}}$. Formula (3) is the prototype of a family of identities for $\varphi(q)^{\operatorname{rank}\mathfrak{g}}$ (*see* § 16); the corresponding automorphism of finite order is a certain natural one of order $h+1$, where h is the Coxeter number of $\mathfrak{g}$. Formula (2) is the prototype of an interesting family of identities related to the exponents of $\mathfrak{g}$ (*see* § 17); the corresponding automorphism of finite order is Kostant's "principal" automorphism, which has order h. [For $\mathfrak{sl}(2, k)$, $h=2$.]

Macdonald's q-identities can be expressed in terms of powers of Dedekind's η-function, in place of the φ-function defined above, because of the Freudenthal-de Vries "strange formula" $\|\rho\|^2=(\dim\mathfrak{g})/24$ for a simple Lie algebra $\mathfrak{g}$; *see* [21]. Analogously, the new identities in paragraphs 16 and 17 can be expressed in terms of the η-function because of new "strange formulas"; *see* below.

16. A formula for $\eta(q)^{\operatorname{rank} \mathfrak{g}}$

Recall from paragraph 15 that $\varphi(q)=\prod_{i>0}(1-q^i)$. Also define $\eta(q)=q^{1/24}\varphi(q)$ (a formal power series in $\mathbf{Z}[[q^{1/24}]]$). Then $\eta(q)$ is essentially Dedekind's η-function.

Let $\mathfrak{g}$ be a simple Lie algebra, so that $\tilde{\mathfrak{g}}$ is the corresponding affine Lie algebra. We shall apply Theorem 13.15 to $\mathbf{s}=(2, 1, \ldots, 1)$ to obtain a formula for $\eta(q)^l$, where l as usual is the rank of $\mathfrak{g}$.

Recall from paragraph 14 the following notation: $\mathfrak{h}_0$ (a Cartan subalgebra of $\mathfrak{g}$ since $j=1$ and $\theta=1$); $\mathfrak{h}_{0,\mathbf{Q}}$; $V=\mathfrak{h}_{0,\mathbf{Q}}\otimes_{\mathbf{Q}}\mathbf{R}$; the scalar product $(.,.)$ on V induced by the Killing form of $\mathfrak{g}$; the identification of V with its dual *via* $(.,.)$; the norm $\|.\|$ on V; $\Phi_+\subset V$; $\rho_0\in V$; $\beta_1, \ldots, \beta_l\in V$ the simple roots; the highest root β_0; the alcove C defined by the condition

$$C=\{x\in V \mid (\beta_i, x)>0 \text{ for } i=1, \ldots, l \text{ and } (\beta_0, x)<1\};$$

$r=2\rho_0\in C$; g, which equals 1/2 (*see* the last Remark in paragraph 14); the affine Weyl group W_a; and $l(w)$ for $w\in W_a$.

Let $\Phi=\pm\Phi_+\subset V$ be the set of roots. For all $\beta\in\Phi$, the *height* of β is the sum of its expansion coefficients in terms of the simple roots. The *Coxeter number* h of $\mathfrak{g}$ is $1+\text{height } \beta_0$. The *exponents* $m_1, \ldots, m_l$ of $\mathfrak{g}$ are positive integers less than h, assumed to be arranged in nondecreasing order, and defined as follows: For all $p=1, \ldots, h$, let η_p be the number of (positive) roots of height p. Then $\eta_p-\eta_{p+1}$ is the multiplicity with which p occurs as an exponent of $\mathfrak{g}$. It is well known that the exponents satisfy the following duality property: If $i=1, \ldots, l$, then $m_i+m_{l+1-i}=h$ (*cf.* [16], (*a*) or [21], p. 121). We have the following straightforward consequences (*cf.* [21], p. 121):

PROPOSITION 16.1. — *For all* $p=1, \ldots, h$,

$$\eta_p+\eta_{h+1-p}=l.$$

PROPOSITION 16.2. — *For all* $p=1, \ldots, h-1$,

$$\eta_p+\eta_{h-p}=l+\text{mult } p,$$

where mult p denotes the multiplicity with which p occurs as an exponent of $\mathfrak{g}$.

(Note that there is a misprint in the statement of the assertion of Proposition 16.2 on p. 121 of [21].)

The automorphism of finite order of $\mathfrak{g}$ associated with $\mathbf{s}=(2, 1, \ldots, 1)$ clearly gives $\mathfrak{g}$ a mod $(h+1)$ grading (*see* paragraph 8 and the discussion before Proposition 13.14). Denote this grading by $\mathfrak{g}=\coprod_{i\in\mathbf{Z}/(h+1)}\mathfrak{g}_i$.

PROPOSITION 16.3. — *For all* $i=1, \ldots, h+1$,

$$\dim \mathfrak{g}_{i\,(\text{mod}\,(h+1))}=l.$$

Proof. — Since $\mathfrak{g}_{h+1(\mathrm{mod}(h+1))}=\mathfrak{h}$, the desired result is true for $i=h+1$. If $\beta\in\Phi_+$ has height i, then the root space corresponding to β lies in $\mathfrak{g}_{i(\mathrm{mod}(h+1))}$. If $\beta\in\Phi_-$ has height $-i$, then the corresponding root space lies in $\mathfrak{g}_{h+1-i(\mathrm{mod}(h+1))}$. Now apply Proposition 16.1.

Q.E.D.

Hence the left-hand side of the formula in Theorem 13.15 is $\varphi(q)^l$. To determine the right-hand side, first note that the sum is over the full affine Weyl group W_a; $d_w=1$; and $N=h+1$. All we have left to determine is t_s.

PROPOSITION 16.4. — *Define*

$$\rho_0'=\frac{1}{2}\sum_{\beta\in\Phi_+}\frac{2\beta}{(\beta,\beta)}\in V,$$

so that ρ_0' *is half the sum of the positive roots for the root system dual to* Φ. *Then* $t_s=\rho_0'/(h+1)$.

Proof. — By Proposition 13.12, all the simple roots $\beta_1,\ldots,\beta_l$ (which may be identitified with $a_1,\ldots,a_l$ in the notation of Proposition 13.12) take the value $s_i/N=1/(h+1)$ on t_s. But ρ_0' is characterized as the element of V such that $(\beta_i,\rho_0')=1$ for all $i=1,\ldots,l$. Hence $t_s=\rho_0'/(h+1)$.

Q.E.D.

By Theorem 13.15, our conclusion is:

THEOREM 16.5. — *In* $\mathbf{Z}[[q]]$,

$$\varphi(q)^l=\sum_{w\in W_a}(-1)^{l(w)}q^{[(h+1)/4](\|w(2\rho_0)-[\rho_0'/(h+1)]\|^2-\|2\rho_0-[\rho_0'/(h+1)]\|^2)}.$$

By analogy with Macdonald's transition from his formula for $\varphi(q)^{\dim\mathfrak{g}}$ to his formula for $\eta(q)^{\dim\mathfrak{g}}$ *via* the "strange formula" $\|\rho_0\|^2=(\dim\mathfrak{g})/24$ of Freudenthal-de Vries (*see* [21]) we are led to conjecture the following new "strange formula":

THEOREM 16.6. — *We have*

$$\frac{h+1}{4}\left\|2\rho_0-\frac{\rho_0'}{h+1}\right\|^2=\frac{l}{24}.$$

If Φ has only one root length, then it turns out that this formula follows easily from the one of Freudenthal-de Vries. Indeed, $(\beta,\beta)h=1$ for all $\beta\in\Phi$ (*cf.* [16] (*d*), Prop. 2.2), and so $2\rho_0=(1/h)\rho_0'$ in this case. However, for $\mathfrak{g}$ of type B_l, C_l, F_4 or G_2, we have had to resort to brute-force case checking using the tables at the end of [4]. There seems to be no point in dragging the reader through the details. But the conclusion is nice:

THEOREM 16.7. — *We have*

$$\eta(q)^l=\sum_{w\in W_a}(-1)^{l(w)}q^{[(h+1)/4]\|w(2\rho_0)-[\rho_0'/(h+1)]\|^2}.$$

Remark. — This formula (or Th. 16.5) is a natural generalization of Euler's pentagonal number formula for $\varphi(q)$ (*cf.* § 15) from the case $\mathfrak{g}=\mathfrak{sl}(2,k)$ to all simple $\mathfrak{g}$.

The right-hand side in Theorem 16.7 can be written in a form more convenient for computation by using the fact that W_a is the semidirect product of the Weyl group W_0 of Φ with the translation subgroup T_0 of W_a (*cf.* [21], Prop. 6.1). Moreover, T_0 is the group of translations by elements of the lattice Λ generated by $\{\beta^{\vee} \mid \beta \in \Phi\}$, where, as usual, $\beta^{\vee} = 2\beta/(\beta, \beta)$ [21], p. 118. For $\lambda \in \Lambda$, denote by $t(\lambda)$ the corresponding translation in W_a. Then for $\sigma \in W_0$ and $\lambda \in \Lambda$, we have

$$t(\lambda)\sigma(2\rho_0) - \frac{\rho_0'}{h+1} = 2\sigma(\rho_0) + \lambda - \frac{\rho_0'}{h+1},$$

and this implies:

PROPOSITION 16.8. — *The right-hand side in Theorem* 16.7 *equals*

$$\sum_{\sigma \in W_0} (-1)^{l(\sigma)} \sum_{\mu \in M} q^{(h+1)\|\sigma(\rho_0) - [\rho_0'/(2(h+1))] + \mu\|^2},$$

where M *is the lattice in* V *generated by* $\{\beta/(\beta, \beta) \mid \beta \in \Phi\}$. *If* Φ *has only one root length, then the exponent of* q *may be replaced by*

$$(h+1)\left\|\sigma(\rho_0) - \frac{h}{h+1}\rho_0 + \mu\right\|^2,$$

and M *is the lattice generated by* $\{h\beta \mid \beta \in \Phi\}$.

Example. — If $\mathfrak{g}$ is of type A_l, $l \geqq 1$, Theorem 16.7 becomes:

$$\eta(q)^l = \sum_{\sigma \in \mathscr{S}_{l+1}} \operatorname{sgn}\sigma \sum_{\substack{\mu_1, \dots, \mu_{l+1} \in (l+1)(l+2)\mathbf{Z} \\ \sum \mu_i = 0}} q^{1/(2(l+1)(l+2)) \sum_{i=1}^{l+1} ((l+2)\sigma(i) - (l+1)i + \mu_i - l/2 - 1)^2},$$

where $\mathscr{S}_{l+1}$ denotes the symmetric group of $\{1, \dots, l+1\}$. For $l = 24$, this is:

$$\sum_{n \in \mathbf{Z}_+} \tau(n) q^n = \eta(q)^{24} = \sum_{\sigma \in \mathscr{S}_{25}} \operatorname{sgn}\sigma \sum_{\substack{\mu_1, \dots, \mu_{25} \in 650\mathbf{Z} \\ \sum \mu_i = 0}} q^{1/1{,}300 \sum_{i=1}^{25} (26\sigma(i) - 25i + \mu_i - 13)^2},$$

where τ is Ramanujan's τ-function (*cf.* [13], Chap. X).

17. The q-identity for the principal specialization

Retain the notation of paragraph 16. In this section, we apply Theorem 13.15 to $\mathbf{s} = (1, \dots, 1)$. The corresponding automorphism of finite order of $\mathfrak{g}$ is inner and has order h, and its centralizer is a Cartan subalgebra (*see* paragraph 8 and the discussion preceding Proposition 13.14). Therefore, it is Kostant's "principal" automorphism, and it induces the Coxeter element of the Weyl group on a suitable Cartan subalgebra (*see* [16], (*a*), (*d*)). We call the q-specialization of type $\mathbf{s}$ the *principal* specialization.

Denote the mod h grading of $\mathfrak{g}$ associated with $\mathbf{s}$ by $\mathfrak{g}=\coprod_{i\in\mathbf{Z}/(h)}\mathfrak{g}_i$. Using Proposition 16.2, we obtain, exactly as in the proof of Proposition 16.3:

PROPOSITION 17.1. — *For all* $i=1, \ldots, h$,

$$\dim \mathfrak{g}_{i(\bmod h)}=l+\operatorname{mult} i.$$

Thus the left-hand side of the formula in Theorem 13.15 is

$$\varphi(q)^l \prod_{\substack{n\in\mathbf{Z}_+\\ i=1,\ldots,l}} (1-q^{m_i+nh}).$$

Also, $\mathrm{N}=h$. Just as in Proposition 16.4, we get:

PROPOSITION 17.2. — *With* ρ_0' *as in Proposition* 16.4, $t_\mathbf{s}=\rho_0'/h$.

Hence Theorem 13.15 yields:

THEOREM 17.3. — *In* $\mathbf{Z}[[q]]$,

$$\varphi(q)^l \prod_{\substack{n\in\mathbf{Z}_+\\ i=1,\ldots,l}} (1-q^{m_i+nh})=\sum_{w\in\mathrm{W}_a}(-1)^{l(w)}q^{(h/4)(\|w(2\rho_0)-(\rho_0'/h)\|^2-\|2\rho_0-(\rho_0'/h)\|^2)}.$$

An easy case check implies;

PROPOSITION 17.4. — *The left-hand side in Theorem* 17.3 *can be expressed uniquely as a shortest possible expression of the following form:*

$$\varphi(q)^{l+1}\prod_i\varphi(q^{d_i})\prod_j\varphi(q^{e_j})^{-1},$$

where the d_i *and* e_j *are distinct divisors of* h *and are greater than* 1.

Case checking using the tables at the end of [4] implies another new "strange formula":

THEOREM 17.5. — *We have*

$$\frac{h}{4}\left\|2\rho_0-\frac{\rho_0'}{h}\right\|^2=\frac{1}{24}(l+1+\sum_i d_i-\sum_j e_j).$$

Remark. — If Φ has only one root length, then this number is 0. It is $(l-1)/24$ for $\mathfrak{g}$ of type B_l or $\mathrm{C}_l\,(l\geqq 2)$, 6/24 for F_4, and 4/24 for G_2.

We can conclude:

THEOREM 17.6. — *We have*

$$\eta(q)^{l+1}\prod_i\eta(q^{d_i})\prod_j\eta(q^{e_j})^{-1}=\sum_{w\in\mathrm{W}_a}(-1)^{l(w)}q^{(h/4)\|w(2\rho_0)-(\rho_0'/h)\|^2}.$$

Remark. — This result (or Th. 17.3) is a natural generalization of Gauss' formula for $\varphi(q)^2/\varphi(q^2)$ (*cf.* § 15) from the case $\mathfrak{g}=\mathfrak{sl}(2, k)$ to all simple $\mathfrak{g}$.

In Table 4, we list the left-hand sides in Theorem 17.6.

TABLE 4

Type of $\mathfrak{g}$	Left-hand side	Type of $\mathfrak{g}$	Left-hand side
$A_l\,(l\geqq 1)$	$\dfrac{\eta(q)^{l+1}}{\eta(q^{l+1})}$	E_7	$\dfrac{\eta(q)^8\,\eta(q^6)\,\eta(q^9)}{\eta(q^2)\,\eta(q^3)\,\eta(q^{18})}$
$B_l\,(l\geqq 2)$	$\dfrac{\eta(q)^{l+1}}{\eta(q^2)}$	E_8	$\dfrac{\eta(q)^9\,\eta(q^6)\,\eta(q^{10})\,\eta(q^{15})}{\eta(q^2)\,\eta(q^3)\,\eta(q^5)\,\eta(q^{30})}$
$C_l\,(l\geqq 2)$	$\dfrac{\eta(q)^{l+1}}{\eta(q^2)}$	F_4	$\dfrac{\eta(q)^5\,\eta(q^6)}{\eta(q^2)\,\eta(q^3)}$
$D_l\,(l\geqq 4)$	$\dfrac{\eta(q)^{l+1}\,\eta(q^{l-1})}{\eta(q^2)\,\eta(q^{2l-2})}$	G_2	$\dfrac{\eta(q)^3\,\eta(q^6)}{\eta(q^2)\,\eta(q^3)}$
E_6	$\dfrac{\eta(q)^7\,\eta(q^4)\,\eta(q^6)}{\eta(q^2)\,\eta(q^3)\,\eta(q^{12})}$		

Remark. — For $\mathfrak{g}=\mathfrak{sl}(n, k)$, Theorem 17.6 gives a formula for $\eta(q)^n/\eta(q^n)$, and for $\mathfrak{g}$ of type B_{n-1} or C_{n-1}, the theorem gives a formula for $\eta(q)^n/\eta(q^2)$. By multiplying both sides by $\eta(q^n)$ or $\eta(q^2)$ and using Euler's pentagonal number formula, we get various expressions for $\eta(q)^n$. Thus Theorems 17.6 and 16.7 give many new formulas for arbitrary positive powers of $\eta(q)$. All these formulas which come from Lie algebras other than $\mathfrak{g}=\mathfrak{sl}(2, k)$ seem to be new. In particular, we get many new formulas for the generating function $\eta(q)^{24}$ of Ramanujan's τ-function (*cf.* [13], Chap. X).

To facilitate computation, we note as in Proposition 16.8:

PROPOSITION 17.7. — *The right-hand side in Theorem* 17.6 *equals*

$$\sum_{\sigma\in W_0}(-1)^{l(\sigma)}\sum_{\mu\in M} q^{h\|\sigma(\rho_0)-(\rho_0'/2h)+\mu\|^2},$$

where M *is the lattice in* V *generated by* $\{\beta/(\beta,\beta)\mid\beta\in\Phi\}$. *If* Φ *has only one root length, then the exponent of* q *may be replaced by*

$$h\|\sigma(\rho_0)-\rho_0+\mu\|^2,$$

and M *is the lattice generated by* $\{h\beta\mid\beta\in\Phi\}$.

Example. — Let $\mathfrak{g}=\mathfrak{sl}(n, k)\,(n\geqq 2)$.

Then Theorem 17.6 states:

$$\frac{\eta(q)^n}{\eta(q^n)}=\sum_{\sigma\in\mathscr{S}_n}\operatorname{sgn}\sigma\sum_{\substack{\mu_1,\,\ldots,\,\mu_n\in n\mathbf{Z}\\ \sum\mu_i=0}} q^{1/2\sum_{i=1}^{n}(\sigma(i)-i+\mu_i)^2},$$

where $\mathscr{S}_n$ is the symmetric group of $\{1,\ldots,n\}$. Changing η to φ, multiplying through by $\varphi(q^n)$, and using Euler's formula for φ, we get:

$$\varphi(q)^n=\sum_{\substack{\sigma\in\mathscr{S}_n\\ v\in\mathbf{Z}}}(-1)^v\operatorname{sgn}\sigma\sum_{\substack{\mu_1,\,\ldots,\,\mu_n\in n\mathbf{Z}\\ \sum\mu_i=0}} q^{(1/2)(nv(3v+1)+\sum_{i=1}^{n}(\sigma(i)-i+\mu_i)^2)}$$

Since $\varphi(q)^{-1} = \sum_{\nu \in \mathbf{Z}_+} p(\nu) q^\nu$, where p is the classical partition function (*see* [1], Chap. 1), we also have:

$$\Big(\sum_{\nu \in \mathbf{Z}_+} p(\nu) q^{n\nu}\Big) \varphi(q)^n = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \sum_{\substack{\mu_1, \ldots, \mu_n \in n\mathbf{Z} \\ \sum \mu_i = 0}} q^{1/2 \sum_{i=1}^{n} (\sigma(i) - i + \mu_i)^2}.$$

When $n=24$, we get:

$$\sum_{\nu \in \mathbf{Z}_+} \tau(\nu) q^\nu = \eta(q)^{24} = \sum_{\substack{\sigma \in \mathcal{S}_{24} \\ \nu \in \mathbf{Z}}} (-1)^\nu \operatorname{sgn} \sigma \sum_{\substack{\mu_1, \ldots, \mu_{24} \in 24\mathbf{Z} \\ \sum \mu_i = 0}} q^{(6\nu+1)^2 + (1/2) \sum_{i=1}^{24} (\sigma(i) - i + \mu_i)^2}$$

and

$$\Big(\sum_{\nu \in \mathbf{Z}_+} p(\nu) q^{24\nu}\Big)\Big(\sum_{\lambda \in \mathbf{Z}_+} \tau(\lambda+1) q^\lambda\Big) = \sum_{\sigma \in \mathcal{S}_{24}} \operatorname{sgn} \sigma \sum_{\substack{\mu_1, \ldots, \mu_{24} \in 24\mathbf{Z} \\ \sum \mu_i = 0}} q^{1/2 \sum_{i=1}^{24} (\sigma(i) - i + \mu_i)^2}.$$

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