Closedness of regular 1-forms on algebraic surfaces

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CLOSEDNESS OF REGULAR 1-FORMS
ON ALGEBRAIC SURFACES (1)

By Niels O. NYGAARD

Introduction

Let $X/k$ be a proper, smooth surface over a perfect field $k$. If $k$ has characteristic 0 it follows
from Hodge theory and the Lefshetz principle that all regular 1-forms on $X$ are closed, i.e.
that the differential

$$d : H^0(X, \Omega^1_{X/k}) \to H^0(X, \Omega^2_{X/k}),$$

vanishes.

In characteristic $p > 0$ the situation is more complicated indeed Mumford [11] and more
recently Raynaud have constructed surfaces with regular 1-forms which are not closed (2). It therefore becomes interesting to look for conditions on $X$ that will ensure the
closedness of regular 1-forms. We relate this question to an invariant defined and studied by
Artin and Mazur in [1], the formal Brauer group, $Br_X$, specially we show that if $Br_X$ is pro-
representable by a $p$-divisible formal group (Barsotti-Tate group) then all the regular 1-forms
are closed, and indeed the whole Hodge to de Rham spectral sequence degenerates
at $E_1$. In a subsequent paper [13] we shall further develop the techniques employed in the
proof of the above statement, and show how these can be used to prove the Rydakov-
Shafarevitch theorem, that $K3$ surfaces have no global vector fields.

We also consider a smooth family of surfaces $f : X \to S$ over an irreducible base scheme of
characteristic $p$, here we show that if there is just one fiber $X_s$ with $p$-divisible formal Brauer
group then the differential

$$d : f_* \Omega^1_{X/S} \to f_* \Omega^2_{X/S},$$
is zero.

(1) This work was supported in part by the Danish Research Council.
(2) Examples have also been constructed by W. Lang [17].
Contents

1. Some properties of the slope spectral sequence.
2. Surfaces over a perfect field.
3. Surfaces over an irreducible scheme.

Acknowledgement

I should like to thank L. Illusie for very useful correspondence during the preparation of this paper. I also thank the referee for pointing out a considerable strengthening of the methods developed in 2.

1. Some properties of the slope spectral sequence

For the construction and the basic properties of the slope spectral sequence we refer to Bloch [3]. Bloch’s construction has been generalized and the restrictions on the relation between the dimension and characteristic has been removed (Illusie [9]), so the restriction in Bloch’s paper will be ignored.

The notation will be as in [3]; the proof of the properties listed below will appear in [9].

Let $F$, $V$ and $d$ denote respectively the Frobenius, the Verschiebung and the differential in the pro-complex $C_{\cdot, X}$, then:

(1.1) $FV = VF = p$.
(1.2) $dF = pFd$, $Vd = pdV$.
(1.3) $FdV = d$.
(1.4) $F$, $V$ and $p$ are injective as maps of pro-sheaves i.e. the transition maps in the pro-system of kernels are 0.
(1.5) Let $n = \dim X$ then $F$ is an automorphism of the pro-sheaf $C^n_{\cdot, X}$.

2. Surfaces over a perfect field

In this section we show that if the formal Brauer group of $X/k$ is pro-representable by a $p$-divisible formal group then the Hodge to de Rham spectral sequence degenerates at $E_1$. If we further assume that $H^2_{\text{crys}}(X/W)$ is torsion free then the Hodge symmetry

$$h^{i,j} = \dim_k H^i(X, \Omega^{j}_{X/k}) = \dim_k H^i(X, \Omega^{j}_{X/k}) = h^{i,j},$$

holds as well.

The following proposition has also been proved by Berthelot (private communication) using results of Mazur and Messing.

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(2.1) Proposition. — Let $X/k$ be a smooth proper variety over a perfect field $k$ of characteristic $p > 0$. Assume that $H^2_{\text{crys}}(X/W)$ is torsion free, then the Picard scheme $\text{Pic}(X)$ is reduced.

Proof. — Consider the exact sequence of Zariski sheaves on $X$:

$$0 \to \omega_r(\mathcal{O}_X) \to \omega_{r+1}(\mathcal{O}_X) \to \mathcal{O}_X \to 0,$$

which gives rise to an exact sequence of finite length $W(k)$-modules

$$\to H^i(X, \omega_r(\mathcal{O}_X)) \to H^i(X, \omega_{r+1}(\mathcal{O}_X)) \to H^{i+1}(X, \omega_r(\mathcal{O}_X)),$$

and hence (using Mittag-Leffler) an exact sequence of $W(k)$-modules

$$\to H^i(X, \omega(\mathcal{O}_X)) \to H^i(X, \omega(\mathcal{O}_X)) \to H^{i+1}(X, \omega(\mathcal{O}_X)).$$

By [12], p. 196, $\text{Pic}(X)$ is reduced if and only if the connecting homomorphism, in the exact sequence above, vanishes, this is equivalent to

$$H^1(X, \omega(\mathcal{O}_X)) \to H^1(X, \mathcal{O}_X),$$

being surjective.

Define the pro-complex $\star C_{\cdot, X}$ by

$$\star C_{\cdot, X} = 0 \to C^0_{\cdot, X} \to C^1_{\cdot, X} \to \ldots \to C^{\dim X}_{\cdot, X} \to 0,$$

since

$$dF \theta = (p \theta) d$$

by (1.2).

this is indeed a complex.

Now define

$$\tilde{V} : C_{\cdot, X} \to \star C_{\cdot, X},$$

by

$$\tilde{V} : C^i_{\cdot, X} \to \star C^i_{\cdot, X} = \left\{ \begin{array}{ll}
V : C^0_{\cdot, X} \to C^0_{\cdot, X+1} & \text{if } i = 0,
\text{id} : C^i_{\cdot, X} \to C^i_{\cdot, X} & \text{if } i > 0.
\end{array} \right.$$
Passing to hypercohomology we obtain an exact sequence of pro-modules

\[ \varprojlim H^i(X, C^\cdot, \mathcal{O}_X) \rightarrow H^i(X, C^\cdot, \mathcal{O}_X) \rightarrow H^{i+1}(X, C^\cdot, \mathcal{O}_X). \]

Since \( H^i(X, C^\cdot, \mathcal{O}_X) \) has finite length over \( \mathcal{O}(k) \) for all \( i, j, r([3], III, Prop. (1.1)) \) it follows from the hypercohomology spectral sequences that \( H^i(X, C) \) and \( H^i(X, *C) \) are pro-systems of modules of finite lengths so by Mittag-Leffler we get an exact sequence

\[ H^i_{\text{crys}}(X/W) \rightarrow \varprojlim H^i(X, *C^\cdot, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X) \rightarrow H^{i+1}_{\text{crys}}(X/W). \]

Since \( H^2_{\text{crys}}(X/W) \) is assumed torsion free the connecting homomorphism

\[ H^1(X, \mathcal{O}_X) \rightarrow H^2_{\text{crys}}(X/W), \]

in the exact sequence above vanishes, i.e.

\[ \varprojlim H^1(X, C^\cdot, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X). \]

is surjective.

We have a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & C^\cdot \otimes_X & \rightarrow & *C^\cdot \otimes_X & \rightarrow & \mathcal{O}_X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}(\mathcal{O}_X) & \rightarrow & \mathcal{O}(\mathcal{O}_X) & \rightarrow & \mathcal{O}_X & \rightarrow & 0.
\end{array}
\]

hence a commutative diagram

\[
\begin{array}{ccccccccc}
H^1_{\text{crys}}(X/W) & \rightarrow & \varprojlim H^1(X, *C^\cdot, \mathcal{O}_X) & \rightarrow & H^1(X, \mathcal{O}_X) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^1(X, \mathcal{O}(\mathcal{O}_X)) & \rightarrow & H^1(X, \mathcal{O}(\mathcal{O}_X)) & \rightarrow & H^1(X, \mathcal{O}_X).
\end{array}
\]

it follows that

\[ H^1(X, \mathcal{O}(\mathcal{O}_X)) \rightarrow H^1(X, \mathcal{O}_X). \]

is surjective as desired.

The next proposition was pointed out by the referee, the proof is based on an idea by Deligne.

\( (2.2) \) \textsc{Proposition.} – \textit{Assume that the differentials in the \( E_1 \) term of the slope spectral sequence vanish then it degenerates at \( E_1 \).}

\textit{Proof.} – We show by induction that the differentials in the \( E_t \) term vanish so assume that the differentials in the \( E_t \) terms \( t = 1, \ldots, s - 1 \) are zero.
This implies that $E^{i,j}_r = H^j(X, C^i_X)$ for all $i, j$ so we must show that
\[ d : H^j(X, C^i_X) \to H^{j-s+1}(X, C^{i+s}_X) \]
vanishes.

Consider the commutative diagram of pro-complexes
\[
\begin{array}{cccccccccc}
C^i_{-1,X} & = 0 & \to & C^i_0 X & \to & \ldots & \to & C^i_1 X & \to & \ldots & \to & C^i_{\dim X} X & \to 0, \\
\uparrow^{\rho V} & & & \uparrow V & & & & \uparrow & & & & \uparrow & \\
\tilde{C}^i_{-1,X} & = 0 & \to & \tilde{C}^i_0 X & \to & \ldots & \to & \tilde{C}^i_1 X & \to & \ldots & \to & \tilde{C}^i_{\dim X} X & \to 0. \\
\end{array}
\]

Consider the hyper cohomology sequences then we have a commutative diagram
\[
\begin{array}{cccccccccc}
E^{i,j}_r(C^i_{-1,X}) & \to & E^{i+s,j-s+1}_r(C^i_{-1,X}) & \to & \ldots & \to & E^{i+s,j-s+1}_{\dim X} X & \to 0, \\
\uparrow V & & \uparrow \alpha & & & & \uparrow & & \uparrow \rho F & & \uparrow \rho F & \uparrow \rho F \\
E^{i,j}_r(C^i_{-1,X}) & \to & E^{i+s,j-s+1}_r(C^i_{-1,X}) & \to & \ldots & \to & E^{i+s,j-s+1}_{\dim X} X & \to 0. \\
\end{array}
\]

Passing to the limit we get a commutative diagram
\[
(2.3)
\begin{array}{cccccccccc}
E^{i,j}_r & \to & E^{i+s,j-s+1}_r & \to & \ldots & \to & E^{i+s,j-s+1}_{\dim X} X & \to 0, \\
\downarrow V & & \uparrow \alpha & & & & \uparrow \rho F & & \uparrow \rho F & \uparrow \rho F \\
E^{i,j}_r(C) & \to & E^{i+s,j-s+1}_r(C) & \to & \ldots & \to & E^{i+s,j-s+1}_{\dim X} X & \to 0. \\
\end{array}
\]

If the differentials in the preceding terms vanish $\alpha$ and $\delta$ are identities so we have a
commutative diagram
\[
\begin{array}{cccccccccc}
H^j(X, C^i_X) & \to & H^{j-s+1}(X, C^{i+s}_X) & \to & \ldots & \to & H^{j-s+1}_{\dim X} X & \to 0, \\
\uparrow V & & \downarrow \rho F & & & & \downarrow \rho F & & \downarrow \rho F & \uparrow \rho F \\
H^j(X, C^i_X) & \to & H^{j-s+1}(X, C^{i+s}_X). \\
\end{array}
\]

By iteration we get
\[
d_s = \rho^n F^n d_s V^n \quad \text{for all } n, \text{ hence,}
\]
\[
\text{Im } d_s \subset \bigcap_n \rho^n H^{j-s+1}(X, C^{i+s}_X) = 0.
\]

(2.4) Theorem. — Let $X/k$ be a surface, proper and smooth over $k$ with $k$ perfect of
characteristic $p > 0$, then the slope spectral sequence degenerates at $E_1$ if and only if
$H^2(X, \mathcal{W}(\mathcal{O}_X))$ is a finitely generated $\mathcal{W}(k)$ module.
Proof. — Assume that the slope spectral sequence degenerates at $E_1$ then $H^2(X, \mathcal{W}^e(\mathcal{O}_X))$ is a quotient of $H^2_{tor}(X/W)$ hence is finitely generated. The proof of the other implication rests on the following Lemma.

(2.5) LEMMA. — Let $d : L \to M$ be a linear map of $\mathcal{W}(k)$ modules. Let $F$ (resp. $V$) be a $\sigma$-linear (resp. $\sigma^{-1}$-linear) endomorphism of $M$ (resp. $L$) [this means $F(\lambda x) = \lambda^n F(x)$ and $V(\lambda y) = \lambda^{-n} V(y)$ where $\lambda \in \mathcal{W}(k)$ and $\sigma$ denotes the Frobenius endomorphism of $\mathcal{W}(k)$]. Assume that $L$ and $M$ are topological $\mathcal{W}(k)$ modules, $d$ is continuous, $M$ is separated and the topology on $L$ is weaker than the $V$-topology (i.e. the topology defined by the submodules $\{V^n L\}$), assume moreover that $F \circ V = d$. Then if the chains

$$
\ker d \subset \ker F d \subset \ldots \subset \ker F^n d \subset \ldots \subset L,
\text{Im } d \subset \text{Im } F d \subset \ldots \subset \text{Im } F^n d \subset \ldots \subset M,
$$

stabilize one has $d = 0$.

Proof. — Assume that both chains are stable at the $n$'th level. Let $x \in \ker F^n d$, then

$$
0 = F^n d x = F^{n+1} d V x \text{ so } V x \in \ker F^{n+1} d = \ker F^n d \text{ i.e. ker } F^n d \text{ is stable under } V \text{ and so } V^n x \in \ker F^n d \text{ hence } dx = F^n d V^n x = 0 \text{ and it follows that }
$$

$$
\ker d = \ker F d = \ldots = \ker F^n d = \ldots \subset L.
$$

Now the commutative diagram

$$
\begin{array}{ccc}
L/\ker d & \overset{F d}{\longrightarrow} & \text{Im } F^n d, \\
\downarrow V & & \downarrow \|
\end{array}
\text{ shows that } V \text{ induces an automorphism on } L/\ker d \text{ which is equivalent to ker } d \text{ being dense in the V-topology. Since the original topology on } L \text{ is weaker than the V-topology, ker } d \text{ is also dense in the original topology. But } d \text{ is continuous and } M \text{ is separated hence ker } d \text{ is also closed and so ker } d = L.
$$

Let us go back to the proof of the Theorem. By (2.2) it is enough to show that the differentials in the $E_1$ term vanish. The $E_1$ term looks as below:

$$
\begin{align*}
H^2(X, \mathcal{W}^e(\mathcal{O}_X)) & \overset{d_{1,2}^1}{\longrightarrow} H^2(X, \mathcal{C}_X^1) \overset{d_{1,2}^2}{\longrightarrow} H^2(X, \mathcal{C}_X^2), \\
H^1(X, \mathcal{W}^e(\mathcal{O}_X)) & \overset{d_{1,1}^1}{\longrightarrow} H^1(X, \mathcal{C}_X^1) \overset{d_{1,1}^2}{\longrightarrow} H^1(X, \mathcal{C}_X^2), \\
H^0(X, \mathcal{W}^e(\mathcal{O}_X)) & \overset{d_{1,0}^1}{\longrightarrow} H^0(X, \mathcal{C}_X^1) \overset{d_{1,0}^2}{\longrightarrow} H^0(X, \mathcal{C}_X^2).
\end{align*}
$$

Let us first show that the differentials in the bottom row are 0. This follows from the fact (1.4) that $p$ is injective on $H^0(X, \mathcal{C}_X^1)$ i.e. these modules are torsion free and the slope spectral sequence degenerates at $E_1$ modulo torsion ([3], III (3.2)). Next consider the differentials

$$
\begin{align*}
d_{1,1}^1 & : H^1(X, \mathcal{W}^e(\mathcal{O}_X)) \to H^1(X, \mathcal{C}_X^1), \\
d_{1,2}^1 & : H^1(X, \mathcal{W}^e(\mathcal{O}_X)) \to H^1(X, \mathcal{C}_X^2), \\
d_{1,0}^1 & : H^0(X, \mathcal{W}^e(\mathcal{O}_X)) \to H^0(X, \mathcal{C}_X^1), \\
d_{1,1}^2 & : H^1(X, \mathcal{W}^e(\mathcal{O}_X)) \to H^1(X, \mathcal{C}_X^2), \\
d_{1,2}^2 & : H^2(X, \mathcal{W}^e(\mathcal{O}_X)) \to H^2(X, \mathcal{C}_X^2), \\
d_{1,0}^2 & : H^0(X, \mathcal{W}^e(\mathcal{O}_X)) \to H^0(X, \mathcal{C}_X^2).
\end{align*}
$$
The modules have separated and complete topologies being limits of the discrete spaces $H^i(X, \mathcal{W}_r(O_X))$ and $H^i(X, C^i_X), \text{ clearly } \partial_{0,i}^i \text{ is continuous. The relation } F \partial_{0,i}^i V = \partial_{0,i}^i \text{ is satisfied by } (1.3) \text{ and the exact sequences }$

$$
H^i(X, \mathcal{W}(O_X)) \xrightarrow{\nu} H^i(X, \mathcal{W}(C_X)) \rightarrow H^i(X, \mathcal{W}_r(O_X)),
$$

show that the $\nu$-topology is finer than the limit topology on $H^i(X, \mathcal{W}(O_X))$ (they are actually identical), so by (2.5) we only have to show that the chains

$$
\ker \partial_{0,i}^i \subset \ker F \partial_{0,i}^i \subset \ldots \subset \ker F^n \partial_{0,i}^i \subset \ldots \subset H^i(X, \mathcal{W}(O_X)),
$$

$$
\operatorname{Im} \partial_{0,i}^i \subset \operatorname{Im} F \partial_{0,i}^i \subset \ldots \subset \operatorname{Im} F^n \partial_{0,i}^i \subset \ldots \subset H^i(X, C^i_X),
$$

stabilize. Now $H^i(X, \mathcal{W}(O_X))$ is finitely generated, for $i = 2$ it is the assumption and for $i = 1$ it is always true as proved in [15], Proposition 4 so the first chain stabilizes.

For the second we have

$$
\operatorname{Im} F^n \partial_{0,i}^i \subset \ker \partial_{1,i}^i \quad \text{for all } n,
$$

so

$$
\operatorname{Im} F^n \partial_{0,i}^i / \operatorname{Im} \partial_{0,i}^i \subset \ker \partial_{1,i}^i / \operatorname{Im} \partial_{1,i}^i = E_{2,i}^1,
$$

and we have $E_{2,i}^1 = E_{\infty,i}^1$ (since $\dim X = 2$) which is a subquotient of $H^2_{\text{crys}}(X/W)$ hence finitely generated so the chain

$$
\operatorname{Im} F \partial_{0,i}^i / \operatorname{Im} \partial_{0,i}^i \subset \ldots \subset \operatorname{Im} F^n \partial_{0,i}^i / \operatorname{Im} \partial_{0,i}^i \subset \ldots \subset E_{\infty,i}^1,
$$

stabilizes which shows that the second chain is stable.

For the differential

$$
\partial_{1,2} : H^2(X, C^2_X) \rightarrow H^3(X, C^3_X),
$$

we use the fact that $F$ is an automorphism of $H^2(X, C^2_X)$ to conclude that the chain of kernels stabilize, namely

$$
\ker \partial_{1,2} = \ker F^n \partial_{1,2} \quad \text{for all } n.
$$

The chain of images stabilizes because $H^2(X, C^2_X) / \operatorname{Im} \partial_{1,2} = E_{\infty,2}^2$ is finitely generated. To conclude that $\partial_{1,2} = 0$ we only need to show that the $\nu$-topology is finer than the limit topology, this follows however from the commutative diagram

$$
\begin{array}{ccc}
H^1(X, C^1_X) & \xrightarrow{\nu} & H^1(X, C^1_X) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^1(X, C^1_X).
\end{array}
$$
The only differential left is
\[ d_{11}^{-1} : \ H^1(X, C_1^j) \to H^1(X, C_2^j). \]

The chain of kernels stabilizes for the same reasons as above, and in order to show that the chain of images stabilizes it is enough to show that \( H^1(X, C_2^j)/\text{Im} \ d_{11}^{-1} \) is finitely generated, this follows however from the exact sequence

\[
E_2^{0,2} = H^1(X, C_2^j)/\text{Im} \ d_{11}^{-1} = E_3^{1,1} = E_\infty^{2,1},
\]

because \( E_2^{0,2} \subset H^2(X, \mathcal{O}_X) \) is finitely generated. This concludes the proof of the Theorem.

Remark. — Some parts of the proof of (2.4) goes through without assuming that \( H^1(X, \mathcal{O}_X) \) is finitely generated or that \( X \) is a surface; in particular that

\[ H^1(X, \mathcal{O}_X) = E_\infty^{0,1}, \]

and

\[ H^0(X, C_1^j) = E_\infty^{1,0}, \]

there results an exact sequence

\[ 0 \to H^0(X, C_1^j) \to H^1\text{crys}(X/W) \to H^1(X, \mathcal{O}_X) \to 0, \]

since \( H^0(X, C_1^j) \) is torsion free by (1.4) and \( H^1(X, \mathcal{O}_X) \) is torsion free by [15], p. 32, we deduce the well known fact that \( H^1\text{crys}(X/W) \) is torsion free.

(2.6) Corollary. — Let \( X/k \) be a smooth proper variety and assume that \( \text{Pic} (X) \) is reduced then the differential

\[ d_{11}^{0,1} : \ H^1(X, \mathcal{O}_X) \to H^1(X, \Omega^1_{X/k}). \]

vanishes.

Proof. — Let \( (E, d) \) denote the slope spectral sequence and \( (E', d') \) the Hodge to de Rham spectral sequence. Since \( C_{1, X} \cong \Omega_{X/k} \) ([3], II (3.1)) we have a map of spectral sequences

\[ (E, d) \to (E', d'), \]

in particular a commutative diagram

\[
\begin{array}{ccc}
H^1(X, \mathcal{O}_X) & \xrightarrow{\delta_{11}^{0,1}} & H^1(X, C_1^j) \\
\downarrow & & \downarrow \\
H^1(X, \mathcal{O}_X) & \xrightarrow{\delta_{11}^{0,1}} & H^1(X, \Omega^1_{X/k}).
\end{array}
\]

By the remark above the horizontal map on top is zero, and the left hand vertical map is surjective since \( \text{Pic} (X) \) is reduced, hence the corollary.
(2.6) Has also been proved by T. Oda in his Harvard thesis [14].

(2.7) Corollary. — Let $X/k$ be a smooth proper surface. Assume that $\text{Br}_X^*$ is pro-represented by a $p$-divisible formal group then the Hodge to de Rham spectral sequence degenerates at $E_1$.

Proof. — By [1], Corollary (4.3), the (covariant) Dieudonné module of $\text{Br}_X^*$ is $H^2(X, \mathcal{W}^*(\mathcal{O}_X))$ so $\text{Br}_X^*$ $p$-divisible implies that $H^2(X, \mathcal{W}^*(\mathcal{O}_X))$ is finitely generated and free [10], and hence by (2.4) the slope spectral sequence degenerates at $E_1$.

Since $H^2(X, \mathcal{W}^*(\mathcal{O}_X))$ is free

$$H^1(X, \mathcal{W}^*(\mathcal{O}_X)) \to H^1(X, \mathcal{O}_X),$$

is surjective.

$$H^2(X, \mathcal{W}^*(\mathcal{O}_X)) \to H^2(X, \mathcal{O}_X),$$

is surjective because $H^3(X, \mathcal{W}^*(\mathcal{O}_X)) = 0$, and

$$H^0(X, \mathcal{W}^*(\mathcal{O}_X)) \to H^0(X, \mathcal{O}_X),$$

because $H^1(X, \mathcal{W}^*(\mathcal{O}_X))$ is free ([15], p. 32) it follows that

$$d_1^{0,i} : H^i(X, \mathcal{O}_X) \to H^i(X, \Omega^1_{k/k}),$$

is zero $i = 0, 1, 2$, by Serre duality the rest of the differentials in the $E_1$ term vanish. A similar argument shows that the higher differentials vanish as well.

(2.8) Proposition. — With the assumptions of (2.7) assume further that $H^2_{\text{crys}}(X/W)$ is torsion free then:

(i) $\dim_k H^i_{\text{DR}}(X/k) = \dim_k H^i_{\text{crys}}(X/W) \otimes K$, $i = 0, 1, 2, 3, 4$, where $K$ is the fraction field of $\mathcal{W}(k)$;

(ii) $h^{i,j} = \dim_k H^i(X, \Omega^j_{X/k}) = \dim_k H^i(X, \Omega^j_{X/k}) = h^{i,j}.$

Proof. — (i) follows from the exact sequences:

$$0 \to H^i_{\text{crys}}(X/W) \otimes k \to H^i_{\text{DR}}(X/k) \to \text{Tor}^W_1(k)(H^{i+1}_{\text{crys}}(X/W), k) \to 0,$$

plus the fact that $H^3_{\text{crys}}(X/W)$ is also torsion free (by Poincaré duality).

To prove (ii) it is enough to show

$$h^{0,1} = h^{1,0}.$$
the other equalities then follow from Serre duality.

\[ h^{0,1} = \dim_k H^1(X, \mathcal{O}_X) = \dim \text{Pic}^0(X), \]

since \( \text{Pic}^0(X) \) is reduced.

\[ \dim H^1_{\text{DR}}(X/k) = h^{0,1} + h^{1,0}, \quad \text{by (2.7)} \]

and

\[ \dim_k H^1_{\text{crys}}(X/W) \otimes K = 2 \dim \text{Pic}^0(X). \]

and the equality follows from (i).

3. Surfaces over an irreducible scheme

In this section we consider a smooth proper \( S \)-scheme \( f : X \to S \) with geometrically irreducible fibers of dimension 2; \( S \) an irreducible \( \mathbb{F}_p \)-scheme such that \( f^* \mathcal{O}_X = \mathcal{O}_S \).

(3.1) Lemma. — Let \( A \) be a local domain of characteristic \( p \) with maximal ideal \( \mathfrak{m} \) and residue field \( k \). Let \( \hat{A} \) be the completion at \( \mathfrak{m} \) and \( L \) the fraction field of \( \hat{A} \). Assume that \( G = \text{Spf} \ A[[t_1, \ldots, t_n]] \) is a connected formal Lie group such that \( G_{\hat{k}} \) is \( p \)-divisible, then the formal Lie group \( G_{\hat{k}} \) is \( p \)-divisible.

Proof. — Let the power series \( f_1, \ldots, f_n \) define multiplication by \( p \) in \( G \), then \( \ker p : G_{\hat{k}} \to G_{\hat{k}} \) is represented by \( L[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n) \) and it is enough to show that this is a finite dimensional \( L \)-vectorspace ([6], p. 47). Since \( G_{\hat{k}} = \text{Spf} \ \hat{k}[[t_1, \ldots, t_n]] \) is \( p \)-divisible \( G_{\hat{k},/\mathfrak{m}} = \text{Spf} \ A/\mathfrak{m}'[[t_1, \ldots, t_n]] \) is \( p \)-divisible for all \( r \geq 1 \) ([6], p. 62) so \( A/\mathfrak{m}'[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n) \) is a finitely generated \( A/\mathfrak{m}' \)-module. Let \( e_1, \ldots, e_s \in A[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n) \) such that

\[ \{ e_1, \ldots, e_s \} \subset k[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n). \]

is a set of generators, it follows from Nakayama’s Lemma that

\[ \{ \overline{e}_1, \ldots, \overline{e}_s \} \subset A/\mathfrak{m}'[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n). \]

generates for all \( r \geq 1 \).

Let \( M \) be the \( A \)-module generated by \( \{ e_1, \ldots, e_s \} \) then

\[ M/\mathfrak{m}' M = A/\mathfrak{m}'[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n). \]

\[ \hat{A}[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n) = \lim A/\mathfrak{m}'[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n) = \lim M/\mathfrak{m}' M = \hat{M}. \]

Since \( M \) is finitely generated \( \hat{M} = M \otimes \hat{A} \) is finitely generated over \( \hat{A} \) so

\[ \hat{A}[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n). \]
and hence

\[ L[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n). \]

is finitely generated.

(3.2) Theorem. — Assume that there is a closed point \( s_0 \in S \) such that the geometric fibre \( Y = X_{s_0} \) has p-divisible formal Brauer group, then the differential

\[ d : f_* \Omega^1_{X/S} \to f_* \Omega^2_{X/S}. \]

is zero.

Proof. — By the smoothness of \( f \), \( f_* \Omega^2_{X/S} \) is a locally free sheaf on \( S \) so the set

\[ F = \{ s \in S | d_s : (f_* \Omega^1_{X/S})_s \to (f_* \Omega^2_{X/S})_s \text{ is zero} \}, \]

is a closed set. We are going to show that the generic points is in \( F \).

\[ \text{Pic}^0(X/S) \] is representable by \([7]\), Theorem (3.1) and since we have assumed that \( Br_Y \) is pro-representable by a \( p \)-divisible formal group \( H^2(Y, \mathcal{W}(\mathcal{O}_Y)) \) is free so \( H^1(Y, \mathcal{O}_Y) \to H^2(Y, \mathcal{W}(\mathcal{O}_Y)) \) is zero hence \( \text{Pic}^0(Y) \) is smooth ([12], p. 196).

By [8], Theorem (3.5) there is a non-empty open set \( s_0 \in \mathcal{U} \subset S \) such that \( \text{Pic}^0(X/\mathcal{U}) \) is smooth and hence \( Br_{X/\mathcal{U}} \) is representable by a formal group which is formally smooth since the fibre dimension is 2 ([1], Cor. (4.1)).

Let \( \{ G[n] \}_n \) be the inductive system of locally free finite groups associated to the formal Lie group \( Br_{X/\mathcal{U}} \) ([6], Prop. (2.6)). Locally on \( \mathcal{U} \) each \( G[n] \) is isomorphic to \( \text{Spec} \mathcal{O}_\mathcal{U}[t_1, \ldots, t_d]/(t_1^d, \ldots, t_d^d) \) where \( d \) is the rank of the conormal bundle of \( Br_{X/\mathcal{U}} \) ([6], Prop. (2.1)).

We can assume \( S = \text{Spec} R \) where \( R = \mathcal{O}_{S,s_0} \), hence over \( \text{Spec} R \), \( Br_{X/R} \) is isomorphic to \( \lim G[n] \) with each

\[ G[n] = \text{Spec} R[t_1, \ldots, t_d]/(t_1^d, \ldots, t_d^d). \]

Since \( X/R \) is smooth the functor \( Br_{X/R} \) is isomorphic to the sheaf \( R^2 f_* \mathbb{G}_m \) on the big etale site of \( \text{Spec} R \) ([1], Prop. (1.7)). By general theorems about sheaf cohomology ([16], Prop. (5.1)) this implies that the formal Brauer group commutes with all base changes. In terms of the inductive system this means that

\[ Br_{X/\gamma} = \lim G[n] \otimes_R T = \lim \text{Spec} \mathcal{O}_T[t_1, \ldots, t_d]/(t_1^d, \ldots, t_d^d). \]

for every \( R \)-scheme \( T \).

Let \( \eta \) be the fraction field of \( R \) and \( L \) the fractional field of \( \bar{R} \), \( \kappa \) is the residue field. By assumption

\[ Br_X = \lim \mathcal{O}_T[t_1, \ldots, t_d]/(t_1^d, \ldots, t_d^d). \]
is $p$-divisible hence (3.1) gives that
$$\text{Br}_{X_L}^c = \lim_{\text{Spec}} L[t_1, \ldots, t_d]/(t_1^p, \ldots, t_d^p),$$
is $p$-divisible and so also $\text{Br}_{X_L}^c$ is $p$-divisible.

(2.4) Then implies that
$$d : H^0(X_L, \Omega^1_{X_L}) \to H^0(X_L, \Omega^2_{X_L}).$$
is zero, and by faithfully flat descent
$$d : H^0(X_\eta, \Omega^1_{X_\eta}) \to H^0(X_\eta, \Omega^2_{X_\eta}),$$
is zero which shows $\text{spec } \eta \in F$.

(3.3) **Corollary.** — With the assumptions of (3.2) assume that all the sheaves $R^if_*\Omega^1_{X/S}$ are locally free on $S$ then the spectral sequence
$$E^1_{i,j} = R^if_*\Omega^1_{X/S} \Rightarrow H^*_\text{DR}(X/S),$$
degenerates at $E_1$.

**Proof.** — In this case the set
$$F = \{ s \in S | (R^if_*\Omega^1_{X/S})_s \Rightarrow H^*_\text{DR}(X/S)_s \text{ degenerates at } E_1 \},$$
is closed and the proof of (3.2) shows that $F$ contains the generic point.

**REFERENCES**


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