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ORBITAL INTEGRALS AND A FAMILY OF GROUPS ATTACHED TO A REAL REDUCTIVE GROUP ⁽¹⁾

BY DIANA SHELSTAD

1. Introduction

In this paper we pursue one of the questions suggested by the formulations in [7] (*cf.* [10]). Our concern will be with transferring orbital integrals from one group (of $\mathbf{R}$ -rational points on a connected reductive linear algebraic group defined over $\mathbf{R}$) to another. In [9] we considered “stable” orbital integrals and obtained a transfer which will be our starting point. We recall some details. Suppose that f is a Schwartz function on the group G , that T is a Cartan subgroup of G and that γ is a regular element in T . Then, following Langlands, we have defined

$$\Phi_f^1(\gamma) = \sum_{\omega} \int_{G/T} f(g \gamma^{\omega} g^{-1}) d\bar{g},$$

where $d\bar{g}$ is a G -invariant measure on G/T (whose normalization we ignore for the present) and ω ranges over the set $\mathcal{D}(T)$ [7] which we may identify simply as the quotient of the imaginary Weyl group for T by the subgroup of those elements realized in G . . . recall that any element of the imaginary Weyl group stabilizes T . Our interest in these stable orbital integrals lies in the fact that the distributions $f \rightarrow \Phi_f^1(\gamma)$ generate the characters attached to L -packets of tempered irreducible representations of G (*cf.* [9]).

Suppose that for each Cartan subgroup T we are given a function Φ^T on the regular elements in T . Then a theorem of [9] provides necessary and sufficient conditions for the existence of a Schwartz function f on G such that $\Phi^T = \Phi_f^1$ for each T . On the other hand, if we fix an L -group (=associate group [8]) for G then we are provided with a quasi-split group G^* and an inner twist ψ from G , the underlying algebraic group for G , to G^* . The map ψ determines embeddings of each Cartan subgroup (of G) in G^* ; these embeddings induce an injection of the set $t(G)$ of conjugacy classes of Cartan subgroups of G in $t(G^*)$. Recall that $t(G)$ is partially ordered (*cf.* [3]); the image of $t(G)$ in $t(G^*)$ forms an “initial segment” of $t(G^*)$ [9]. We say that an element γ' of G^* originates from the regular element γ of G if γ' is the image of γ under one of the embeddings in G^* of the Cartan subgroup containing γ .

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Now combining this and the characterization theorem, we can effect a transfer of stable orbital integrals from G to G^* in the following sense: given a Schwartz function f on G there is a Schwartz function f' on G^* such that $\Phi_{f'}^1(\gamma') = \Phi_f^1(\gamma)$ if γ' originates from the regular element γ in G , with $\Phi_{f'}^1(\gamma') = 0$ if γ' does not originate in G .

We come then to our present problem. First, we replace Φ_f^1 by an "unstable" orbital integral. If κ assigns to each ω in $\mathcal{D}(T)$ a value, either 1 or -1 , then again following Langlands, we set

$$\Phi_f^\kappa(\gamma) = \sum_{\omega \in \mathcal{D}(T)} \kappa(\omega) \int_{G/T} f(gt^\omega g^{-1}) dg,$$

for regular γ in T . Global considerations (for example, the suitable grouping of some terms on one side of the Trace Formula (*cf.* [5], § 5, for SL_2) suggest that we consider those κ described in [7]; we recall the appropriate definitions and observations in Paragraphs 2,3. Briefly, as described in [7], $\mathcal{D}(T)$ can be embedded in a quotient of the module generated by the coroots of T in G and κ is a quasicharacter on this quotient. . . the domain of κ is thus larger than $\mathcal{D}(T)$. From now on we assume that κ is of such type and call Φ_f^κ a κ -orbital integral. In Paragraph 4 we will describe the invariance, smoothness and "jump" properties (which we find easier to work with than "germ expansions") of κ -orbital integrals.

The triple (G, T, κ) determines, via an L-group construction, a quasi-split group H of same rank as G , but possibly of lower dimension [7]. We will recall the construction in Paragraph 5, remarking now only the fact that T can be embedded in H and $\mathcal{D}_H(T)$ transferred to G ; κ is trivial on the image of $\mathcal{D}_H(T)$. An imprecise version of a question of Langlands asks whether the κ -orbital integrals for G transfer to stable orbital integrals on H . To proceed to a more careful formulation we observe that the L-group construction provides not only H but also some ancillary data, including a quasi-split group G^* and an inner twist ψ from G to G^* . The data yield embeddings of the Cartan subgroups of H in G^* and a map from $t(H)$ into $t(G^*)$; recalling the map of $t(G)$ into $t(G^*)$ determined by ψ we obtain then a notion of a Cartan subgroup of H originating in G . For example, using the notation of [3] for $t(\)$, we may have:

and obtain three conjugacy classes of Cartan subgroups in H originating in G (case G nonsplit, noncompact form of type C_2 . . . H of type $A_1 \times A_1$). Suppose that T' originates from T (our given Cartan subgroup). Then the transfer of Φ_f^κ to T' depends on the choice of map from T' to T . Thus we have to qualify our notion of an element γ' of H originating from a regular element γ of G . We will do this by choosing a set $\mathcal{J} = \{i_m : T'_m \rightarrow T_m, m=0, 1, \dots, N\}$ of embeddings such that T_0 is our given group T and $T'_0, \dots, T'_N$ form a complete set of representatives for the conjugacy classes which originate in G (*see*

Paragraph 6 for technical assumptions). We then say that γ' originates from $\gamma \in T_m$ with respect to $\mathcal{J}$ if γ' is stably conjugate to $i_m^{-1}(\gamma)$; that is, if γ' is obtained from $i_m^{-1}(\gamma)$ by the action of an element of $\mathcal{A}(T_m)$ (cf. [7], recalled also in Paragraph 2). Also attached to $\mathcal{J}$ is a transfer of κ to each of the Cartan subgroups $T_1, \dots, T_N$ (cf. Paragraph 7).

We come then to the main problem, that of finding a factor Δ so that for each Schwartz function f on G there is a Schwartz function f' on H satisfying:

- (1) $\Phi_{f'}^1(\gamma') = \Delta(\gamma) \Phi_f^*(\gamma)$ if γ' originates from the regular element γ in G with respect to $\mathcal{J}$ and
- (2) $\Phi_{f'}^1 \equiv 0$ on those Cartan subgroups of H which do not originate in G .

Thus Δ is to be a function on the regular elements of $\bigcup_{m=0}^N T_m$. On each Cartan subgroup T_m we fix a system of positive imaginary roots. We may consider, at least formally,

$$\prod_{\substack{\alpha \text{ positive} \\ \text{imaginary} \\ \text{not from } H}} (e^{\alpha/2} - e^{-\alpha/2}) \prod_{\substack{\alpha \text{ positive} \\ \text{not imaginary} \\ \text{not from } H}} |e^{\alpha/2} - e^{-\alpha/2}|$$

(the conditions on α are made precise in Paragraph 7).

This expression can be interpreted as a function Δ_m on T_m if half the sum of the positive imaginary roots “not from H ” lifts to a character on T_m . That will be the major part of our assumption (8.1). In prescribing a candidate for Δ we insert parameters $\varepsilon_0, \dots, \varepsilon_N$, each equal to 1 or -1 ; thus our candidate will be the function $\Delta_H^G = \Delta_H^G(\varepsilon_0, \dots, \varepsilon_N)$ defined by $\{\varepsilon_m \Delta_m; m=0, \dots, N\}$. The existence (for some choice of $\varepsilon_0, \dots, \varepsilon_N$) of a “transfer of orbital integrals” in the sense of the last paragraph is then independent of our choice of $\mathcal{J}$ and the systems of positive imaginary roots. In Theorem 8.3 we show that $\gamma' \rightarrow \Delta_H^G(\gamma) \Phi_f^*(\gamma)$ is well-defined (although, in general, neither Δ_H^G nor Φ_f^* alone transfers to H in this way).

Our main result, Theorem 10.2, is a set of necessary and sufficient conditions on the choices for $\varepsilon_0, \dots, \varepsilon_N$ in order that $\Delta_H^G = \Delta_H^G(\varepsilon_0, \dots, \varepsilon_N)$ provide a transfer of orbital integrals. Suppose that the classes of T'_m and T'_n are adjacent in the lattice $t(H)$. Then we attach to the pair (m, n) a signature $\varepsilon_\kappa(m, n)$ obtained from values of κ and a signature $\varepsilon_+(m, n)$ obtained by evaluating some determinants. Our conditions are:

$$\varepsilon_m \varepsilon_n = \varepsilon_\kappa(m, n) \varepsilon_+(m, n).$$

In Paragraph 11 we begin a study of the consistency of these equations as the pair (m, n) varies. After some remarks, suggesting a general procedure, and two examples we can conclude that if the derived group of G is isogenous to a product of groups each of rank at most two, then there is indeed a choice of $\varepsilon_0, \dots, \varepsilon_N$ for which $\Delta_H^G(\varepsilon_0, \dots, \varepsilon_N)$ provides a transfer of orbital integrals.

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NOTATION. — We continue with the notation of [9], except that now σ denotes complex conjugation and we further generalize the notion of Cayley transform (cf. Paragraph 3). By

the character module of a torus we will mean the group of rational characters, with multiplication written additively; roots will be rational characters, rather than linear functionals on the Lie algebra (as in [9] and the present section).

2. The set $\mathcal{D}(T)$

Let T be a maximal torus in G , defined over R . We recall from [7] that

$$\mathcal{A}(T) = \{g \in G : \text{ad } g/T \text{ is defined over } R\},$$

and $\mathcal{D}(T) = G \backslash \mathcal{A}(T) / T$. If M is the centralizer in G of the maximal R -split torus in T then $\mathcal{A}(T) = G \cdot \text{Norm}(M, T)$ [9]. Hence we may as well regard $\mathcal{D}(T)$ as $\Omega(M, T) \backslash \Omega(M, T)$, $\Omega(M, T)$ being the Weyl group of T in M (. . . the “imaginary Weyl group of T ”) and $\Omega(M, T)$ the subgroup of $\Omega(M, T)$ consisting of those elements which can be realized in M .

We need to recall some facts from [7]. We will use σ to denote the non-trivial element of the Galois group of C over R and $H^*()$ to denote the cohomology of $\langle 1, \sigma \rangle$. If $g \in \mathcal{A}(T)$ then $\sigma(g^{-1})g \in T$ so that $g \rightarrow (1 \rightarrow 1, \sigma \rightarrow \sigma(g^{-1})g)$ yields a map of $\mathcal{A}(T)$ into the 1-cocycles for T . This map induces a bijection between $\mathcal{D}(T)$ and those elements of $H^1(T)$ which are annihilated by the natural map of $H^1(T)$ into $H^1(G)$. Such elements of $H^1(T)$ lie in a subgroup $\mathcal{E}(T)$ obtained as follows. Let $\tilde{G}$ be the simply-connected covering group of the derived group of G , π the natural homomorphism of $\tilde{G}$ into G and $\tilde{T}$ the inverse image of T under π . Then $\mathcal{E}(T)$ is the image of $H^1(\tilde{T})$ under the homomorphism into $H^1(T)$ induced by π .

To continue with [7], we denote the character module of T by $L(T)$ and set $\tilde{L}(T) = \text{Hom}(L(T), Z)$. In the usual manner we identify $\tilde{L}(\tilde{T})$ with the submodule $\langle \tilde{\Xi} \rangle$ of $\tilde{L}(T)$ generated by the set $\tilde{\Xi}$ of coroots for T in G . Tate-Nakayama duality then establishes a canonical isomorphism between $\mathcal{E}(T)$ and the image under the natural homomorphism of $H^{-1}(\langle \tilde{\Xi} \rangle)$ into $H^{-1}(\tilde{L}(T))$ or, just as well, between $\mathcal{E}(T)$ and the quotient of $\{\lambda \in \langle \tilde{\Xi} \rangle : \sigma\lambda = -\lambda\}$ by

$$\mathcal{L}(T) = \{\lambda \in \langle \tilde{\Xi} \rangle : \lambda = \sigma\mu - \mu, \text{ some } \mu \text{ in } \tilde{L}(T)\}.$$

Hence $\mathcal{D}(T)$ is identified as a collection of cosets of $\mathcal{L}(T)$ in $\langle \tilde{\Xi} \rangle$; we shall call this the T-N identification.

As for realizing T-N explicitly we will need only an (unpublished) observation of Langlands; we state it as a proposition as we will use it in several places. Recall that a root α is imaginary if and only if $\sigma\alpha = -\alpha$ or, equivalently, $\sigma\tilde{\alpha} = -\tilde{\alpha}$. Assume now that α is imaginary; ω_α , the Weyl reflection with respect to α , lies in $\Omega(M, T)$. In the case α is compact (cf. [9]) ω_α lies in $\Omega(M, T)$.

PROPOSITION 2.1. — *In the case α is noncompact the image under T-N of $\Omega(M, T)\omega_\alpha$ is $\tilde{\alpha} + \mathcal{L}(T)$.*

The proof is straightforward. Indeed, fix a homomorphism (over $\mathbf{R}$) of SL_2 in $\mathbf{G}$ as in [9]. Then the image of $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is a 1-cocycle of $\mathbf{T}$ attached to ω_α in the manner earlier. It is now a matter of reviewing the $\mathbf{T} - \mathbf{N}$ identification explicitly (cf. [6]); we omit the details.

3. Characters on $\mathcal{D}(\mathbf{T})$

Following [7] we will consider a quasicharacter κ on $\langle \Xi^\sim \rangle$, trivial on $\mathcal{L}(\mathbf{T})$. Note that the restriction of κ (as quasicharacter on $\langle \Xi^\sim \rangle / \mathcal{L}(\mathbf{T})$) to $\mathcal{D}(\mathbf{T})$ takes only the values ± 1 . We will often refer to κ as a “character on $\mathcal{D}(\mathbf{T})$ ” [although, in general, the domain is larger and κ is not determined by its restriction to $\mathcal{D}(\mathbf{T})$].

If $\omega \in \mathcal{A}(\mathbf{T})$ then clearly

$$\kappa^\omega(\lambda^\sim) = \kappa(\omega^{-1} \lambda^\sim), \quad \lambda^\sim \in \langle \Xi^\sim \rangle,$$

defines a character on $\mathcal{D}(\mathbf{T}^\omega)$; here, as usual, ω acts on $\langle \Xi^\sim \rangle$ by the contragredient of the adjoint action. On the other hand, we will often write $\kappa(\omega)$ for the value of κ on the coset $\Omega(\mathbf{M}, \mathbf{T})\omega$ in $\mathcal{D}(\mathbf{T})$. If $\omega_1 \in \mathcal{A}(\mathbf{T})$ and $\omega_2 \in \mathcal{A}(\mathbf{T}^{\omega_1})$ then $\omega_2 \omega_1 \in \mathcal{A}(\mathbf{T})$ and:

PROPOSITION 3.1:

$$\kappa(\omega_2 \omega_1) = \kappa(\omega_1) \kappa^{\omega_1}(\omega_2).$$

Proof. — Note that

$$\sigma(\omega_1^{-1} \omega_2^{-1}) \omega_2 \omega_1 = \sigma(\omega_1^{-1}) \omega_1 \omega_1^{-1} (\sigma(\omega_2^{-1}) \omega_2) \omega_1.$$

Since the $\mathbf{T} - \mathbf{N}$ identification respects the action of $\mathcal{A}(\mathbf{T})$ the assertion is now clear.

Suppose that α is an imaginary root of $\mathbf{T}$ in $\mathbf{G}$. Provided that there is a noncompact root among the elements $\omega\alpha$, ω in the imaginary Weyl group of $\mathbf{T}$ [or, just as well, ω in $\Omega_0(\mathbf{G}, \mathbf{T})$, the elements realized in $\mathcal{A}(\mathbf{T})$], we can find $s \in \mathbf{G}$ such that $\sigma(s^{-1})s$ realizes the Weyl reflection ω_α [9]. In the case that α itself is noncompact we have called s a Cayley transform with respect to α [9]. It is convenient now to drop this requirement on α : thus, as long as α is imaginary and $\sigma(s^{-1})s$ realizes ω_α we will call s a *Cayley transform with respect to α* . The assertions of Proposition 2.7 in [9] remain true; in particular, $\mathbf{T}_s$, the image of $\mathbf{T}$ under s , is defined over $\mathbf{R}$.

PROPOSITION 3.2. — If $\kappa(\alpha^\sim) = 1$ then

$$\kappa^s(\lambda^\sim) = \kappa(s^{-1} \lambda^\sim), \quad \lambda^\sim \in \langle \Xi_s^\sim \rangle,$$

defines a character on $\mathcal{D}(\mathbf{T}_s)$.

Here $\Xi_s^\sim$ denotes the set of coroots for $\mathbf{T}_s$ in $\mathbf{G}$.

Proof. — We have only to show that if $\mu^\sim \in L^\sim(T_s)$ and $\sigma\mu^\sim - \mu^\sim \in \langle \Xi_s^\sim \rangle$ then $\kappa^s(\sigma\mu^\sim - \mu^\sim) = 1$. But

$$s^{-1}\sigma\mu^\sim - s^{-1}\mu^\sim = \sigma\sigma(s^{-1})s(s^{-1}\mu^\sim) - s^{-1}\mu^\sim = \sigma\omega_\alpha(s^{-1}\mu^\sim) - s^{-1}\mu^\sim,$$

which differs from $\sigma(s^{-1}\mu^\sim) - s^{-1}\mu^\sim$ by an integral multiple of $\alpha^\sim$. Hence the proposition is proved.

Finally, we include some simple computations needed in the next section.

PROPOSITION 3.3. — *Suppose that α is a noncompact imaginary root for which $\kappa(\alpha^\sim) = 1$. Then:*

- (i) $\kappa^{\omega_\alpha} = \kappa$;
- (ii) $\kappa(\omega\omega_\alpha) = \kappa(\omega)$, $\omega \in \mathcal{A}(T)$; and
- (iii) if s is a Cayley transform with respect to α then $\kappa^s(\omega) = \kappa(s^{-1}\omega s)$ for any $\omega \in \mathcal{A}(T_s)$ which normalizes T_s .

Proof:

- (i) $\kappa^{\omega_\alpha}(\lambda^\sim) = \kappa(\lambda^\sim)\kappa(\omega_\alpha\lambda^\sim - \lambda^\sim) = \kappa(\lambda^\sim)$, $\lambda^\sim \in \langle \Xi^\sim \rangle$;
- (ii) $\kappa(\omega\omega_\alpha) = \kappa(\omega_\alpha)\kappa^{\omega_\alpha}(\omega) = \kappa(\alpha^\sim)\kappa(\omega) = \kappa(\omega)$ (cf. Props. 3.1, 2.1);
- (iii) Proposition 4.6 of [9] and Proposition 3.1 show that it is enough to prove (iii) in the case where ω realizes the Weyl reflection with respect to an imaginary root β of T_s .

Suppose that β is compact. Then $\kappa^s(\omega_\beta) = 1$. Proposition 4.6 of [9] shows that either $\omega_{s^{-1}\beta}$ or $\omega_{s^{-1}\beta}\omega_\alpha$ is realized in G . Since $\kappa(\omega_{s^{-1}\beta}\omega_\alpha) = \kappa(\omega_{s^{-1}\beta})$ (ii) we obtain $\kappa^s(\omega_\beta) = \kappa(\omega_{s^{-1}\beta}) = \kappa(s^{-1}\omega_\beta s)$.

Suppose that β is noncompact. Again an argument as in Proposition 4.6 of [9] shows that if $\omega_{s^{-1}\beta}$ is realized in G then so is ω_β . Hence if ω_β is not realized in G we get

$$\kappa^s(\omega_\beta) = \kappa^s(\beta^\sim) = \kappa(s^{-1}\beta^\sim) = \kappa(\omega_{s^{-1}\beta}) = \kappa(s^{-1}\omega_\beta s).$$

On the other hand, if ω_β is realized in G we may argue as in the previous paragraph and the proof is completed.

4. Definition and properties of Φ_f^*

We come then to orbital integrals. Fix a Schwartz function f on G . As in [9], if T is a Cartan subgroup of G , dt a Haar measure on T , dg a Haar measure on G and γ a regular element of T we set

$$\Phi_f(\gamma, dt, dg) = \int_{G/T} f(g\gamma g^{-1}) d\bar{g},$$

$d\bar{g}$ denoting the quotient measure arising from dt and dg . Recall that if $\omega \in \mathcal{A}(T)$ then $\Phi_f(\gamma^\omega, (dt)^\omega, dg)$ depends only on the class of ω in $\mathcal{D}(T)$. Hence we may define

$$\Phi_f^*(\gamma, dt, dg) = \sum_{\omega \in \mathcal{D}(T)} \kappa(\omega) \Phi_f(\gamma^\omega, (dt)^\omega, dg)$$

(cf. [5]); recall that $\gamma^\omega = \omega \gamma \omega^{-1}$. It is clear that

$$\Phi_f^\kappa(\gamma, \alpha dt, \beta dg) = \beta/\alpha \Phi_f^\kappa(\gamma, dt, dg), \quad \alpha, \beta > 0.$$

PROPOSITION 4.1:

$$\Phi_f^{\kappa^\omega}(\gamma^\omega, (dt)^\omega, dg) = \kappa(\omega) \Phi_f^\kappa(\gamma, dt, dg), \quad \omega \in \mathcal{A}(\mathbf{T}).$$

Proof:

$$\begin{aligned} \sum_{\omega' \in \mathcal{D}(\mathbf{T}^\omega)} \kappa^\omega(\omega') \Phi_f((\gamma^\omega)^{\omega'}, ((dt^\omega)^{\omega'}, dg) &= \sum_{\omega' \in \mathcal{D}(\mathbf{T}^\omega)} \frac{\kappa^\omega(\omega')}{\kappa(\omega' \omega)} \kappa(\omega' \omega) \Phi_f(\gamma^{\omega' \omega}, (dt)^{\omega' \omega}, dg) \\ &= \kappa(\omega) \sum_{\omega'' \in \mathcal{D}(\mathbf{T})} \kappa(\omega'') \Phi_f(\gamma^{\omega''}, (dt)^{\omega''}, dg), \end{aligned}$$

as desired, since Proposition 3.1 shows that

$$\frac{\kappa^\omega(\omega')}{\kappa(\omega' \omega)} = \frac{1}{\kappa(\omega)} = \kappa(\omega).$$

Fix a system $\mathbf{I}^+$ of positive roots for $\mathbf{T}$ in $\mathbf{M}$; that is, a system of positive imaginary roots for $\mathbf{T}$. As in [9] we define

$$\mathbf{R}_\mathbf{T}(\gamma) = |\det(\text{Ad } \gamma - 1)_{\mathfrak{g}/\mathfrak{m}}|^{1/2} \prod_{\alpha \in \mathbf{I}^+} (1 - \alpha(\gamma^{-1})),$$

and then set

$$\Psi_f^\kappa(\gamma) = \Psi_f^\kappa(\gamma, dt, dg) = \mathbf{R}_\mathbf{T}(\gamma) \Phi_f^\kappa(\gamma, dt, dg).$$

PROPOSITION 4.2. — Ψ_f^κ extends to a Schwartz function on

$$\mathbf{T}_0 = \{\gamma \in \mathbf{T} : \alpha(\gamma) \neq 1, \alpha \in \mathbf{I}^+\}.$$

Proof. — The assertion follows immediately from [2], for $\Psi_f^\kappa(\gamma) = \sum_{\omega \in \mathcal{D}(\mathbf{T})} \kappa(\omega) \Psi_f^\omega(\gamma)$ where $\Psi_f^\omega(\gamma) = \mathbf{R}_\mathbf{T}(\gamma) \Phi_f(\gamma^\omega)$ which can be written as $c \Lambda(\gamma) \mathring{F}_f(\gamma^\omega)$ where c is a constant, Λ a unitary character on $\mathbf{T}$ and $\mathring{F}_f$ is the function of Harish-Chandra [12]; here we are using representatives ω [for the classes in $\mathcal{D}(\mathbf{T})$] which lie in $\text{Norm}(\mathbf{M}, \mathbf{T})$.

Thus, like the function $\mathring{F}_f$, Ψ_f^κ (and each derivative) “jumps” across each wall $\alpha=1$, $\alpha \in \mathbf{I}^+$. We discuss these “jumps” following the usual procedure (cf. [2]): α will be a root in $\mathbf{I}^+$, γ_0 an element of $\mathbf{T}$ such that $\alpha(\gamma_0)=1$ and $\beta(\gamma_0) \neq 1$ if $\beta \neq \pm \alpha$, γ_v will denote $\gamma_0 \exp i v H_\alpha$, where H_α is the coroot (as element of $\mathfrak{t}$, the Lie algebra of $\mathbf{T}$) attached to α , and D will be an invariant differential operator on $\mathbf{T}$.

LEMMA 4.3. — If $\kappa(\alpha^\vee) = -1$ then

$$\lim_{v \downarrow 0} D \Psi_f^\kappa(\gamma_v) = \lim_{v \uparrow 0} D \Psi_f^\kappa(\gamma_v).$$

Proof. — If all the roots $\omega\alpha$, ω an element of the imaginary Weyl group, are compact then the result follows immediately from [2].

Suppose now that α is noncompact. By [2] again, (and an earlier paper cited in [2]), we have only to show that under the assumption $D^{\omega\alpha} = D$ the jump for $\hat{D}$, as defined in [9], paragraph 4, is zero. Recall that $\hat{D}$, introduced because of the awkward transformation of R_T under the imaginary Weyl group, is the image of D under the automorphism induced by $H \rightarrow H + s(H)I$, $H \in \mathfrak{t}$, where $s = 1/2 \sum_{\alpha \in I^+} \log \alpha$. Because $\kappa(\alpha^\sim) = -1$ we have $\kappa(\omega_\alpha) = -1$ (Prop. 2.1) and so ω_α is not realized in G . Hence to compute

$$\lim_{v \downarrow 0} \hat{D} \Psi_f^\kappa(\gamma_v) - \lim_{v \uparrow 0} \hat{D} \Psi_f^\kappa(\gamma_v),$$

we may replace Ψ_f^κ by Ψ where

$$\Psi(\gamma) = R_T(\gamma) \sum_{\delta} (\kappa(\delta) \Phi_f(\gamma^\delta) + \kappa(\delta\omega_\alpha) \Phi_f(\gamma^{\delta\omega_\alpha})),$$

and δ , an element of Norm (M, T) satisfying $\delta\alpha = \alpha$, ranges over a complete set of representatives for the classes in $\mathcal{D}(T)$ containing such an element (cf. [9], §4). But $\kappa^{\omega_\alpha}(\delta) = \kappa(\delta)$. To prove this, a simple argument shows that it is enough to consider the case that δ is a reflection; then the proof is immediate (cf. Paragraph 3). Thus we have

$$\kappa(\delta\omega_\alpha) = \kappa(\omega_\alpha) \kappa(\delta) = \kappa(\alpha^\sim) \kappa(\delta) = -\kappa(\delta).$$

Hence $\Psi(\gamma^{\omega_\alpha}) = (1 - \omega_\alpha, 1)(\gamma) \Psi(\gamma)$. Since $D^{\omega_\alpha} = D$ we obtain immediately that

$$\lim_{v \downarrow 0} \hat{D} \Psi_f^\kappa(\gamma_v) - \lim_{v \uparrow 0} \hat{D} \Psi_f^\kappa(\gamma_v) = 0,$$

as desired.

Finally, suppose that α is compact but that $\omega\alpha$ is noncompact. Then since

$$\Psi_f^\kappa(\gamma) = \kappa(\omega) \Psi_f^{\kappa^\omega}(\gamma^\omega),$$

[using the positive system $(I^+)^\omega$ to define $\Psi_f^{\kappa^\omega}$] the proof is easily completed.

We come then to the other possibility, namely $\kappa(\alpha^\sim) = 1$. We have already observed that, regardless of the value of $\kappa(\alpha^\sim)$, if all $\omega\alpha$ are compact then

$$\lim_{v \downarrow 0} D \Psi_f^\kappa(\gamma_v) = \lim_{v \uparrow 0} D \Psi_f^\kappa(\gamma_v).$$

For the remaining case we proceed in steps. Suppose first that α is noncompact and that s is a Cayley transform with respect to α , *standard* in the sense of [9], Paragraph 2. Since $\kappa(\alpha^\sim) = 1$ the character κ^s is well-defined (Prop. 3.2). We claim that

$$(1) \quad \lim_{v \downarrow 0} \hat{D} \Psi_f^\kappa(\gamma_v) - \lim_{v \uparrow 0} \hat{D} \Psi_f^\kappa(\gamma_v) = 2i \hat{D}^s \Psi_f^{\kappa^s}(\gamma_0).$$

The notation is that of [9]. Thus we assume that the system I^+ is adapted to α (. . . if β is imaginary and $\langle \beta, \alpha \rangle > 0$ then $\beta \in I^+$); R_{T_s} , and hence Ψ_f^{κ} , is defined relative to $I_s^+ = \{ \beta: s^{-1} \beta \in I^+ \}$; $\hat{D}$ and $\hat{D}^s$ are defined relative to I^+ and I_s^+ respectively. For the choice of Haar measure on T_s we refer to [9].

To prove the claim we again recall the computations of [9], Paragraph 4. First, on the left-hand side of (1) Ψ_f^{κ} may be replaced by Ψ where $\Psi(\gamma) = \sum_{\omega} \kappa(\omega) \Psi_f(\gamma^\omega)$ with ω an element of $\text{Norm}(\mathbf{M}, \mathbf{T})$ satisfying $\omega\alpha = \pm\alpha$, ranging over a complete set of representatives for those classes in $\mathcal{D}(\mathbf{T})$ which contain such an element. Fix ω such that $\omega\alpha = \alpha$. Then by Proposition 4.5 of [9] we have

$$\lim_{v \downarrow 0} \hat{D} \Psi_f^\omega(\gamma_v) - \lim_{v \uparrow 0} \hat{D} \Psi_f^\omega(\gamma_v) = \text{id}(\alpha) \hat{D}^s \Psi_f^{s\omega s^{-1}}(\gamma_0),$$

where $d(\alpha) = 2$ if ω_α can be realized in G and $d(\alpha) = 1$ otherwise. If ω_α can be realized in G then we obtain

$$\lim_{v \downarrow 0} \hat{D} \Psi_f^\omega(\gamma_v) - \lim_{v \uparrow 0} \hat{D} \Psi_f^\omega(\gamma_v) = 2i \sum_{\omega\alpha = \alpha} \kappa(\omega) \hat{D}^s \Psi_f^{s\omega s^{-1}}(\gamma_0) = 2i \hat{D}^s \Psi_f^{\kappa}(\gamma_0),$$

since, by Proposition 3.3, $\kappa(s\omega s^{-1}) = \kappa(\omega)$. If ω_α is not realized in G and $\omega\alpha = \alpha$ then $\omega\omega_\alpha$ and ω lie in distinct classes of $\mathcal{D}(\mathbf{T})$. However $\kappa(\omega\omega_\alpha) = \kappa(\omega)$. We now argue again as in [9]. First, we may assume that $D^{\omega_\alpha} = D$. Then it follows that the term on the left-hand side of (1) coming from $\omega\omega_\alpha$ equals that for ω . By applying Lemma 4.6 of [9] we obtain the formula (1).

We continue with the assumption that α is noncompact but allow s to be any Cayley transform with respect to α . Then s may be written as $\omega_0 s_0$, where s_0 is a standard transform (with respect to α) and $\omega_0 \in \mathcal{A}(T_{s_0})$. We know that

$$\kappa^{s_0}(\omega_0) \Phi_f^{\kappa}(\gamma^{\omega_0}) = \Phi_f^{\kappa^{s_0}}(\gamma), \quad \gamma \in (T_{s_0})_{\text{reg}}.$$

Also, by definition,

$$R_{T_s}(\gamma^{\omega_0}) = R_{T_{s_0}}(\gamma), \quad \gamma \in T_{s_0},$$

and $\hat{D}^s = (\hat{D}^{s_0})^{\omega_0}$. Hence

$$\lim_{v \downarrow 0} \hat{D} \Psi_f^\omega(\gamma_v) - \lim_{v \uparrow 0} \hat{D} \Psi_f^\omega(\gamma_v) = 2i \kappa^{s_0}(\omega_0) \hat{D}^s \Psi_f^{\kappa}(\gamma_0^s).$$

Now we come to the general case. Thus we will assume that $\sigma(s^{-1})s$ realizes ω_α , with α possibly compact. Suppose that $\omega\alpha$ is noncompact. Then s may be written in the form $\omega_0 s_0$ where s_0 is a standard transform with respect to $\omega\alpha$ and $\omega_0 \in \mathcal{A}(T_{s_0})$. But

$$\Psi_f^\omega(\gamma) = \kappa(\omega) \Psi_f^{\omega}(\gamma^\omega), \quad \gamma \in T_{\text{reg}},$$

the “R-” function in the definition of Ψ_f^{ω} being relative to $(I^+)^{\omega}$, a system adapted to $\omega\alpha$. It is then easy to check that

$$\hat{D} \Psi_f^\omega(\gamma_v) = \kappa(\omega) \hat{D}^\omega \Psi_f^{\omega}(\gamma_v^\omega),$$

$\widehat{D}^\omega$ being defined relative to $(I^+)^\omega$. We have then

$$(2) \quad \lim_{v \downarrow 0} \widehat{D} \Psi_f^\kappa(\gamma_v) - \lim_{v \uparrow 0} \widehat{D} \Psi_f^\kappa(\gamma_v) = 2i\kappa(\omega) \kappa^{s_0\omega}(\omega_0) \widehat{D}^s \Psi_f^{\kappa^s}(\gamma_0^s),$$

where terms on the right-hand side are defined relative to the positive system $((I^+)^\omega)^{\omega_0} = I_s^+$.

We wish to give an (intrinsic) interpretation of $\kappa(\omega) \kappa^{s_0\omega}(\omega_0)$ as a “ κ -signature” for s . We continue to assume that α is an imaginary root, that $\kappa(\alpha^\vee) = 1$ and that s is an element of $\mathbf{G}$ such that $s_\sigma = \sigma(s^{-1})s$ realizes ω_α . We write $\mathbf{G}_\alpha$ for the image of the appropriate real form of SL_2 under one of the standard homomorphisms attached to α (cf. [9]); $\mathbf{G}_\alpha$ is independent of the choices made in defining such a homomorphism. Our first observation is that we may modify s_σ by an element of $\mathbf{G}_\alpha$ to obtain a 1-cocycle for $\mathbf{T}$ trivial in $\mathbf{G}$ (. . . we are considering the cohomology of just $\langle 1, \sigma \rangle$, as before). Indeed, suppose that α is noncompact. Then $s = \omega_0 s_0$ where $s_0 \in \mathbf{G}_\alpha$ and $\omega_0 \in \mathcal{A}(\mathbf{T}_{s_0})$. We write ω_0 as $g_0 \omega'_0$ where $g_0 \in \mathbf{G}$ and ω'_0 normalizes $\mathbf{T}_{s_0}$ and centralizes the maximal $\mathbf{R}$ -split torus in $\mathbf{T}_{s_0}$. Then setting $\omega_1 = s_0^{-1} \omega'_0 s_0$ we have $\omega_1^{-1} \mathbf{G}_\alpha \omega_1 = \mathbf{G}_\alpha$ and hence

$$s_\sigma = \sigma(\omega_1^{-1}) \sigma(s_0^{-1}) s_0 \omega_1 = \sigma(\omega_1^{-1}) \omega_1 \omega_1^{-1} \sigma(s_0^{-1}) s_0 \omega_1,$$

where $\sigma(\omega_1^{-1}) \omega_1$ is a 1-cocycle for $\mathbf{T}$ (and 1-coboundary for $\mathbf{G}$) and $\omega_1^{-1} \sigma(s_0^{-1}) s_0 \omega_1$ is an element of $\mathbf{G}_\alpha$. Now suppose that α is compact but that $\omega\alpha$ is noncompact. Then we may write s as $\omega_0 s_\omega \omega$ where $s_\omega \in \mathbf{G}_{\omega\alpha}$ is such that $\sigma(s_\omega^{-1}) s_\omega$ realizes $\omega_{\omega\alpha}$ and $\omega_0 \in \mathcal{A}(\mathbf{T}_{s_\omega})$. Decomposing ω_0 as before we find that we may assume that $\omega_0 = 1$. Then

$$s_\sigma = \sigma(\omega^{-1}) \omega \cdot \omega^{-1} \sigma(s_\omega^{-1}) s_\omega \omega,$$

where $\sigma(\omega^{-1}) \omega$ is a 1-cocycle for $\mathbf{T}$ (and 1-coboundary for $\mathbf{G}$) and $\omega^{-1} \sigma(s_\omega^{-1}) s_\omega \omega \in \mathbf{G}_\alpha$. This justifies our claim.

Suppose now that we decompose s_σ in two ways, say $s_\sigma = w_1 t_1 = w_2 t_2$, where $w_1, w_2 \in \mathbf{G}_\alpha$ and t_1, t_2 are 1-cocycles for $\mathbf{T}$ and 1-coboundaries for $\mathbf{G}$. We claim that the images of (the cohomology classes of) t_1, t_2 under T-N differ by an element of $\mathbf{Z} \alpha^\vee$ and hence $\kappa(t_1) = \kappa(t_2)$. To prove the claim we have only to note that the classes of t_1 and t_2 differ by an element of $H^1(\mathbf{T} \cap \mathbf{G}_\alpha)$; such an element maps under T-N into $\mathbf{Z} \alpha^\vee$.

It is now immediate that if $s_\sigma \in t_\sigma \mathbf{G}_\alpha$ where t_σ is a 1-cocycle of $\mathbf{T}$ trivial in $\mathbf{G}$ then we may define the “ κ -signature” $\varepsilon_\kappa(s)$ of s as $\kappa(t_\sigma)$.

In (2) we wrote s as $\omega_0 s_0 \omega$ where $\omega\alpha$ is noncompact, $s_0 \in \mathbf{G}_{\omega\alpha}$ and $\omega_0 \in \mathcal{A}(\mathbf{T}_{s_0})$. To compute s_σ we may assume that ω_0 normalizes $\mathbf{T}_{s_0}$. Set $\omega_1 = s_0^{-1} \omega_0 s_0 \omega$. Then

$$\varepsilon_\kappa(s) = \kappa(\sigma(\omega_1^{-1}) \omega_1) = \kappa(\omega_1)$$

in our usual notation. On the other hand

$$\kappa(\omega) \kappa^{s_0\omega}(\omega_0) = \kappa(\omega) \kappa^\omega(s_0^{-1} \omega_0 s_0) = \kappa(\omega_1).$$

Hence $\kappa(\omega) \kappa^{s_0\omega}(\omega_0) = \varepsilon_\kappa(s)$. We conclude:

LEMMA 4.4. — Suppose $\kappa(\alpha^\vee) = 1$. Then:

(i) if all $\omega\alpha$ are compact we have

$$\lim_{v \downarrow 0} D \Psi_f^\kappa(\gamma_v) = \lim_{v \uparrow 0} D \Psi_f^\kappa(\gamma_v);$$

(ii) if $\sigma(s^{-1})s$ realizes ω_α then

$$\lim_{v \downarrow 0} \hat{D} \Psi_f^\alpha(\gamma_v) - \lim_{v \uparrow 0} \hat{D} \Psi_f^\alpha(\gamma_v) = 2i \varepsilon_\alpha(s) \hat{D}^\alpha \Psi_f^{\alpha^*}(\gamma_0^\alpha).$$

Here, we recall, the terms $\hat{D}$, Ψ_f^α are defined relative to a system I^+ of positive imaginary roots adapted to α and the terms $\hat{D}^\alpha$, $\Psi_f^{\alpha^*}$ relative to $I_s^+ = \{\beta : s^{-1}\beta \in I^+\}$.

5. (T, κ) -groups

To establish notation and introduce the groups of [7] we recall some more material from [7] (and [8]). Our data will be a connected reductive group G over R , a maximal torus T in G , also defined over R , and a quasicharacter κ on the module generated by the coroots for T , trivial on the submodule $\mathcal{L}(T)$ (cf. Paragraph 2).

We begin with an L-group ${}^L G$ for G . Thus fix a pair (G^*, ψ) , where G^* is a group quasi-split over R and $\psi: G \rightarrow G^*$ is an isomorphism (over C) such that $\sigma(\psi^{-1})\psi$ is inner. In G^* fix a Borel subgroup B^* over R and a maximal torus T^* over R , contained in B^* . To abbreviate notation we use L for the character module for T^* and $L^\sim$ for its dual; $\Sigma \subset L$ will be the set of simple roots for T^* in B^* and $\Sigma^\sim$ the corresponding set of coroots. Fix a triple $({}^L G^0, {}^L B^0, {}^L T^0)$, where ${}^L G^0$ is a connected reductive group over C , ${}^L B^0$ is a Borel subgroup of ${}^L G^0$ and ${}^L T^0$ is a maximal torus contained in ${}^L B^0$, such that the character module for ${}^L T^0$ is $L^\sim$ and the set of simple roots for ${}^L T^0$ in ${}^L B^0$ is $\Sigma^\sim$. For each $\alpha^\sim \in \Sigma^\sim$ fix a root vector $X_{\alpha^\sim}$ in the Lie algebra of ${}^L G^0$. The element σ acts on T^* , L , $L^\sim$ and ${}^L T^0$; we denote also by σ the action on ${}^L G^0$ which extends that on ${}^L T^0$ and satisfies $\sigma X_{\alpha^\sim} = X_{\sigma\alpha^\sim}$, $\alpha^\sim \in \Sigma^\sim$. The semi-direct product of ${}^L G^0$ by the Weil group of C/R , with $1 \times \sigma$ acting by σ and $C^* \times 1$ acting trivially, defines an object in the category $\mathcal{G}^\sim(R)$ of [8]; this object will be our L-group ${}^L G$.

Next, we use the pair (T, κ) to construct another object ${}^L H$ in $\mathcal{G}^\sim(R)$. We denote by σ_T the action of σ on T , $L(T)$ and $L^\sim(T)$. Fix $x \in G^*$ such that $\psi_x = \text{ad } x \circ \psi$ maps T to T^* . Thus ψ_x induces an isomorphism of $L^\sim(T)$ with $L^\sim$ by which we transfer σ_T to $L^\sim$; by the same means we transfer κ to a quasicharacter on $\langle \Sigma^\sim \rangle$; this new quasicharacter, κ^* , is trivial on $\mathcal{L} = \{\lambda^\sim \in \langle \Sigma^\sim \rangle : \lambda^\sim = \mu^\sim - \sigma_T \mu^\sim, \text{ some } \mu^\sim \in L^\sim\}$ and so is σ_T -invariant. Let ${}^L H^0$ be the connected reductive subgroup of ${}^L G^0$ generated by ${}^L T^0$ and the 1-parameter subgroups defined by those roots of ${}^L T^0$ in ${}^L G^0$ on which κ^* is trivial. Fix a Borel subgroup of ${}^L H^0$ containing ${}^L T^0$ and let $\Sigma_H^\sim$ be the set of simple roots for ${}^L T^0$ in this group. Since κ^* is σ_T -invariant the set of all roots of ${}^L T^0$ in ${}^L H^0$ is preserved by σ_T . We write σ_T as a product $\omega \cdot \sigma_H$, with $\omega \in \Omega({}^L H^0, {}^L T^0)$, the Weyl group of ${}^L T^0$ in ${}^L H^0$, and σ_H induced by an automorphism of $\Sigma_H^\sim$. For each $\alpha^\sim \in \Sigma_H^\sim$ choose a root vector $Y_{\alpha^\sim}$ in the Lie algebra of ${}^L H^0$; we denote also by σ_H that extension of σ_H to ${}^L H^0$ satisfying $\sigma_H Y_{\alpha^\sim} = Y_{\sigma_H \alpha^\sim}$, $\alpha^\sim \in \Sigma_H^\sim$. The semi-direct product of ${}^L H^0$ by the Weil group of C/R , with $1 \times \sigma$ acting by σ_H and $C^* \times 1$ acting trivially, defines an object ${}^L H$ in $\mathcal{G}^\sim(R)$; the isomorphism class of ${}^L H$ in $\mathcal{G}^\sim(R)$ depends only on G , T and κ .

We come then to the groups attached to G : we call a quasi-split group H over R a (T, κ) -group for G if the object ${}^L H$ described above is an L-group for H . Up to isomorphism over R there is exactly one (T, κ) -group for G .

6. Cartan subgroups

We change notation slightly to write (T_0, κ_0) for the fixed Cartan subgroup and quasicharacter; $\mathbf{H}$ will now be a (T_0, κ_0) -group for $\mathbf{G}$. In this section we embed the Cartan subgroups of $\mathbf{H}$ in $\mathbf{G}^* \dots$ and some of them in $\mathbf{G}$. The basis for our discussion is a result (unpublished) of Langlands.

For once and for all we fix (in the notation of the last section):

- (i) $\psi, \mathbf{G}^*, \mathbf{B}^*, \mathbf{T}^*$ and hence ${}^L\mathbf{G}^0; \{X_{\alpha^-}\}$ and hence ${}^L\mathbf{G}$;
- (ii) an element x of $\mathbf{G}^*$ such that $\psi_x = \text{ad } x \circ \psi$ maps $\mathbf{T}_0$ to $\mathbf{T}^*$, and hence ${}^L\mathbf{H}^0$;
- (iii) a Borel subgroup of ${}^L\mathbf{H}^0$ and hence the action of $\sigma_{\mathbf{H}}$ on ${}^L\mathbf{T}^0; \{Y_{\alpha^-}\}$ and hence the action of $\sigma_{\mathbf{H}}$ on ${}^L\mathbf{H}^0$ and the object ${}^L\mathbf{H}$.

Recalling that $L^\vee$ is the dual of the character module for $\mathbf{T}^*$ we make the canonical identification of $\mathbf{T}^*$ with $L^\vee \otimes \mathbf{C}^*$. By construction, $L^\vee$ is also the dual of the character module for some torus in $\mathbf{H}$ defined over $\mathbf{R}$ (and containing a torus maximal among the $\mathbf{R}$ -split tori in $\mathbf{H}$). Thus we can identify $\mathbf{T}^*$ (as complex torus) together with the action of $\sigma_{\mathbf{H}}$ (induced from that of $\sigma_{\mathbf{H}}$ on $L^\vee$) as a maximal torus in $\mathbf{H}$, defined over $\mathbf{R}$. Recall that the action of $\Omega(\mathbf{G}, \mathbf{T}^*)$ (respectively, $\Omega(\mathbf{H}, \mathbf{T}^*)$) on $L^\vee$ coincides with that of $\Omega({}^L\mathbf{G}^0, {}^L\mathbf{T}^0)$ [respectively, $\Omega({}^L\mathbf{H}^0, {}^L\mathbf{T}^0)$]. Hence $\Omega(\mathbf{H}, \mathbf{T}^*)$ is a subgroup of $\Omega(\mathbf{G}^*, \mathbf{T}^*)$. We remark that on $\mathbf{T}^*$, $\sigma_{\mathbf{T}} = \omega_1 \sigma_{\mathbf{H}}$, $\omega_1 \in \Omega(\mathbf{H}, \mathbf{T}^*)$; $\sigma_{\mathbf{G}} = \omega_2 \sigma_{\mathbf{T}}$, $\omega_2 \in \Omega(\mathbf{G}^*, \mathbf{T}^*)$ and so $\sigma_{\mathbf{G}} = \omega_3 \sigma_{\mathbf{H}}$, $\omega_3 \in \Omega(\mathbf{G}^*, \mathbf{T}^*)$.

We come now to the embeddings. Let $\mathbf{T}'$ be a maximal torus in $\mathbf{H}$ defined over $\mathbf{R}$. We pick $h \in \mathbf{H}$ such that $\text{ad } h$ maps $\mathbf{T}'$ to $\mathbf{T}^*$. Composing $\text{ad } h$ with the identity on $\mathbf{T}^*$ (as map over $\mathbf{C}$, from a subgroup of $\mathbf{H}$ to $\mathbf{G}^*$) we obtain an embedding $j(h)$ of $\mathbf{T}'$ in $\mathbf{G}^*$, defined over $\mathbf{C}$. According to Langlands (unpublished) there exists $g \in \mathbf{G}^*$ such that $j(g, h) = \text{ad } g^{-1} \circ j(h)$ is defined over $\mathbf{R}$. (The proof proceeds as follows. Choose an element $\gamma = \exp X$, $X \in \mathfrak{t}'$, such that $h\gamma h^{-1} \in \mathbf{T}^*$ is regular in $\mathbf{G}^*$ and lies in the derived group of $\mathbf{G}^*$. Consider the natural projection of the simply-connected covering group $(\mathbf{G}^*)^\sim$ onto the derived group. There is an element γ_0 in the preimage of $h\gamma h^{-1}$ whose conjugacy class in $(\mathbf{G}^*)^\sim$ is defined over $\mathbf{R}$. By [11] this class contains an $\mathbf{R}$ -rational point, say γ_1 . Let $g(h\gamma h^{-1})g^{-1}$ be the image of γ_1 in $\mathbf{G}^*$. Then $\text{ad } g \circ j(h)$ is defined over $\mathbf{R}$.) If both $j(g, h)$ and $j(g', h')$ map $\mathbf{T}'$ into $\mathbf{G}^*$ over $\mathbf{R}$ then the action of $j(g', h') \circ j(g, h)^{-1}$ on $\mathbf{T}$, the image of $\mathbf{T}'$ under $j(g, h)$, can be realized by an element of $\mathbf{G}^*$. Clearly this element lies in $\mathcal{A}(\mathbf{T})$. Hence the image of $\mathbf{T}'$ is determined up to conjugacy under $\mathbf{G}^*$ (cf. [9]). It follows easily that if $\mathbf{T}'$ and $\mathbf{U}'$ are conjugate in $\mathbf{H}$ under $\mathbf{H}$ then their images in $\mathbf{G}^*$ are conjugate under $\mathbf{G}^*$. We conclude then that the embeddings $j(\cdot, \cdot)$ induce a map from the set $t(\mathbf{H})$ of conjugacy classes of Cartan subgroups of $\mathbf{H}$ to $t(\mathbf{G}^*)$. This map preserves the usual ordering (cf. [9], § 2) and, in fact, maps adjacent classes to adjacent classes. However it need not be injective. On the other hand, our twist $\psi: \mathbf{G} \rightarrow \mathbf{G}^*$ induces an embedding of $t(\mathbf{G})$ in $t(\mathbf{G}^*)$ (cf. [9]). Thus we have a map from a subset of $t(\mathbf{H})$ into $t(\mathbf{G})$ (preserving adjacency). The domain is non-empty for, according to [7], the image contains the conjugacy class of $\mathbf{T}_0$, our fixed Cartan subgroup of $\mathbf{G}$.

While the map above is canonical [given the data in (i), (ii), (iii)] we will need to examine the correspondence of individual Cartan subgroups, where the choices will be of importance

(temporarily). First, we will say that a Cartan subgroup T' of H originates in G if its conjugacy class lies in the domain of the map into $t(G)$. Clearly, if T' originates in G then T' is embedded in G , over R , by a map $\psi^{-1} \circ \text{ad } g'' \circ j(g', h)$, $g', g'' \in G^*$, $h \in H$; that is, by a map of the form

$$i(g, h) = \psi \circ \text{ad } g^{-1} \circ \text{id} \circ \text{ad } h, \quad g \in G^*, \quad h \in H.$$

Moreover, $i(g, h) = i(g', h')$ if and only if $g' = w_1 g$, $h' = w_2 h$ where both w_1, w_2 realize some element ω of $\Omega(H, T^*)$; both $i(g, h)$ and $i(g'', h'')$ embed T' in G (over R) if and only if $i(g'', h'') = \text{ad } g_0 \circ i(g, h)$ for some $g_0 \in \mathcal{A}(T)$, T denoting the image of T' under $i(g, h)$.

We now fix a set $\mathcal{I} = \{i_0, \dots, i_N\}$ of these embeddings $i(g, h)$, denoting the domain of i_m by T'_m and the range by T_m (T_0 remains our fixed torus). We assume:

(i) $T'_0, \dots, T'_N$ form a complete set of non-conjugate groups among the Cartan subgroups of H originating in G ;

(ii) i_0 is of the form $i(x, \cdot)$, where x is the element fixed in (ii) at the beginning of this section and

(iii) if T_m is conjugate to T_n then $T_m = T_n$.

That (ii) is possible is indicated in [7] (the argument is similar to that we reported earlier); (iii) is only for convenience.

We consider an embedding $i_m: T'_m \rightarrow T_m$. Write L_m for $L(T_m)$, $L_m^\sim$ for $L^\sim(T_m)$, Ξ_m for the roots of T_m in G , $\Xi_m^\sim$ for the coroots and $\mathcal{L}_m$ for the module $\mathcal{L}(T_m)$. Clearly i_m induces isomorphism between $L(T'_m)$ and L_m and between $L^\sim(T'_m)$ and $L_m^\sim$. We claim that under these maps the coroots for T'_m are embedded in $\Xi_m^\sim$ and the roots in Ξ_m . Moreover these embeddings commute with the action of σ and if α' maps to α in Ξ_m and γ' to γ in T_m then $\alpha'(\gamma') = \alpha(\gamma)$. To obtain the embedding of the coroots we write i_m as $i(g_m, h_m)$; then $\text{ad } h_m$ maps the coroots for T'_m to the coroots for T^* in H and $\text{ad } g_m \circ \psi$ maps $\Xi_m^\sim$ to the coroots for T^* in G . Now we need only recall that a coroot for T^* in G^* (respectively, H) is a root for ${}^L T^0$ in ${}^L G^0$ (respectively, ${}^L H^0$). For the correspondence of roots, if we identify L with $(L^\sim)^\sim$ then a root α of T^* in H is identified with

$$\lambda^\sim \rightarrow \frac{2\langle \alpha^\sim, \lambda^\sim \rangle}{\langle \lambda^\sim, \lambda^\sim \rangle}, \quad \lambda^\sim \in L^\sim,$$

where we use a positive definite bilinear form $\langle \cdot, \cdot \rangle$ on $L^\sim \otimes \mathbb{Q}$ invariant under $\Omega({}^L G^0, {}^L T^0)$. But then α is also identified as a root of T^* in G^* . The rest follows easily.

The map $\omega \rightarrow i_m \circ \omega \circ i_m^{-1}$ yields an embedding of $\Omega(H, T'_m)$ into $\Omega(G, T_m)$ compatible, in the obvious sense, with the map on roots. We will denote by $\Omega_0(G, T_m)$ the subgroup of $\Omega(G, T_m)$ consisting of those elements which commute with σ ; that is, those elements which can be realized in $\mathcal{A}(T_m)$. The map above embeds $\Omega_0(H, T'_m)$ in $\Omega_0(G, T_m)$.

We continue with a set $\mathcal{I}$ of embeddings satisfying (i)-(iii). If T' is any Cartan subgroup of H (originating in G) an $\mathcal{I}$ -embedding of T' in G will be a map of the form $i_m \circ \text{ad } h$ where h is an element of $\mathcal{A}(T')$ mapping T' to T'_m . Also we will say that an element γ' in T' originates from a regular element γ of G with respect to $\mathcal{I}$ if γ is the image of γ' under some $\mathcal{I}$ -embedding; γ' is then regular in H .

LEMMA 6.1. — *If γ originates from regular elements $\gamma_1 \in T_m$ and $\gamma_2 \in T_n$ (with respect to $\mathcal{J}$) then $T_m = T_n$ and there exists ω in the image of $\Omega_0(\mathbf{H}, T'_m)$ in $\Omega_0(\mathbf{G}, T_m)$ such that $\gamma_2 = \gamma_1^\omega$.*

Proof. — That $T_m = T_n$ follows from [9], Theorem 2.1 and the condition (iii) satisfied by $\mathcal{J}$. The rest is immediate.

We remark that whether or not an element of $\mathbf{H}$ originates in G_{reg} is independent of the choice for $\mathcal{J}$; however the collection of elements (if non-empty) from which it originates is not.

7. Transferring κ_0

We have fixed $\mathcal{J} = \{i_0, \dots, i_N\}$. It is now an easy matter to transfer κ_0 to a character κ_m on $\mathcal{D}(T_m)$. Indeed, choose $h \in \mathbf{H}$ such that $\text{ad } h$ maps T'_m to T'_0 . Then $\bar{h} = i_0 \circ \text{ad } h \circ i_m^{-1}$ maps T_m to T_0 and $\Xi_m^\sim$ to $\Xi_0^\sim$. Thus we have immediately a quasi-character κ_m on $\langle \Xi_m^\sim \rangle$. That κ_m is trivial on $\mathcal{L}_m$ follows from:

PROPOSITION 7.1. — $\bar{h}$ maps $\mathcal{L}_m$ to $\mathcal{L}_0 + \text{Ker } \kappa_0$.

Proof. — Let $\lambda^\sim \in L_m^\sim$ be such that $\sigma\lambda^\sim - \lambda^\sim \in \langle \Xi_m^\sim \rangle$. Then

$$\bar{h}(\sigma\lambda^\sim - \lambda^\sim) = (\sigma(\bar{h}\lambda^\sim) - \bar{h}\lambda^\sim) + \sigma(\sigma(\bar{h})\bar{h}^{-1}(\bar{h}\lambda^\sim) - \bar{h}\lambda^\sim).$$

But $\sigma(\bar{h})\bar{h}^{-1}$ lies in the image of $\Omega(\mathbf{H}, T'_0)$ in $\Omega(\mathbf{G}, T_0)$. Hence the second term is a sum of coroots for T_0 each coming from $\mathbf{H}$. This forces the second term to lie in $\text{Ker } \kappa_0$ because, by choosing $i_0 = i(x, \cdot)$, we have arranged that $\alpha^\sim \in \Xi_0^\sim$ come from $\mathbf{H}$ if and only if $\alpha^\sim \in \text{Ker } \kappa_0$. It follows now that the first term lies in $\langle \Xi_0^\sim \rangle$ and hence in $\mathcal{L}_0$. This completes the proof.

We have to check that κ_m is well-defined. Suppose that h is replaced by h' . Then:

PROPOSITION 7.2. — $\bar{h}'\lambda^\sim \in \bar{h}\lambda^\sim + \text{Ker } \kappa_0$, $\lambda^\sim \in \langle \Xi_m^\sim \rangle$.

Proof.

$$\bar{h}'\lambda^\sim = \bar{h}\lambda^\sim + \bar{h}(\bar{h}^{-1}\bar{h}'\lambda^\sim - \lambda^\sim).$$

Since $\bar{h}^{-1}\bar{h}'$ lies in the image of $\Omega(\mathbf{H}, T'_m)$ in $\Omega(\mathbf{G}, T_m)$ the assertion follows easily.

For future use we note:

PROPOSITION 7.3:

(i) a coroot $\alpha^\sim$ in $\Xi_m^\sim$ lies in the image of the coroots for T'_m (that is, “comes from $\mathbf{H}$ ”) if and only if $\kappa_m(\alpha^\sim) = 1$;

(ii) if ω lies in the image of $\Omega(\mathbf{H}, T'_m)$ in $\Omega(\mathbf{G}, T_m)$ then $\kappa_m^\omega = \kappa_m$.

Proof. — The assertion in (i) is immediate since it is true for $m=0$ (cf. the proof of Proposition 7.1).

For (ii), let $\lambda^\sim \in \langle \Xi_m^\sim \rangle$. Then $\omega\lambda^\sim - \lambda^\sim$ lies in the span of the image in $\Xi_m^\sim$ of the coroots for T'_m . Hence, by (i), $\kappa_m^\omega = \kappa_m$.

The converse of (ii) is false. To clarify this, extend κ_0^* (κ_0 shifted to $\langle \Sigma^\sim \rangle$) in some way to a quasicharacter on $L^\sim$. Identify this extension as an element of ${}^L T^0 \dots {}^L H^0$ is the connected component of the identity in the centralizer in ${}^L G^0$ of this element. The condition $\kappa_m^\omega = \kappa_m$ is that the action of ω shifted to ${}^L T^0$ be realized in the (full) centralizer of our element.

We have defined $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ and $\Omega_0(\mathbf{G}, \mathbf{T}_m)$ (Paragraph 6); i_m induces an embedding of $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$. If ω_1, ω_2 lie in the image then we have defined $\kappa_m(\omega_1), \kappa_m(\omega_2)$; by Propositions 3.1 and 7.3, $\kappa_m(\omega_1 \omega_2) = \kappa_m(\omega_1) \kappa_m(\omega_2)$. Clearly also $\kappa_m(\omega_i) = \pm 1$. We will need further information.

Let $\mathbf{T}$ be a maximal torus, over $\mathbf{R}$, in a connected reductive group $\mathbf{G}$ over $\mathbf{R}$. Let Ω be the Weyl group of $\mathbf{T}$ in $\mathbf{G}$ and Ω_0 the subgroup of Ω consisting of those elements realized in $\mathcal{A}(\mathbf{T})$. Let $\mathbf{S}$ be the maximal $\mathbf{R}$ -split torus in $\mathbf{T}$ and $\mathbf{M}$ be the centralizer of $\mathbf{S}$ in $\mathbf{G}$. The imaginary Weyl group of $\mathbf{T}$, denoted here by Ω_1 , is the Weyl group of $\mathbf{T}$ in $\mathbf{M}$; we have $\Omega_1 \subset \Omega_0$. Let $\mathcal{W}$ be the restricted Weyl group attached to the pair $(\mathbf{G}, \mathbf{S})$. Restriction to $\mathbf{S}$ defines a surjective homomorphism from Ω_0 to $\mathcal{W}$ (this follows easily from Theorem 2.1 of [9]); the kernel is Ω_1 . We will classify the elements of Ω_0 according to image in $\mathcal{W}$. First we recall the structure of $\mathcal{W}$. According to [4], $\mathcal{W}$ is generated by the reflections with respect to certain (useful) roots of $(\mathbf{G}, \mathbf{S})$. To describe the reflections needed we assume $\mathbf{G}$ simple. For convenience we exclude for the present the case that $\mathbf{G}$ is of type G_2 . Then if $\tilde{\alpha}$ is a root of $(\mathbf{G}, \mathbf{S})$ the set of roots proportional to $\tilde{\alpha}$ is $\{\pm \tilde{\alpha}\}, \{\pm 1/2 \tilde{\alpha}, \pm \tilde{\alpha}\}$ or $\{\pm \tilde{\alpha}, \pm 2\tilde{\alpha}\}$ [4]. We assume that $1/2 \tilde{\alpha}$ is not a root. We call $\tilde{\alpha}$ of type (A), (B') or (C) accordingly as:

- (A) $\tilde{\alpha}$ coincides with some (real) root of $(\mathbf{G}, \mathbf{T})$;
- (B') $\tilde{\alpha}$ is not a root of $(\mathbf{G}, \mathbf{T})$; $2\tilde{\alpha}$ is not a root of $(\mathbf{G}, \mathbf{S})$, or
- (C) $\tilde{\alpha}$ is not a root of $(\mathbf{G}, \mathbf{T})$; $2\tilde{\alpha}$ is a root of $(\mathbf{G}, \mathbf{S})$.

Suppose that $\tilde{\alpha}$ is of type B' and choose a root λ of $(\mathbf{G}, \mathbf{T})$ whose restriction to $\mathbf{S}$ is $\tilde{\alpha}$. Then $\lambda \neq \sigma\lambda$ (σ denotes complex conjugation) and $\lambda + \sigma\lambda$ is not a root so that $\langle \lambda, \sigma\lambda \rangle \geq 0$. An argument on $\langle \lambda, \lambda \rangle$ shows that $\langle \lambda, \sigma\lambda \rangle$ is independent of the choice of λ . If $\langle \lambda, \sigma\lambda \rangle > 0$ then comparison with the definitions of [4] shows that $\tilde{\alpha}$ cannot be useful in the sense of [4]. We call $\tilde{\alpha}$ of type B if $\tilde{\alpha}$ is of type B' and $\langle \lambda, \sigma\lambda \rangle = 0$ for each λ .

Suppose now that $\tilde{\alpha}$ is of type C. Choose a root λ of $(\mathbf{G}, \mathbf{T})$ whose restriction to $\mathbf{S}$ is $\tilde{\alpha}$ and a root μ whose restriction is $2\tilde{\alpha}$. If $\langle \lambda, \sigma\lambda \rangle > 0$ then $\langle \mu, \mu \rangle \geq 3\langle \lambda, \lambda \rangle$. Since we have excluded systems of type G_2 we conclude that $\langle \lambda, \sigma\lambda \rangle \leq 0$, and moreover that $\lambda + \sigma\lambda$ is a root of $\mathbf{T}$.

The reflections $\omega_\alpha, \tilde{\alpha}$ of type A, B or C, generate $\mathcal{W}$. We call $\omega \in \Omega_0$ of type A (respectively, B, C) if its image in $\mathcal{W}$ is a reflection of type A (respectively, B, C).

We return to the tori $\mathbf{T}'_m$ in $\mathbf{H}$ and $\mathbf{T}_m$ in $\mathbf{G}$. Let $\mathbf{S}'_m$ be the maximal $\mathbf{R}$ -split torus in $\mathbf{T}'_m$ and $\mathbf{S}_m$ the maximal $\mathbf{R}$ -split torus in $\mathbf{T}_m$. Then i_m maps $\mathbf{S}'_m$ to $\mathbf{S}_m$ and induces an embedding of the set of roots of $(\mathbf{H}, \mathbf{S}'_m)$ in the set of roots of $(\mathbf{G}, \mathbf{S}_m)$ [since each root of $\mathbf{S}'_m$ (respectively, $\mathbf{S}_m$) is the restriction of a root of $\mathbf{T}'_m$ (respectively, $\mathbf{T}_m$)]. Let $\mathcal{W}'_m$ be the restricted Weyl group attached to $(\mathbf{H}, \mathbf{S}'_m)$ and $\mathcal{W}_m$ be the group for $(\mathbf{G}, \mathbf{S}_m)$. Then i_m

induces an embedding of $\mathcal{W}'_m$ in $\mathcal{W}_m$; the image of the reflection with respect to a root of S'_m is the reflection with respect to its image in the roots of S_m and also

$$\begin{array}{ccc} \Omega_0(\mathbf{H}, \mathbf{T}'_m) & \rightarrow & \Omega_0(\mathbf{G}, \mathbf{T}_m) \\ \downarrow & & \downarrow \\ \mathcal{W}'_m & \longrightarrow & \mathcal{W}_m \end{array}$$

is commutative.

We come then to computing κ_m as a character on the image of $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$.

PROPOSITION 7.4. — *If ω lies in the image of the imaginary Weyl group of $\mathbf{T}'_m$ then $\kappa_m(\omega) = 1$.*

Proof. — Let $\omega = \omega_{\alpha_1} \dots \omega_{\alpha_r}$ where $\alpha_1, \dots, \alpha_r$ are imaginary roots of $\mathbf{T}_m$ coming from $\mathbf{H}$. Then

$$\kappa_m(\omega) = \kappa_m(\omega_{\alpha_1}) \kappa_m^{\omega_{\alpha_1}}(\omega_{\alpha_2} \dots \omega_{\alpha_r}) = \kappa_m(\omega_{\alpha_1} \dots \omega_{\alpha_r})$$

since $\kappa_m(\omega_{\alpha_1}) = 1$ if α_1 is compact, $\kappa_m(\omega_{\alpha_1}) = \kappa_m(\check{\alpha}_1) = 1$ if α_1 is noncompact and, in either case, $\kappa_m^{\omega_{\alpha_1}} = \kappa_m$. Induction now completes the argument.

We conclude from this proposition that $\kappa_m(\omega)$ depends only on the image of ω in $\mathcal{W}_m$. Assume now that $\mathbf{G}$ is simple. If $\mathbf{G}$ is of type G_2 then direct computation shows that $\kappa_m(\omega) = 1$ for all ω in the image of $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$. Suppose that $\mathbf{G}$ is not of type G_2 ; then neither is any simple factor (of the simply-connected covering of the derived group) of $\mathbf{H}$. If $\tilde{\alpha}'$ is a root of S'_m of type A and α is its image in $\mathbf{G}$ then $2\tilde{\alpha}$ is not a root (by an argument as in [13], §1.1); $1/2\tilde{\alpha}$ may be a root. If $1/2\tilde{\alpha}$ is not a root then $\tilde{\alpha}$ is of type A; if $1/2\tilde{\alpha}$ is a root then $1/2\tilde{\alpha}$ is of type C. If $\tilde{\alpha}'$ is of type B then $1/2\tilde{\alpha}$ is not a root. Also $\tilde{\alpha}$ cannot be of type A; hence $\tilde{\alpha}$ is of type B or C. If $\tilde{\alpha}'$ is of type C then so also is $\tilde{\alpha}$.

LEMMA 7.5:

- (i) *If ω is the image of an element of $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ of type A or C then $\kappa_m(\omega) = 1$, and*
- (ii) *if ω is the image of an element of type B then $\kappa_m(\omega) = 1$ if ω is also of type B; otherwise $\kappa_m(\omega) = -1$.*

Proof. — In case (i) there is a real root λ such that ω_λ has the same image in $\mathcal{W}_m$ as ω . Hence $\kappa_m(\omega) = \kappa_m(\omega_\lambda) = 1$ since ω_λ can be realized in $\mathbf{G}$.

In case (ii), suppose that ω has image $\omega_{\tilde{\alpha}}$ in $\mathcal{W}_m$ and that λ is a root of $\mathbf{T}_m$ such that $\langle \lambda, \sigma\lambda \rangle = 0$ and the restriction of λ to S_m is $\tilde{\alpha}$. Then ω has the same image in $\mathcal{W}_m$ as $\omega_\lambda \omega_{\sigma\lambda}$. If $\tilde{\alpha}$ is of type B then $\{\pm\lambda, \pm\sigma\lambda\}$ are the only roots of $\mathbf{T}_m$ in the plane determined by $\lambda, \sigma\lambda$. Hence $\omega_\lambda \omega_{\sigma\lambda}$ can be realized in $\mathbf{G}$ and $\kappa(\omega) = 1$. The only other possible type for $\tilde{\alpha}$ is C; then

$$\omega_\lambda \omega_{\sigma\lambda} = \omega_{\lambda - \sigma\lambda} \omega_{\lambda + \sigma\lambda}$$

and

$$\kappa_m(\omega_\lambda \omega_{\sigma\lambda}) = \kappa_m(\omega_{\lambda - \sigma\lambda}) = \kappa_m((\lambda - \sigma\lambda)^\vee) = -1$$

by Proposition 2.1, since $\lambda - \sigma\lambda$ is not from $\mathbf{H}$ and must be noncompact (by an examination of the root systems of type C_2). This completes the argument.

8. A factor and an assumption

Again we consider one of the embeddings $i_m: \mathbf{T}'_m \rightarrow \mathbf{T}_m$. We fix, for once and for all, a positive system for the imaginary roots of $\mathbf{T}_m$ in $\mathbf{G}$ and use the induced system for the imaginary roots of $\mathbf{T}'_m$ in $\mathbf{H}$. Recalling the "R"-function of Paragraph 4 we set

$$R_m(\gamma) = \prod_{\substack{\alpha \text{ imaginary} \\ \alpha > 0}} (1 - \alpha(\gamma^{-1})) \prod_{\substack{\alpha \text{ not imaginary} \\ \alpha > 0}} |(\alpha(\gamma))^{1/2} - (\alpha(\gamma))^{-1/2}|,$$

for $\gamma \in \mathbf{T}_m$, α denoting a root of $\mathbf{T}_m$ in $\mathbf{G}$, and

$$R'_m(\gamma') = \prod_{\substack{\alpha' \text{ imaginary} \\ \alpha' > 0}} (1 - \alpha'(\gamma'^{-1})) \prod_{\substack{\alpha' \text{ not imaginary} \\ \alpha' > 0}} |(\alpha'(\gamma'))^{1/2} - (\alpha'(\gamma'))^{-1/2}|,$$

for $\gamma' \in \mathbf{T}'_m$, α' denoting a root of $\mathbf{T}'_m$ in $\mathbf{H}$; the second product in each expression is to be interpreted as in Paragraph 4. Next we set

$$\mathfrak{t}_m = \frac{1}{2} \sum_{\substack{\alpha \text{ imaginary} \\ \alpha > 0}} \alpha \quad \text{and} \quad \mathfrak{t}'_m = \frac{1}{2} \sum_{\substack{\alpha' \text{ imaginary} \\ \alpha' > 0}} \alpha';$$

$\mathfrak{t}_m \in L_m \otimes \mathbf{Q}$ and $\mathfrak{t}'_m \in L(\mathbf{T}'_m) \otimes \mathbf{Q}$. Using i_m we transfer $\mathfrak{t}'_m$ to $L_m \otimes \mathbf{Q}$, again writing $\mathfrak{t}'_m$. Our assumption will be

$$\mathfrak{t}_m - \mathfrak{t}'_m \in L_m.$$

(8.1) and

$$\mathfrak{t}_0 - \mathfrak{t}'_0 - \bar{h}(\mathfrak{t}_m - \mathfrak{t}'_m) \in \langle \Xi_0 \rangle,$$

for some $h \in \mathbf{H}$ such that $\bar{h} = i_0 \circ \text{ad } h \circ i_m^{-1}$ maps $\mathbf{T}_m$ to $\mathbf{T}_0$.

As before, Ξ_m is the set of roots of $\mathbf{T}_m$. Clearly (8.1) is independent of the choice of h and the positive systems for imaginary roots. The second part of the assumption is a consequence of the first in all but a few cases. Those cases where (8.1) fails will be dealt with in another paper.

On transferring R'_m to $\mathbf{T}_m$ (without change in notation) we may define a function Δ_m on the regular elements of $\mathbf{T}_m$ by

$$\Delta_m(\gamma) = \frac{(\mathfrak{t}_m - \mathfrak{t}'_m)(\gamma) R_m(\gamma)}{R'_m(\gamma)}.$$

LEMMA 8.2. — *If ω is in the image of $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$ then*

$$\Delta_m(\gamma^\omega) = \kappa_m(\omega) \Delta_m(\gamma), \quad \gamma \in (\mathbf{T}_m)_{\text{reg}}.$$

Proof. — Suppose that ω is the image of ω' . Then

$$\Delta_m(\gamma^\omega) = \frac{\varepsilon(\omega)}{\varepsilon(\omega')} \Delta_m(\gamma),$$

where $\varepsilon(-)$ denotes the signature with respect to imaginary roots; that is, $\varepsilon(\omega) = (-1)^n$ where n is the number of positive imaginary roots α of $\mathbf{T}_m$ in $\mathbf{G}$ for which $\omega\alpha$ is negative, and $\varepsilon(\omega')$ is similarly defined relative to the imaginary roots of $\mathbf{T}'_m$ in $\mathbf{H}$. To show that $\varepsilon(\omega)/\varepsilon(\omega') = \kappa_m(\omega)$ we proceed in steps. We remark first that $\varepsilon(-)$ does not depend on the choice of positive system for the imaginary roots.

(i) Consider the signature of ω with respect to all roots of $\mathbf{T}_m$ in $\mathbf{G}$ (and some choice of positive system). This signature coincides with the determinant of ω (on $L_m \otimes \mathbf{C}$) since $\omega \in \Omega(\mathbf{G}, \mathbf{T}_m)$; we denote it by $\det \omega$. Similarly we consider the signature $\det \omega'$ of ω' with respect to all roots of $\mathbf{T}'_m$. Clearly $\det \omega = \det \omega'$ since the result is true if we replace ω' by any reflection in $\Omega(\mathbf{H}, \mathbf{T}'_m)$.

(ii) Because ω preserves real roots also, we can consider the signature $\eta(\omega)$ of ω relative to the real roots of $\mathbf{T}_m$ in $\mathbf{G}$ (. . . and similarly the real signature $\eta(\omega')$ of ω' in $\mathbf{H}$). We claim that $\det \omega = \varepsilon(\omega) \eta(\omega)$, $\det \omega' = \varepsilon(\omega') \eta(\omega')$. To prove this we choose systems of positive roots in the following way. Take a system of positive roots for $\mathbf{T}_m$ in $\mathbf{G}$ with the property that if $\alpha > 0$ and $\sigma\alpha \neq -\alpha$ then $\sigma\alpha > 0$. Use the induced systems for the real roots of $\mathbf{T}_m$, the imaginary roots of $\mathbf{T}_m$, all roots of $\mathbf{T}'_m$, the imaginary roots of $\mathbf{T}'_m$, etc. Since $\omega(\sigma\alpha) = \sigma(\omega\alpha)$ the claim follows.

We will prove the lemma [in (v)] by showing that $\eta(\omega)/\eta(\omega') = \kappa_m(\omega)$.

(iii) To compute $\eta(-)$ we use restricted roots. As before, let $\mathbf{S}_m$ be the maximal $\mathbf{R}$ -split torus in $\mathbf{T}_m$. Each root $\tilde{\alpha}$ of $(\mathbf{G}, \mathbf{S}_m)$ is the restriction to $\mathbf{S}_m$ of some root of $\mathbf{T}_m$; we define $m(\tilde{\alpha})$ to be the number of roots of $\mathbf{T}_m$ whose restriction to $\mathbf{S}_m$ is $\tilde{\alpha}$. Recall that restriction to $\mathbf{S}_m$ also defines a surjective homomorphism from $\Omega_0(\mathbf{G}, \mathbf{T}_m)$ to $\mathscr{W}_m$, the restricted Weyl group attached to $(\mathbf{G}, \mathbf{S}_m)$. We denote by $\tilde{\omega}$ the image in $\mathscr{W}_m$ of $\omega \in \Omega_0(\mathbf{G}, \mathbf{T}_m)$. Finally, we set $\tilde{\alpha} > 0$ if $\tilde{\alpha}$ is the restriction of a positive root of $\mathbf{T}_m$, using an ordering for the roots of $\mathbf{T}_m$ as in (ii).

For any $\tau \in \mathscr{W}_m$ we define

$$\tilde{\eta}(\tau) = \sum_{\tilde{\alpha} > 0} (-1)^{m(\tilde{\alpha}) n_\tau(\tilde{\alpha})},$$

where $n_\tau(\tilde{\alpha}) = 0$ if $\tau\tilde{\alpha} > 0$ and $n_\tau(\tilde{\alpha}) = 1$ if $\tau\tilde{\alpha} < 0$. If $\omega \in \Omega_0(\mathbf{G}, \mathbf{T}_m)$ then $\eta(\omega) = \tilde{\eta}(\tilde{\omega})$.

(iv) To compute $\tilde{\eta}$, we note that $\tilde{\eta}$ is a quadratic character on $\mathscr{W}_m$ since η is a quadratic character on $\Omega_0(\mathbf{G}, \mathbf{T}_m)$. We will then need to calculate just $\tilde{\eta}(\omega_{\tilde{\alpha}})$ assuming $\omega_{\tilde{\alpha}} \in \mathscr{W}_m$ (and $\tilde{\alpha} > 0$).

If $\tilde{\beta} > 0$ and $\omega_{\tilde{\alpha}}(\tilde{\beta}) < 0$, set $\tilde{\gamma} = -\omega_{\tilde{\alpha}}(\tilde{\beta})$. Then $\tilde{\gamma} > 0$ and $\omega_{\tilde{\alpha}}(\tilde{\gamma}) < 0$; also $m(\tilde{\gamma}) = m(\tilde{\beta})$. Since $\tilde{\gamma} = \tilde{\beta}$ if and only if $\tilde{\beta}$ is proportional to $\tilde{\alpha}$ we conclude that

$$\tilde{\eta}(\omega_{\tilde{\alpha}}) = (-1)^{a(\tilde{\alpha})} \quad \text{where} \quad a(\tilde{\alpha}) = \sum_{\substack{\tilde{\beta} > 0 \\ \tilde{\beta} \text{ prop. to } \tilde{\alpha}}} m(\tilde{\beta}).$$

To determine the parity of $a(\tilde{\alpha})$ we may assume $\mathbf{G}$ semisimple and simply-connected (by replacing the group by the simply-connected covering of its derived group) and consider each simple factor of $\mathbf{G}$ separately. It is convenient to exclude factors of type G_2 and deal with them separately later. Thus we assume that $\mathbf{G}$ is simple and not of type G_2 . Suppose that $\tilde{\alpha}$ is a root of $\mathbf{S}_m$ for which $1/2\tilde{\alpha}$ is not a root. To generate $\mathcal{W}_m$ we need only $\omega_{\tilde{\alpha}}$, for those $\tilde{\alpha}$ which are of type A, B or C (cf. Paragraph 7).

If $\tilde{\alpha}$ is of type A then $m(\tilde{\alpha})$ is odd and $2\tilde{\alpha}$ is not a root. Thus $\tilde{\eta}(\omega_{\tilde{\alpha}}) = -1$. If $\tilde{\alpha}$ is of type B then $m(\tilde{\alpha})$ is even and $\tilde{\eta}(\omega_{\tilde{\alpha}}) = 1$. If $\tilde{\alpha}$ is of type C then again $m(\tilde{\alpha})$ is even. However, $m(2\tilde{\alpha})$ is odd. Hence $\tilde{\eta}(\omega_{\tilde{\alpha}}) = -1$.

(v) We come now to the proof of the lemma. A straight forward argument shows that we may assume that $\mathbf{G}$ is simple. If then $\mathbf{G}$ is of type G_2 direct computation shows that

$$\frac{\eta(\omega)}{\eta(\omega')} = \frac{\tilde{\eta}(\tilde{\omega})}{\tilde{\eta}(\tilde{\omega}')} = \kappa_m(\omega) = 1 \quad \text{for all } \omega.$$

Suppose that $\mathbf{G}$ is not of type G_2 ; then neither is any simple factor (of the simply-connected covering of the derived group) of $\mathbf{H}$. It is enough to consider ω' of type A, B or C (cf. § 7). If ω' is of type A or C then we know that ω is of type A or C. Hence

$$\frac{\tilde{\eta}(\tilde{\omega})}{\tilde{\eta}(\tilde{\omega}')} = \kappa_m(\omega) = 1.$$

If ω' is of type B then we have that ω is of type B or C. If ω is of type B then again

$$\frac{\tilde{\eta}(\tilde{\omega})}{\tilde{\eta}(\tilde{\omega}')} = \kappa_m(\omega) = 1.$$

However, if ω is of type C then

$$\frac{\tilde{\eta}(\tilde{\omega})}{\tilde{\eta}(\tilde{\omega}')} = -1;$$

this is exactly the case where $\kappa_m(\omega) = -1$. The lemma is therefore proved.

We now define a function $\Delta_H^G = \Delta_H^G(\varepsilon_0, \dots, \varepsilon_N)$ on the regular elements in $\bigcup_{m=0}^N T_m$ by $\Delta_H^G(\gamma) = \varepsilon_m \Delta_m(\gamma)$, if γ is a regular element in T_m ; ε_m is a constant, either 1 or -1 . We also write just $\Phi_f^*(\gamma, \dots)$ for $\Phi_{f^m}^*(\gamma, \dots)$.

We summarize our choices once again: a set $\mathcal{J} = \{i_m: T'_m \rightarrow T_m, m=0, \dots, N\}$ of embeddings of tori as in Paragraph 6, on each T_m a positive system for the imaginary roots, and parameters $\varepsilon_0, \dots, \varepsilon_N$.

Let f be a Schwartz function on $\mathbf{G}$ and assume fixed Haar measures on $T_0, \dots, T_N$ (denoted generically by dt) and $\mathbf{G}$ (denoted dg). If $\gamma' \in \mathbf{H}$ originates from the regular element γ of $\mathbf{G}$ with respect to $\mathcal{J}$ set

$$\Phi(\gamma') = \Delta_H^G(\gamma) \Phi_f^*(\gamma, dg, dt).$$

Then:

THEOREM 8.3. — Φ is a well-defined function on the elements of H which originate from regular elements of G . If γ' is such an element and lies in the Cartan subgroup T' of H then

$$\Phi((\gamma')^{\omega'}) = \Phi(\gamma') \quad \text{for } \omega' \in \mathcal{A}(T').$$

Proof. — This follows from Lemmas 4.1, 6.1 and 8.2.

9. Transferring orbital integrals

We continue with the notation of the last section. Our aim now is to write down conditions on $\varepsilon_0, \dots, \varepsilon_N$ (necessary for generic f) to ensure the existence of a Schwartz function f' on H such that

$$(1) \quad \Phi_{f'}^1(\gamma', dt', dh) = \Delta_H^G(\gamma) \Phi_f^*(\gamma, dt, dg),$$

if γ' originates from the regular element γ of G and

$$(2) \quad \Phi_{f'}^1(\gamma', dt', dh) = 0,$$

on Cartan subgroups H of G which do not originate in G .

Here dt' is to be obtained from dt via an $\mathcal{J}$ -embedding; for each measure dg we pick a Haar measure dh on H subject only to the conditions: if $(dh)'$ corresponds to $(dg)'$ and $(dg)' = \beta dg$, $\beta > 0$, then $(dh)' = \beta dh$, and if H is a torus then $dh = dt'$.

Before proceeding, we note that a change in $\mathcal{J}$ or the positive systems for imaginary roots causes at most a sign change on the right-hand side of (1); this change may as well be effected by adjusting $\varepsilon_0, \dots, \varepsilon_N$ instead.

Let T' be a Cartan subgroup of H . Then we set:

$$\Phi^{T'}(\gamma', dt', dh) = \begin{cases} \Delta_H^G(\gamma) \Phi_f^*(\gamma, dt, dg), \\ \text{if } \gamma' \in T' \text{ originates from } \gamma \in T_{\text{reg}}, \\ 0 \text{ if } T' \text{ does not originate in } G. \end{cases}$$

Then:

$$(I) \quad \Phi^{T'}(\gamma', \alpha dt', \beta dh) = \beta/\alpha \Phi^{T'}(\gamma', dt', dh), \quad \alpha, \beta > 0,$$

and

$$(II) \quad \Phi^{(T')^{\omega'}}((\gamma')^{\omega'}, (dt')^{\omega'}, dh) = \Phi^{T'}(\gamma', dt', dh), \quad \omega \in \mathcal{A}(T').$$

We want to check whether $\{\Phi^{T'}\}$ satisfies the remaining conditions of [9], Theorem 4.7. From (I) and (II) above it follows that we may fix dt and dh and assume that either T' is one of the Cartan subgroups T'_m and $\gamma = i_m(\gamma')$ or T' does not originate in G .

We dispose first of the case that T' does not originate in G . Then nor does $(T')^{s'}$, for any Cayley transform s' with respect to a noncompact imaginary root of T' (cf. [9], § 2). Hence III, III a, III b of Theorem 4.7 in [9] are satisfied.

We will write Φ_m for Φ^{T_m} and set

$$\Psi_m(\gamma) = R'_m(\gamma') \Phi_m(\gamma') = \varepsilon_m(\iota_m - \iota'_m)(\gamma) R_m(\gamma) \Phi_{f^m}^*(\gamma);$$

$\Psi_m(\gamma)$ is defined on the regular elements in T_m . For the next few paragraphs we omit the subscript m from $T_m, T'_m, \Phi_m, \Psi_m, i_m, R_m, R'_m, \iota_m$ and ι'_m ; we write I for the set of imaginary roots for T_m , I' for the imaginary roots of T'_m and sometimes identify I' with its image in I .

From Proposition 4.2 we obtain that Ψ is a Schwartz function on

$$T^{(0)} = \{\gamma \in T: \alpha(\gamma) \neq 1, \alpha \in I\}.$$

To satisfy III of Theorem 4.7 in [9] we have to show that Ψ extends to a Schwartz function on

$$T^{(1)} = \{\gamma \in T: \alpha(\gamma) \neq 1, \alpha \in I'\}.$$

According to a standard argument (cf. [13], § 8.4) it is sufficient to show:

PROPOSITION 9.1. — *If $\alpha \in I - I'$ and $\gamma_0 \in T$ is such that $\beta(\gamma_0) = 1$ only if $\beta = \pm \alpha$ then*

$$\lim_{v \downarrow 0} D \Psi(\gamma_v) = \lim_{v \uparrow 0} D \Psi(\gamma_v),$$

where $\gamma_v = \gamma_0 \exp v H_\alpha$, H_α denoting the coroot attached to α (as element of $\mathfrak{t}$), and D is any invariant differential operator on T .

Proof. — Since $\kappa(\alpha^\vee) = -1$ this follows immediately from Lemma 4.3.

We come next to III a of Theorem 4.7 in [9]. Because H is quasi-split this condition is vacuous. Indeed:

LEMMA 9.2. — *If G is a quasi-split group over $\mathbf{R}$ then the following is true for any Cartan subgroup T of G : if α is an imaginary root for T then there exists ω in the imaginary Weyl group for T such that $\omega\alpha$ is noncompact.*

Proof. — We may assume that G is semisimple and simply-connected. By [9], Proposition 4.11 it is sufficient to show that for each imaginary root α of T there exists $g \in G$ such that $\sigma(g^{-1})g$ realizes ω_α , the Weyl reflection with respect to α .

Let $\gamma_0 \in T$ be such that $\alpha(\gamma_0) = 1$ and $\beta(\gamma_0) \neq 1$ if $\beta \neq \pm \alpha$. Set C to be the connected component of the identity in the centralizer of γ_0 in G ; recall that C is of type A_1 and C contains T as fundamental Cartan subgroup. Let $\psi: C \rightarrow C'$ be an inner twist taking C to a quasi-split form C' and such that the restriction of ψ to T is defined over $\mathbf{R}$ (cf. [8]). Let s be a Cayley transform with respect to a (noncompact) root $\psi(\alpha)$ of $\psi(T)$ and set $\lambda = \text{ad } s \circ \psi$. Then clearly the automorphism $\sigma(\lambda^{-1})\lambda$ of T realizes ω_α . Now choose an $\mathbf{R}$ -rational point t in the image of T under λ such that $\gamma = \lambda^{-1}(t)$ is regular in G . Then $\sigma(\gamma) = \gamma^{\omega_\alpha}$ so that the conjugacy class of γ in G is defined over $\mathbf{R}$. But then,

by [11], this class contains an $\mathbf{R}$ -rational point, say $g\gamma g^{-1}$. Clearly $(\sigma(g^{-1})g)\gamma(\sigma(g^{-1})g)^{-1} = \gamma^{\omega_\alpha}$. Since γ is regular in $\mathbf{T}$ this implies that $\sigma(g^{-1})g$ realizes ω_α and so the lemma is proved.

We come then to the condition III b of [9], Theorem 4.7. Suppose that α' is a noncompact root in I' . Then α , the image of α' in I , may be compact. . . in fact it may happen that each $\omega\alpha$, ω in the imaginary Weyl group of $\mathbf{T}$, is compact.

PROPOSITION 9.3. — *Let s' be a Cayley transform with respect to α' . Then we can find a noncompact root among the $\omega\alpha$ if and only if $(\mathbf{T}')_{s'}$ originates in $\mathbf{G}$.*

Proof. — Suppose that $(\mathbf{T}')_{s'}$ originates in $\mathbf{G}$. Then an $\mathcal{J}$ -embedding $i^{(s')}$ of $(\mathbf{T}')_{s'}$ in $\mathbf{G}$ yields a map $i^{(s')} \circ \text{ad } s' \circ i^{-1}$ on $\mathbf{T}$ which can be realized by an element of $\mathbf{G}$, say s . Clearly $\sigma(s^{-1})s$ realizes ω_α and we are done. Conversely, suppose that $\omega\alpha$ is noncompact in $\mathbf{G}$ and that s is a Cayley transform with respect to α (in our general sense). Then $i^{(s)} = \text{ad } s \circ i \circ \text{ad } (s')^{-1}$ is defined over $\mathbf{R}$; by choosing s suitably we can ensure that $i^{(s)}$ is an $\mathcal{J}$ -embedding. Hence $(\mathbf{T}')_{s'}$ originates in $\mathbf{G}$ and the proposition is proved.

Suppose now that α' (noncompact in $\mathbf{H}$) is a root for which all $\omega\alpha$ are compact. Suppose that $\gamma'_0 \in \mathbf{T}'$ is such that $\pm\alpha'$ are the only roots in I' annihilating γ'_0 . It is possible that $\beta(\gamma_0) = 1$ where β lies outside I' (as usual, γ_0 is the image of γ'_0); nevertheless, for small v , $\gamma_v = \gamma_0 \exp i v H_\alpha$ lies in $\mathbf{T}^{(1)}$ so that $\Psi(\gamma_v)$ is well-defined. To satisfy condition III b for the present α' we have to show

$$\lim_{v \downarrow 0} D\Psi(\gamma_v) = \lim_{v \uparrow 0} D\Psi(\gamma_v),$$

for each D . If γ_0 is annihilated by no root outside I' then it is immediate (cf. Lemmas 4.3 and 4.4). To obtain this formula in general we have only to apply the usual argument (cf. [13], §8.4).

The remaining case provides us with the conditions on $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N$. Here we have a noncompact imaginary root α' for which some root in the imaginary Weyl group orbit of α is noncompact. Suppose that γ'_0 is an element for which $\alpha'(\gamma'_0) = 1$ and $\beta'(\gamma'_0) \neq 1$ if $\beta' \neq \pm\alpha'$. Once again a straight forward argument shows that we may assume that $\pm\alpha$ are the only roots which annihilate γ_0 .

We return to writing $\mathbf{T}_m$ for $\mathbf{T}$, Φ_m for Φ , etc. Fix a Cayley transform s' with respect to α' . Recall that $(\mathbf{T}'_m)_{s'}$ originates in $\mathbf{G}$ (Prop. 9.3). Whatever our choice for s' , $(\mathbf{T}'_m)_{s'}$ originates from the same torus, say $\mathbf{T}_n$, among $\mathbf{T}_0, \dots, \mathbf{T}_N$. Since to verify III b we are free to make any choice for s' we may assume that $(\mathbf{T}'_m)_{s'}$ is $\mathbf{T}'_n$. Thus we have:

$$\begin{array}{ccc} \mathbf{T}'_m & \xrightarrow{i_m} & \mathbf{T}_m \\ \downarrow s' & & \\ \mathbf{T}'_n & \xrightarrow{i_n} & \mathbf{T}_n \end{array}$$

We denote by s the map $i_n \circ s' \circ i_m^{-1}$; s can be realized by an element of $\mathbf{G}$ and $\sigma(s^{-1})s$ realizes ω_α .

We need now to label our chosen systems of positive imaginary roots; we denote by I_m^+ the system for T_m and by $(I_m^+)'$ the induced system for T_m' . Because I_m^+ need not be adapted to α we pick a system J^+ which is. Then the induced system $(J^+)'$ is adapted to α' . We denote by R^* the R -function defined by J^+ and by ι^* one-half the sum of the roots in J^+ . We write

$$R = \varepsilon(I_m^+, J^+) (\iota^* - \iota) R^*; \quad \varepsilon(I_m^+, J^+) = \pm 1.$$

Similarly we define $(R^*)'$, $(\iota^*)'$ and $\varepsilon((I_m^+)', (J^+)')$. As before, we will often transfer functions and operators from T_m' to T_m without change in notation. We have to compute

$$\lim_{v \downarrow 0} \hat{D}((R^*)' \Phi_m)(\gamma_v) - \lim_{v \uparrow 0} \hat{D}((R^*)' \Phi_m)(\gamma_v),$$

... III b will be satisfied if and only if the result is

$$2i \hat{D}^s((R^*)'_s \Phi_n)(\gamma_0^s).$$

We summarize our calculations in:

PROPOSITION 9.4:

- (a) $\hat{D}((R^*)' \Phi_m) = \varepsilon_m \varepsilon(I_m^+, J^+) \varepsilon((I_m^+)', (J^+)') (\iota^* - (\iota^*)') \hat{D}(R^* \Phi_{f_m}^s),$
- (b) $(\iota^* - (\iota^*)')(\gamma_0) = (\iota_s^* - (\iota_s^*)')(\gamma_0^s),$
- (c) $(\iota_s^* - (\iota_s^*)')(\gamma_0^s) \hat{D}^s(R_s^* \Phi_{f_s}^s)(\gamma_0^s) = \varepsilon_n \varepsilon(I_n^+, J_s^+) \varepsilon((I_n^+)', (J_s^+)') \hat{D}^s((R^*)'_s \Phi_n)(\gamma_0^s).$

Note that (b) utilizes the second part of our assumption (8.1). Lemma 4.4 now shows that III b is satisfied provided

$$(9.5) \quad \varepsilon_m \varepsilon_n = \varepsilon_{\kappa_m}(s) \varepsilon(I_m^+, J^+) \varepsilon((I_m^+)', (J^+)') \varepsilon(I_n^+, J_s^+) \varepsilon((I_n^+)', (J_s^+)').$$

Recall that $\varepsilon_{\kappa_m}(s)$, the κ_m -signature of s , was defined in Paragraph 4.

10. Transferring orbital integrals (cont.)

We come now to some explicit calculations and our main result (Theorem 10.2). Suppose that T_m' and T_n' are a pair among $\{T_0', \dots, T_N'\}$ for which there is some Cayley transform (in our general sense) from T_m' to T_n' . This means just that the conjugacy class of T_n' succeeds that of T_m' in the lattice $t(H)$ (more briefly, " T_n' succeeds T_m' "). The left-hand side of (9.5) depends, apparently, on the choice (α') of root to define the Cayley transform, choice (s') of Cayley transform and choice (J^+) of positive system adapted to the image in G of that root. We will check that the choices have no effect. Let

$$\begin{aligned} \varepsilon_{\kappa_0}(m, n) &= \varepsilon_{\kappa_m}(s), \\ \varepsilon_+(m, n) &= \varepsilon(I_m^+, J^+) \varepsilon((I_m^+)', (J^+)') \varepsilon(I_n^+, J_s^+) \varepsilon((I_n^+)', (J_s^+)'). \end{aligned}$$

Although we have omitted it in notation, $\varepsilon_{\kappa_0}(m, n)$ and $\varepsilon_+(m, n)$ may depend on the choices above . . . it is only their product which we claim to be independent. Our equations (9.5) are now:

$$(10.1) \quad \varepsilon_m \varepsilon_n = \varepsilon_{\kappa_0}(m, n) \varepsilon_+(m, n).$$

Summarizing Paragraph 9 we have:

THEOREM 10.2. — *If $\varepsilon_m \varepsilon_n = \varepsilon_{\kappa_0}(m, n) \varepsilon_+(m, n)$ whenever T'_n succeeds T'_m ($m, n = 0, 1, \dots, N$) then the factor $\Delta_G^H = \Delta_G^H(\varepsilon_0, \dots, \varepsilon_N)$ has the property that for each Schwartz function f on G there exists a Schwartz function f' on H such that:*

$$(1) \quad \Phi_f^1(\gamma', dt', dh) = \Delta_H^G(\gamma) \Phi_{f'}^*(\gamma, dt, dg),$$

if γ' originates from $\gamma \in G_{\text{reg}}$ and

$$(2) \quad \Phi_{f'}^1(\dots) \equiv 0,$$

on Cartan subgroups of H which do not originate in G .

The notation has been explained in Paragraph 9. The converse is also true: if the equations are not satisfied then we can find functions f for which there is no f' satisfying (1) and (2). Of interest for character identities is the following: if both f' and f'' are attached to f as in the theorem then any of the (tempered) characters χ_ϕ of [9] takes the same value on f' and f'' and, conversely, we can always replace f' by a function on which each χ_ϕ takes the same value (cf. [9], Lemma 5.3).

It remains now to prove our claim of the first paragraph; $\alpha', \alpha, s', s, J^+$ and $(J^+)'$ are as at the end of Paragraph 9.

PROPOSITION 10.2:

$$\varepsilon(I_m^+, J^+) \varepsilon(I_n^+, J_s^+) = \frac{1}{2} (|\{\beta: \langle \beta, \alpha \rangle \neq 0, \text{ both } \beta \text{ and } \omega_\alpha(\beta) \in I_m^+\}|) \\ + |\{\beta: \beta \in I_m^+, \langle \beta, \alpha \rangle = 0 \text{ and } \beta \notin I_n^+\}|.$$

The proof is straightforward; we omit the details.

COROLLARY 10.3. — *Neither $\varepsilon(I_m^+, J^+) \varepsilon(I_n^+, J_s^+)$ nor $\varepsilon((I_m^+)', (J^+)') \varepsilon((I_n^+)', (J^+)'_s)$ depends on the choice for J^+ .*

We will need the following:

LEMMA 10.5. — *Let G be a connected reductive group over $\mathbf{R}$, T a Cartan subgroup of G and α, β imaginary roots of T for which there exist Cayley transforms. Suppose that the image of T under some (and hence every) Cayley transform with respect to α is G -conjugate to the image under some transform with respect to β . Then there exists ω in $\Omega_0(G, T)$ [that is, an element ω of $\Omega(G, T)$ realized in $\mathcal{A}(T)$] mapping α to β .*

It is clear that, conversely, if α and β are so related then the image of T under a Cayley transform with respect to α is G -conjugate to the image under any Cayley transform with

respect to β . Lemma 9.2 thus says that if G is quasi-split then the $\mathcal{A}(T)$ -orbits of imaginary roots of T parametrize the successors in the lattice $t(G)$ of the conjugacy class of T .

Proof. — As usual, we denote by $G^\sim$ the simply-connected covering of the derived group of G : two maximal tori in G , defined over $\mathbf{R}$, are stably conjugate if and only if their preimages in $G^\sim$ are stably conjugate in $G^\sim$ and so the natural projection induces a bijection between $t(G^\sim)$ and $t(G)$. Hence it is enough to prove the lemma in the case G is simply-connected, semi simple . . . clearly, we can then assume G simple, as well. Finally, by the results of Paragraph 2 in [9] we can assume G quasi-split.

The rest of our proof is a case-by-case study. In several places we will use the following. Let T_0 be a fundamental Cartan subgroup of G and $\Delta = \{\alpha_1, \dots, \alpha_r\}$ an ordered set of imaginary roots for T_0 with the property that $T = s T_0 s^{-1}$, where $s = s_r s_{r-1} \dots s_1$, s_1 is a Cayley transform with respect to α_1 and, for $i \geq 2$, s_i is a Cayley transform with respect to $s_{i-1} \dots s_1 \alpha_i$. Then $\alpha_0 = s^{-1} \alpha$, $\beta_0 = s^{-1} \beta$ are imaginary roots of T_0 , perpendicular to Δ . Suppose that there exists $\omega_0 \in \Omega(G, T_0)$ such that $\beta_0 = \omega_0 \alpha_0$, $\sigma \omega_0 = \omega_0 \sigma$ and ω_0 fixes $\alpha_1, \dots, \alpha_r$. Then clearly $\omega = s \omega_0 s^{-1}$ has the properties required in the lemma.

We summarize now the (elementary) argument for each type. The roots for T_0 are labelled as in [1]; we transfer roots from T_0 to T (via s) without change in notation.

(A_n) We have only to consider SL_{n+1} and special unitary groups (of maximal index). In the case of SL_{n+1} only the roots $e_{2i-1} - e_{2i}$ of (the usual) T_0 are imaginary and it is easy to find ω_0 . In the case of unitary groups all the roots of T_0 are imaginary and again ω_0 is easily found.

(B_n, C_n) We give an argument for C_n which adapts immediately to the case B_n . Consider each pair of (imaginary) roots in T_0 as possibilities for $\{\alpha_0, \beta_0\}$. In the cases $\{2e_i, 2e_j\}$, $\{e_i \pm e_j\}$, $\{e_i - e_j, e_i - e_k\}$, $j \neq k$, and $\{e_i - e_j, e_k - e_l\}$ with i, j, k, l distinct and $e_i + e_j, e_k + e_l \notin \Delta$ the choice of ω_0 is easy. In the case $\{e_i - e_j, e_k - e_l\}$ with i, j, k, l distinct and both $e_i + e_j, e_k + e_l$ lying in Δ , we have on T that $\sigma e_i = e_j, \sigma e_k = e_l$ so that $\omega = \omega_{e_i - e_k} \omega_{e_j - e_l}$ commutes with σ and maps α to β . Next we observe that $\{e_i - e_j, e_k - e_l\}$ with i, j, k, l distinct and $e_i + e_j \in \Delta, e_k + e_l \notin \Delta$, is not a possibility (by counting the number of long imaginary roots in the images of T under Cayley transforms with respect to $e_i - e_j, e_k - e_l$). Similarly $\{e_i - e_j, 2e_k\}$ is not possible. The remaining pairs are similarly dealt with.

(D_n) Here we have to consider (i) the groups $Spin(2m, 2m)$, $Spin(2m, 2m+2)$ (where fundamental Cartan subgroups are compact) and (ii) $Spin(2m+1, 2m+1)$, $Spin(2m-1, 2m+1)$ (where fundamental Cartan subgroups are not compact). Again we examine each pair of imaginary roots in T_0 . In the case $\{e_i - e_j, e_i - e_l\}$, $j \neq l$, ω_0 is easily found. In the case $\{e_i \pm e_j\}$, suppose that there is some e_k not appearing in the roots in Δ . Then for both (i) and (ii) the choice of ω_0 is easy. In the same case, suppose that every e_k appears in a root of Δ and that for some pair (l, p) both $e_l + e_p$ and $e_l - e_p$ lie in Δ . Then on T , $\sigma e_l = e_l$ and $\sigma e_j = -e_j$ so that $\omega_{e_j + e_l} \omega_{e_j - e_l}$ will do for ω . Finally, suppose that every index appears in the roots of Δ (except i, j) and that if $e_l \pm e_p$ belongs to Δ then $e_l \mp e_p$ does not. Then we must be in the case of a well-known example for

Spin $(2m, 2m)$ (cf. [12]) where twists by $e_i + e_j, e_i - e_j$ lead to non-conjugate Cartan subgroups.

Next we consider the case $\{e_i - e_j, e_k - e_l\}$ with i, j, k, l distinct. If either both or neither $e_i + e_j, e_k + e_l$ belong to Δ then we find ω as before (cf. the argument for C_n). We claim that if exactly one of these roots belongs to Δ then $\{e_i - e_j, e_k - e_l\}$ is not a possibility. We justify this by performing Cayley transforms on $\mathbf{T}$ with respect to $e_i - e_j$ and $e_k - e_l$ and then calculating the "root spaces" attached to the images (cf. [12]); these spaces are easily seen to be non-conjugate in the sense of [12] [for both types (i), (ii)].

The remaining cases are now easily examined.

(E₆) There are two groups to consider: the simply-connected split form, whose fundamental Cartan subgroup is not compact and the simply-connected quasi-split form with compact fundamental Cartan subgroup. We investigate the second first.

If both roots $e_i + e_j, \pm 1/2(e_i + e_j) \pm \dots$ are imaginary then clearly we can find an element of $\mathcal{A}(\mathbf{T})$ mapping the former to the latter. A simple inductive argument then shows that we can assume that Δ contains only roots of the form $e_i \pm e_j$. We have now only to show that for any pair among $\{e_i \pm e_j\}$, $1 \leq j < i \leq 5$, we can find an ω as desired. For pairs $\{e_i - e_j, e_i - e_l\}$, j, l distinct, this is immediate. In the case of $\{e_i \pm e_j\}$, there is some e_k not appearing in the roots of Δ and so we can argue as for the case D_n . In the case of $\{e_i - e_j, e_k - e_l\}$, with i, j, k, l distinct we again argue as before if either both or neither of $e_i + e_j, e_k + e_l$ belong to Δ . Suppose that $e_i + e_j \in \Delta, e_k + e_l \notin \Delta$. Then $\Delta = \{e_i + e_j\}$ and the root $1/2(e_i - e_j + e_k - e_l + \dots)$ is imaginary in $\mathbf{T}$ and perpendicular to neither $e_i - e_j$ nor $e_k - e_l$. Hence ω is easily found. The remaining cases are handled similarly.

To investigate the other form of type E₆ we make the appropriate definition of "inverse Cayley transform" with respect to a *real* root of $\mathbf{T}$ (generalizing the usual notion). It follows easily that we have only to check that if there are inverse Cayley transforms with respect to the real roots α, β which lead to conjugate Cartan subgroups then β is of the form $\omega\alpha$, with $\omega \in \mathcal{A}(\mathbf{T})$ (or, just as well, with ω in G). To make this check we set up the analogue of Δ among the (real) roots of the split Cartan subgroup of G . As before, we can assume that this set contains only roots of the form $e_i \pm e_j$ and consider candidates for α, β . The argument is analogous to that of the previous paragraph; we omit the details.

(E₇) We can assume that Δ contains only roots of the form $e_i \pm e_j$, for if $\mathbf{T} = \mathbf{T}_0$, any two roots of $\mathbf{T}$ can be connected by an element of $\mathcal{A}(\mathbf{T})$ and so we can restrict our attention to the case Δ contains $e_8 - e_7$. We have then to consider just pairs from $\{e_i \pm e_j\}$, $1 \leq j < i \leq 6$, as candidates for α, β . For a pair $\{e_i - e_j, e_i - e_l\}$, i, j, l distinct, ω is easily found. Consider a pair $\{e_i + e_j, e_i - e_j\}$. Our previous arguments show how to find ω in all but the case where Δ has three elements $e_8 - e_7, e_k \star e_l, e_m \star e_n$ where k, l, m, n are distinct from i, j and $\star$ denotes some choice of $\pm$. For this $\mathbf{T}$ we perform Cayley transforms by $e_i - e_j$ and $e_i + e_j$ and count the number of real roots in the images; this enables us to exclude this case. Next we consider a pair $\{e_i - e_j, e_k - e_l\}$, with i, j, k, l distinct. Again if either both or neither $e_i + e_j, e_k + e_l$ lie in Δ then we can find $\omega \dots$ and similar arguments apply if we change either or both signs in $\{e_i - e_j, e_k - e_l\}$. The remaining case requires several arguments; we find it easier to use numerical indices. Suppose that $\Delta = \{e_8 - e_7, e_1 + e_2\}$. We exclude the pair

$\{e_2 - e_1, e_4 - e_3\}$ by counting the number of imaginary roots in the image of $\mathbf{T}$ under a Cayley transform with respect to $e_2 - e_1, e_4 - e_3$, respectively. It follows similarly that $\{e_2 - e_1, e_4 + e_3\}$ is not a possibility. Suppose now that $\Delta = \{e_8 - e_7, e_1 + e_2, e_6 - e_5\}$. Then the pair $\{e_2 - e_1, e_4 + e_3\}$ is excluded (. . . this time counting real roots in the images). On the other hand, consider $\{e_2 - e_1, e_4 - e_3\}$. The root $1/2((e_8 - e_7) + (e_2 - e_1) + (e_4 - e_3) + (e_6 - e_5))$ is imaginary and perpendicular to neither $e_2 - e_1$ nor $e_4 - e_3$. Hence we can find ω in $\mathcal{A}(\mathbf{T})$ mapping $e_2 - e_1$ to $e_4 - e_3$. Suppose that $\Delta = \{e_8 - e_7, e_1 + e_2, e_6 \pm e_5\}$. Then for each pair $\{e_2 - e_1, e_4 - e_3\}, \{e_2 - e_1, e_4 + e_3\}$ we can construct a root as above and so obtain ω . We can now easily complete the argument.

(E₈) Once again we can assume that Δ contains only roots of the form $e_i \pm e_j$ and investigate just pairs among $\{e_i \pm e_j\}$. The arguments are similar to those for E₇ and so we omit the details.

(F₄) For the pairs $\{e_i \pm e_j\}, \{e_i - e_j, e_i - e_l\}, \{e_i - e_j, e_k - e_l\}$ and $\{1/2(e_1 \pm e_2 \dots), e_i\}$, i, j, k, l distinct, we find ω easily. The pairs $\{e_i - e_j, e_k\}$ are eliminated (by counting short imaginary roots in the images of $\mathbf{T}$ under . . .) and the argument then easily completed.

(G₂) Here we need only observe that if $\mathbf{T}$ is compact then the G-conjugacy class of image of $\mathbf{T}$ under a Cayley transform depends just on the length of the root used.

Lemma 10.5 is thus proved.

Returning to $\varepsilon_{\kappa_0}(m, n)$ and $\varepsilon_+(m, n)$ we have now that we may replace α' only by $\omega' \alpha'$, $\omega' \in \Omega_0(\mathbf{H}, \mathbf{T}'_m)$ [since we have required α' to be noncompact there is further restriction on ω' (cf. [9], Lemma 4.2) but we do not need this explicitly]. Thus s' may be replaced only by $t' = \omega'_0 s' \omega'$ where $\omega' \in \Omega_0(\mathbf{H}, \mathbf{T}'_m)$ and $\omega'_0 \in \Omega_0(\mathbf{H}, \mathbf{T}'_n)$; s is then replaced by $\omega_0 s \omega$ where ω is the image of ω' in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$ and ω_0 the image of ω'_0 in $\Omega_0(\mathbf{G}, \mathbf{T}_n)$. A straightforward computation shows that $\varepsilon_{\kappa_0}(m, n)$ is multiplied by $\kappa_m(\omega) \kappa_n(\omega_0)$ and $\varepsilon_+(m, n)$ by $\varepsilon(\omega)/\varepsilon(\omega') \cdot \varepsilon(\omega_0)/\varepsilon(\omega'_0)$ (in the notation of Lemma 8.2). Hence, by the proof of Lemma 8.2, $\varepsilon_{\kappa_0}(m, n) \varepsilon(m, n)$ is unchanged.

11. Application of Theorem 10.2

As an immediate corollary of Theorem 10.2 we obtain:

PROPOSITION 11.1. — *If the ordering on $t(\mathbf{H})$ is linear (that is, if the derived group of $\mathbf{H}$ is trivial, of type A_n or of type E_6) or if $\mathbf{G}$ has split rank one then given some ε_M there is a choice for $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{M-1}, \varepsilon_{M+1}, \dots, \varepsilon_N$ for which the factor $\Delta_G^H(\varepsilon_0, \dots, \varepsilon_N)$ provides a transfer of orbital integrals in the sense of Theorem 10.2.*

We would like to remove this assumption on $\mathbf{H}$ (. . . or $\mathbf{G}$). Here we just describe some reductions and, as application, check that the conclusion of Proposition 11.1 remains valid under the assumption that the derived group of $\mathbf{G}$ is isogenous to a product of groups each of which has rank at most two, with $\mathbf{H}$ (or, more precisely, κ_0) arbitrary. Recall that we admit only those pairs $(\mathbf{G}, \mathbf{H})$ which satisfy the condition (8.1); in particular, for each $\mathbf{T}_m$ one half of the sum of the positive imaginary roots not coming from $\mathbf{H}$ defines a character on $\mathbf{T}_m$.

We have thus to investigate the consistency of the equations (10.1) as the pair (m, n) varies. The following observation will allow us to consider just consistency around subsets of $t(H)$ of the form:

PROPOSITION 11.2. — Suppose that G is a connected reductive group over $\mathbf{R}$ and that T' and T'' are Cartan subgroups of G succeeded by the same Cartan subgroup. Then both T' and T'' succeed some Cartan subgroup.

Proof. — A straightforward argument brings us to the case where G is simply-connected, simple and quasi-split (cf. the proof of Lemma 10.6). We have then only to examine the possibilities for $t(G)$. This is easily done using the lists in [12]; we omit the details.

Suppose now that T'_m is fixed and ε_m chosen as 1. Suppose also that T'_{n_1} and T'_{n_2} are non-conjugate Cartan subgroups which succeed T'_m and that ε_{n_1} and ε_{n_2} are defined so that (10.1) holds; that is,

$$\varepsilon_{n_i} = \varepsilon_{x_0}(m, n_i) \varepsilon_+(m, n_i) \quad \text{for } i = 1, 2.$$

Finally, suppose that T'_p succeeds both T'_{n_1} and T'_{n_2} . Then both

$$\varepsilon_{x_0}(m, n_i) \varepsilon_{x_0}(n_i, p) \varepsilon_+(m, n_i) \varepsilon_+(n_i, p), \quad i = 1, 2,$$

are candidates for ε_p . Proposition 11.2 and a simple inductive argument allow us to conclude:

LEMMA 11.3. — Given some ε_M there is a choice for $\varepsilon_0, \dots, \varepsilon_{M-1}, \varepsilon_{M+1}, \dots, \varepsilon_N$ for which $\Delta_G^H(\varepsilon_0, \dots, \varepsilon_N)$ provides a transfer of orbital integrals if and only if

$$\varepsilon_{x_0}(m, n_1) \varepsilon_{x_0}(n_1, p) \varepsilon_+(m, n_1) \varepsilon_+(n_1, p) = \varepsilon_{x_0}(m, n_2) \varepsilon_{x_0}(n_2, p) \varepsilon_+(m, n_2) \varepsilon_+(n_2, p),$$

for each 4-tuple (m, n_1, n_2, p) as above.

To compute terms, let α'_i be a noncompact root of T'_m for which there is a Cayley transform, say s'_i , with respect to α'_i taking T'_m to T'_{n_i} ($i = 1, 2$); let α_i, s_i be the images in G . Similarly, let β'_i be a noncompact root of T'_{n_i} for which there is a Cayley transform, say t'_i , with respect to β'_i taking T'_{n_i} to T'_p , and β_i, t_i be the images in G . Thus we have

Note that it may happen that $T_{n_1} = T_{n_2}$. Choose a positive system I^+ for the imaginary roots of T_p and system J_i^+ for T_m adapted to α_i such that $(J_i^+)_{s_i}$ is adapted to β_i and $((J_i^+)_{s_i})_{t_i} = I^+$, $i = 1, 2$. Then the product of all terms in (11.4) of the form $\varepsilon_+(\ , \)$ is

$$\varepsilon_+ = \varepsilon(J_1^+, J_2^+) \varepsilon((J_1^+)', (J_2^+)'),$$

[see Paragraph 10 for the definition of $\varepsilon_+(\ , \)$]. Note that ε_+ depends only on the isogeny class of the derived part of G . The same is true for the remaining terms in (11.4), for these are the signatures of the Cayley transforms s_i, t_i ($i = 1, 2$): to compute the signature of, say s_1 , choose $\tilde{s}_1$ in the preimage of s_1 in $\tilde{G}$ (the simply-connected covering of the derived group of G). Then $\tilde{s}_1$ is a Cayley transform in $\tilde{G}$ and its signature (regarding κ_m as a character for $\tilde{G}$) is the same as that of s_1 . Indeed if $\sigma(\tilde{s}_1^{-1})\tilde{s}_1 \in \tilde{t}_\sigma G_\alpha$ then $\sigma(s_1^{-1})s_1 \in t_\sigma G_\alpha$, where t_σ is the image of $\tilde{t}_\sigma$ in G ; by definition, $\kappa_m(\tilde{t}_\sigma) = \kappa_m(t_\sigma)$. We will write ε_* for the product of the signatures of the s_i, t_i ($i = 1, 2$).

Our second observation is that we need only verify (11.4) in the case that α_1, α_2 are roots for the same simple factor of $\tilde{G}$. It remains then to examine the various simple types. . . here we will examine just the simple systems of rank two (only for the split forms of type C_2, G_2 is there something to prove). For the reduction, we argue as follows. Suppose that α_i is a root for the simple factor G_i of $\tilde{G}$, $i = 1, 2$. Recalling the comment of the third paragraph of Paragraph 9 we may assume that i_{n_1}, i_{n_2} and i_p have been chosen in such a way that we may take $\alpha'_1 = \beta'_2, \alpha'_2 = \beta'_1$ and $\tilde{s}_1 = \tilde{t}_2, \tilde{s}_2 = \tilde{t}_1$ with $\tilde{s}_i$ lying in the factor G_i of $\tilde{G}$.

Then clearly the κ_{n_1} -signature of $\tilde{t}_1$ is the same as the κ_m -signature of $\tilde{s}_2$ and the κ_{n_2} -signature of $\tilde{t}_2$ is the same as the κ_m -signature of $\tilde{s}_1$. This implies that $\varepsilon_* = 1$. On the other hand, the positive systems J_1^+, J_2^+ are equal so that $\varepsilon_+ = 1$ also, as desired.

Suppose now that the Lie algebra of the derived group of G is the split form of type G_2 . There will be consistency problems only if H also has split rank two. Since such an H must contain (a copy of) the fundamental Cartan subgroup of G we may restrict our attention to the case that T_0 is a fundamental Cartan subgroup. We list the roots of T_0 as $\alpha = e_1 - e_2, \beta = -2e_1 + e_2 + e_3, \alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta$ and their negatives, and the dual system as $\tilde{\alpha} = e_1 - e_2, \tilde{\beta} = 1/3(-2e_1 + e_2 + e_3)$, etc. The possibilities for κ_0 are given in the following table:

	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} + 3\tilde{\beta}$	$2\tilde{\alpha} + 3\tilde{\beta}$	$\tilde{\alpha} + \tilde{\beta}$	$\tilde{\alpha} + 2\tilde{\beta}$
1.	1	1	1	1	1	1
κ_0^1	1	-1	-1	-1	-1	1
κ_0^2	-1	1	-1	1	-1	-1
κ_0^3	-1	-1	1	-1	1	-1

The characters κ_0^2, κ_0^3 are of the form $(\kappa_0^1)^\omega, \omega \in \Omega(G, T_0)$. It follows that we need consider only the case $\kappa_0 = \kappa_0^1$. Then, on fixing embeddings of the Cartan subgroups of H into G according to the prescription of Paragraph 6, we can identify α and $3\alpha + 2\beta$ as the roots from H . . . H is thus a group of type $A_1 \times A_1$. Note that the condition (8.1) is satisfied. As usual, we will denote the preimage of α by α' and the preimage of $3\alpha + 2\beta$

by $(3\alpha + 2\beta)'$. On the Cartan subgroup T'_0 both α' and $(3\alpha + 2\beta)'$ are noncompact; we may assume that we have labelled the roots of T_0 so that both α and $3\alpha + 2\beta$ are noncompact. We compute first the term ε_+ . For J_1^+ we must take the positive system with simple roots $3\alpha + 2\beta$ and $-(\alpha + \beta)$ and for J_2^+ the system with simple roots α and β ; it follows that $\varepsilon_+ = 1$ (to conform with our earlier notation we write α as α_1 and $3\alpha + 2\beta$ as α_2). In computing the signature of s_1 , we have only to write s_1 as $s'_1 \omega$, where s'_1 is a standard transform with respect to α_1 followed by a real conjugation and $\omega \in \Omega(\mathbf{G}, T_0)$ fixes α_1 . Then the κ_0 -signature of s_1 is $\kappa_0(\omega)$ (cf. Paragraph 4). But the only possibilities for ω are 1 and ω_{α_2} , both of which are annihilated by κ_0 . Similarly all the other signatures to be computed are one and so we obtain $\varepsilon_* = 1$ and (11.4) is satisfied.

The case that the Lie algebra of the derived group of G is of type C_2 is more instructive. Again we may assume that T_0 is the fundamental Cartan subgroup. We list the roots of T_0 as $\alpha = e_1 - e_2$, $\beta = 2e_2$, $\alpha + \beta$, $2\alpha + \beta$ and their negatives and the dual system as $\alpha^\sim = e_1 - e_2$, $\beta^\sim = e_2$, etc. The possibilities for κ_0 are:

	$\alpha^\sim$	$\beta^\sim$	$\alpha^\sim + 2\beta^\sim$	$\alpha^\sim + \beta^\sim$
1	1	1	1	1
κ_0^1	1	-1	1	-1
κ_0^2	-1	1	-1	-1
κ_0^3	-1	-1	-1	1

Only κ_0^1 gives a group $\mathbf{H}$ of rank 2. In this case we can identify the roots α and $\alpha + \beta$ as the roots of $\mathbf{H}$. . . $\mathbf{H}$ is again of type $A_1 \times A_1$ and the assumption (8.1) is satisfied. We may as well take T_0 , or, more precisely, its Lie algebra, as in [12] [we are assuming that $\mathfrak{g}$ is $\mathfrak{sp}(2, \mathbf{R})$] and label the roots in the usual way. Then, on T_0 , α is compact and $\alpha + \beta$ noncompact, whereas the preimages α' , $(\alpha + \beta)'$ are both noncompact. Again to conform with earlier notation we write α as α_1 and $\alpha + \beta$ as α_2 . For J_1^+ we must take the system with simple roots $\alpha + \beta$ and $-\beta$ and for J_2^+ the system with simple roots α and β . It follows that $\varepsilon_+ = -1$. As before, the signatures of s_2 , t_1 and t_2 are all easily shown to be one. We have then to show that s_1 has negative signature. If we write s_1 as $s'_1 \omega$, where s'_1 is a standard transform with respect to $\underline{\alpha_2}$ (noncompact) followed perhaps by a real conjugation and $\omega \in \Omega(\mathbf{G}, T_0)$ takes α_1 to α_2 then $\kappa_0(\omega)$ is the signature of s_1 (cf. Paragraph 4). Clearly ω is either ω_β or $\omega_{2\alpha + \beta} \omega_\alpha$. But $\kappa_0(\omega_{2\alpha + \beta} \omega_\alpha) = \kappa_0(\omega_{2\alpha + \beta})$ since α comes from $\mathbf{H}$; both β and $2\alpha + \beta$ are noncompact so that

$$\kappa_0(\omega_\beta) = \kappa_0(\beta^\sim) = -1$$

and

$$\kappa_0(\omega_{2\alpha + \beta}) = \kappa_0((2\alpha + \beta)^\sim) = \kappa_0(\alpha^\sim + \beta^\sim) = -1.$$

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