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# A GENERAL COMPARISON THEOREM WITH APPLICATIONS TO VOLUME ESTIMATES FOR SUBMANIFOLDS

By Ernst HEINTZE and Hermann KARCHER (1)

## 1. Introduction

To get a lower bound for the cut locus distance on a compact Riemannian manifold following Klingenberg [9], Cheeger [4] proved the existence of a lower bound for the length of a closed geodesic in terms of an upper bound of the diameter and lower bounds for the volume and the curvature. We generalize and sharpen this result—with a more direct proof, e. g. not using Toponogov's triangle comparison theorem—to the following inequality between data of M and a compact submanifold N:

$$\operatorname{vol}(\mathrm{M}) \leqq \int_{\mathrm{N}} f_\delta(d(\mathrm{M}), \mathrm{H}(p))\, d\operatorname{vol}_{\mathrm{N}} \leqq \operatorname{vol}(\mathrm{N}) . f_\delta(d(\mathrm{M}), \Lambda),$$

where $\delta$ is a lower curvature bound for M (in fact Ricci curvature bound if N is a point or a hypersurface), $d(\mathrm{M})$ is the diameter of M, $\mathrm{H}(p)$ is the length of the mean curvature normal $\eta$ of N at $p$, $\Lambda = \max \mathrm{H}(p)$ and $f_\delta$ is given explixitly in paragraph 2. It grows monotonically with $d(\mathrm{M})$ and $\mathrm{H}(p)$.

For $\delta > 0$ the inequality—if slightly weakened—allows a considerable simplification (without changing the cases of equality, if $\dim \mathrm{N} = n > 0$):

$$\operatorname{vol}(\mathrm{M}^m) \leqq \frac{\operatorname{vol}(\mathrm{S}^m_\delta)}{\operatorname{vol}(\mathrm{S}^n_1)} \int_{\mathrm{N}} (\delta + \mathrm{H}^2(p))^{n/2}\, d\operatorname{vol}_{\mathrm{N}}$$

$$\leqq \frac{\operatorname{vol}(\mathrm{S}^m_\delta)}{\operatorname{vol}(\mathrm{S}^n_{\delta+\Lambda^2})} . \operatorname{vol}(\mathrm{N}^n) \qquad (n > 0),$$

where $\mathrm{S}_\delta$ denotes a sphere of curvature $\delta$. Equality implies that both M and N are of constant curvature, and in fact all space forms can occur for N whereas for M only those

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can occur with $d(\mathrm{M}) = \pi/2\sqrt{\delta}$. With a simple limit arguement the Fenchel-Willmore-Chen inequality ([6], [14], [5]) for compact N in $\mathbf{R}^m$:

$$\int_{\mathrm{N}} |\mathrm{H}(p)|^n \, d\,\mathrm{vol}_{\mathrm{N}} \geqq \mathrm{vol}(\mathrm{S}_1^n),$$

is a consequence of our inequality.

Simplifications also in the case $\delta \leqq 0$ are possible under further assumptions, e. g. for minimal submanifolds ($\mathrm{H} \equiv 0$), *see* 2.3. In particular if $\gamma$ is a closed geodesic, 2.3 gives

$$\text{length } (\gamma) \geqq 2\pi \frac{\mathrm{vol}\,\mathrm{M}^m}{\mathrm{vol}\,\mathrm{S}_1^m} \cdot \left( \frac{\sqrt{|\delta|}}{\sin h(\sqrt{|\delta|} \,.\, d(\mathrm{M}))} \right)^{m-1}.$$

The main tool for proving the inequality is an extension of well known Jacobi field estimates. In fact we prove a very general comparison theorem for the length and volume distortion of the normal exponential map of a submanifold containing as special cases the Rauch [11] and Berger [1] estimates (§ 3). The equality discussion rests on the following Theorem 4.5: *Let* N, M *be compact Riemannian manifolds,* N *isometrically immersed and totally umbilic in* M. *Assume that all planes of* M *containing a tangent vector to a geodesic segment which is normal to* N *and has no focal points have the same sectional curvature* $\delta > 0$. *Then* M *has constant curvature* $\delta$. We emphasize that for dim $\mathrm{N} \geqq 2$ the assumptions $\delta > 0$ and M compact are essential. The proof is given in paragraph 5. The usual Codazzi equation arguments cannot be used. We first extend a result of Warner ([12], Th. 3.2) (who showed that kernel $d\exp_p$—at regular conjugate points of constant multiplicity $\geqq 2$—is tangent to the conjugate set) and conclude that the mean curvature vector field must be parallel. Then we derive that N has constant curvature and finally obtain constant curvature for M.

## 2. The inequality between volume and diameter of a compact Riemannian manifold M and the volume of a compact submanifold N

Let N, M be compact riemannian manifolds, N isometrically immersed in M and M connected. For each $p \in \mathrm{M}$ there exists a distance minimizing geodesic from $p$ to N which hits N perpendicularly; its length is clearly not longer than the diameter of M and it is also not larger than the first focal distance of N in the direction of the geodesic. Therefore the exponential map of the normal bundle $\nu(\mathrm{N})$ of N in M is surjective, even if we restrict $\exp_\nu$ to that subset U of normal vectors $\xi$, which are not longer than $d(\mathrm{M})$ or the focal distance in direction $\xi$. In paragraph 3 we describe the canonical riemannian metric on $\nu(\mathrm{N})$. For this the projection $\pi : \nu \to \mathrm{N}$ is a riemannian submersion. Therefore we can apply Fubini's Theorem to evaluate the following volume integral by first integrating over the fibres of $\nu$ and then over the base N:

2.0.1: $$\mathrm{vol}(\mathrm{M}) \leqq \int_{\mathrm{U}} |\det(d\exp_\nu)| \,.\, d\,\mathrm{vol}_{\nu(n)}.$$

We first state the *curvature assumptions:*

(K) *If* N *is a hypersurface or a point, assume that the Ricci curvatures of* M *for the tangent vectors of distance minimizing normal geodesics are* $\geqq \delta$.

*If* N *has arbitrary codimension, assume that the planes of* M *containing a tangent vector of a geodesic segment which minimizes the distance to* N *have sectional curvatures* $\geqq \delta$.

*Let* H *be the length of the mean curvature normal* $\eta$ *and let* $\Lambda$ *be an upper bound for* H.

We use the abbreviation

2.0.2:
$$\mathbf{s}_\delta(r)=\begin{cases}\delta^{-1/2}\sin\delta^{1/2}r & \text{if } \delta>0\\ r & \text{if } \delta=0\\ |\delta|^{-1/2}\sin h|\delta|^{1/2}r & \text{if } \delta<0\end{cases}$$
$$\mathbf{c}_\delta(r)=\mathbf{s}'_\delta(r).$$

Then Corollaries 3.3.1 and 3.3.2 state that for each $\xi\in \mathrm{U}$ (put $r=|\xi|$) we have

$$|\det(d\exp_\nu)_\xi|.r^{m-n-1}\leqq\left(\mathbf{c}_\delta(r)-\left\langle\eta,\frac{\xi}{r}\right\rangle\mathbf{s}_\delta(r)\right)^n.\mathbf{s}_\delta(r)^{m-n-1},$$

where $m=\dim \mathrm{M}$ and $n=\dim \mathrm{N}$. Now 2.0.1 gives

$$\mathrm{vol}(\mathrm{M})\leqq\int_{\mathrm{U}}\left(\mathbf{c}_\delta(r)-\left\langle\eta,\frac{\xi}{r}\right\rangle\mathbf{s}_\delta(r)\right)^n.\mathbf{s}_\delta(r)^{m-n-1}.r^{-(m-n-1)}\,d\,\mathrm{vol}_{\nu(\mathrm{N})}.$$

Since we do not want to make very specific assumptions about the second fundamental form but assume only bounds on $|\eta|$, we cannot use the description of U in terms of focal distances. However 3.3.1 shows that the first zero $z(\eta,\xi)$ of the integrand is an upper bound for the focal distance in direction $\xi$ ("conjugate" if $n=0$). Therefore we enlarge U correspondingly, now apply Fubini and abbreviate the obtained fibre integral as

2.0.3:

$$f_\delta(d(\mathrm{M}),\mathrm{H}(p))=\int_{\mathrm{S}^{m-n-1}}^{\min(d(\mathrm{M}),z(\eta,\xi))}\int_0(\mathbf{c}_\delta(r)-\langle\eta(p).\,\xi\rangle\mathbf{s}_\delta(r))^n\mathbf{s}_\delta(r)^{m-n-1}\,dr\,d\xi,$$

where $d\xi$ is the standard volume for the sphere $\mathrm{S}^{m-n-1}$.

REMARKS. — 1. By definition $f_\delta$ seems to depend on $\eta$, but the spherical integration eliminates the direction dependence and leaves a function of $|\eta|=\mathrm{H}$. Of course $f_\delta$ also depends on the dimensions $n$, $m$.

2. If one is interested in the volume of tubes of radius R around N, then one only has to change the upper bound of the radial integration to $\min(\mathrm{R},z(\eta,\xi))$.

The above explanations prove the first inequality of the following Theorem, but we emphasize that the Jacobi field estimates of paragraph 3 are the essential part of the proof.

2.1. THEOREM. — *Let, as above,* N *be isometrically immersed in* M *and make the curvature assumptions* (K). *Then*

$$\begin{aligned}\mathrm{vol}(\mathrm{M}) &\leqq \int_{\mathrm{N}} f_\delta(d(\mathrm{M}), \mathrm{H})\, d\,\mathrm{vol}_{\mathrm{N}} \\ &\leqq \mathrm{vol}(\mathrm{N}) . f_\delta(d(\mathrm{M}), \Lambda).\end{aligned}$$

For the proof of the last inequality we use.

2.1.1. PROPOSITION. — $f_\delta(d(\mathrm{M}), \mathrm{H})$ *is monoton increasing in* H.

*Proof.* — We write $\eta = \mathrm{H}.e$, $|e| = 1$ and note, that the functions

$$\int_0^{\min(d(\mathrm{M}),\, z(\eta,\, \xi))} (c_\delta - \mathrm{H}.\langle e, \xi\rangle s_\delta)^n . s_\delta^{m-n-1}\, dr$$

and

$$\int_0^{\min(d(\mathrm{M}),\, z(\eta,\, -\xi))} (c_\delta + \mathrm{H}.\langle e, \xi\rangle s_\delta)^n . s_\delta^{m-n-1}\, dr$$

depend differentiably on H except for those values of H where $d(\mathrm{M}) = z(\eta, \xi)$ or $d(\mathrm{M}) = z(\eta, -\xi)$. If the upper bound of the integral is $d(\mathrm{M})$, then only the monotonicity of the integrand matters; if the upper bound is $z(\eta, \xi)$, then the integrand vanishes there and the derivative of $z$ is not needed. If $\langle e, \xi \rangle > 0$, then the integrand is *not* monoton increasing in H, however in this case $z(\eta, \xi) < z(\eta, -\xi)$. Therefore, since

$$(x - \mathrm{H}\,y)^n + (x + \mathrm{H}\,y)^n = 2 \sum_{2k \leqq n} \binom{n}{2k} x^{n-2k} y^{2k} \mathrm{H}^{2k},$$

the monotonicity of the sum of the two integrands for the directions $\xi$ and $-\xi$ follows and thus the monotonicity of $f_\delta$.

We give two special cases of 2.1.

2.2. THEOREM. — *If under the assumptions* 2.1 *one has* $\delta > 0$, *then*

$$\mathrm{vol}(\mathrm{M}) \leqq \frac{\mathrm{vol}(\mathrm{S}_\delta^m)}{\mathrm{vol}(\mathrm{S}_1^n)} . \int_{\mathrm{N}} (\delta + \mathrm{H}(p)^2)^{n/2}\, d\,\mathrm{vol}_{\mathrm{N}} \qquad \textit{for} \quad n > 0,$$

*and*

$$\mathrm{vol}(\mathrm{M}) \leqq \mathrm{vol}(\mathrm{S}_\delta^m) \qquad \textit{for} \quad n = 0.$$

*In particular,* if $|\eta| \leqq \Lambda$ *and* $n > 0$ *then*

$$\frac{\mathrm{vol}(\mathrm{M})}{\mathrm{vol}(\mathrm{S}_\delta^m)} \leqq \frac{\mathrm{vol}(\mathrm{N})}{\mathrm{vol}(\mathrm{S}_{\delta+\Lambda^2}^n)}$$

($\mathrm{S}_\delta$ *denotes a sphere of curvature* $\delta$).

Note that this gives a lower bound for the volume of submanifolds of bounded mean curvature which is sharp for small spheres $\mathrm{N} = \mathrm{S}_{\delta+\Lambda^2}^n$ in spheres $\mathrm{M} = \mathrm{S}_\delta^m$. For a very special case *see* Heim [8].

The case $n = 0$ had been proved by Bishop [2].

2.3. THEOREM. — *If under the assumptions* 2.1 N *is also minimal* ($\eta = 0$), *then*

$$\mathrm{vol}(M^m) \leqq \mathrm{vol}(N^n).\mathrm{vol}(S_1^{m-n-1}).\int_0^{\min(d(M),\,\pi/2\sqrt{\delta})} c_\delta(r)^n.s_\delta(r)^{m-n-1}\,dr,$$

*where the number* $\pi/2\sqrt{\delta}$ *has to be deleted if* $\delta \leqq 0$ *and has to be replaced by* $\pi/\sqrt{\delta}$ *if* $n = 0$ *and* $\delta > 0$.

2.3.1. COROLLARY. — *Inequality* 2.3 *can be interpreted as a lower bound for* $d(M)$, *e.g.*: *if* $\delta = 0$ *then*

$$\mathrm{vol}(M) \leqq \mathrm{vol}(N).\mathrm{vol}(S_1^{m-n-1}).\frac{d(M)^{m-n}}{m-n},$$

*if* $n = 0$ *and* $\delta > 0$ *and* $\mathrm{vol}(M) \geqq \frac{1}{2}\mathrm{vol}(S_\delta^m)$ *then* $d(M) \geqq \frac{1}{2}\frac{\pi}{\sqrt{\delta}}$ [10].

In the last case either $d(M) > (1/2)\,\pi/\sqrt{\delta}$ then M is homeomorphic to $S^m$ by Grove-Shiohama [7], or $d(M) = (1/2)\,\pi/\sqrt{\delta}$, hence $\mathrm{vol}(M) = (1/2)\,\mathrm{vol}(S^m)$, then M is isometric to the projective space P $\mathbf{R}^m$, *see* [10] or our equality discussion.

2.3.2. COROLLARY (Improvement of Cheeger's inequality). — *If* N *is a closed geodesic, then*

$$\mathrm{length}(N) \geqq 2\pi.\frac{\mathrm{vol}(M^m)}{\mathrm{vol}(S_1^m)}.s_\delta\left(\min\left(d(M), \frac{\pi}{2\sqrt{\delta}}\right)\right)^{-m+1}$$

$$(\textit{if } \delta > 0{:}) \geqq \frac{2\pi}{\sqrt{\delta}}\frac{\mathrm{vol}(M^m)}{\mathrm{vol}(S_\delta^m)} \quad (\textit{sharp for } M = S_\delta^m).$$

2.3.3. *Remark*. — If $\pi.(\max K)^{-1/2}$ is not a lower bound for the cut locus distance (e.g. in case $\max K \leqq 0$) then there exists a closed geodesic whose length is twice the minimal cut locus distance. Therefore 2.3.2 gives a lower bound for the cut locus distance of a compact riemannian manifold. For example for compact oriented surfaces of genus $g$ with curvature between $-1$ and 0 one has from 2.3.2

$$(\text{cut locus distance}) \geqq \frac{\pi.(g-1)}{\sinh d(M)}.$$

*Proofs*. — 2.3 is immediate from 2.1 and the definition of $f_\delta$ and can of course be generalized to submanifolds with $|\eta| = \text{Const}$. For 2.3.2 recall

$$(m-1).\mathrm{vol}(S_1^m) = 2\pi.\mathrm{vol}(S_1^{m-2}).$$

To obtain 2.2 first replace in the fibre integral $\min(d(M), z(\eta, \xi))$ by $z(\eta, \xi)$ since the integrand of $f_\delta$ remains positive up to there. Then observe that *each* fibre integral can be recognized as the fibre integral which one gets if one computes the volume of $S_\delta^m$ with the normal bundle of a small sphere $S_{\delta+H(p)^2}$ in $S_\delta^m$. Therefore the fibre integral at $p \in N$ equals

$$\mathrm{vol}(S_\delta^m)/\mathrm{vol}(S_{\delta+H^2(p)}^n), \text{ which proves 2.2.}$$

We can also deduce

2.4. Corollary (Fenchel-Willmore-Chen). — *Let* $N^n$ *be a compact submanifold of* $\mathbf{R}^m$ *with mean curvature normal of length* H, *then*

$$\operatorname{vol}(S_1^n) \leqq \int_N H^n \, d \operatorname{vol}_N.$$

*Proof.* — Consider $\mathbf{R}^m$ as the tangent space of a very large sphere $S_\delta^m$, map N by the exponential map into $S_\delta^m$, apply 2.2 with $M = S_\delta^m$ and take the limit $\delta \to 0$. This proof, however, will not give the equality discussion. The inequality can also be derived by comparing the volume of a ball of radius R in $\mathbf{R}^m$ with the volume of a tube of radius R around N in $\mathbf{R}^m$.

## 3. Estimates for the length and volume distortions of the normal exponential map

3.1. As before, let N be isometrically immersed in M and let $\pi : \nu(N) \to N$ be the normal bundle of this immersion. The aim of this paragraph is to estimate in terms of lower curvature bounds

$$|\det(d\exp_\nu)_\xi| = \frac{|d\exp_\nu u_1 \wedge \ldots \wedge d\exp_\nu u_m|}{|u_1 \wedge \ldots \wedge u_m|},$$

where $\{u_1, \ldots, u_m\}$ is a basis of the tangent space $T_\xi \nu(N)$. To do this we actually prove (with the same amount of work) a comparison Theorem where curvature inequalities for two manifolds are assumed and obtain our estimate by specializing one of the manifolds to be of constant curvature. Furthermore the same proof gives in certain cases a comparison Theorem for the distortion of $r$-dimensional volumes ($r < m$) by $\exp_\nu$, i. e. for quotients $|d\exp_\nu u_1 \wedge \ldots \wedge d\exp_\nu u_r| / |u_1 \wedge \ldots \wedge u_r|$. In fact our Theorem 3.2 contains as special cases the Rauch and Berger comparison theorems ($r = 1$) as well as the top-dimensional volume estimates ($r = m$) which we used in paragraph 2.

First we describe the canonical metric for $\nu(N)$ and the differential of the normal exponential map in terms of N-Jacobi fields. This information has to be used again in paragraph 5.

3.1.1. Induced from the metric of M we have the normal connection $D^\perp$ for $\nu(N)$. We use it to split the tangent bundle as a sum of the "vertical" and the "horizontal" bundle, $T\nu(N) = \mathscr{V} + \mathscr{H}$: the vertical tangent vectors are tangent to the fibres (and killed by $\pi_*$), the horizontal tangent vectors are tangent to curves in $\nu(N)$ which—considered as vector-fields along their base curves—are $D^\perp$-parallel. Since each vertical tangent space $\mathscr{V}_\xi$ can canonically and hence isometrically be identified with the fibre $\nu_{\pi(\xi)}(N)$ by parallel translation in the fibre and since each horizontal tangent space $\mathscr{H}_\xi$ can canonically and hence isometrically be identified with $T_{\pi(\xi)} N$ via $\pi_*$ we have the canonical metric for $\nu(N)$ defined by

$$\|u\|^2 = \|\pi_* u\|^2 + \|u_{\text{ver}}\|^2, \qquad u \in T_\xi \nu(N).$$

Then, clearly, the above splitting is orthogonal, $\pi : \nu(N) \to N$ is a riemannian submersion and, if we represent $u$ as $u = (d/dt)\,\xi(t)\,|_0$, then, $\| u_{\mathrm{ver}} \| = \| (D^{\perp}/dt)\,\xi \|$. This splitting fits also nicely with the description of $d\exp_\nu$ using N-Jacobi fields:

3.1.2. We call a Jacobi field Y along a normal geodesic $s \to \exp_\nu s.\xi$, $\xi \in \nu_p(N)$, an *N-Jacobi field*, if it comes from a variation of geodesics normal to N, i. e.:

$$Y(s) = \frac{d}{dt} \exp_\nu s.\xi(t)\,|_{t=0}.$$

Directly from this definition it is clear that N-Jacobi fields describe the differential of $\exp_\nu$: If $u \in T_\xi\,\nu(N)$ is represented as $(d/dt)\,\xi(t)\,|_{t=0}$, then

$$d\exp_\nu . u = Y(1).$$

Two facts are important:

(i) N-Jacobi fields can also be characterized by their initial conditions:

Y is an N-Jacobi field if and only if

$$Y(0) \in T_p N, \quad \frac{D}{ds} Y(0) - S_\xi . Y(0) \perp T_p N,$$

where $S_\xi$ is the Weingarten map of N (for the normal $\xi$). Proof:

$$Y(0) = (\pi \circ \xi)^{\cdot}(0), \; \frac{D}{ds} Y(0) = \frac{D}{dt} \xi(0) = \frac{D^{\perp}}{dt} \xi(0) + S_\xi . (\pi \circ \xi)^{\cdot}(0).$$

(ii) The tangent vector $u \in T_\xi\,\nu(N)$ determines the "linear" vectorfield U along $s \to s.\xi(0)$ given by

$$U(s) = \frac{d}{dt} s.\xi(t)\,|_{t=0} = A(s) + s.B(s), \qquad U(1) = u,$$

where A is horizontal and B is vertical. And $d\exp_\nu$ maps this linear vectorfield onto the N-Jacobi field $Y(s) = d\exp_\nu . U(s)$. We call Y *the N-Jacobi field associated with u*. Note

$$\pi_* A(s) = Y(0), \qquad |B(s)| = \left| \left( \frac{D}{ds} Y(0) \right)^{\perp} \right| \qquad \text{and} \qquad |u|^2 = |A(0)|^2 + |B(0)|^2.$$

Now, for $r$ linearly independent of these "linear" vectorfields, say $U_1, \ldots, U_r$, it is easy to compute $|U_1(s) \wedge \ldots \wedge U_r(s)|$. On the other hand, the Jacobi equation controls the corresponding $Y_i(s)$ and in this way we obtain our information about length and volume distortions of $d\exp_\nu$. We will only consider Jacobi fields which are orthogonal to a normal geodesic because $\exp_\nu$ is a radial isometry (Gauss-Lemma).

As a last preliminary, we give two useful specializations.

3.1.3. The Jacobi equation along a geodesic depends only on the sectional curvatures of planes containing $c'$. In particular, if these sectional curvatures have the constant value $\delta$ then the Jacobi equation reduces already to $(D^2/ds^2)\,Y+\delta.Y = 0$ and the solutions are in terms of parallel translation $P_s$ along $c$:

$$Y(s) = P_s.\left(\mathbf{c}_\delta(s).Y(0)+\mathbf{s}_\delta(s).\frac{D}{ds}Y(0)\right),$$

*see* 2.0.2 for definition of $\mathbf{c}_\delta$ and $\mathbf{s}_\delta$.

From this explicit solution of the Jacobi equation it is clear that the riemannian metric of M is uniquely determined by the canonical metric on $\nu(N)$, the constant $\delta$ and the second fundamental tensor of N in M. A detailed but somewhat lengthy formulation is given in Lemma 5.6.

3.1.4. If in addition N is totally umbilic with mean curvature normal $\eta$

$$(\text{i.e.}\ \langle \eta, \xi \rangle.\langle X, Y \rangle = \langle S_\xi X, Y \rangle)$$

then N-Jacobi fields along $c$ are given as

$$Y(s) = P_s((\mathbf{c}_\delta(s)-\langle \eta, \xi \rangle.\mathbf{s}_\delta(s)).Y(0)+\mathbf{s}_\delta(s).E),$$

where $\xi = c'/|c'|$ and $E \perp N_p$ is the normal component of $(D/ds)\,Y(0)$.

This equation contains precise information about the focal points of N; for example if $\xi \in \nu(N)$ is a focal point, then $E = 0$ hence $\ker(d\exp_\nu|_\xi) = \mathscr{H}_\xi$.

3.2. THE COMPARAISON THEOREM.

NOTATIONS. — Let N be isometrically immersed in M, $\xi \in \nu_p N$ a unit vector $Y_i(s) = d\exp_\nu U_i(s)$ $(i = 1, \ldots, r)$ be linearly independent N-Jacobi fields along $c(s) = \exp s.\xi$ (*see* 3.1.2); let $k(s)$ be the minimum of the sectional curvatures of planes containing $c'(s)$ and $K(s)$ be the maximum; let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the Weingarten map $S_\xi$. Consider a second such situation $\overline{N} \subset \overline{M}$ etc.

ASSUMPTIONS. — Let $s_0$ be not larger than the *first focal distance of N in direction* $\xi$ and make one of the following assumptions (*a*), (*b*), (*c*) or (*d*) for $0 \leqq s \leqq s_0$:

(*a*) $k(s) \geqq \overline{K}(s)$, $2 \leqq r+1 \leqq \dim \overline{M} \leqq \dim M$, $\dim N = \dim \overline{N} = 0$;

(*b*) $k(s) \geqq \overline{K}(s)$, $2 \leqq r+1 \leqq \dim \overline{M} \leqq \dim M$;

$$\operatorname{codim} N = \operatorname{codim} \overline{N} = 1,\ \max \lambda_i \leqq \min \overline{\lambda}_i.$$

(*c*) $k(s) \geqq \overline{K}(s)$, $r+1 = \dim M = \dim \overline{M}$, $\dim N = \dim \overline{N}$, $\lambda_i \leqq \overline{\lambda}_i$ for some ordering of these eigenvalues;

(*d*) $\operatorname{Ric}(c', c') \geqq (m-1).\delta$, $\overline{M}$ of constant curvature $\delta$, $r+1 = \dim M = \overline{M}$ and either $\dim N = \dim \overline{N} = 0$ or $= m-1$ and $\overline{N}$ totally umbilic and $\operatorname{tr} S_\xi \leqq \operatorname{tr} \overline{S}_\xi$.

Then we have the following comparison between $r$-dimensional volume distortions under $d\exp_\nu$ for $0 \leqq s \leqq s_0$:

3.2.1. *Main inequality:*

$$\frac{|Y_1(s)\wedge\ldots\wedge Y_r(s)|}{|U_1(s)\wedge\ldots\wedge U_r(s)|} \leqq \frac{|\overline{Y}_1(s)\wedge\ldots\wedge \overline{Y}_r(s)|}{|\overline{U}_1(s)\wedge\ldots\wedge \overline{U}_r(s)|}.$$

3.2.2. The proof will give necessary and sufficient conditions for *equality* to hold. We are interested only in the following special cases:

If $\bar{k}(s) = \bar{K}(s) = \delta$ for $s \leqq s_0$ then equality in (*c*) is equivalent to $k(s) = K(s) = \delta$ and $\lambda_i = \bar{\lambda}_i$ $(i = 1, \ldots, n)$.

Equality in (*d*) is equivalent to $k(s) = K(s) = \delta$ for $s \leqq s_0$, $\operatorname{tr} S_\xi = \operatorname{tr} \bar{S}_\xi$ and N umbilic for the normal $\xi$.

3.2.3. *Note that:*

$$|U_1(s)\wedge\ldots\wedge U_r(s)| = \begin{cases} s^r.|Y_1'(0)\wedge\ldots\wedge Y_r'(0)| & \text{in case } (a), \\ |Y_1(0)\wedge\ldots\wedge Y_r(0)| & \text{in case } (b), \\ s^{\operatorname{codim} N - 1}.\text{Const.} & \text{in cases } (c),\ (d). \end{cases}$$

Therefore (*a*) resp. (*b*) are generalizations of the Rauch resp. Berger estimate. Note also that (*c*) and (*d*) give comparisons between the focal distances of N and $\bar{N}$; under our dimension assumptions they are stronger than that in Warner ([13]), Th. 3). Finally, the restriction in (*c*) to the top-dimensional volume $(r = m-1)$ is essential, since an example of Warner ([13], p. 353) shows that a length comparison is *not* true up to the first focal point of N in general.

3.3. The Corollaries which were needed in paragraph 2 are obtained if $\bar{M}$ has constant sectional curvature $\delta$:

3.3.1. Corollary [to 3.2 (*c*)]. — *With the notation in* 3.2 *assume* $k(s) \geqq \delta$ *for s not larger than the first focal distance of* N *in direction* $\xi$ $(|\xi| = 1)$, *then*

$$|\det(d\exp_\nu)_{s.\xi}|.s^{m-n-1} \leqq \prod_{i=1}^{n}(\mathbf{c}_\delta(s)+\lambda_i.\mathbf{s}_\delta(s)).\mathbf{s}_\delta^{m-n-1}(s)$$
$$\leqq (\mathbf{c}_\delta(s)-\langle\eta,\xi\rangle.\mathbf{s}_\delta(s))^n.\mathbf{s}_\delta(s)^{m-n-1},$$

*where* $\eta$ *is the mean curvature normal of* N.

Equality in the last inequality is equivalent to

$$k(s) = K(s) = \delta \qquad \text{and} \qquad \lambda_i = -\langle\eta,\xi\rangle \quad (i = 1, \ldots, n).$$

Also, the first zero $z(\eta, \xi)$ of the last estimate is an upper bound for the first *focal distance* of N in direction $\xi$.

*Proof.* — Apply 3.2 (*c*) to N and a local submanifold $\bar{N}$ in a space of constant curvature $\delta$ which has at one point the same second fundamental tensor as N at $\pi(\xi)$. The second inequality then uses the geometric-arithmetic-mean inequality.

3.3.2. Corollary [explicit reformulation of 3.2 (*d*)]. — *For s not larger than the first focal distance of* N *in direction* $\xi$ $(|\xi| = 1)$ *holds*

$$|\det(d\exp_\nu)_{s.\xi}|.s^{m-n-1} \leqq (\mathbf{c}_\delta(s)-\langle\eta,\xi\rangle\mathbf{s}_\delta(s))^n.\mathbf{s}_\delta(s)^{m-n-1}.$$

*Again, the first zero* $z(\eta, \xi)$ *of the right hand function is an upper bound for the first focal distance* ($n = m-1$) *resp. conjugate distance* ($n = 0$) *of* N *in direction* $\xi$ (*with* $\delta \leqq \operatorname{Ric}(c', c')/m-1$). *Equality as in* 3.2.2.

3.4. *Proof.* — The arguments generalize Bishop's volume estimate and the Jacobi field estimates of Rauch and Berger.

3.4.1. Put:

$$f(s) = \frac{|Y_1(s)\wedge \ldots \wedge Y_r(s)|}{|U_1(s)\wedge \ldots \wedge U_r(s)|},$$

$$\bar{f}(s) = \frac{|\bar{Y}_1(s)\wedge \ldots \wedge \bar{Y}_r(s)|}{|\bar{U}_1(s)\wedge \ldots \wedge \bar{U}_r(s)|}.$$

Since $\lim_{s\to 0} f(s) = \lim_{s\to 0} \bar{f}(s) = 1$ it suffices to prove $(\log f)' \leqq (\log \bar{f})'$. Because of the mentioned curvature-independent behaviour of $|U_1 \wedge \ldots \wedge U_r|$ in (3.2.3) it suffices to prove

$$(\star) \qquad (\log |Y_1(s)\wedge \ldots \wedge Y_r(s)|)' \leqq (\log |\bar{Y}_1(s)\wedge \ldots \wedge \bar{Y}_r(s)|)',$$

for $s$ smaller than the first focal distances of N and $\bar{\text{N}}$; now $f(s) \leqq \bar{f}(s)$ as long as $(\star)$ holds and, consequently, the first focal point of $\bar{\text{N}}$ does not come earlier than that of N.

3.4.2. We fix $s_1$ (smaller than those focal distances along the geodesics $c$, $\bar{c}$ under consideration) and, after taking linear combinations with constant coefficients we assume that $\bar{Y}_1(s_1), \ldots, \bar{Y}_r(s_1)$ are orthonormal. Similarly $Y_1(s_1), \ldots, Y_r(s_1)$ can be replaced by $r$ linearly independent combinations without changing the logarithmic derivative, a freedom needed at the end.

3.4.3. We denote by $I_{s_1}$ the indexform of $c/[0, s_1]$ with respect to N. Its fundamental property is that, before the first focal point (!) one has:

If Y is an N-Jacobi field and X is any $C^1$-Vector field along $c$ with $X(s_1) = Y(s_1)$ and $X(0) \in N_{c(0)}$ then $I_{s_1}(Y, Y) \leqq I_{s_1}(X, X)$; equality if and only if $X = Y$.

3.4.4. For N-Jacobi fields and likewise for $\bar{\text{N}}$-Jacobi fields one has

$$\langle Y, Y' \rangle (0) = \langle Y, S_{c'(0)}.Y \rangle (0),$$

therefore

$$\begin{aligned} &\log(|Y_1 \wedge \ldots \wedge Y_r|)'(s_1) \\ &= \sum_{i=1}^{r} \langle Y_i, Y_i' \rangle (s_1) \quad [\{Y_i(s_1)\} \text{ orthonormal}] \\ &= \sum_{i=1}^{r} (\langle Y_i, S_{c'} Y_i \rangle (0) + \int_0^{s_1} (\langle Y_i', Y_i' \rangle - \langle R(Y_i, c')c', Y_i \rangle ds \\ &= \sum_{i=1}^{r} I_{s_1}(Y_i, Y_i). \end{aligned}$$

So, we have to compare the indexforms on M and $\bar{\text{M}}$ to prove $(\star)$.

3.4.5. Choose a linear isometric injection

$$\iota_{s_1}: \ \overline{M}_{\bar{p}} \to M_p \quad (p = c(0), \ \bar{p} = \bar{c}(0)) \text{ which satisfies,}$$

$$\iota_{s_1}(\bar{c}'(0)) = c'(0),$$

$$\iota_{s_1}(\overline{N}_{\bar{p}}) \subset N_p \quad [\text{no condition in } (a), (b), (d)],$$

$$\iota_{s_1}(\overline{V}_{s_1}) = V_{s_1} \quad [\text{no condition in } (c), (d)],$$

where $V_{s_1}$ (resp. $\overline{V}_{s_1}$) is that $r$-dimensional linear subspace of $M_p$ (resp. $\overline{M}_{\bar{p}}$) which is obtained by parallel translation along $c$ to $c(0)$ [along $\bar{c}$ to $\bar{c}(0)$] of the span of the $Y_i(s_1)$ [of the $\overline{Y}_i(s_1)$].

Clearly, *in case* (*c*) one can in addition assume that $\iota_{s_1}$ maps, independently of $s_1$, eigenspaces of $S_{\bar{c}'}$ onto eigenspaces of $S_{c'}$ in such a way that $\bar{\lambda}_j \geqq \lambda_j$ ($j = 1, \ldots, n$), hence for all $\overline{X} \in \overline{N}_{\bar{p}}$ $\langle \overline{SX}, \overline{X} \rangle \geqq \langle S\,\iota_{s_1}\overline{X}, \iota_{s_1}\overline{X} \rangle$. If we want to prove the inequalities in 3.3 using only $|\eta| = |\bar{\eta}|$, then we note that there exist (locally) submanifolds $\overline{N}$ in spaces of constant curvature, which have on *one* of their tangent spaces $T_{\bar{p}}\overline{N}$ any prescribed second fundamental tensor, in particular the same as N at $p$. To eliminate the unknown eigenvalues, use $\det(A) \leqq (1/n \operatorname{trace} A)^n$ (if $A \geqq 0$).

3.4.6. After the choices in 3.4.2 and 3.4.5 have been made, define vectorfields $W_i$ along $c$ as follows

$$W_i(s) := P_s \circ \iota_{s_1} \circ \overline{P}_{-s}.\overline{Y}_i(s) \qquad (0 \leqq s \leqq s_1),$$

where $P_s$ and $\overline{P}_{-s}$ denote parallel translation along $c$ (from 0 to $s$) and along $\bar{c}$ (from $s$ to 0). Obviously we have

$$|W_i(s)| = |\overline{Y}_i(s)|, \qquad |W_i'(s)| = |\overline{Y}_i'(s)|.$$

We now assume (*see* 3.4.2), by taking a suitable linear combination, that

$$Y_i(s_1) = W_i(s_1) \qquad (i = 1, \ldots, r),$$

since span $\{W_i(s_1)\}$ = span $\{Y_i(s_1)\}$, as follows from $\iota_{s_1}(\overline{V}_{s_1}) = V_{s_1}$. Finally we get

$$W_i(0) \in \iota_{s_1}(\overline{N}_{\bar{p}}) \subset N_p \qquad \text{from} \quad \overline{Y}_i(0) \in \overline{N}_{\bar{p}}.$$

Therefore we have from 3.4.4 and 3.4.3:

$$(\log|Y_1 \wedge \ldots \wedge Y_r|)'(s_1) = \sum_{i=1}^{r} I_{s_1}(Y_i, Y_i) \leqq \sum_{i=1}^{r} I_{s_1}(W_i, W_i).$$

It remains to show

$$\sum_{i=1}^{r} I_{s_1}(W_i, W_i) \leqq \sum_{i=1}^{r} \overline{I}_{s_1}(\overline{Y}_i, \overline{Y}_i).$$

3.4.7. The curvature assumptions and the definition of $W_i$ imply in cases (*a*), (*b*), (*c*) immediately:

$$\int_0^{s_1}(|W_i'|^2-K(c'\wedge W_i).|c'|^2.|W_i|^2)\,ds$$
$$\leqq \int_0^{s_1}(|\overline{Y}_i'|^2-\overline{K}(\overline{c}'\wedge\overline{Y}_i).|\overline{c}'|^2.|\overline{Y}_i|^2)\,ds.$$

In case (*d*) the constant curvature of $\overline{M}$ implies that the $\overline{Y}_i$ and hence the $W_i$ are, up to a *common* factor, orthonormal; therefore, after summation the Ricci curvature assumptions on M imply the desired inequality. This proves case (*a*); (*b*) and (*c*) follow, since the eigenvalue assumptions imply $\langle SW_i, W_i\rangle(0) \leqq \langle \overline{SY_i}, \overline{Y}_i\rangle(0)$; for (*d*) we need again that the $\{\overline{Y}_i\}$ and the $\{W_i\}$ are, up to a common factor, orthonormal, so that the trace assumption suffices.

3.4.8. *Equality discussion.* — If equality holds, then $I_{s_1}(Y_i, Y_i) = I_{s_1}(W_i, W_i)$ implies that all the $W_i$ are Jacobi fields; this forces corresponding sectional curvatures of M and $\overline{M}$ along $c$ and $\overline{c}$ to be equal, which we will use in paragraph 4 if $\overline{M}$ has constant curvature. Furthermore, $I_{s_1}(W_i, W_i) = \overline{I}_{s_1}(\overline{Y}_i, \overline{Y}_i)$ [in cases (*b*) and (*c*)] implies equality of corresponding eigenvalues of $S_v$ and $S_{\overline{v}}$. If the inequality is in case (*c*) simplified to contain only the trace (3.3.1), then equality in $\det A = (1/n \text{ trace } A)^n$ forces all eigenvalues to be equal, i. e. N must be totally umbilic at $p$ in M. Since equality in (*d*) forced all the sectional curvatures of M of planes containing $c'$ to be $= \delta$, we can now apply part (*c*) of the theorem so that equality in (*d*) also forces N to be totally umbilic at $p$ in M.

*Remark.* — Warner [13] proved a length comparison theorem for *N-Jacobi fields* which holds at least up to the first focal point of a suitably choosen hypersurface $\tilde{N}$ in M but which does in general *not* hold up to the first focal point of N, example 13, p. 353. We give an example of a Jacobi field on a homogeneous space for which one does not have a length comparison on any interval $[0, \varepsilon]$:

The space is $S^3$ with a left invariant metric (a general Berger sphere) which is obtained by changing the biinvariant metric $\langle\ ,\ \rangle$ with a left invariant endomorphism field A to $g(\ ,\ ) = \langle A\ ,\ \rangle$. We denote the eigenvalues by $a > b\,(= 1) > c > 0$ and the corresponding left invariant eigenfields by $E_1$, $E_2$, $E_3$. Then the following is true [16]: The integral curves of $E_2$ are unit speed closed geodesics along which one has the following exponentially growing Jacobi field

$$Z(t) = C.\sinh\alpha t.E_1(t) + \cosh\alpha t.E_3(t),$$

where

$$\alpha^2 = \frac{(a-b)(b-c)}{ac} \quad \text{and} \quad C = \alpha.\frac{c}{b-c}.$$

The sectional curvature

$$K(E_2, E_3) = \frac{2b+2c-3a}{4} + \frac{(b-c)^2}{4a} = \delta$$

is always the *minimum* of the sectional curvatures of the metric $g$ and $\delta > 0$ for example if $3a \leqq 2b+2c$.

On the sphere of constant curvature $\delta$ choose a unit speed geodesic; normal Jacobi fields along it are in terms of parallel fields V, W given as

$$Y(t) = \cos\sqrt{\delta}\, t.V(t) + \frac{1}{\sqrt{\delta}} \sin\sqrt{\delta}\, t.W(t).$$

Choose the initial data to obtain a comparison situation

$$|Y(0)| = |Z(0)|, \qquad \langle Y, Y' \rangle (0) = 0 = \langle Z, Z' \rangle (0), \qquad |Y'(0)| = |Z'(0)|.$$

Then one might expect a comparison $|Z(t)| \leqq |Y(t)|$ at least on some interval. However one checks easily for all $t \neq 0 : |Z(t)| > |Y(t)|$! (Differentiate $f(t) = |Z(t)|^2 - |Y(t)|^2$ twice, observe $f(0) = f'(0) = 0$ because of the initial data and $f''(t) = C_1.\cosh 2\alpha t - C_2.\cos 2\sqrt{\delta}\, t$ with $f''(0) = C_1 - C_2 = 0$ since $|Y'(0)| = |Z'(0)|$ and with $K(E_2, E_3) = \delta$ also $\langle Y, Y'' \rangle (0) = \langle Z, Z'' \rangle (0)$.)

Our proof of Theorem 3.2 when applied to a single Jacobi field (case $r = 1$) leaves room for improvements. For example one does not need $\max \lambda_i \leqq \min \overline{\lambda}_i$ but only $\langle \overline{Y}, \overline{Y}' \rangle (0) \rangle \geqq \max \lambda_i . |\overline{Y}(0)|^2$.

Also the hypersurface N perpendicular to the geodesic $c$ can be choosen so as to have a suitable Weingarten map A; the only condition for the symmetric endomorphism $A : c'(0)^{\perp} \to c'(0)^{\perp}$ with eigenvalues $\lambda_i$ are:

$$A.Y(0) = Y'(0) \qquad \text{and} \qquad \langle \overline{Y}, \overline{Y}' \rangle (0) \rangle \geqq \max \lambda_i . |\overline{Y}(0)|^2.$$

Then $|\overline{Y}(t)| \geqq |Y(t)|$ up to the first focal point of a hypersurface N with Weingarten map A at $c(0)$. Such an A does not exist in our example, but one obtains sharp estimates if $Y(0)$ and $Y'(0)$ are linearly dependent.

## 4. Equality Discussion

If we have equality in 2.1 or 2.2 then all the upper estimates which were used in the derivation must be sharp. Therefore we have parts (i) and (ii) of the following proposition immediately from 3.4.8 and (iii) from the definition of $f_\delta$ in 2.0.2.

4.1. PROPOSITION. — *Let* N *be isometrically immersed in* M *and make the curvature assumptions* (K) *of paragraph* 2. *Then equality in* 2.1 *is equivalent to the following set of statements:*

(i) *all planes of* M *containing a tangent vector to a geodesic segment which minimizes the distance to* N *have the same sectional curvature* $\delta$ ;

(ii) N *is totally umbilic in* M ($\alpha(X, Y) = \eta . \langle X, Y \rangle$);

(iii) *the normal geodesics minimize the distance to* N *up to the minimum between* $d$(M) *and the first focal distance. In particular the immersion of* N *in* M *is an embedding and* N *is connected. Equality in* 2.2 *equivalent to* (i), (ii) *and*

(iii′) *the normal geodesics minimize the distance to* N *up to their first focal point.*

With 4.1, 3.1.4 and 4.4 we will prove the following theorem which gives the first part of the equality discussion.

4.2. THEOREM. — (i) *if* $\delta \leqq 0$, *then equality does not occur in* 2.1;

(ii) *if* $\delta > 0$, *then equality in* 2.1 *implies equality in* 2.2 *with the only exception where* N *is a point in real projective space* (*there* 2.1 *gives equality*, 2.2 *does not*).

4.4. LEMMA. — *Assume for* $N \subset M$ *property* 4.1 (iii) *and* dim $N > 0$. *Then, a geodesic which minimizes the distance from* N *up to* $d$(M) *has focal endpoints.*

*Proof:*

4.4.1. Let $c$ be a geodesic which minimizes the distance from N up to length $d$(M), let $p$ be its endpoint. Then we have or every $n \in N$:

$$d(p, n) \leqq d(M) \quad \text{(definition of diameter)},$$
$$d(p, n) \geqq d(p, N) = d(M) \quad \text{(choice of } p\text{)}.$$

In addition, every minimizing geodesic from $p$ to N is also perpendicular to N, since otherwise there would be shorter curves.

4.4.2. Assume, that $p$ is *not* a focal point. By continuity all normals close to $c'(0)$ also have their first focal distance larger than $d$(M), hence [with 4.1 (iii)!] everything in 4.4.1 holds for them. Since their endpoints are not focal we obtain a piece of hypersurface P such that:

(i) all points of P have distance $d$(M) from N, hence all minimizing goedesics are perpendicular both to N and P;

(ii) each point of P has minimal geodesic connections to all points of N.

As soon as N has more than two points this is a contradiction.

*Proof of* 4.2. — If N is a point in (i) or (ii), then M is of constant curvature $\delta$ and a ball of radius $d$(M) in the universal covering $\tilde{M}$ would be a fundamental domain for M, but this is impossible if $\delta \leqq 0$ since the ball is strictly convex and similarly if $\delta > 0$ and $d(M) \neq \pi/2\sqrt{\delta}$. In the exceptional case, obviously, $M = RP^m$.

If N is not a point, then (ii) follows immediately from 4.4, compare 4.1 (iii) to (iii′). If $\delta \leqq 0$ and $N \subset M$ satisfies 4.1 (i), (ii) then it is clear from 3.1.4 that for at least one of the normals $\xi$, $-\xi$ there does not exist a focal point. Then 4.1 (iii) and 4.4 give a contradiction.

The second part of the equality discussion (i. e. of 2.2) rests on the following theorem which will be proved in paragraph 5.

4.5. THEOREM. — *Let* N *and* M *be compact riemannian manifolds,* N *isometrically immersed and totally umbilic in* M. *Assume that all planes of* M *containing a tangent vector to a geodesic segment which is normal to* N *and has no focal points have the same sectional curvature* $\delta > 0$.

*Then* M *has constant curvature* $\delta$ *and, of course, the totally umbilic submanifolds are well known.*

4.5.1. *Remark.* — The preceeding result is trivial if N is a point. locally true if N is a curve (5.7) but *locally false* if $\dim N \geqq 2$: for *any* compact manifold N choose a $\cos(\sqrt{\delta}\, t)$-warped metric on the product $N \times (-\pi/2\sqrt{\delta}, \pi/2\sqrt{\delta})$, then all assumptions except compactness are satisfied. If $\delta \leqq 0$ then these warped product metrics can even be made complete, for example product metrics on $N \times \mathbf{R}^k$ if $\delta = 0$.

4.6. THEOREM (Equality discussion of 2.2). — (i) *if* $\eta \not\equiv 0$, $n \neq 0$, *then equality in* 2.2 *is equivalent to:* $M = S^m_\delta$, $|\eta| =$ Const. *and* $N = S^n_{\delta+|\eta|^2}$ *with standard embedding in* $S^m_\delta$;

(ii) *if* $\eta = 0$, *then equality in* 2.2 *is equivalent to:* $(M, N) = (S^m_\delta, S^n_\delta)/\Gamma$, *where* $\Gamma$ *acts reducibly on* $\mathbf{R}^{m+1}$ [*leaving span* $(S^n_\delta)$ *invariant*]. *All space forms* $S^n|\Gamma$ *can occur for* N. *If* $\Gamma$ *is not trivial, then* $d(M) = \pi/2\sqrt{\delta}$. *If* $n = 0$, *then* $\Gamma$ *is trivial.*

*Proof:*

4.6.1. Equality in 2.2 implies (because of 4.5) constant curvature for M, hence $M = S^m_\delta$ or $M = S^m_n/\Gamma$. We first treat the simply connected case and come to non-simply connected M in 4.6.4.

4.6.2. The case $\dim N = 1$ must be handled seperately, since 4.1 (ii) is void for $n \leqq 1$. We consider M as sphere in $\mathbf{R}^{m+1}$. Then $(\delta + H^2(p))^{1/2}$ is the curvature $\varkappa$ of N as a curve in $\mathbf{R}^{m+1}$, so that 2.2 reduces to the Fenchel-Borsuk inequality:

$$2\pi \leqq \int_N |\varkappa|\, ds.$$

Equality in this Fenchel-Borsuk inequality holds if and only if N is a convex curve in some 2-dimensional plane. In our case N is also a spherical curve, hence a circle [in particular $H(p) =$ Const.].

4.6.3. If $\dim N > 1$ then, since N is totally umbilic by 4.1 (ii), we have the standard conclusion that $H =$ Const. (which follows also from 5.5) and that $N = S^n_{\delta+H^2}$ together with the standard embedding in $S^m_\delta$.

4.6.4. If equality in 2.2 holds for a non simply connected M then, obviously, we have equality also if we lift N to the universal covering $S^m_\delta$ of M. The lift $\tilde{N}$ is connected because of 4.1 (iii). If $\eta \neq 0$, then $\tilde{N}$ is a small sphere invariant under the deck group $\Gamma$ of M in $S^m_\delta$; but then the midpoint of $\tilde{N}$ is a fixed point of $\Gamma$, so the group must be trivial. If $\dim N = 0$, then 4.1 (iii) and this fixed point argument also forces $\Gamma$ to be trivial. If $\eta = 0$, then $\tilde{N}$ is a totally geodesics sphere in $S^m_\delta$ and invariant under $\Gamma$, so $\Gamma$ acts reducibly on $\mathbf{R}^{m+1}$. If $S^n/\Gamma$ is any space form, choose $m = 2n+1$ and take the representation corresponding to $S^n/\Gamma$ twice.

4.6.5. If a quotient of $S^m$ has diameter $> \pi/2$, then there exist points $p, q \in S^m$ such that the distance from $p$ to the orbit $\Gamma.q$ is $> \pi/2$, i. e. $\Gamma.q$ lies in a ball of radius $< \pi/2$. The midpoint of the smallest (convex!) ball containing $\Gamma.q$ is a fixed point of $\Gamma$, hence $\Gamma$ is trivial. On the other hand, if $\Gamma$ acts reducibly on $S^m$, then any pair of points from orthogonal invariant subspaces has a distance $\pi/2$ in $S^m$ and in $S^m/\Gamma$; for reducible actions we therefore have $d(S^m/\Gamma) = \pi/2$.

This completes the proof of Theorem 4.6.

## 5. Proof of Theorem 4.5

The heart of the proof consists of Lemmas 5.3 (iii), 5.4 and 5.5, which show (in case $\dim N > 1$) that $\eta$ is parallel, that $D^{\perp}$-parallel displacement in $\nu(N)$ is locally independent of the path and that N has constant curvature $\delta+|\eta|^2$. Note that these statements, which we will prove in this order, correspond to the Codazzi, Ricci and Gauss equations for "small" spheres $S^n_{\delta+|\eta|^2}$ in spheres $S^m_\delta$ of curvature $\delta$. Together with the more technical Lemma 5.6 they immediately will imply the theorem. If $\dim N \leqq 1$, the theorem is locally true and follows already from Lemma 5.6.

To prove that $\eta$ is parallel, we first extend a result of Warner ([12], Th. 3.2) (*see* also Whitehead [14]) to the case $\dim N > 0$.

Let N, M be riemannian manifolds, N isometrically immersed in M and $F \subset \nu(N)$ the set of first focal points of the normal exponential map. Assume that the multiplicity of the first focal points is a constant, say $k$, so that F is a hypersurface in $\nu(N)$.

5.1. Theorem. — *If* $k > 1$, *then*

$$\ker(d\exp_{\nu/\xi}) \subset T_\xi F \quad \text{for all } \xi \in F.$$

5.1.1. Corollary. — $\ker d\exp_\nu$ *is an integrable distribution in* TF.

*Proof of* 5.1. — For any map $f: N \to M$ between riemannian manifolds one has from the differential $df$ the tension field $D^2 f$, defined for local vector fields X, Y on N by

$$D^2 f(X, Y) = D^M_X(df\ Y) - df\ D^N_X Y,$$

where $D^N$ and $D^M$ are the covariant derivations in N and M, the one in M generalized to vector fields along maps into M. Working with symmetric connections, one has the usual symmetry of second derivatives

$$D^2 f(X, Y) = D^2 f(Y, X),$$

which will be essential later on, when applied to $f = \exp_\nu$.

Now, let $\xi$ be a focal point, $X \in T_\xi F$ and $Y \in \ker(d\exp_{\nu/\xi})$. Then

$$(\star) \qquad D^2 \exp_\nu(X, Y) \in \operatorname{image}(d\exp_\nu),$$

since X can be represented as the tangent vector of a curve $\xi(s)$ in F and Y can be extended to a vector field $Y(s)$ along this curve with $Y(s) \in \ker(d\exp_\nu)$. Since $k > 1$ and $T_\xi F$ is a hyperplane in $T_\xi \nu(N)$, we can find a non-zero $Z \in \ker(d\exp_{\nu/\xi}) \cap T_\xi F$. Let $Y \in \ker(d\exp_{\nu/\xi})$ be decomposed into $Y = V + X$, $X \in T_\xi F$ and V tangent to the geodesic $\gamma(t) = \exp t\xi$. We claim $V = 0$. Indeed, by the symmetry of $D^2 \exp_\nu$ and $(\star)$:

$$D^2 \exp_\nu(V, Z) = D^2 \exp_\nu(Z, Y) - D^2 \exp_\nu(X, Z) \in \operatorname{image}(d\exp_\nu).$$

Thus it follows from the definition of the tension field that also

$$D_V^M(d\exp_\nu Z(t)) \in \operatorname{image} d\exp_\nu,$$

where $Z(t)$ is any extension of Z along $\gamma(t)$. Choosing for $Z(t)$ the "linear" vectorfield such that $J(t) = d\exp_\nu Z(t)$ is the N-Jacobi field associated with Z (*see* 3.1.2) we get on one hand $J(1) = 0$ and on the other hand

$$|V| \cdot \frac{D}{dt} J(1) = D_V^M d\exp_\nu Z(t) \in \operatorname{image}(d\exp_\nu).$$

But this forces V to be zero by the next Lemma, finishing the proof of the Theorem.

5.2. LEMMA. — *Let* N *be isometrically immersed in* M *and* $\mathcal{J}$ *the set of all* N-*Jacobi fields along the geodesic* $\gamma(t) = \exp t\xi$, $\xi \in \nu(N)$. *Let* $t_0 \in \mathbf{R}$ *and put* $V_1 = \{ Y(t_0)/Y \in \mathcal{J} \}$ *and* $V_2 = \{ Y'(t_0)/Y \in \mathcal{J}$ *with* $Y(t_0) = 0 \}$.
*Then* $V_1$ *is orthogonal to* $V_2$ (*and clearly* $\dim V_1 + \dim V_2 = \dim M$).

*Proof.* — Since $\langle Y_1(t), Y_2'(t) \rangle - \langle Y_1'(t), Y_2(t) \rangle$ is constant for any Jacobi fields along $\gamma$, it is zero for N-Jacobi fields (the Weingarten map is symmetric). Thus $\langle Y_1(t_0), Y_2'(t_0) \rangle = 0$ if $Y_1, Y_2 \in \mathcal{J}$ and $Y_2(t_0) = 0$.

We now come to the proof of the Theorem. Let F denote as usual the set of first focal points, $D^\perp$ the induced connection in the normal bundle and $\eta$ the mean curvature vector field of the immersion.

5.3. LEMMA. — *If, in addition to the assumptions of Theorem* 4.5, $\dim N > 1$, *then:*

(i) $D^\perp$-*parallel curves in* F *are mapped under* $\exp_\nu$ *onto point curves;*

(ii) F *is invariant under* $D^\perp$-*parallel displacement along curves in* N;

(iii) $\eta$ *is* $D^\perp$-*parallel.*

*Proof.* — Note that by 3.1.4 $\ker(d\exp_\nu)$ consists, at a focal point, of the horizontal subspace. This proves (i) and shows also that Corollary 5.1.1 applies. Now, let $\xi : [0, 1] \to \nu(N)$ be a $D^\perp$-parallel curve along $c = \pi \circ \xi$ with $\xi(0) \in F$. Obviously the set of $t$ for which $\xi(t) \in F$ is closed. But it is also open. Indeed if $\xi(t_0) \in F$, for some $t_0 \in [0, 1]$ and $K \subset F$ is an integral manifold of $\ker(d\exp_\nu)$ through $\xi(t_0)$, then there exists a curve $\zeta(t)$ in K with $\zeta(t_0) = \xi(t_0)$ and $\pi \circ \zeta = \pi \circ \xi$ because $\pi : K \to N$ is a local diffeomorphism. But $\zeta(t)$ in K implies $(D^\perp/dt)\zeta = 0$ (3.1.4), hence $\zeta = \xi$ in a neighbourhood of $t_0$. Therefore $\xi(t) \in F$ in this neighbourhood and thus $\xi(t) \in F$ for all $t \in [0, 1]$. This proves (ii) and shows also that the N-Jacobi fields $Y_t$ associated with $\dot{\xi}(t)$

vanish at 1. From the explicit formula for N-Jacobi fields in 3.1.4 we conclude that $\langle \xi, \eta \circ c \rangle$ is constant and therefore from (ii) that $\langle \xi, \eta \circ c \rangle$ is constant for any $D^{\perp}$-parallel vector field $\xi$ along any curve $c$ in N. This finishes the proof of (iii).

5.4. LEMMA. — *If the assumptions of Theorem* 4.5 *hold, then* $D^{\perp}$*-parallel displacement in* $\nu$ (N) *is locally independent of the path.*

*Proof.* — We may assume dim N > 1. Let $p_0 \in N$ and $\xi \in F$ with $\pi\xi = p_0$. Let $c$ be a closed curve in N starting at $p_0$, which is homotopic to zero. Parallel displacement of $\xi$ with respect to $D^{\perp}$ around any closed curve of a fixed differentiable homotopy gives a curve $\xi(t)$ in $N^{\perp}_{p_0}$ with $\xi(1) = \xi$ [since $\xi(1)$ is obtained by parallel displacement around a point curve]. By 5.3 (i) $\exp \xi(t) = \exp \xi$ for all $t$. Since ker $(d \exp_\nu)$ is horizontal, $\xi(t)$ must be constant. Hence $\xi(t) \equiv \xi$ and the lemma is proved.

5.5. LEMMA. — *If in addition to the assumptions of Theorem* 4.5 dim N > 1, N *is of constant curvature* $\delta + |\eta|^2$.

*Proof.* — Let $p_0 \in N$. We will explicitly construct a homothety between a neighbourhood of $p_0$ with an open subset of an $n$-dimensional sphere in $M_{q_0}$, $q_0$ a focal point of N.

Put $N^* = \exp_\nu F$. It follows from 5.3 (ii) that $N^* = \exp_\nu (F \cap N^{\perp}_{p_0})$ and from 3.1.4 that $\exp_\nu : F \cap N^{\perp}_{p_0} \to M$ is an immersion. Furthermore, if we put $N^*_\xi = d\exp_\nu (T_\xi F)$ for any $\xi \in F$, then 5.3 shows that $N^*_\xi$ remains unchanged if we $D^{\perp}$-parallel translate $\xi$ along curves in N.

If $\eta = 0$, $F \cap N_{p_0}$ is a sphere of radius $\pi/2\sqrt{\delta}$ around 0 as follows from 3.1.4. Therefore by the Gauss-Lemma we have $\gamma(1) \perp N^*_\xi$ for all geodesics $\gamma(t) = \exp t\xi$ with $\xi \in F \cap N^{\perp}_{p_0}$. If $\eta \neq 0$ only the geodesics in the direction of $\pm\eta$ hit $N^*$ orthogonally as follows again from 3.1.4 and the Gauss-Lemma.

Now, let $\xi_0 \in N^{\perp}_{p_0}$ be an arbitrary unit vector if $\eta = 0$ and $\xi_0 = \eta(p_0)/\|\eta(p_0)\|$ if $\eta \neq 0$. Extend $\xi_0$ to a $D^{\perp}$-parallel vector field $\xi$ in a neighbourhood U of $p_0$, which is possible by Lemma 5.4. Of course $\xi(p) = \eta(p)/\|\eta(p)\|$ if $\eta \neq 0$ by 5.3 (iii). Let $s_0 \in (0, \pi/\sqrt{\delta})$ be determined by $\sqrt{\delta} \cot \sqrt{\delta}\, s_0 = |\eta|$. Then $q_0 = \exp s_0 . \xi_0$ is the first focal point along $\exp t\xi$. Let $\gamma_p(t) = \exp t\xi(p)$. Then also $\gamma_p(s_0) = q_0$ by 5.3 (ii) and $\dot\gamma_p(s_0) \perp N^*_{s_0\xi_0}$ by the above discussion. Thus the map $\Phi$ from U into

$$S = \{ x \in M_{q_0} / \|x\| = 1,\ x \perp N^*_{s_0\xi_0} \},$$

the normal sphere of $N^*_{s_0\xi_0}$, given by $\Phi(p) = \dot\gamma_p(s_0)$ is a well defined differentiable map. Note that dim S = dim N. We claim that $\Phi$ is an isometry up to a constant factor. Indeed, if $c(t)$ is a curve in U and $Y_t(s)$ the N-Jacobi field along $\gamma_{c(t)}(s)$ associated with $\dot\xi \circ c(t)$, i.e. $Y_t(s) = (\partial\alpha/\partial t)(s, t)$, $\alpha(s, t) = \exp s . \xi(c(t))$, then $\dot{\widehat{\Phi \circ c}}(t) = (D/ds) Y_t(s_0)$. Since $Y_t(s_0) = 0$, 3.1.4 shows

$$Y_t(s) = \left( \cos \sqrt{\delta}\, s - |\eta| \frac{\sin \sqrt{\delta}\, s}{\sqrt{\delta}} \right) P_s(\dot c(t)),$$

so that

$$|\Phi_*(\dot{c}(t))| = \left|\frac{D}{ds}Y_t(s_0)\right| = \sin\sqrt{\delta}\,s_0.\left(\sqrt{\delta}+\frac{|\eta|^2}{\sqrt{\delta}}\right).|\dot{c}(t)| = \sqrt{\delta+|\eta|^2}.|\dot{c}(t)|.$$

This finishes the proof. Note also that $\Phi$ gives a global isometry if $\eta \neq 0$, an alternative proof of 4.6 (i).

5.6. LEMMA. — *Let* M, $\overline{M}$ *be complete riemannian manifolds and* $N \subset M$, $\overline{N} \subset \overline{M}$ *be embedded submanifolds with the induced metrics. Let* $\widetilde{\nu(N)}$ *and* $\widetilde{\nu(\overline{N})}$ *be open starshaped neighbourhoods of the zero sections in* $\nu(N)$ *and* $\nu(\overline{N})$, *respectively, on which the normal exponential maps are injective and of maximal rank.*

Let $\delta \in \mathbf{R}$ and assume that $K(\sigma) = \delta$ for all tangent planes $\sigma$ to M and $\overline{M}$ which contain a tangent vector of a geodesic $\exp tv$ with $t \in [0, 1]$ and $v \in \widetilde{\nu(N)}$ or $\widetilde{\nu(\overline{N})}$, respectively.

Let $\Psi : \nu(N) \to \nu(\overline{N})$ be a vector bundle isomorphism which maps $\widetilde{\nu(N)}$ into $\widetilde{\nu(\overline{N})}$. Denote by $\Phi$ the induced map between N and $\overline{N}$ and put $\tilde{\Psi} = \exp_{\nu(\overline{N})} \circ \Psi \circ (\exp_{\nu(N)/\widetilde{\nu(N)}})^{-1}$. Then:

I. $\Psi$ is an isometry if and only if:

(i) $\Phi$ is an isometry;

(ii) $|\Psi(v)| = v$ for all $v \in \nu(N)$;

(iii) $\Psi$ maps $D^\perp$-parallel vectorfields along curves in N onto $D^\perp$-parallel vectorfields.

II. $\tilde{\Psi}$ is an isometry if and only if $\Psi$ is an isometry and:

(iv) for all $p \in N$ and $v, w \in N_p$:

$$\bar{\alpha}(\Phi_* v, \Phi_* w) = \Psi(\alpha(v, w)),$$

where $\alpha$ and $\bar{\alpha}$ denote the second fundamental forms of N in M and $\overline{N}$ in $\overline{M}$, respectively.

*Remark.* — If N and $\overline{N}$ are both totally umbilic or 1-dimensional, (iv) is equivalent to:

(iv)′ $\Psi \circ \eta = \bar{\eta} \circ \Phi$ ($\eta$ and $\bar{\eta}$ the mean curvature vector fields).

*Proof.* — The first part follows directly from the definition of the riemannian metrics for the normal bundles and it is clear also that the conditions in the second part are necessary. That $\overline{\Psi}$ is an isometry if $\Psi$ is an isometry and (iv) holds is precisely the content of the discussion in 3.1.3.

5.7. END OF PROOF OF THEOREM 4.5. — The low dimensional cases follow quite trivial and don't even use $\delta > 0$. Indeed, if $\dim N = 0$ the last Lemma already shows that M except on a set of measure zero (the cut locus) has constant curvature $\delta$. Therefore M itself is of constant curvature $\delta$ by continuity of the curvature function. If $\dim N = 1$ and $p_0 \in N$, we choose locally an orthonormal $m$-frame $v_1, \ldots, v_m$ ($m = \dim M$) with $v_1 \in TN$ and $D^\perp v_i = 0$ for $i \geqq 2$. Put $w_{ik} = \langle (D/dt)\, v_i, v_k \rangle = -w_{ki}$ and integrate the corresponding Frenet differential equation in a space of constant curvature $\delta$ to obtain a curve $\overline{N}$ with orthonormal $m$-frame $\bar{v}_1, \ldots, \bar{v}_m$. Since $w_{ik} = 0$ for $i, k \geqq 2$ we get a bundle isometry between the normal bundles by identifying $v_i$ with $\bar{v}_i$, $i \geqq 2$. This bundle isometry also satisfies (iv)′ of Lemma 5.6 because $\langle (\mathrm{D}/dt)\, v_1, v_i \rangle = \langle (\mathrm{D}/dt)\, \bar{v}_1, \bar{v}_i \rangle$. Thus we may conclude as above that M is of constant curvature.

Now, let dim $\mathrm{N} > 1$. For any $p_0 \in \mathrm{N}$ there exists by Lemma 5.5 an open neighbourhood U and an isometry $\Phi : \mathrm{U} \to \overline{\mathrm{U}}$, $\overline{\mathrm{U}}$ an open subset of a "small" sphere $\overline{\mathrm{N}}$ of curvature $\delta + |\,\eta\,|^2$ in a sphere $\overline{\mathrm{M}}$ of curvature $\delta$. By Lemma 5.4 we may assume that $\nu(\mathrm{U})$ is parallizable with respect to $\mathrm{D}^{\perp}$. Choosing next a linear isometry between $\mathrm{N}^{\perp}_{p_0}$ and $\overline{\mathrm{N}}^{\perp}_{\Phi(p_0)}$ which maps $\eta\,(p_0)$ onto $\bar{\eta}(p_0)$ ($\bar{\eta}$ the mean curvature vector field of $\overline{\mathrm{N}}$ in $\overline{\mathrm{M}}$) and extending this by parallel translation, Lemma 5.6 can be applied, yielding constant curvature $\delta$ for all points in exp ((U)) within focal distance. Thus, by continuity as above, M has constant curvature $\delta$.

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