ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 10, nº 4 (1977), p. 547-558
<http://www.numdam.org/item?id=ASENS_1977_4_10_4_547_0>

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Ann. scient. Éc. Norm. Sup.
4 série, t. 10, 1977, p. 547 à 558.

AMPLITUDE INEQUALITIES FOR COMPLEXES

BY BIRGER IVERSEN

Throughout this paper we consider bounded complexes $X^{\bullet}$ of modules over a local noetherian ring A. The amplitude of $X^{\bullet}$ is defined by

$$\operatorname{amp} X^{\bullet} = \sup\{ i \mid H^i(X^{\bullet}) \neq 0\} - \inf\{ i \mid H^i(X^{\bullet}) \neq 0\}.$$

The main result is that for a bounded complex $L^{\bullet}$ of finitely generated free modules and a bounded complex $X^{\bullet}$ of finitely generated modules with $H^{\bullet}(L^{\bullet}) \neq 0$ and $H^{\bullet}(X^{\bullet}) \neq 0$,

$$\operatorname{amp} L^{\bullet} \otimes X^{\bullet} \geqq \operatorname{amp} X^{\bullet}$$

provided A is equicharacteristic.

The case where $L^{\bullet}$ is a resolution of a module P and $X^{\bullet}$ is a complex of the form $A \xrightarrow{\alpha} A$, states that if α is a non zero divisor on P then α is a non zero divisor in A. This is Auslanders "zero divisor conjecture" proved by Peskine and Szpiro [9] in case of local rings of char. $p > 0$ or rings essentially of finite type over a field. The general equicharacteristic case is due to the existence of big Cohen Macaulay modules, Hochster [7]. For other special cases of the amplitude inequality related to the notion of "Tor-rigidity" *see* the discussion at the end of paragraph 3.

In case A has a dualizing complex, the amplitude inequality has the following, dual form

$$\dim L^{\bullet} \otimes X^{\bullet} \geqq \dim X^{\bullet} - \operatorname{proj.amp} L^{\bullet}.$$

If $H^{\bullet}(L^{\bullet})$ has finite length and $X^{\bullet} = A$ this says:

$$\operatorname{proj.amp} L^{\bullet} \geqq \dim A,$$

which is Peskines and Szpiro's "new intersection theorem" [10]. In return our proof is based on this theorem. Another special case of the intersection theorem can be found in Foxby [4], whom we also have to thank for some of the results in paragraph 1.

A dual form of the intersection theorem in case $X^{\bullet} = A$ is the following "generalized Bass conjecture": Let $E^{\bullet}$ be a bounded complex of injective modules with finitely generated cohomology. If $H^{\bullet}(E^{\bullet}) \neq 0$, then $E^{\bullet}$ has length $\geqq \dim A$.

In case A is an arbitrary regular local ring, we prove the sharper inequality

$$\operatorname{amp} L^{\bullet} \otimes X^{\bullet} \geqq \operatorname{amp} L^{\bullet} + \operatorname{amp} X^{\bullet},$$

the dual form of this is

$$\dim L^{\bullet} \otimes X^{\bullet} + \dim A \geqq \dim L^{\bullet} + \dim X^{\bullet}$$

generalizing Serre's result [12] to complexes.

1. Depth of a complex

Throughout, A denotes a local noetherian ring with maximal ideal m and residue field k. By a module is understood an A-module and by a complex is understood a complex of A-modules.

For a power series $f(t) = \Sigma\, a_i\, t^i$ with **Z**-coefficients we put $\nu_t(f(t)) = \inf\{ i \mid a_i \neq 0 \}$.

DEFINITION 1.1. — Let $X^{\bullet}$ be a complex with finitely generated cohomology modules. If $X^{\bullet}$ is bounded below define

$$\mu(X^{\bullet}, t) = \Sigma_i \dim_k \operatorname{Ext}^i(k, X^{\bullet})\, t^i,$$

and if $X^{\bullet}$ is bounded above define

$$\beta(X^{\bullet}, t) = \Sigma_i \dim_k \operatorname{Tor}_i(k, X^{\bullet})\, t^i.$$

If $X^{\bullet}$ is bounded and $H^{\bullet}(X^{\bullet}) \neq 0$ define the *depth* of $X^{\bullet}$ by

$$\operatorname{depth} X^{\bullet} = \nu_t(\beta(X^{\bullet}, t)\, \mu(X^{\bullet}, t)).$$

We remark that depth $X^{\bullet}$ may be negative and that this concept extends the usual notion of depth of a module.

If moreover the series $\mu(X^{\bullet}, t)$, resp. $\beta(X^{\bullet}, t)$ is finite we say that $X^{\bullet}$ has finite *injective amplitude* resp. *projective amplitude* and we define

$$\operatorname{inj.amp} X^{\bullet} = -\nu_t(\mu(X^{\bullet}, t)\, \mu(X^{\bullet}, t^{-1})),$$

$$\operatorname{proj.amp} X^{\bullet} = -\nu_t(\beta(X^{\bullet}, t)\, \beta(X^{\bullet}, t^{-1})).$$

PROPOSITION 1.2. — *Let* $L^{\bullet}$ *be a bounded above complex with finitely generated cohomology modules and* $X^{\bullet}$ *a bounded below complex of injective modules with finitely generated cohomology. Then:*

$$\mu(\operatorname{Hom}^{\bullet}(L^{\bullet}, X^{\bullet}), t) = \beta(L^{\bullet}, t)\, \mu(X^{\bullet}, t).$$

If $L^{\bullet}$ *and* $X^{\bullet}$ *are bounded, then:*

$$\beta(\operatorname{Hom}^{\bullet}(L^{\bullet}, X^{\bullet}), t) = \mu(L^{\bullet}, t)\, \mu(X^{\bullet}, t^{-1}).$$

Proof. — For the first formula we may assume that $L^{\bullet}$ is a complex of finitely generated free modules. We have isomorphisms of complexes

$$\operatorname{Hom}^{\bullet}(k, \operatorname{Hom}^{\bullet}(L^{\bullet}, X^{\bullet})) \xrightarrow{\sim} \operatorname{Hom}^{\bullet}(k \otimes L^{\bullet}, X^{\bullet}) \xrightarrow{\sim} \operatorname{Hom}^{\bullet}(k \otimes L^{\bullet}, \operatorname{Hom}^{\bullet}(k, X^{\bullet})),$$

from which the first formula follows.

To prove the second formula choose a bounded below complex $E^{\bullet}$ of injective modules quasi-isomorphic to $L^{\bullet}$. For a finitely generated module M consider the natural map

$$\mathrm{Hom}^{\bullet}(E^{\bullet}, X^{\bullet}) \otimes M \to \mathrm{Hom}^{\bullet}(\mathrm{Hom}^{\bullet}(M, E^{\bullet}), X^{\bullet}).$$

Choose a presentation of M by finitely generated free modules to see that it is an isomorphism. This has the consequence that $\mathrm{Hom}^{\bullet}(E^{\bullet}, X^{\bullet})$ is a complex of flat modules. It is now easy to conclude by taking $M = k$ and noting that

$$\mathrm{Hom}^{\bullet}(\mathrm{Hom}^{\bullet}(k, E^{\bullet}), X^{\bullet}) = \mathrm{Hom}^{\bullet}(\mathrm{Hom}^{\bullet}(k, E^{\bullet}), \mathrm{Hom}^{\bullet}(k, X^{\bullet})).$$

Q. E. D.

PROPOSITION 1.3. — *Let* $L^{\bullet}$ *be a bounded complex of finitely generated free modules with* $H^{\bullet}(L^{\bullet}) \neq 0$. *Then:*

$$\operatorname{depth} L^{\bullet} + \operatorname{proj.amp} L^{\bullet} = \operatorname{depth} A.$$

Proof. — Put $L^{\bullet\vee} = \mathrm{Hom}^{\bullet}(L^{\bullet}, A)$. As is easily seen, we have $\beta(L^{\bullet\vee}, t) = \beta(L^{\bullet}, t^{-1})$. From the first formula in 1.2 we get

$$\mu(L^{\bullet\vee}, t) = \beta(L^{\bullet}, t)\mu(A, t)$$

or applying this to $L^{\bullet\vee}$,

$$\mu(L^{\bullet}, t) = \mu(A, t)\beta(L^{\bullet}, t^{-1}).$$

Finally multiply this with $\beta(L^{\bullet}, t)$ and apply ν_t.

Q. E. D.

REMARK 1. 4. — In case $H^{\bullet}(L^{\bullet})$ has finite length we have

$$\operatorname{depth} L^{\bullet} = -\operatorname{amp} L^{\bullet}$$

and 1.3 specializes to a well known acyclicity lemma [9] (1.9).

PROPOSITION 1.5. — *Let* $E^{\bullet}$ *be a bounded complex of injective modules with finitely generated cohomology modules and* $H^{\bullet}(E^{\bullet}) \neq 0$. *Then:*

$$\operatorname{depth} E^{\bullet} + \operatorname{inj.amp} E^{\bullet} = \operatorname{depth} A.$$

Proof. — From the last formula in 1.2 we get with $X^{\bullet} = A$,

$$\beta(E^{\bullet}, t) = \mu(A, t)\mu(E^{\bullet}, t^{-1})$$

multiply this with $\mu(E^{\bullet}, t)$ and apply ν_t to get the result.

Q. E. D.

REMARK 1. 6. — In case $E^{\bullet}$ is an injective resolution of a finitely generated module M, we get the well known fact that

$$\operatorname{inj.dim} M = \operatorname{depth} A.$$

If $X^{\cdot}$ is a complex with $H^{\cdot}(X^{\cdot}) \neq 0$ and of finite projective amplitude and finite injective amplitude we have from the proof of 1.9 and 1.4:

$$\beta(X^{\cdot}, t) = \mu(A, t)\mu(X^{\cdot}, t^{-1})$$

and

$$\mu(X^{\cdot}, t) = \mu(A, t)\beta(X^{\cdot}, t^{-1}).$$

Foxby: Playing these two formulas against each other we find that $\mu(A, t)$ has the form t^d that is A is necessarily Gorenstein. Added in proof: Foxby's results on Poincaré series has appeared in *Math. Scand.*, Vol. 70, 1977, pp. 1-19.

PROPOSITION 1.7. — *Let* $L^{\cdot}$ *be a bounded complex of finitely generated free modules and* $X^{\cdot}$ *a bounded complex with finitely generated cohomology modules. If* $H^{\cdot}(L^{\cdot}) \neq 0$ *and* $H^{\cdot}(X^{\cdot}) \neq 0$, *then:*

$$\operatorname{depth} L^{\cdot} \otimes X^{\cdot} + \operatorname{depth} A = \operatorname{depth} L^{\cdot} + \operatorname{depth} X^{\cdot}.$$

Proof. — We have $L^{\cdot} \otimes X^{\cdot} = \operatorname{Hom}(L^{\cdot\vee}, X^{\cdot})$ thus by the first part of 1.2:

$$\mu(L^{\cdot} \otimes X^{\cdot}, t) = \beta(L^{\cdot\vee}, t)\mu(X^{\cdot}, t).$$

In particular with $X^{\cdot} = A$:

$$\mu(L^{\cdot}, t) = \beta(L^{\cdot\vee}, t)\mu(A, t),$$

thus we deduce:

$$\mu(A, t)\mu(L^{\cdot} \otimes X^{\cdot}, t) = \mu(L^{\cdot}, t)\mu(X^{\cdot}, t),$$

combine this with the obvious

$$\beta(L^{\cdot} \otimes X^{\cdot}, t) = \beta(L^{\cdot}, t)\beta(X^{\cdot}, t)$$

and apply ν_t.

Q. E. D.

REMARK 1. 8. — In case $X^{\cdot} \otimes L^{\cdot}$ has cohomology of finite length, we deduce from 1. 3, 1.4 and 1.7, that

$$\operatorname{amp} L^{\cdot} \otimes X^{\cdot} = \operatorname{proj.amp} L^{\cdot} - \operatorname{depth} X^{\cdot},$$

which contains the last theorem in Serre's notes [11].

We shall now assume that A has a dualizing complex $D^{\cdot}$. This we will represent as a bounded complex of injective modules. For a bounded complex $X^{\cdot}$ we put $X^{\cdot D} = \operatorname{Hom}^{\cdot}(X^{\cdot}, D^{\cdot})$.

PROPOSITION 1.9. — *Suppose the local ring* A *has a dualizing complex* $D^{\cdot}$. *Then for a bounded complex* $X^{\cdot}$ *with finitely generated cohomology and* $H^{\cdot}(X^{\cdot}) \neq 0$:

$$\operatorname{depth} X^{\cdot} = \operatorname{depth} X^{\cdot D}.$$

If $X^{\bullet}$ *has finite projective amplitude*

$$\mathrm{proj.amp}\, X^{\bullet} = \mathrm{inj.amp}\, X^{\bullet D}.$$

If $X^{\bullet}$ *has finite injectives amplitude.*

$$\mathrm{inj.amp}\, X^{\bullet} = \mathrm{proj.amp}\, X^{\bullet D}.$$

Proof. — We may normalize $D^{\bullet}$ such that

$$\mu(D^{\bullet}, t) = 1$$

and the result follows from 1.2.

Q. E. D.

2. Dimension of a complex

In this section we shall generalize the notion of dimension to complexes.

LEMMA 2.1. — *Let* $X^{\bullet}$ *be a bounded below complex with finitely generated cohomology modules. Then:*

$$\inf\{ i \mid \mathrm{Ext}^i(k, X^{\bullet}) \neq 0\} = \inf\{ i \mid R^i\Gamma_m(X^{\bullet}) \neq 0\},$$

where $R^{\bullet}\Gamma_m$ *denotes the local cohomology with support in* m.

Proof. — Let us call a complex of injective modules $E^{\bullet}$ a minimal injective complex if $E^{\bullet}$ is bounded below and for all $n \in \mathbf{Z}$:

$$\mathrm{Ker}(\partial^n) \to E^n$$

is an essential extension. Using the construction of Hartshorne [6] (I.4.6), one sees that $X^{\bullet}$ admits a quasi-isomorphism into a minimal injective complex $E^{\bullet}$. Note first that the complex $\mathrm{Hom}^{\bullet}(k, E^{\bullet})$ has zero differential. Whence:

$$\inf\{ i \mid \mathrm{Ext}^i(k, E^{\bullet}) \neq 0\} = \inf\{ i \mid \mathrm{Hom}(k, E^i) \neq 0\}.$$

Let j denote the above integer. By decomposing E^i into indecomposable injectives, one sees that $\Gamma_m(E^i) = 0$ for $i < j$. On the other hand $\mathrm{Hom}(k, E^j)$ is a non trivial submodule of the kernel of

$$\Gamma_m(\partial^j): \quad \Gamma_m(E^j) \to \Gamma_m(E^{j+1}).$$

Q. E. D.

DEFINITION 2.2. — Let $X^{\bullet}$ be a bounded complex with finitely generated cohomology and $H^{\bullet}(X^{\bullet}) \neq 0$. Define the dimension of $X^{\bullet}$:

$$\dim X^{\bullet} = \sup\{ i \mid R^i\Gamma_m(X^{\bullet}) \neq 0\} - \sup\{ i \mid H^i(X^{\bullet}) \neq 0\}.$$

In case amp $X^{\bullet} = 0$ this coincides with the dimension concept for modules, *see* [5] (6.4). It is proved in 2.5 that dim $X^{\bullet} \geqq 0$ in general.

The following theorem expresses the basic duality between the concepts of dimension and amplitude.

THEOREM 2.3. — *Suppose* A *has a dualizing complex* $D^{\bullet}$. *For a bounded* $X^{\bullet}$ *with finitely generated cohomology modules and* $H^{\bullet}(X^{\bullet}) \neq 0$,

$$\dim X^{\bullet} = \operatorname{depth} X^{\bullet} + \operatorname{amp} X^{\bullet D}.$$

Proof. — From the definitions and 2.1 follows, that

$$\dim X^{\bullet} - \operatorname{depth} X^{\bullet} = \operatorname{amp} R^{\bullet}\Gamma_m(X^{\bullet}).$$

Grothendieck's local duality theorem [6] (V.6.2) states that ($i \in \mathbf{Z}$):

$$R^i\Gamma_m(X^{\bullet}) = \operatorname{Hom}(H^{-i}(X^{\bullet D}), I)$$

when I denotes an injective envelope of k. From this follows:

$$\operatorname{amp} R^{\bullet}\Gamma_m(X^{\bullet}) = \operatorname{amp} X^{\bullet D}.$$

Q. E. D.

PROPOSITION 2.4. — *Suppose* A *has a dualizing complex* $D^{\bullet}$. *Let* $X^{\bullet}$ *be a bounded complex with finitely generated cohomology modules and* $H^{\bullet}(X^{\bullet}) \neq 0$. *Put*

$$c = \inf\{ i \mid H^i(X^{\bullet}) \neq 0 \}.$$

Then:

$$\dim X^{\bullet D} \leqq \inf\{ \dim A/q \mid q \in \operatorname{Ass} H^c(X^{\bullet}) \}.$$

Moreover $\dim X^{\bullet D} = 0$ *if and only if* $m \in \operatorname{Ass}(H^c(X^{\bullet}))$.

Proof. — By 2.3 and 1.9 it will suffice to prove that

$$\operatorname{amp} X^{\bullet} + \operatorname{depth} X^{\bullet} \leqq \inf\{ \dim A/q \mid q \in \operatorname{Ass} H^c(X^{\bullet}) \}.$$

So let $q \in \operatorname{Ass} H^c(X)$. We have

$$\operatorname{Hom}_{A_q}(k(q), H^c(X_q^{\bullet})) = \operatorname{Ext}^c_{A_q}(k(q), X_q^{\bullet}).$$

So by lemma 2.6 below we have

$$\operatorname{Ext}^{c+d}(k, X) \neq 0, \qquad d = \dim A/q.$$

From this follows:

$$\operatorname{amp} X^{\bullet} + \operatorname{depth} X^{\bullet} \leqq d.$$

For the last statement note that $\operatorname{amp} X^{\bullet} + \operatorname{depth} X^{\bullet} = 0$ if and only if $\operatorname{Ext}^c k, X^{\bullet}) \neq 0$.

Q. E. D.

COROLLARY 2.5. — *Let* $X^{\bullet}$ *be a bounded complex with finitely generated cohomology and* $H^{\bullet}(X^{\bullet}) \neq 0$. *Then:*

$$0 \leqq \dim X^{\bullet} \leqq \dim \operatorname{Supp} X^{\bullet},$$

where $\operatorname{Supp} X^{\bullet} = \cup_i \operatorname{Supp} H^i(X^{\bullet})$.

Proof. — Passing to the completion of A does not change the three integers above. Thus we may assume that A has a dualizing complex $D^{\bullet}$. To prove the first inequality note by 2.3 and 1.9:

$$\dim X^{\bullet} = \operatorname{depth} X^{\bullet D} + \operatorname{amp} X^{\bullet D}$$

which is easily seen to be positive. The second follows from 2.4 and the fact that $\operatorname{Supp} X^{\bullet} = \operatorname{Supp} X^{\bullet D}$.

Q. E. D.

LEMMA 2.6. — *Let* $X^{\bullet}$ *be a bounded below complex with finitely generated cohomology modules. For prime ideals* $p \subsetneq q$ *with no prime ideal lying properly between* p *and* q, *we have for* $i \in \mathbf{Z}$:

$$\operatorname{Ext}^{i}_{A_p}(k(p), X^{\bullet}_p) \neq 0 \quad \Rightarrow \quad \operatorname{Ext}^{i+1}_{A_q}(k(q), X^{\bullet}_q) \neq 0.$$

Proof. — This is a generalization of a well known lemma of Bass [2] (3.1). The proof given there extends immediately to the general case.

Q. E. D.

3. The amplitude inequality

In this section we shall prove the amplitude inequality for equicharacteristic local rings. It will be proved by means of the "new intersection theorem" of Peskine and Szpiro [10] and Roberts [11]. In the simplest case it says:

let $P^{\bullet}$ be a bounded complex of finitely generated free modules over an equicharacteristic local noetherian ring A. If $H^{\bullet}(P^{\bullet}) \neq 0$ and consists of modules of finite length, then:

3.1 $$\operatorname{proj.amp} P^{\bullet} \geqq \dim A.$$

Concerning the attribution of the "new intersection theorem", the same reservation as made in the introduction concerning "Auslanders zero-divisor conjecture" is in order here.

THEOREM 3.2. — *Let* A *be equicharacteristic. For a bounded complex* $L^{\bullet}$ *of finitely generated free modules and a bounded complex* $X^{\bullet}$ *with finitely generated cohomology with* $H^{\bullet}(L^{\bullet}) \neq 0$ *and* $H^{\bullet}(X^{\bullet}) \neq 0$:

$$\operatorname{amp} L^{\bullet} \otimes X^{\bullet} \geqq \operatorname{amp} X^{\bullet}.$$

Proof. — Passing to the completion of A we may assume that A has a dualizing complex $D^{\bullet}$. Let $L^{\bullet}$ be normalized such that

$$H^0(L^{\bullet}) \neq 0, \quad L^i = 0 \qquad \text{for} \quad i > 0$$

and $X^{\bullet}$ normalized such that

$$H^0(X^{\bullet}) \neq 0, \quad X^i = 0, \qquad i < 0.$$

We shall proceed by induction on dim A. Consider the E_1-term of the spectral sequence of the double complex (underlying) $L^{\cdot} \otimes X^{\cdot}$:

$$\ldots \to L_2 \otimes H^p \to L_1 \otimes H^p \to L_0 \otimes H^p,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\ldots \to L_2 \otimes H^1 \to L_1 \otimes H^1 \to L_0 \otimes H^1,$$

$$\ldots \to L_2 \otimes H^0 \to L_1 \otimes H^0 \to L_0 \otimes H^0.$$

CASE 1. — $\dim \operatorname{Supp} L^{\cdot} \otimes H^0 \neq 0$. We have to prove that $H^i(L^{\cdot} \otimes X^{\cdot}) \neq 0$ for some $i \leqq 0$. Choose $p \in \operatorname{Supp} L^{\cdot} \otimes H^0$ with $p \neq m$. Localize A at p and apply the induction hypothesis to $L^{\cdot}_p \otimes X^{\cdot}_p$.

CASE 2. — $\dim \operatorname{Supp} L^{\cdot} \otimes H^0 = 0$. Put $I = \operatorname{Ann} H^0$ and consider $L^{\cdot} \otimes A/I$. As it is easily seen, this complex has comology of finite length thus by 3.1:

$$\operatorname{proj.amp} L^{\cdot} \geqq \dim H^0(X^{\cdot}),$$

on the other hand by 2.4 we have

$$\dim H^0(X^{\cdot}) \geqq \dim X^{\cdot D}.$$

We can now conclude by the following inequalities ($L^{\cdot\vee} = \operatorname{Hom}^{\cdot}(L^{\cdot}, A)$):

$$\begin{aligned}
&\operatorname{amp} L^{\cdot} \otimes X^{\cdot} - \operatorname{amp} X^{\cdot}\\
&\quad = \dim L^{\cdot\vee} \otimes X^{\cdot D} - \operatorname{depth} L^{\cdot} \otimes X^{\cdot} - \operatorname{amp} X^{\cdot}\\
&\quad \geqq - \operatorname{depth} L^{\cdot} \otimes X^{\cdot} - \operatorname{amp} X^{\cdot}\\
&\quad = - \operatorname{depth} L^{\cdot} - \operatorname{depth} X^{\cdot} + \operatorname{depth} A - \operatorname{amp} X^{\cdot}\\
&\quad = \operatorname{proj.amp} L^{\cdot} - \dim X^{\cdot D}
\end{aligned}$$

Q. E. D.

We shall end this paragraph by showing that if $P \neq 0$ has a finite resolution by finitely generated free modules $L^{\cdot}$ and P is *rigid*, that is for any finitely generated module M and $i \in \mathbf{N}$

$$\operatorname{Tor}_i(P, M) = 0 \quad \Rightarrow \quad \operatorname{Tor}_{i+1}(P, M) = 0,$$

then for any bounded complex $X^{\cdot}$ as above

$$\operatorname{amp} L^{\cdot} \otimes X^{\cdot} \geqq \operatorname{amp} X^{\cdot}.$$

To see this choose a bounded above complex $F^{\cdot}$ of finitely generated free modules and a quasi-isomorphism $F^{\cdot} \to X^{\cdot}$, then we have a quasi-isomorphism $P \otimes F^{\cdot} \to L^{\cdot} \otimes X^{\cdot}$. We shall now prove that $H^n(P \otimes F^{\cdot}) = 0$ implies $H^n(F^{\cdot}) = 0$. To see this we simply have to apply Lemma 3.3 below to the cononical map $\operatorname{Cok.}(\partial^{n-1}) \to F^{n+1}$.

The special case where $\operatorname{amp} X^{\cdot} \otimes L^{\cdot} = 0$ I have learned from Fulton, the still more special case where $X^{\cdot}$ is a complex of free modules is due to Auslander [1] (4.1). Let us finally recall [1], [8], that a finitely generated module over a regular local ring is rigid.

LEMMA 3.3. — *Let* $P \neq 0$ *be a finitely generated rigid module and* $f: N \to F$ *a linear map where* N *is finitely generated and* F *is a finitely generated free module. If* $f \otimes 1_P$ *is a monomorphism, then* f *is a monomorphism.*

Proof. — Factor $f = ig$ where i is a monomorphism and g an epimorphism. It follows that $g \otimes 1_P$ is an isomorphism and that $i \otimes 1_P$ is a monomorphism. This implies $\mathrm{Tor}_1(\mathrm{Cok}(i), P) = 0$ and therefore $\mathrm{Tor}_2(\mathrm{Cok}(i), P) = 0$ by rigidity. From this we conclude $\mathrm{Tor}_1(\mathrm{Im}(f), P) = 0$. From this and the exact sequence

$$0 \to \mathrm{Ker}(f) \to N \to \mathrm{Im}(f) \to 0,$$

we conclude $P \otimes \mathrm{Ker}(f) = 0$ and whence $\mathrm{Ker}(f) = 0$.

Q. E. D.

4. The intersection theorem

The following theorem is the dual form of the amplitude inequality.

THEOREM 4.1. — *Let* A *be equicharacteristic. For a bounded complex* $L^{\cdot}$ *of finitely generated free modules and a bounded complex* $X^{\cdot}$ *with finitely generated cohomology modules with* $H^{\cdot}(L^{\cdot}) \neq 0$ *and* $H^{\cdot}(X^{\cdot}) \neq 0$, *we have*

$$\dim X^{\cdot} \otimes L^{\cdot} \geqq \dim X^{\cdot} - \mathrm{proj.amp}\, L^{\cdot}.$$

Proof. — Passing to the completion we may assume A has a dualizing complex $D^{\cdot}$. We have

$$\begin{aligned}\dim X^{\cdot} \otimes L^{\cdot} &= \mathrm{amp}\, X^{\cdot D} \otimes L^{\cdot\vee} + \mathrm{depth}\, X^{\cdot} \otimes L^{\cdot}\\ &= \mathrm{amp}\, X^{\cdot D} \otimes L^{\cdot\vee} + \mathrm{depth}\, X^{\cdot} + \mathrm{depth}\, L^{\cdot} - \mathrm{depth}\, A\\ &= \mathrm{amp}\, X^{\cdot D} \otimes L^{\cdot\vee} + \mathrm{depth}\, X^{\cdot} - \mathrm{proj.amp}\, L^{\cdot}\\ &\geqq \mathrm{amp}\, X^{\cdot D} + \mathrm{depth}\, X^{\cdot} - \mathrm{proj.amp}\, L^{\cdot}\\ &= \dim X^{\cdot} - \mathrm{proj.amp}\, L^{\cdot}.\end{aligned}$$

Q. E. D.

The special case of 4.1 with $K^{\cdot} = A$ has the following dual form.

COROLLARY 4.2 (Generalized Bass conjecture). — *Let* A *be equicharacteristic. Then for any bounded complex*

$$0 \to E^0 \to E^1 \to \ldots \to E^d \to 0$$

of injective modules with finitely generated cohomology and $H^{\cdot}(E^{\cdot}) \neq 0$ *we have* $d \geqq \dim A$.

Proof. — Passing to the completion of A and replacing $E^{\cdot} \otimes \hat{A}$ with an injective resolution we may assume that A has a dualizing complex $D^{\cdot}$. We have

$$\dim E^{\cdot D} + \mathrm{proj.amp}\, E^{\cdot D} \geqq \dim A$$

on the other hand:

$$\begin{aligned}&\dim E^{\cdot D} + \mathrm{proj.amp}\, E^{\cdot D}\\ &\quad = \mathrm{amp}\, E^{\cdot} + \mathrm{depth}\, E^{\cdot} + \mathrm{inj.amp}\, E^{\cdot} \leqq d.\end{aligned}$$

Q. E. D.

REMARK 4. 3. — The "Generalized Bass conjecture" is easily verified by means of Hochster's big Cohen Macaulay modules [7]. One proceeds as in [3] (1.4).

REMARK 4. 4. — Theorem 4. 1 is trivially true if proj. amp. $L^{\bullet} \geqq \dim A$. By duality, 3.2 is true in case proj.amp $L^{\bullet} \geqq \dim A$. One checks easily that 3.2 is valid in case proj.amp $L^{\bullet} \leqq 1$, Thus the amplitude inequality is valid in case $\dim A \leqq 2$. Note also, if $\dim \operatorname{Supp} X^{\bullet} = 0$, then:

$$\operatorname{amp} L^{\bullet} \otimes X^{\bullet} - \operatorname{amp} X^{\bullet} = \operatorname{proj.amp} L^{\bullet} \geqq 0$$

as it follows from 1.8.

REMARK 4. 5. — Consider a class of local rings, stable under completion, localization and formation of quotient rings. Then the following three "conjectures" are equivalent:

Peskine-Szpiro's intersection property 3.1;

the amplitude inequality 3.2;

the Generalized Bass conjecture 4.2.;

This follows by "reversing" the proof of 4.2.

REMARK 4. 6. — For a bounded complex $L^{\bullet}$ of finitely generated free modules with $H^{\bullet}(L^{\bullet}) \neq 0$ put:

$$\operatorname{gr} L^{\bullet} = \inf\{ i \mid H^i(L^{\bullet\vee}) \neq 0\} + \sup\{ i \mid H^i(L^{\bullet}) \neq 0\}.$$

From the fact that $\operatorname{proj.amp} L^{\bullet\vee} = \operatorname{proj.amp} L^{\bullet}$ follows:

$$\operatorname{gr} L^{\bullet} = \operatorname{proj.amp} L^{\bullet} - \operatorname{amp} L^{\bullet\vee}.$$

In case A is Gorenstein it follows from 2.3 that

$$\dim A = \dim L^{\bullet} + \operatorname{gr} L^{\bullet}.$$

Note, however that

$$\dim L^{\bullet} \otimes X^{\bullet} \geqq \dim X^{\bullet} - \operatorname{gr} L^{\bullet}$$

is *not* generally valid since its dual form is (as the reader easily varifies):

$$\operatorname{amp} L^{\bullet} \otimes X^{\bullet} \geqq \operatorname{amp} X^{\bullet} + \operatorname{amp} L^{\bullet}$$

which is not generally valid: take for $L^{\bullet}$ a complex of the form $A \to A$ and for $X^{\bullet}$ a single module.

5. Regular local rings

In this section we use Serre's intersection theorem to prove a sharper amplitude inequality for complexes over an arbitrary regular local ring.

THEOREM 5.1. — *Let* A *be a regular local ring,* $X^{\bullet}$ *and* $Y^{\bullet}$ *bounded complexes of finitely generated free modules. If* $H^{\bullet}(X^{\bullet}) \neq 0$ *and* $H^{\bullet}(Y^{\bullet}) \neq 0$, *then:*

$$\operatorname{amp} X^{\bullet} \otimes Y^{\bullet} \geqq \operatorname{amp} X^{\bullet} + \operatorname{amp} Y^{\bullet}.$$

Proof. — Let $X^\cdot$ and $Y^\cdot$ be normalized such that

$$H^0(X^\cdot) \neq 0,\ H^i(X^\cdot) = 0 \quad \text{for} \quad i < 0,$$

$$H^0(Y^\cdot) \neq 0,\ H^i(Y^\cdot) = 0 \quad \text{for} \quad i < 0.$$

Note that we have

$$\sup\{\, i \mid H^i(X^\cdot \otimes Y^\cdot) \neq 0 \,\} = \sup\{\, i \mid H^i(X^\cdot) \neq 0 \,\} + \sup\{\, i \mid H^i(Y^\cdot) \neq 0 \,\}.$$

Thus the amplitude inequality is equivalent to $H^i(X^\cdot \otimes Y^\cdot) \neq 0$ for some $i \leqq 0$.

We shall now procede by induction on dim A. The case dim A = 0 follows from the Künneth formula.

CASE 1. — $\operatorname{Supp} H^0(X^\cdot) \cap \operatorname{Supp} H^0(Y^\cdot) \neq \{ m \}$. Choose

$$p \in \operatorname{Supp} H^0(X^\cdot) \cap \operatorname{Supp} H^0(Y^\cdot) \quad \text{with} \quad p \neq m.$$

Localize at p and notice that the normalization of $X^\cdot$ and $Y^\cdot$ has not been destroyed. By the induction hypothesis

$$H^i(X^\cdot_p \otimes Y^\cdot_p) \neq 0 \quad \text{for some} \quad i \leqq 0.$$

CASE 2. — $\operatorname{Supp} H^0(X^\cdot) \cap \operatorname{Supp} H^0(Y^\cdot) = \{ m \}$. By Serre's intersection theorem ([12], V, Th. 3):

$$\dim H^0(X^\cdot) + \dim H^0(Y^\cdot) \leqq \dim A.$$

By 2.4 we have

$$\dim X^{\cdot\vee} \leqq \dim H^0(X^\cdot) \quad \text{and} \quad \dim Y^{\cdot\vee} \leqq \dim H^0(Y^\cdot)$$

and consequently

$$\dim X^{\cdot\vee} + \dim Y^{\cdot\vee} \leqq \dim A.$$

Using 2.3 and 1.7 we get

$$\begin{aligned}
&\operatorname{amp} X^\cdot \otimes Y^\cdot - \operatorname{amp} X^\cdot - \operatorname{amp} Y^\cdot \\
&\quad = \dim X^{\cdot\vee} \otimes Y^{\cdot\vee} - \dim X^{\cdot\vee} - \dim Y^{\cdot\vee} \\
&\qquad - \operatorname{depth} X^\cdot \otimes Y^\cdot + \operatorname{depth} X^\cdot + \operatorname{depth} Y^\cdot \\
&\quad = \dim X^{\cdot\vee} \otimes Y^{\cdot\vee} - \dim X^{\cdot\vee} - \dim Y^{\cdot\vee} + \dim A \\
&\quad \geqq \dim A - \dim X^{\cdot\vee} - \dim Y^{\cdot\vee} \geqq 0.
\end{aligned}$$

Q. E. D.

COROLLARY 5.2. — *With the notation above*

$$\dim A + \dim X^\cdot \otimes Y^\cdot \geqq \dim X^\cdot + \dim Y^\cdot.$$

Proof:

$$\begin{aligned}
\dim X^\cdot \otimes Y^\cdot &= \operatorname{amp} X^{\cdot\vee} \otimes Y^{\cdot\vee} + \operatorname{depth} X^\cdot \otimes Y^\cdot \\
&\geqq \operatorname{amp} X^{\cdot\vee} + \operatorname{depth} X^\cdot + \operatorname{amp} Y^{\cdot\vee} + \operatorname{depth} Y^\cdot - \operatorname{depth} A \\
&= \dim X^\cdot + \dim Y^\cdot - \dim A.
\end{aligned}$$

Q. E. D.

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(Manuscrit reçu le 24 mai 1977.)

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