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THE K_2 OF RINGS WITH MANY UNITS

BY WILBERD VAN DER KALLEN

Introduction

In this paper we give generalisations of Matsumoto's theorem on the presentation of the K_2 of an infinite field (symplectic and non-symplectic cases). The main results are as follows. Matsumoto's theorem still holds for a local ring with infinite residue field, both in the symplectic and the non-symplectic cases. Moreover, for Milnor's K_2 (type SL_n with $n \geqq 3$), the result even applies with a local ring whose residue field has more than five elements.

Our approach can be described as follows. "Compute first in general position, then extrapolate from there." This is also known as the group chunk method (*cf.* [1]) and it is only feasible for groups that contain sufficiently many elements in general position. In the "general position" stage of our computations we need the existence of inverses of certain coordinates. Therefore the relevant notion of "general position" is that certain coordinates should be units. In [11], where we used a similar approach, "general position" corresponded to a certain piece of a column being unimodular. One may view the present case as the one where the piece of the column has length 1. To get sufficiently many elements in general position we assume that the ring satisfies certain conditions, similar to the strongest of Bass's stable range conditions. Roughly speaking, most elements of the ring should be units and there should be plenty of them, as in infinite fields. We give two proofs for type SL_n, $n \geqq 3$. In the first proof, which applies to a local ring with infinite residue field, we view SL_n as a Chevalley group and ignore the usual representation by n by n matrices. This first proof is easy. The main reason is that, when proving the necessary properties in general position, we accept losing ground, i. e. we accept that later results are proved for fewer elements than earlier results. (More and more expressions are required to have invertible values. The number of such expressions remains finite, but not bounded.) In the second proof, which is much more computational, we work with matrices and try to recover, by extrapolation, the ground we lose, so that we still get useful results when there are only many, not very many, elements in general position. (This second proof works for a local ring with more than three elements in its residue field.) We only need to study the situation for 3 by 3 matrices in detail because we can invoke "stability for K_2" to pass from $n = 3$ to $n > 3$.

To make the computations in the 3 by 3 case, the presentation of Matsumoto is not convenient. There is a richer presentation (more generators, more relations) which gives a more suitable calculus. It is the presentation given by Dennis and Stein for the K_2 of a discrete valuation ring or a homomorphic image of such a ring (*see* [8]). This presentation, which for fields amounts to the same as Matsumoto's, was put in a more manageable form by H. Maazen and J. Stienstra [16]. They also showed how to use it in the Steinberg group if certain coordinates are units. (In their application the relevant coordinates are congruent to 1 modulo the Jacobson radical.) Among other things Maazen and Stienstra thus extended the above result of Dennis and Stein to all local rings with residue field $\mathbf{F}_p$, p prime. Our method extends the result of Dennis and Stein to local rings with more than three elements in the residue field, so it has now been proved for all local rings.

Our paper is organized as follows. In section 1 we introduce the condition on R which is used in sections 2, 3, 4. In sections 2, 3, 4 we rewrite Matsumoto's original proof in the spirit of group chunks. In section 5 we prepare for the case of 3 by 3 matrices by discussing that part of the argument that can be understood in terms of 2 by 2 matrices. (We mostly ignore "symplectic" phenomena in the 2 by 2 case, as they don't pass to 3 by 3.) In section 6 we prove the presentation for the 3 by 3 case, i. e. for $K_2(3, R)$. In section 7 we indicate how one can apply [11] or [23]. In section 8 we compare the competing presentations (Matsumoto versus Dennis-Stein). In section 9 we introduce norm residue symbols, as an obvious application of the main result.

I wish to thank Jan Stienstra and Henk Maazen for the many discussions leading to the present paper (*cf.* [12]). I am also indebted to Jan Strooker who introduced us to the subject of algebraic K-theory.

1. Unit-irreducible rings and the primitive criterion

1.1. Rings are commutative and have a unit.

1.2. EXAMPLE. — Let Aff denote the affine line. If k is an infinite field, the set k-rational points in Aff, denoted by Aff (k), is an irreducible topological space, when endowed with the Zariski-topology. Let R be a local ring with infinite residue field k. Then Aff (R) can be endowed with an irreducible topology obtained from the Zariski-topology on Aff (k) by taking inverse images with respect to the natural map Aff (R) $\rightarrow$ Aff (k). Similarly Aff^n (R) can be endowed with an irreducible topology. As a set, Aff^n (R) is nothing else than R^n. So we can transfer the topology to R^n. A polynomial $f(X_1, \ldots, X_n)$ over R gives a function $R^n \rightarrow R$. The inverse image under f of the set of units in R is an open subset of R^n, and every open subset of R^n is a union of subsets that are obtained in this way.

1.3. DEFINITIONS. — Let R be a ring. If $f(X) \in R[X]$ is a polynomial, we put $D(f) = \{ r \in R \mid f(r) \text{ is a unit in } R \}$. The topology on R which has the D (f) as a basis is called the *unit-topology*. In other words, it is the weakest topology on R such that:

(i) the set R* of units in R is open;

(ii) every polynomial in one variable over R defines a continuous map from R into R.

The ring R is called U-*irreducible* if it is an irreducible space when endowed with the unit-topology. (Recall that this means that the intersection of two non-empty open subsets is again non-empty.)

1.4. R is U-irreducible if and only if the following holds: given $f(X)$, $g(X) \in R[X]$ and $r, s \in R$ such that $f(r)$, $g(s)$ are units, there is $t \in R$ such that $f(t)$, $g(t)$ are units [or such that $f(t)g(t)$ is a unit].

1.5. EXAMPLES. — (*a*) If R is an infinite field, then R is U-irreducible, because any nontrivial polynomial has only finitely many zeroes. Finite fields clearly fail to be U-irreducible.

(*b*) Let I be a set of indices, finite or infinite. Let $(R_i)_{i\in I}$ be a family of U-irreducible rings. Then $\prod_{i\in I} R_i$ is U-irreducible. A similar result holds for a direct limit over a directed family of U-irreducible rings.

(*c*) R is U-irreducible if and only if R/Rad (R) is U-irreducible, where Rad (R) denotes the Jacobson radical.

(*d*) Let R be semi-local. Then R is U-irreducible if and only if all its residue fields are infinite. This is the example one should keep in mind in paragraphs 2, 3, 4.

(*e*) Let R be a (commutative) von Neumann regular ring. Then R is U-irreducible if and only if all its residue fields are infinite.

1.6. LEMMA. — *Let* R *be* U-*irreducible and* I *an ideal in* R. *Then* R/I *is* U-*irreducible.*

Proof. — It is sufficient to show that the projection $p : R \to R/I$ is continuous with respect to the unit-topologies. Say $\bar{f}(X)$ is the projection in $(R/I)[X]$ of $f(X) \in R[X]$. Let $v \in p^{-1}(D(\bar{f}))$. There is $u \in R$ such that $uf(v)-1 \in I$. Define $g(X) \in R[X]$ by $g(X) = uf(X)-uf(v)+1$. Then $D(g)$ is an open neighbourhood of v in $p^{-1}(D(\bar{f}))$. So p is continuous.

1.7. COROLLARY. — *If* R *is* U-*irreducible then all its residue fields are infinite.*

1.8. LEMMA. — *Let* R *be* U-*irreducible,* (a, b) *unimodular* (*i. e.* $aR+bR = R$). *There is* $r \in R$ *such that* $a+br \in R^*$. *So* R *satisfies the strongest of Bass's stable range conditions.*

Proof. — Choose $s \in R$ such that $D(aX+bs)$ is non-empty. Then choose $t \in D(X) \cap D(aX+bs)$. Take $r = st^{-1}$.

1.9. It is necessary, in order that R be U-irreducible, that R has infinite residue fields and that it satisfies Bass's stable range conditions. But even that is not enough. The ring of totally real algebraic integers in **C** provides a counter-example (due to H. W. Lenstra). (One has only one point in $D(1+X^2)$, as the norm in **Z** of $1+\alpha^2$ is larger than 1 for a non-zero totally real algebraic integer α. It easily follows that the ring is not U-irreducible. We omit the proof that it satisfies Bass's stable range conditions. It is clear that all residue fields are infinite [*see* also 5.4 (5)].) If one takes for R the ring of all algebraic integers in **C** then R is U-irreducible. (Again this example is due to H. W. Lenstra and again we omit the proof.)

1.10. DEFINITION. – A polynomial $f(X) \in R[X]$ is called *primitive* if its coefficients generate the unit ideal. We say that R satisfies the *primitive criterion* if every primitive polynomial $f(X)$ has a unit in its image (i. e. there is $r \in R$ such that $f(r) \in R^*$).

1.11. LEMMA. – *If* R *satisfies the primitive criterion then* R *is* U-*irreducible.*

Proof. – Let $f(X)$, $g(X) \in R[X]$ so that $D(f)$, $D(g)$ are non-empty. Clearly f, g are primitive. Therefore $f(X)g(X)$ has nontrivial image in $(R/M)[X]$ for every maximal ideal M of R. So $f(X)g(X)$ is primitive and $D(f(X)g(X))$ is non-empty.

1.12. It seems unlikely that all U-irreducible rings satisfy the primitive criterion, but we don't know an example to the contrary.

1.13. *Examples.* – (*a*) Let R be a commutative ring. Let S be the multiplicative system in $R[X]$ consisting of primitive polynomials. Then $S^{-1}R[X]$ satisfies the primitive criterion and its maximal spectrum is the same as the one of R. In particular, any maximal spectrum can occur for a ring which satisfies all of Bass's stable range conditions (compare [9]). The proofs are easy.

(*b*) Let R be the ring of continuous complex valued functions on a 1-complex K. Then R satisfies the primitive criterion. Hint: say K is the unit segment. Let $f_0(t), \ldots, f_n(t)$ generate the unit ideal in R. We have to find $g(t) \in R$ such that $f_0(t)g(t)^n + \ldots + f_n(t)$ is free of zeroes on K. For $p \in K$ there is a compact connected subset $V(p)$ of the unit disc in **C**, with area $\geqq 3$, so that $f_0(p)a^n + f_1(p)a^{n-1} + \ldots + f_n(p) \neq 0$ for $a \in V(p)$. One can use the same $V(p)$ in a neighbourhood of p.

(*c*) Let R be a topological ring. (Not necessarily with the unit-topology.) Let A be a dense subring, and suppose that the units in A form an open subset of A in the induced topology. If R is U-irreducible then A is U-irreducible. If R satisfies the primitive criterion, then A satisfies the primitive criterion.

(*d*) Let R be the ring of rational functions $f(X) \in \mathbf{C}(X)$ whose poles lie outside the unit segment. Then R satisfies the primitive criterion [use (*b*) and (*c*)].

1.14. It follows from [24] that the ring of real valued continuous functions on the unit segment is not U-irreducible [compare with 1.13 (*b*)].

1.15. From now on (until 5.1) we assume that R is U-irreducible.

1.16. DEFINITION. – The *unit-topology* on R^n is the weakest topology which makes polynomial maps $R^n \to R$ continuous ($n > 1$). In other words, a basis for the unit-topology on R^n consists of the

$$D(f), \qquad f \in R[X_1, \ldots, X_n], \qquad \text{where } D(f) = \{(r_1, \ldots, r_n) \in R^n \mid f(r_1, \ldots, r_n) \in R^*\}.$$

Alternatively, consider R^n as the set $Aff^n(R)$ of R-valued points in affine n-space. Say Aff^n_R is the scheme Spec $(R[X_1, \ldots, X_n])$. Then $Aff^n(R)$ consists of the sections of the natural morphism $Aff^n_R \to$ Spec (R). Let V be an open subscheme of Aff^n_R. The set V (R) of sections of V $\to$ Spec (R) can be considered as a subset of $Aff^n(R)$, hence of R^n. If I is an ideal that describes the complement of V, then $V(R) = \bigcup_{f \in I} D(f)$. So the V (R) also form a basis for the unit-topology on R^n.

1.17. LEMMA. — R^n *is irreducible.*

Proof. — For $n = 1$ this is our assumption (1.15). Let $n > 1$ and let $D(f)$, $D(g)$ be non-empty in R^n. For $r \in R$ let f_r denote the map $v \mapsto f(v, r)$ from R^{n-1} into R. Choose r such that $D(f_r)$ is non-empty in R^{n-1} and, similarly, choose s such that $D(g_s)$ is non-empty. By induction we may assume there is $v \in R^{n-1}$ such that $f(v, r)$, $g(v, s)$ are units. Applying the case $n = 1$ we may now choose $t \in R$ such that $f(v, t)$, $g(v, t)$ are units.

1.18. DEFINITION. — Let X be a scheme defined over R [i. e. one is given a morphism X $\to$ Spec (R)]. As in 1.16 a basis of the *unit-topology* on X (R) [= set of sections of X $\to$ Spec (R)] will consist of the V (R) with V open in X. Clearly, if Y is an open subscheme of X, the unit-topology on Y (R) is induced from X (R). Also, if $f : X \to Z$ is a morphism of R-schemes, the induced map X (R) $\to$ Z (R) is continuous.

1.19. *Remark.* — We call the topology on X (R) the unit-topology (instead of Zariski topology), so as not to confuse it with the Zariski topology on X itself. For instance, X (R) being irreducible is not the same as X being irreducible.

2. The geometry of the big cell

2.1. In Matsumoto's argument it is essential that every element of the elementary group can be written as a product of certain specific generators. As we want to "start from general position" we have to prove something stronger: if an element x is in general position it can be written as a product of certain generators, with certain subproducts still in general position.

2.2. Let Φ be an irreducible reduced root system and let $G(\Phi, R)$ be the group of R-rational points of the simply connected Chevalley-Demazure group scheme associated to Φ. Let $St(\Phi, R)$ be the corresponding Steinberg group (*cf.* [19]). We have the natural "projection" $St(\Phi, R) \to G(\Phi, R)$, sending $x_\alpha(r)$ to $e_\alpha(r)$ ($r \in R$, $\alpha \in \Phi$.) The image of this projection is $E(\Phi, R)$, the elementary subgroup. Actually one expects $E(\Phi, R) = G(\Phi, R)$, at least for most examples of U-irreducible rings [*see* [20] for conditions which ensure $E(\Phi, R) = G(\Phi, R)$]. We are interested in the kernel $K_2(\Phi, R)$ of the projection $St(\Phi, R) \to E(\Phi, R)$.

2.3. Recall that the big cell Ω of $G(\Phi, R)$ consists of elements x which can be written in "lower-diagonal-upper" form, i. e.:

$$x = e_{-\alpha_1}(u_1) \ldots e_{-\alpha_n}(u_n) h_{\alpha_1}(t_1) \ldots h_{\alpha_l}(t_l) e_{\alpha_1}(v_1) \ldots e_{\alpha_n}(v_n).$$

Here $\alpha_1, \ldots, \alpha_l$ are the simple roots, $\alpha_{l+1}, \ldots, \alpha_n$ the remaining positive roots, u_i, v_i are in R, the t_j are in R^*, the $e_\alpha(r)$ are the usual generators, $h_\alpha(t) = w_\alpha(t)\, w_\alpha(-1)$, $w_\alpha(t) = e_\alpha(t)\, e_{-\alpha}(-t^{-1})\, e_\alpha(t)$ (*cf.* [19]). The order of the positive roots is fixed, once and for all. We call u_i, v_i, t_j the coordinates of x and also denote them by $u_i(x)$, $v_i(x)$, $t_j(x)$.

2.4. We provide $G(\Phi, R)$ with the unit-topology, as indicated in 1.18 [note that $G(\Phi, R)$ is the group of sections of $G_\Phi \underset{\text{Spec}\,\mathbf{Z}}{\times} \text{Spec}(R) \to \text{Spec}(R)$, where G_Φ is the group scheme over **Z**.] For instance, for $SL_m(R)$ this just means that we use the topology induced from the unit-topology on R^{m^2}. The maps $x \mapsto x^{-1}$ and $x \mapsto gx$ (fix $g \in G(\Phi, R)$) are continuous (*see* 1.18). The coordinates u_i, v_i, t_j on Ω induce an embedding $\Omega \to R^{2n+l}$. Both $\Omega \to R^{2n+l}$ and $\Omega \to G(\Phi, R)$ correspond to open immersion of schemes (*see* [5], Prop. 1), so Ω is an irreducible topological space (use 1.17 and 1.18).

2.5. DEFINITION. — The elements of type $e_\alpha(t)$ with α or $-\alpha$ simple, $t \in R^*$, are called *basic*. Let V be a subset of $E(\Phi, R)$. We define a *path from x to y inside* V as a sequence of *steps* $s_1, \ldots, s_m$ such that:

(*a*) The s_i are basic;

(*b*) $xs_1 s_2 \ldots s_m = y$;

(*c*) $x, xs_1, \ldots, xs_1 \ldots s_m$ are in V.

We call x the *starting point* of the path and y the *end point*. (Note that these "paths" are discrete.) We say that V is *path connected* if for every x and y in V there is a path from x to y inside V.

2.6. PROPOSITION. — *Any open subset of Ω is path connected.*

Proof. — We will define subsets $F_0, \ldots, F_M$ of $E(\Phi, R)$ such that:

(A) F_0 consists of basic elements;

(B) $F_i \subseteq F_{i+1}$;

(C) F_i is invariant under $x \mapsto x^{-1}$.

We say that $s_1, \ldots, s_m$ define a path of *order p* from x to y inside V if:

(*a*) the s_i are in F_p;

(*b*) $xs_1 \ldots s_m = y$;

(*c*) $x, xs_1, \ldots, xs_1 \ldots s_m \in V$.

So a path of order zero is just a path in the sense of 2.5. Let V be an open subset of Ω. The idea is to show first that one can join any x, y in V by a path of order M inside V. Next that they can be joined by a path of order $M-1$, and so on. As instructive example we take Φ of type G_2, leaving most details of the simpler cases to the reader.

Let:

F_0 be the set of basic elements;

$F_1 = \{ e_\gamma(t) \mid t \in R,\ \gamma \text{ has height } \pm 1 \}$;

$F_2 = \{ e_\gamma(t) \mid t \in R, \text{ height } (\gamma) \in \{ -5, -1, 1, 5 \}\}$;
$F_3 = \{ e_\gamma(t) \mid t \in R, \gamma \text{ is a root} \}$;
$F_4 = F_3 \cup \{ h_\gamma(t) \mid t \in R^*, \gamma \text{ is a simple root} \}$.

First we have to show that x, y in V can be joined by a path of order 4. We claim it suffices to show that for $x \in V$ the set A_x of z in V which can be reached from x (by a path of order 4 inside V) contains a non-empty open subset.

For, then A_x intersects A_y, by irreducibility, so x can be joined with y *via* a point in $A_x \cap A_y$ $(x, y \in V)$. We refer to this argument as "starting from both ends". Let us show that A_x is a neighbourhood of x. We may assume $x = 1$, as one can shift *via* $z \mapsto x^{-1} z$. Consider the standard expression

$$e_{-\alpha_1}(u_1) \ldots e_{-\alpha_n}(u_n) h_{\alpha_1}(t_1) \ldots h_{\alpha_l}(t_l) e_{\alpha_1}(v_1) \ldots e_{\alpha_n}(v_n)$$

for an element z in Ω. We can view the factors in this expression as steps in a path of order 4 from 1 to z. This path need not lie inside V, but it does for $z = 1$. The set of $(u_1, \ldots, u_n, t_1, \ldots, t_l, v_1, \ldots, v_n)$ for which the path lies inside V is an open subset of R^{2n+l}, corresponding with an open subset of Ω, contained in A_x $(x = 1, l = 2, n = 6)$. We thus have proved existence of paths of order 4. Next we have to show that one can join points in V by paths of order 3 inside V. By the previous result we may restrict ourselves to joining pairs of the form x, xs, where $s \in F_4$. If it happens that $s \in F_3$, there is nothing to prove. Say $s = h_\gamma(u)$. It suffices to show that there is an open neighbourhood N of 1 in R such that one can join x with $xh_\gamma(t)$ for $t \in N$. (Start from both ends.) Again we may assume $x = 1$. Consider the expression

$$e_{-\gamma}(-a(1+ap)^{-1}) e_\gamma(p) e_{-\gamma}(a) e_\gamma(-p(1+ap)^{-1}) \qquad \text{for} \quad h_\gamma(1+ap)$$

(it is defined when $1+ap \in R^*$). View its factors as steps in a path again. There is an open neighbourhood W of the origin in R^2 such that for $(a, p) \in W$ the path lies inside V. As $(0, 0) \in W$ there is $a \in R^*$ with $(a, 0) \in W$ (irreducibility). Fix such an a and solve $1+ap = t$ for p. The solution, $p = (t-1) a^{-1}$, is a rational function $p(t)$ of t. The values of t for which $(a, p(t))$ is in W, form an open neighbourhood N of 1 in R, as required. Thus we mastered the construction of paths of order 3. Now we have to construct paths of order 2 inside V from x to xs, with $s \in F_3$, $x \in V$, $xs \in V$. For example, let $s = e_{\alpha+\beta}(u)$, where α, β are the two simple roots, α shorter than β. We can start from both ends again, and it suffices to find a neighbourhood N of zero in R so that one can join x with $x\, e_{\alpha+\beta}(t)$ for $t \in N$. And again we may assume $x = 1$. Put

$$g(a, b) = e_\alpha(a) e_\beta(b) e_\alpha(-a) e_\beta(-b),$$

$$z = z(a, b, c, p, q, r, s) = g(a, p) g(b, q) g(c, r) e_{3\alpha+2\beta}(s).$$

As z is a product of factors from F_2 (plug in the definition of g) we get a path of order 2 from 1 to z. This path lies inside V when $(a, b, c, p, q, r, s) = (0, \ldots, 0)$. By irreducibility we can choose a, b, c such that the path lies inside V for $p = q = r = s = 0$ and such that simultaneously the "determinant" $abc(a-b)(b-c)(c-a)$ is invertible.

Fix such a, b, c. The determinant occurs (as the determinant of a system of equations in p, q, r) when one tries to solve

$$z(a, b, c, p, q, r, s) = e_{\alpha+\beta}(t) \quad \text{for} \quad p, q, r, s$$

(*see* [10], § 33 for explicit formulas). As the determinant is invertible, the equations can be solved and one finds solutions $p(t)$, $q(t)$, $r(t)$, $s(t)$, rational in t, with $p(0) = q(0) = r(0) = s(0) = 0$. As before the existence of N easily follows. This finishes the proof when $s = e_{\alpha+\beta}(u)$. For $s = e_{2\alpha+\beta}(u)$ or $s = e_{3\alpha+\beta}(u)$ one can use essentially the same proof [same $z(a, b, c, p, q, r, s)$]. Of course the cases $s = e_{-\alpha-\beta}(u)$, $s = e_{-2\alpha-\beta}(u)$, $s = e_{-3\alpha-\beta}(u)$ are dealt with in an analogous fashion. So much for paths of order 2. To get back to order 1, exploit that $e_{3\alpha+2\beta}(a^4 p) = [e_\beta(p), [e_\beta(a), e_\alpha(a)]]$, where $[x, y]$ stands for $xyx^{-1}y^{-1}$. Clearly the right hand side can be written as a product of factors from F_1 and thus we can use the same type of argument as before. Finally, to get a path of order 0, start from both ends and use irreducibility once more.

Hopefully the reader has got the idea by now. For Φ of type different from G_2, let H be the highest height. Put F_0 = the set of basic elements,

$$F_p = \{ e_\alpha(t) \mid t \in R, \ -p \leqq \text{height}(\alpha) \leqq p \} \quad \text{for} \quad p \leqq H,$$
$$F_{H+1} = F_H \cup \{ h_\gamma(t) \mid t \in R^*, \ \gamma \text{ is simple root} \}.$$

The proof can now be left to those readers who checked the argument for G_2 (for type SL_n it can be left to any reader).

3. Statement of the results

3.1. Recall that the Steinberg group St (Φ, R) is generated by the $x_\alpha(t)$ $(\alpha \in \Phi, t \in R)$. Defining relations are:

(R 1)
$$x_\alpha(t) x_\alpha(u) = x_\alpha(t+u),$$

(R 2)
$$[x_\alpha(t), x_\beta(u)] = \prod_{i, j > 0} x_{i\alpha+j\beta}(N_{ij\alpha\beta} t^i u^j),$$

where $\alpha, \beta \in \Phi$, $\alpha+\beta \neq 0$, the $N_{ij\alpha\beta}$ are known integers (independent of t, u and even R); the product at the right hand side of (R 2) is taken in some prescribed order. (The order in this product only matters for type G_2.) In case Φ is of type A_1, i. e. G $(\Phi, R) = SL_2(R)$, one has to replace (R 2) by (R 2′):

(R 2′)
$$x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t) x_\alpha(u) x_\alpha(-t) x_{-\alpha}(t^{-1}) x_\alpha(-t) = x_{-\alpha}(-t^{-2} u),$$

where α is a simple root, $t \in R^*$, $u \in R$. (Of course this relation actually holds for any root and the factors $x_\alpha(\pm t)$ near the middle cancel.)

We denote $x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t)$ by $\tilde{w}_\alpha(t)$ and $\tilde{w}_\alpha(t) \tilde{w}_\alpha(-1)$ by $\tilde{h}_\alpha(t)$, for $\alpha \in \Phi$, $t \in R^*$. [Recall that $w_\alpha(t)$, $h_\alpha(t)$ denote the respective images in E (Φ, R).] The group $K_2(\Phi, R)$ (*see* 2.2) contains elements $\{ t, u \}_\alpha$, central in St (Φ, R), defined by $\{ t, u \}_\alpha = \tilde{h}_\alpha(t) \tilde{h}_\alpha(u) (\tilde{h}_\alpha(tu))^{-1}$, for $\alpha \in \Phi$, $t, u \in R^*$.

3.2. The following relations are standard (*see* [17], § 5; [18], Lemma 9.8):

(*a*) $$\{x, y\}_\alpha \{xy, z\}_\alpha = \{x, yz\}_\alpha \{y, z\}_\alpha;$$

(*b*) $$\{1, 1\}_\alpha = 1;$$

(*c*) $$\{x, y\}_\alpha = \{x^{-1}, y^{-1}\}_\alpha;$$

(*d*) $$\{x, y\}_\alpha = \{x, -xy\}_\alpha;$$

(*e*) $\{w, y\}_\alpha = \{w, (1-w)y\}_\alpha$ (where, of course, w, $1-w$, y have to be units).

Consequences are (*cf.* [17], § 5; [21], Prop. 1.1):

(*f* 1) $$\{x, y^2\}_\alpha \{x, z\}_\alpha = \{x, y^2 z\}_\alpha;$$

(*f* 2) $$\{x, y^2\}_\alpha \{z, y^2\}_\alpha = \{xz, y^2\}_\alpha;$$

(*f* 3) $$\{x, y^{-1}\}_\alpha = \{y, x\}_\alpha = \{x^{-1}, y\}_\alpha;$$

(*f* 4) $$\{x, y\}_\alpha \{x, -y^{-1}\}_\alpha = \{x, -1\}_\alpha;$$

(*f* 5) $$\{x, y^2\}_\alpha = \{x, y\}_\alpha \{y, x\}_\alpha^{-1} = \{x^2, y\}_\alpha;$$

(*f* 6) $\{x, y\}_\alpha = \{x(1-y), y\}_\alpha$. (Again assuming that both sides make sense.);

(*f* 7) $$\{x, 1\}_\alpha = \{1, x\}_\alpha = 1.$$

Also, *if there is* $\beta \in \Phi$ *with* $2(\alpha, \beta)/(\beta, \beta) = -1$, one has:

(*g*) $$\{x, yz\}_\alpha = \{x, y\}_\alpha \{x, z\}_\alpha.$$

Here (,) is an inner product, invariant under the Weyl group. Put

$$\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta).$$

Then:

(*h*) $$[\tilde{h}_\alpha(u), \tilde{h}_\beta(v)] = \{u, v^{\langle \alpha, \beta \rangle}\}_\alpha,$$

whence

(*i*) $$\{u, v^{\langle \alpha, \beta \rangle}\}_\alpha \{v, u^{\langle \beta, \alpha \rangle}\}_\beta = 1;$$

(*j*) $$\{u, v\}_\alpha \{v, u\}_{-\alpha} = 1;$$

(*k*) $$x_\alpha(t)\, x_{-\alpha}(u) = x_{-\alpha}(u(1+tu)^{-1})\, \tilde{h}_\alpha(1+tu) \{-t^{-1}, 1+tu\}_\alpha^{-1}\, x_\alpha(t(1+tu)^{-1}),$$

if t, $1+tu \in R^*$ (*cf.* [21], Prop. 2.7).

Remarks. — (1) Some authors write $\{t, u\}_\alpha$ where we write $\{t, u\}_{-\alpha}$. For symplectic types this makes a difference [compare (*j*)].

(2) When (*g*) holds, one can simplify (*a*), (*d*), (*e*). Then (*b*), (*c*), (*f*) become obvious. Note also that (*f* 3), (*g*) imply (*a*).

3.3. Recall that Φ is of *symplectic type* if there is a root α for which no root β exists with $\langle \alpha, \beta \rangle = -1$. We call such α a *long symplectic root*. Long symplectic roots are:

(1) the roots of A_1 (i. e. in SL_2);

(2) the long roots in B_2 or C_2. ($B_2 = C_2$);

(3) the long roots in C_l, $l \geqq 3$.

In a root system of symplectic type (irreducible and reduced, as always) there is exactly one long symplectic simple root. (We fix an ordering of Φ.) Let α_0 be this root. Then all $\{ t, u \}_\alpha$, with α simple, can be expressed in the form $\{ t^m, u^n \}_{\alpha_0}^p$ with m, n, p integers which only depend on α. This one sees by following the Dynkin diagram, starting from α_0, and applying 3.2 (f 3), (i) along the way. So the $\{ t, u \}_{\alpha_0}$ are sufficient to describe all others. One checks that relations 3.2(a) through (g), for a simple root α distinct from α_0, follow from relations 3.2 (a) through (e) for α_0, *via* the relation $\{ t, u \}_\alpha = \{ t^m, u^n \}_{\alpha_0}^p$. (Use that mn is even.)

Our purpose is to prove:

3.4. THEOREM. — *Let* R *be* U-*irreducible*, Φ *an irreducible reduced root system of symplectic type with long symplectic simple root* α_0. *Then* $K_2(\Phi, R)$ *is isomorphic to the abelian group presented by:*

generators: $\{ t, u \}_{\alpha_0}$ *with* $t, u \in R^*$;

relations: 3.2 (a) *through* (e) *with* $\alpha = \alpha_0$.

3.5. *Remarks.* — (1) Of course the symbols $\{ t, u \}_{\alpha_0}$ in the presentation correspond to the elements $\{ t, u \}_{\alpha_0}$ in $K_2(\Phi, R)$.

(2) Matsumoto didn't include relation 3.2 (d) when stating his result. The reason is that 3.2 (d) is an easy consequence of the other relations when R is a field. In section 8 we will show that 3.2 (d) still follows from the other ones (for U-irreducible R) (*see* 8.1). However, this proof of 3.2 (d) is very technical and the proof of the theorem is much more natural when we keep 3.2 (d) in the list.

3.6. If Φ is not of symplectic type we have relation 3.2 (g) for all $\alpha \in \Phi$. Then (a) through (f) can be simplified in the usual way. Again we choose a long simple root α_0. (If all roots have the same length, call them long.) Note that α_0 need no longer be unique. As in 3.3 we can express $\{ t, u \}_\alpha$, for α simple, in terms of $\{\ ,\ \}_{\alpha_0}$. The non-symplectic version of our result is:

3.7. THEOREM. — *Let* R *be* U-*irreducible*, Φ *a non-symplectic irreducible reduced root system*, α_0 *a long simple root. Then* $K_2(\Phi, R)$ *is isomorphic to the abelian group presented by:*

generators: $\{ t, u \}_{\alpha_0}$ *with* $t, u \in R^*$;

relations: 3.2 (a), (d), (e), (g) *with* $\alpha = \alpha_0$.

3.8. *Remarks.* — *See* 3.5 for Remarks.

4. Proof of Theorems 3.4 and 3.7

4.1. NOTATIONS. — In section 4 let US (Φ, R) denote the group defined by the presentation mentioned in the relevant theorem, i. e. Theorem 3.4 if Φ is of symplectic type, Theorem 3.7 otherwise. Recall that we have in $K_2(\Phi, R)$ formulas $\{t, u\}_\alpha = \{t^m, u^n\}_{\alpha_0}^p$ connecting $\{\ ,\ \}_\alpha$ with $\{\ ,\ \}_{\alpha_0}$. In US (Φ, R) we also have elements called $\{r, s\}_{\alpha_0}$ and it is thus natural to define $\{t, u\}_\alpha$ as $\{t^m, u^n\}_{\alpha_0}^p$ in US (Φ, R). Recall that $\alpha_1, \ldots, \alpha_l$ are the simple roots (α_0 is one of them). Let H_i denote the subgroup of $E(\Phi, R)$ generated by the $h_{\alpha_i}(t)$, $t \in R^*$. Then $t \mapsto h_{\alpha_i}(t)$ defines an isomorphism from R^* into H_i (recall that we use the simply connected group scheme). Let H denote the direct product of the H_i, and view it as a subgroup of $E(\Phi, R)$. Let H act trivially on US (Φ, R). Then

$$\left(\prod_{i=1}^{l} h_{\alpha_i}(t_i), \prod_{j=1}^{l} h_{\alpha_j}(u_j)\right) \mapsto \left(\prod_{i<j} \{t_j, u_i^{\langle \alpha_j, \alpha_i \rangle}\}_{\alpha_j}\right) \left(\prod_i \{t_i, u_i\}_{\alpha_i}\right)$$

is a 2-cocycle, hence defines an extension

$$1 \to US(\Phi, R) \to \tilde{H} \xrightarrow{p} H \to 1.$$

Using the cross-section (set-theoretic) of p, corresponding to the 2-cocycle, one obtains elements $\tilde{h}_{\alpha_i}(t) \in \tilde{H}$ such that $\tilde{h}_{\alpha_i}(t)\, \tilde{h}_{\alpha_i}(u)\, (\tilde{h}_{\alpha_i}(tu))^{-1} = \{t, u\}_{\alpha_i}$ in H and such that 3.2 (*h*) also holds in $\tilde{H}$. (Note that actually 3.2 (*a*) through (*i*) hold in H, with α and β simple roots.) Of course US (Φ, R) is central in $\tilde{H}$. For details (*see* [17]). The 2-cocycle was chosen such that one has a homomorphism $\pi : \tilde{H} \to St(\Phi, R)$ defined by

$$\pi(\tilde{h}_{\alpha_i}(t)) = \tilde{h}_{\alpha_i}(t).$$

4.2. We do not want to enlarge $\tilde{H}$ to a group which also covers the Weyl group, as is done in [17]. This would not help at all to get "general position" formulas, as the Weyl group is something discrete. If one is not satisfied with "general position" one tends to end up with complicated formulas (*cf.* [6]). (For fields the formulas are still manageable; the trouble comes in when one doesn't have a Bruhat-decomposition.)

4.3. NOTATIONS. — Let U^- be the subgroup of St (Φ, R) generated by the $x_\alpha(t)$ with α negative. Let U^+ be the subgroup of St (Φ, R) generated by the $x_\alpha(t)$ with α positive. Recall that U^+, U^- are isomorphic with their images in $E(\Phi, R)$. Let us identify them with these images so that we get coordinate functions u_i, v_i on U^-, U^+ respectively. When $\alpha_i = \beta$ we also write $u_{-\beta}$, v_β for u_i, v_i respectively. Fix a simple root α. Then put

$$U^-_{(-\alpha)} = \{x \in U^- \mid u_{-\alpha}(x) = 0\},$$
$$U^+_{(\alpha)} = \{x \in U^+ \mid v_\alpha(x) = 0\}.$$

This defines subgroups of St (Φ, R), both of them normalized by the $x_\alpha(t)$, $x_{-\alpha}(t)$. If $x \in U^-$ then $x x_{-\alpha}(-u_{-\alpha}(x)) \in U^-_{(-\alpha)}$. In analogy with the definition of Ω we let $\tilde{\Omega}$ be the direct product $U^- \times \tilde{H} \times U^+$ (direct product as sets). We define $\pi : \tilde{\Omega} \to St(\Phi, R)$

by $\pi(u, \tilde{h}, v) = u\,\pi(\tilde{h})\,v$. [Recall we have already $\pi : \tilde{H} \to \mathrm{St}(\Phi, R)$.] We define $p : \tilde{\Omega} \to \Omega$ by composing π with the projection $\mathrm{St}(\Phi, R) \to E(\Phi, R)$. (The composite map has Ω as its image.) We provide $\tilde{\Omega}$ with the weakest topology which makes p continuous. Next we want to define partially defined maps $L(g) : \tilde{\Omega} \to \tilde{\Omega}$ such that $\pi(L(g)(z)) = g\,\pi(z)$ when z is in the domain of $L(g)$. [Here $g \in \mathrm{St}(\Phi, R)$.] We put

$$\begin{aligned} &L(x_\alpha(t))(u\,x_{-\alpha}(w), \tilde{h}, v) \\ &\quad = (x_\alpha(t)\,u\,x_\alpha(-t)\,x_{-\alpha}(w(1+tw)^{-1}), \tilde{h}_\alpha(1+tw)\{-t^{-1}, 1+tw\}_\alpha^{-1}\tilde{h}, x_\alpha(t')\,v), \end{aligned}$$

where α is simple, $t \in R^*$, $u \in U^-_{(-\alpha)}$, $w \in R$, $\tilde{h} \in \tilde{H}$, $v \in U^+$, $1+tw \in R^*$, t' is such that $e_\alpha(t(1+tw)^{-1})\,p(\tilde{h}) = p(\tilde{h})\,e_\alpha(t')$. [Compare 3.2 (*k*).] We also put

$$L(x_{-\alpha}(t))(u, \tilde{h}, v) = (x_{-\alpha}(t)\,u, \tilde{h}, v),$$

where α is simple, $t \in R$, $u \in U^-$, $\tilde{h} \in \tilde{H}$, $v \in U^+$.

Similarly we put

$$\begin{aligned} &R(x_{-\alpha}(w))(u, \tilde{h}, x_\alpha(t)\,v) \\ &\quad = (u\,x_{-\alpha}(w'), \tilde{h}\,\tilde{h}_\alpha(1+tw)\{w, 1+tw\}_\alpha^{-1}, x_\alpha(t(1+tw)^{-1})\,x_{-\alpha}(-w)\,v\,x_{-\alpha}(w)) \end{aligned}$$

if α is simple, $w \in R^*$, $u \in U^-$, $h \in \tilde{H}$, $t \in R$, $v \in U^+_{(\alpha)}$, $1+tw \in R^*$, w' is such that $e_{-\alpha}(w')\,p(\tilde{h}) = p(\tilde{h})\,e_{-\alpha}(w(1+tw)^{-1})$.

And we put

$$R(x_\alpha(t))(u, \tilde{h}, v) = (u, \tilde{h}, v\,x_\alpha(t))$$

if α is simple, $t \in R$, $u \in U^-$, $\tilde{h} \in \tilde{H}$, $v \in U^+$.

4.4. Lemma. — *Let α be simple, $t \in R^*$, $z \in \tilde{\Omega}$. Then $L(x_\alpha(t))(z)$ is defined if and only if $e_\alpha(t)\,p(z) \in \Omega$.*

Proof. — We can always write z in the form $(u x_{-\alpha}(w), \tilde{h}, v)$ with $u \in U^-_{(-\alpha)}$. If $L(x_\alpha(t))(z)$ is defined, it is an element of $\tilde{\Omega}$ and $e_\alpha(t)\,p(z) = p(L(x_\alpha(t))(z)) \in \Omega$. Conversely, assume $e_\alpha(t)\,p(z) \in \Omega$. We have to show that $1+tw$ is a unit. If it isn't a unit, pass to a residue field where it vanishes. One finds, with R replaced by the residue field, that

$$e_\alpha(t)\,p(z) \in U^-\,e_\alpha(t)\;e_{-\alpha}(-t^{-1})\,p(\tilde{H})\,U^+ = U^-\,w_\alpha(t)\,p(\tilde{H})\,U^+,$$

which is disjoint from Ω by the Bruhat-decomposition. This contradicts $e_\alpha(t)\,p(z) \in \Omega$.

4.5. Let α be simple. It is clear that $L(x_\alpha(t))$ defines a homeomorphism from an open subset of $\tilde{\Omega}$ to an open subset of $\tilde{\Omega}$. The map $L(x_{-\alpha}(t))$ defines a homeomorphism from $\tilde{\Omega}$ to $\tilde{\Omega}$. It is easy to see that $R(x_{-\alpha}(w))(z)$ is defined for $z \in \tilde{\Omega}$ with $p(z)\,e_{-\alpha}(w) \in \Omega$. (Compare the Lemma.) Just as $\pi(L(g))(z) = g\,\pi(z)$ one has $\pi(R(g)(z)) = \pi(z)\,g$, but we will not use this. [We will use that $p(R(g)(z))$ is the same as the image of $\pi(z)\,g$ in Ω.]

4.6. *See* ([18], § 12) for an exposition of left-right aspects. The crucial result is (*cf.* [17], lemma 7.1):

LEMMA. — *Let* α, β *be simple*, $t, w \in R^*$, $y, z \in R$. *Then the maps* $L(x_\alpha(t))$, $L(x_{-\alpha}(z))$ *commute with the maps* $R(x_\beta(y))$, $R(x_{-\beta}(w))$ *as far as the composites are defined* [*e. g.* $L(x_\alpha(t))\,R(x_{-\beta}(w))(u, \tilde{h}, v) = R(x_{-\beta}(w))\,L(x_\alpha(t))(u, \tilde{h}, v)$ *if both sides are defined*].

Proof. — For combinations like $L(x_{-\alpha}(z))$ with $R(x_{-\beta}(w))$ this is a trivial consequence of the definitions: $L(x_{-\alpha}(z))\,R(x_{-\beta}(w))(u, \tilde{h}, v)$ has the same projection in Ω as $R(x_{-\beta}(w))\,L(x_{-\alpha}(z))(u, \tilde{h}, v)$, and their $\tilde{H}$-components look the same. Remains the combination $L(x_\alpha(t))$ with $R(x_{-\beta}(w))$. If $\alpha \neq \beta$ then it is still easy: one notes that $R(x_{-\beta}(w))$ doesn't affect the coordinate $u_{-\alpha}$ and that $L(x_\alpha(t))$ leaves the coordinate v_β alone. Consequently the $\tilde{H}$-components of

$$L(x_\alpha(t))\,R(x_{-\beta}(w))(u, \tilde{h}, v) \qquad \text{and} \qquad R(x_{-\beta}(w))\,L(x_\alpha(t))(u, \tilde{h}, v)$$

still look the same. So we are left with the combination $L(x_\alpha(t))$, $R(x_{-\alpha}(w))$. In other words, our computation explains why one only needs relations that come from rank 1 or from the action of an $h_{\alpha_i}(u)$ on the rank 1 subgroup corresponding to α_j [as expressed in 3.2 (*g*), (*h*)]. Now let us deal with the remaining combination. Again the $\tilde{H}$-components are the relevant ones, as the images in $E(\Phi, R)$ still coincide. Say $(u, \tilde{h}, v) = (\hat{u}\, x_{-\alpha}(a), \tilde{h}, x_\alpha(b)\,\hat{v})$ with $\hat{u} \in U^-_{(-\alpha)}$, $\hat{v} \in U^+_{(\alpha)}$, $a, b \in R$. Say $r = \alpha(p(\tilde{h}))$, i. e. $p(\tilde{h})\,x_\alpha(1) = x_\alpha(r)\,p(\tilde{h})$. If α is a long symplectic root then r is a square and therefore $\{x, r\}_\alpha \{w, x\}_\alpha = \{wr^{-1}, x\}_\alpha$ for $x \in R^*$. [Use 3.2 (*f*).] If α is not a long symplectic root then the same relation holds, so we may use it in any case.

Put $A = 1+bw+at+abtw+r^{-1}tw$, $B = A/(1+bw)(1+at)$, $C = a+abw+r^{-1}w$. We get

$$\begin{aligned} &L(x_\alpha(t))\,R(x_{-\alpha}(w))(u, \tilde{h}, v) \\ &\quad = L(x_\alpha(t))(\hat{u}\, x_{-\alpha}(C(1+bw)^{-1}), \tilde{h}\,\tilde{h}_\alpha(1+bw)\{w, 1+bw\}_\alpha^{-1}, \star) \\ &\quad = (\star, \tilde{h}_\alpha(A(1+bw)^{-1})\{-t^{-1}, A(1+bw)^{-1}\}_\alpha^{-1}\,\tilde{h}\,\tilde{h}_\alpha(1+bw)\{w, 1+bw\}_\alpha^{-1}, \star). \end{aligned}$$

One sees from 3.2 (*h*), (*f* 1), (*g*) that $[\tilde{h}_\alpha(z), \tilde{h}] = \{z, r\}_\alpha$. So the $\tilde{H}$-component is

$$\{-t^{-1}, A(1+bw)^{-1}\}_\alpha^{-1}\{w, 1+bw\}_\alpha^{-1}\{A(1+bw)^{-1}, 1+bw\}_\alpha\{1+bw, r\}_\alpha^{-1}\,\tilde{h}_\alpha(A)\,\tilde{h}.$$

Similarly we get the following $\tilde{H}$-component for $R(x_{-\alpha}(w))\,L(x_\alpha(t))(u, \tilde{h}, v)$:

$$\{-t^{-1}, 1+at\}_\alpha^{-1}\{A(1+at)^{-1}, r\}_\alpha^{-1}\{1+at, A(1+at)^{-1}\}_\alpha\{w, A(1+at)^{-1}\}_\alpha^{-1}\,\tilde{h}_\alpha(A)\,\tilde{h}.$$

So if Q is the quotient of the first $\tilde{H}$-component by the second, then

$$\begin{aligned} Q &= \{-t^{-1}, (1+at)B\}_\alpha^{-1}\{r^{-1}w, 1+bw\}_\alpha^{-1} \\ &\quad \times\{(1+at)B, 1+bw\}_\alpha\{-t^{-1}, 1+at\}_\alpha\{r^{-1}w, (1+bw)B\}_\alpha\{1+at, B(1+bw)\}_\alpha^{-1} \\ &= \{-t^{-1}(1+at), B\}_\alpha^{-1}\{1+at, B\}_\alpha\{r^{-1}w(1+bw), B\}_\alpha\{1+bw, B\}_\alpha^{-1} \\ &\quad \times\{1+at, B\}_\alpha^{-1}\{B, 1+bw\}_\alpha. \end{aligned}$$

Now $1-B = -r^{-1}tw(1+at)^{-1}(1+bw)^{-1}$, so

$$Q = \{r^{-1}w(1+bw)^{-1}, B\}_\alpha^{-1}\{(1+bw)^2, B\}_\alpha^{-1}\{r^{-1}w(1+bw), B\}_\alpha$$

which is 1. This proves the Lemma. (Compare 5.12.)

4.7. We are going to construct a group G from the "group chunk" $\tilde{\Omega}$.

DEFINITIONS. — Let f, g be two maps each defined on an open dense subset of $\tilde{\Omega}$ and with values in $\tilde{\Omega}$. We call f, g *equivalent* if they coincide on an open dense subset of $\tilde{\Omega}$ (necessarily contained in the intersection of the two domains). We denote the class of f by $[f]$.

Let G be the set of classes that have at least one representative f such that

(i) f has an open image and is a homeomorphism;

(ii) if β is a simple root and $w \in R^*$ then $R(x_{-\beta}(w))f$ coincides with $fR(x_{-\beta}(w))$ where both are defined. Similarly, for $y \in R$, the composite maps $fR(x_\beta(y))$ and $R(x_\beta(y))f$ coincide where both are defined;

(iii) if $s \in US(\Phi, R)$ then f commutes with the map $(u, \tilde{h}, v) \mapsto (u, \tilde{hs}, v)$;

(iv) there is $x \in St(\Phi, R)$ such that $\pi(f(u, \tilde{h}, v)) = x\pi(u, \tilde{h}, v)$ whenever the left hand side is defined.

We write $p(x) = \varphi(f)$ when x is as in (iv). We also write $\varphi([f]) = p(x)$. (Note that x only depends on the class of f.)

A representative f which satisfies (i) through (iv) is called *special*. The composite of two elements of G is formed as follows. Let $[f]$, $[g] \in G$ have special representatives f, g respectively. Put $h = f \circ g$, i. e. $h(x) = f(g(x))$ whenever the latter is defined. We claim that h is again special so that its class $[h]$ is again an element of G. We put $[f] \circ [g] = [h]$. (The class of h does not depend on the particular choice of representatives.) In order to see that h is special, note that if, for example, $gR(x_{-\beta}(w))$ and g are both defined at some point t, the map $R(x_{-\beta}(w))g$ must also be defined at that point, because $p(g(t))e_{-\beta}(w) = \varphi(g)p(t)e_{-\beta}(w) = p(gR(x_{-\beta}(w))(t)) \in \Omega$. (Compare 4.5.)

It is easy to see that with the above composition G is a group and $\varphi : G \to E(\Phi, R)$ a homomorphism. It is also clear from preceding results that G contains the classes of $L(x_\alpha(t))$, $L(x_{-\alpha}(z))$ for α simple, $t \in R^*$, $z \in R$. Further, if $s \in US(\Phi, R)$ the class of the map $(u, \tilde{h}, v) \mapsto (u, \tilde{hs}, v)$ is in G and even in the kernel of φ.

4.8. LEMMA. — *Let $[f]$, $[g] \in G$ with special representatives f, g respectively. If f and g coincide at one point they coincide at any point in the intersection of their domains. In particular,* $[f] = [g]$. (Compare [18], Lemma 12.8.)

Proof. — Say $f(y) = g(y)$ and let z be in the domain of both f and g. We have to show that $f(z) = g(z)$. Let V be the intersection of the domains of f and g. If $R(x_{-\beta}(w))(y) \in V$ then

$$fR(x_{-\beta}(w))(y) = R(x_{-\beta}(w))f(y) = R(x_{-\beta}(w))g(y) = gR(x_{-\beta}(w))(y),$$

so f and g also agree at $R(x_{-\beta}(w))(y)$. Similarly, if y looks like $(u, \tilde{h}, v)$ and if $s \in US(\Phi, R)$, then f also agrees with g at $(u, \tilde{h}s, v)$. Now use that there is a path from $p(y)$ to $p(z)$ inside the open set $p(V)$ (*see* Prop. 2.6) to join z with y and to pass from the equality $f(y) = g(y)$ to the equality $f(z) = g(z)$.

4.9. NOTATION. – Let $d \in \tilde{H}$. Let $L(d)$ be the map $(u, \tilde{h}, v) \mapsto (\pi(d)\, u\, \pi(d^{-1}), d\tilde{h}, v)$ from $\tilde{\Omega}$ to $\tilde{\Omega}$. Its class is an element of G, denoted by $[d]$, and $L(d)$ is a special representative of $[d]$. (Compare Proof of 4.6.)

4.10. LEMMA [*cf.* [17], *lemma* 6.11 (*d*)]. – *The kernel of* φ *consists of the* $[d]$ *with* $d \in US(\Phi, R)$. *These* $[d]$ *are central elements of* G.

Proof. – The latter statement is part of the definition of G. For the first we only need to test at one point, by Lemma 4.8. If $\varphi(f) = 1$, then $f(u, \tilde{h}, v) = (u, s\tilde{h}, v)$ for some $u \in U^-$, $\tilde{h} \in \tilde{H}$, $v \in U^+$, $s \in US(\Phi, R)$ and we see that $[f]$ must be $[s]$.

4.11. NOTATIONS. – Let α be simple. Put $x_{-\alpha}(t) = [L(x_{-\alpha}(t))]$, for $t \in R$. Choose $\tilde{w}_\alpha = [L(x_\alpha(1))]\,[L(x_{-\alpha}(-1))]\,[L(x_\alpha(1))]$ in G. So $\varphi(\tilde{w}_\alpha) = w_\alpha(1)$. Put $x_\alpha(t) = \tilde{w}_\alpha\, x_{-\alpha}(-t)\, \tilde{w}_\alpha^{-1}$. (We will come back to this definition in 4.13.) More generally, if γ is a positive root, choose $\tilde{w}_\gamma \in G$ with $\varphi(\tilde{w}_\gamma) = w_\gamma(1)$. Let $x_{-\gamma}(t)$ denote the class of the map $(u, \tilde{h}, v) \mapsto (x_{-\gamma}(t)\, u, \tilde{h}, v)$ and put $x_\gamma(t) = \tilde{w}_\gamma\, x_{-\gamma}(-t)\, \tilde{w}_\gamma^{-1}$.

4.12. LEMMA. – *The Steinberg relations hold in* G.

Proof. – Let α be simple. Then $x_{-\alpha}$ is obviously additive and therefore x_α is additive. Next consider $\tilde{w}_\alpha(t)\, x_{-\alpha}(u)\, \tilde{w}_\alpha(-t)$, where $\tilde{w}_\alpha(t) = x_\alpha(t)\, x_{-\alpha}(-t^{-1})\, x_\alpha(t)$. Up to a central element we have $\tilde{w}_\alpha(t) = \tilde{w}_\alpha\,[\tilde{h}_\alpha(t^{-1})]$. Also $[\tilde{h}_\alpha(t^{-1})]\, x_{-\alpha}(u)\, [\tilde{h}_\alpha(t^{-1})]^{-1}$ is easily seen to be $x_{-\alpha}(t^2 u)$. [Test for instance at (1, 1, 1) and then apply 4.8.] So $\tilde{w}_\alpha(t)\, x_{-\alpha}(u)\, \tilde{w}_\alpha(t)^{-1} = x_\alpha(-t^2 u)$ or $\tilde{w}_\alpha(-t)\, x_\alpha(-t^2 u)\, \tilde{w}_\alpha(t) = x_{-\alpha}(u)$, as required. This shows that the Steinberg relations hold in the rank 1 case.

Choose $a \in R^*$ with $1 - a^2 \in R^*$. Let γ be a positive root. Choose $\tilde{h}_{-\gamma} \in \tilde{H}$ such that $p(\tilde{h}_{-\gamma}) = h_{-\gamma}(a)$. Also choose $\tilde{h}_\gamma \in \tilde{H}$ with $p(\tilde{h}_\gamma) = h_\gamma(a)$. Then $\varphi([\tilde{h}_{-\gamma}]) = h_{-\gamma}(a)$ and $[[\tilde{h}_{-\gamma}], x_{-\gamma}(t(a^2-1)^{-1})] = x_{-\gamma}(t)$, as one sees by testing at (1, 1, 1). So $x_{-\gamma}(t)$ satisfies Steinberg's normalisation (*cf.* [22], 9.2) and therefore $x_\gamma(t)$ does it too. (Recall Steinberg's trick which tells that "$\varphi(x) = \varphi(x')$ and $\varphi(y) = \varphi(y')$" implies "$[x, y] = [x', y']$", because ker (φ) is central. Thus

$$x_\gamma(t) = [\tilde{w}_\gamma [\tilde{h}_{-\gamma}]\, \tilde{w}_\gamma^{-1}, \tilde{w}_\gamma\, x_{-\gamma}(-t(a^2-1)^{-1})\, \tilde{w}_\gamma^{-1}] = [[\tilde{h}_\gamma], x_\gamma(t(a^2-1)^{-1})].)$$

Because of this normalisation we have $\tilde{w}_\gamma\, x_\beta(t)\, \tilde{w}_\gamma^{-1} = x_\delta(\varepsilon t)$ when

$$w_\gamma(1)\, e_\beta(t)\, w_\gamma(1)^{-1} = e_\delta(\varepsilon t), \qquad \varepsilon = \pm 1.$$

(Use Steinberg's trick again.) Those Steinberg relations which involve only negative roots are easy [test at (1, 1, 1)] and by means of the $\tilde{w}_\gamma$ we can reduce the arbitrary case to that situation (see [3], 1.5 Th. 2 and 1.7 Cor. 2.)

4.13. We have defined $x_\alpha(t)$ as $\tilde{w}_\alpha x_{-\alpha}(-t)\tilde{w}_\alpha^{-1}$. If α is simple and $t \in R^*$ we also have $[L(x_\alpha(t))]$. One expects $x_\alpha(t) = [L(x_\alpha(t))]$, which is indeed the case. For, by Steinberg's normalisation and Steinberg's trick, $x_\alpha(t) = [[\tilde{h}_\alpha], [L(x_\alpha(t(a^2-1)^{-1}))]]$. By testing at $(1, 1, x_\alpha(t(a^2-1)^{-1}))$ the required equality follows.

Similarly one sees that $x_{-\alpha}(t^{-1}-1)\,x_\alpha(-1)\,x_{-\alpha}(1-t)\,x_\alpha(-t^{-1}) = [\tilde{h}_\alpha(t)]$ by testing at (1, 1, 1). [Note that the analogous identity holds in St (Φ, R) by 3.2 (*f* 7), (*k*).]

By Lemma 4.12 we have a homomorphism τ from St (Φ, R) to G sending $x_\gamma(t)$ to $x_\gamma(t)$. It sends $\tilde{h}_\alpha(u)$ to $[\tilde{h}_\alpha(u)]$ (α simple, $u \in R^*$). Let $x = x_{\gamma_1}(t_1) \ldots x_{\gamma_r}(t_r) \in K_2(\Phi, R)$ with γ_i equal to a simple root or its negative and $t_i \in R^*$, $i = 1, \ldots, r$. (Any $x \in K_2(\Phi, R)$ can be written this way, as St (Φ, R) is generated by the $\tilde{w}_\alpha(1)$ and the $x_\alpha(t)$ with α simple, $t \in R^*$.) As $\varphi\tau(x) = 1$ we have $\tau(x) = [d]$ for some $d \in US(\Phi, R)$. Also,

$$\pi(L(x_{\gamma_1}(t_1)) \ldots L(x_{\gamma_r}(t_r))(u, \tilde{h}, v)) = x\,\pi(u, \tilde{h}, v)$$

whenever the left hand side is defined. By Lemma 4.8 the left hand side is $\pi((u, d\tilde{h}, v))$ or $\pi(d)\,\pi(u, \tilde{h}, v)$. We see that $\pi(d) = x$. Therefore the map $x \mapsto \tau(x) = [d] \mapsto d$ is the inverse of the restriction of π to US (Φ, R). (Use $\tau(\tilde{h}_\alpha(u)) = [\tilde{h}_\alpha(u)]$.) Theorems 3.4 and 3.7 follow.

5. The rank 1 case

5.1. We start all over again, with a slightly more general type of ring, but restricting ourselves to type SL_n. Again we prove our intermediate results in big subsets ("general position"), but these subsets are no longer open dense. In fact our ring may now be finite so that we now have a discrete situation. Section 5 is still easy, section 6 will be messier.

5.2. Let us recall some notations and terminology from [12]. Let k be a positive integer. The commutative ring R is called k-fold stable if, given k pairs (a_i, b_i) of elements in R with $a_i R + b_i R = R$, there is $r \in R$ such that each of the $a_i + b_i r$ is a unit. A pair $(a, b) \in R^2$ with $aR + bR = R$ is called unimodular. (Unimodularity can be checked in the residue fields of R. This will often be the convenient way.) The group D (R) is defined by the following presentation:

Generators are the symbols $\langle a, b \rangle$ with $a, b \in R$ such that $1 + ab \in R^*$, where R^* is the group of units of R.

Relations are:

(D 0) the group D (R) is abelian;

(D 1) $\langle a, b \rangle \langle -b, -a \rangle = 1$;

(D 2) $\langle a, b \rangle \langle a, c \rangle = \langle a, b+c+abc \rangle$;

(D 3) $\langle a, bc \rangle = \langle ab, c \rangle \langle ac, b \rangle$.

Here it is assumed that the left hand side makes sense, e. g. in (D 2) one needs $a, b, c \in R$ such that $1+ab$, $1+ac \in R^*$.

For $n \geqq 2$ we have a Steinberg group $\mathrm{St}(n, R) = \mathrm{St}(A_{n-1}, R)$ (*cf.* § 3). As usual the generators are denoted by $x_{ij}(r)$ and we put $w_{ij}(u) = x_{ij}(u)\, x_{ji}(-u^{-1})\, x_{ij}(u)$, $h_{ij}(u) = w_{ij}(u)\, w_{ij}(-1)$, for $u \in R^*$, $i \neq j$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n$. [We do not write $\tilde{h}_{ij}(u)$ for the element of $\mathrm{St}(n, R)$ so that h_{ij} is analogous to $\tilde{h}_\alpha$ of section 3, not h_α of section 2.] We put $\{u, v\}_{ij} = h_{ij}(u)\, h_{ij}(v)\, (h_{ij}(uv))^{-1}$, for $u, v \in R^*$. If $1+ab \in R^*$ we define an element of $\mathrm{St}(n, R)$ by

$$\langle a, b \rangle_{ij} = x_{ji}(-b(1+ab)^{-1})\, x_{ij}(a)\, x_{ji}(b)\, x_{ij}(-a(1+ab)^{-1})\, h_{ij}^{-1}(1+ab).$$

If a is also a unit, $\langle a, b \rangle_{ij} = \{-a^{-1}, 1+ab\}_{ij}^{-1}$ [*see* 3.2 (*k*)]. Note that $\langle a, b \rangle_{ij} \in K_2(n, R) = \ker(\mathrm{St}(n, R) \to E(n, R))$.

5.3. THEOREM. — *Let* R *be* 3-*fold stable:*

(i) $K_2(2, R)$ *is a central subgroup of* $\mathrm{St}(2, R)$, *generated by the* $\{u, v\}_{ij}$ *with* $u, v \in R^*$, $ij = 12$ *or* 21;

(ii) *there is a homomorphism* $\tau : K_2(2, R) \to D(R)$, *sending* $\langle a, b \rangle_{12}$ *to* $\langle a, b \rangle$ *for* $a, b \in R$ *with* $1+ab \in R^*$.

5.4. EXAMPLES AND REMARKS. — (1) Note that $\{u, v\}_{12}\, \{v, u\}_{21} = 1$ by 3.2 (*j*).

(2) In the next sections we will see that $\ker \tau$ is just the kernel of the standard map $K_2(2, R) \to K_2(R)$.

(3) A semi-local ring is *k*-fold stable if and only if each of its residue fields has at least $k+1$ elements. This is the example to keep in mind. For (commutative) von Neumann regular rings we have the same criterion (that all residue fields should contain at least $k+1$ elements). Of course U-irreducible rings are also *k*-fold stable (*cf.* 1.8).

(4) The analogues of 1.5 (*b*), (*c*), 1.6 hold for *k*-fold stability. [For the case $k = 1$, *see* [24]. For $k \geqq 1$, use that if I is an ideal in the *k*-fold stable ring R, and if $ax+by-1 \in I$, there is $r \in R$ such that $(a+r(ax+by-1), b)$ is unimodular, by ibid. Theorem 1.]

(5) It has been shown by H. W. Lenstra that the ring of totally real algebraic integers in **C** is *k*-fold stable for any $k \geqq 1$. Compare this with 1.9.

(6) In general $\tau : K_2(2, R) \to D(R)$ is not injective. If one wants to get an injective map, and thus a presentation for $K_2(2, R)$, one has to use a different list of relations. Of course this list should contain only relations that are known to hold for the $\langle a, b \rangle_{12}$ (given that the ring is 3-fold stable). The following list of relations will do. [That they hold follows from two facts: R is commutative and $K_2(2, R)$ is central in $\mathrm{St}(2, R)$. For the second fact see part (i) of the theorem.]

(A) $\langle t, u \rangle_{12}$ commutes with $\langle a, b \rangle_{12}$;

(B) $\langle t+u, v \rangle_{12} = \langle u, v \rangle_{12}\, \langle t, v(1+uv)^{-1} \rangle_{12}\, \{(1+tv+uv)(1+uv)^{-1}, 1+uv\}_{12}$, where $\{p, q\}_{12}$ stands for $\langle pq-q, q^{-1} \rangle_{12}^{-1}$;

(C) $\langle t, u+v \rangle_{12} = \langle t, u \rangle_{12}\, \langle t(1+ut)^{-1}, v \rangle_{12}\, \{1+tu, (1+tu+tv)(1+tu)^{-1}\}_{12}$;

(D) $\langle u, t^{-2}v \rangle_{12} = \langle ut^{-2}, v \rangle_{12}\, \langle uv, t^{-2} \rangle_{12}$;

(E) $\langle u, v \rangle_{12} = \langle v(1+uv)^{-1}, -u \rangle_{12}$.

Compare [7], § 9. The construction of a map τ, based on this list, is quite similar to what follows, but more cumbersome.

5.5. The proof of the theorem is given in the remainder of this section. So R will be 3-fold stable. (We do not use this in the next two lemmas.)

5.6. For $u, v \in R^*$ put $\{u, v\} = \langle (u-1)v^{-1}, v \rangle \in D(R)$.

LEMMA (*cf.* [16]). — *The following relations hold in* D (R), *whenever the left hand side is defined:*

$$\langle a, 0 \rangle = 1, \quad \langle a, 1 \rangle = 1, \quad \langle 0, a \rangle = 1, \quad \langle -1, a \rangle = 1;$$
$$\langle b, a \rangle \langle c, a \rangle = \langle b+c+abc, a \rangle;$$
$$\langle a, b \rangle = \langle -b, a(1+ab)^{-1} \rangle;$$
$$\{u, -u\} = 1, \quad \{u, 1-u\} = 1;$$
$$\{tu, v\} = \{t, v\}\{u, v\};$$
$$\{t, uv\} = \{t, u\}\{t, v\};$$
$$\{u, v\}\{v, u\} = 1;$$
$$\langle a, b \rangle \langle b, a \rangle = \{1+ab, -1\}.$$

Proof. — Use

$$\langle a, 0 \rangle^2 = \langle a, 0 \rangle = \langle 0, -a \rangle^{-1};$$
$$\langle a, 1 \rangle^2 = \langle a, 1 \rangle = \langle -1, -a \rangle^{-1};$$
$$\langle b, a \rangle \langle c, a \rangle = (\langle -a, -b \rangle \langle -a, -c \rangle)^{-1};$$
$$\langle a, b \rangle \langle -a(1+ab)^{-1}, b \rangle = \langle 0, b \rangle;$$
$$\{u, -u\} = \langle -(u-1)u^{-1}, -u \rangle = \langle u, (u-1)u^{-1} \rangle^{-1}$$
$$= \langle -(u-1)u^{-1}, u(1+u-1)^{-1} \rangle^{-1};$$
$$\{uv, -uv\} = \{u, -u\}\{u, v\}\{v, u\}\{v, -v\}.$$

5.7. LEMMA. — *Let* $u, v \in R^*$. *Then*

$$\langle u+v, -v^{-1} \rangle_{ij} = w_{ij}(u)\, w_{ij}(v)\, h_{ij}(-uv^{-1})^{-1} = \{u, -v\}_{ij}^{-1},$$

which is central.

Proof. — Write out $w_{ij}(u)\, w_{ij}(v)$ and follow the obvious path. Centrality follows from [18], Corollary 9.3.

5.8. LEMMA. — *Any element of* St (2, R) *can be written in normal form* $hx_{12}(a)\, x_{21}(b)\, x_{12}(c)$ *where h is in the subgroup* H *of* St (2, R) *generated by the* $h_{ij}(t)$.

Remarks. — Once this lemma is proved, it is easy to see that $K_2(2, R) \subset H$ and part (i) of the theorem follows as in the proof of [18], Theorem 9.11. Note that the normal form in the lemma is not unique.

Proof of lemma. — Let X be the set of elements that can be written in normal form. It suffices to show that X is invariant under right multiplication by $x_{21}(1)$ and $x_{12}(u)$, as these generate St (2, R). Consider $hx_{12}(a)\,x_{21}(b)\,x_{12}(c)\,x_{21}(1) = y$. By threefold stability choose t such that t, $1+bt$, $1+c-t$ are units. Then

$$y = h\,x_{12}(a)\,x_{12}(\star)$$
$$\times\langle b, t\rangle_{21}\,h_{21}(\star)\,x_{21}(\star)\,x_{21}(\star)\,\langle c-t, 1\rangle_{12}\,h_{12}(\star)\,x_{12}(\star)$$

and, *via* Lemma 5.7, it easily follows that $y \in X$ (*cf.* [18], Lemma 9.2). Here, and in the sequel, we use stars where the precise form of an expression is irrelevant.

5.9. The proof of part (ii) of the theorem is based on setting up a calculus with normal forms, or rather with equivalence classes of normal forms. We will describe the proof in a different language, viz. with partially defined maps (*cf.* § 4).

DEFINITIONS. — Let $1 \to D(R) \to \tilde{H} \xrightarrow{\sigma} R^* \to 1$ be the central extension defined by the 2-cocycle $\{t, u\}$ (*cf.* 4.1). We use the suggestive notation $\tilde{h}_{12}$ for the cross section of σ that corresponds to the 2-cocycle. So $\tilde{h}_{12}(t)\,\tilde{h}_{12}(u) = \tilde{h}_{12}(tu)\,\{t, u\}$ and D (R) is a central subgroup of $\tilde{H}$. [Recall that $h_{12}(t)$ is an element of St (2, R), with $h_{12}(t)\,h_{12}(u) = h_{12}(tu)\,\{t, u\}_{12}$. So $\tilde{h}_{12}$ mimics h_{12}.] The *chunck* C is the set of triples $(x_{21}(a), \tilde{h}, x_{12}(b))$ with $a, b \in R$, $\tilde{h} \in \tilde{H}$. We will simply write $x_{21}(a)\,\tilde{h}x_{12}(b)$ for $(x_{21}(a), \tilde{h}, x_{12}(b))$. (As $x_{ij}(\star)$, $\tilde{h}$ are not in the same group it will be clear where one should put the commas.)

We define $\pi : \tilde{H} \to E(2, R)$ by $\pi(\tilde{h}) = d_{12}(\sigma(\tilde{h}))$. Here $d_{12}(t)$ is the diagonal matrix with diagonal (t, t^{-1}). We define the map $\pi : C \to E(2, R)$ by

$$\pi(x_{21}(a)\,\tilde{h}x_{12}(b)) = e_{21}(a)\,\pi(\tilde{h})\,e_{12}(b).$$

We also define $\pi : St(2, R) \to E(2, R)$ by $\pi(x_{ij}(r)) = e_{ij}(r)$. So there are three maps π, all with codomain E (2, R). The first and the last one are homomorphisms. (C is not a group.)

5.10. As in section 4 we model left and right multiplications in the Steinberg group by partially defined maps from the chunck into itself. Note that $\pi(C) = \Omega$ is the big cell of SL (2, R), i. e. it consists of the matrices $(a_{ij}) \in SL(2, R)$ with $a_{11} \in R^*$.

DEFINITIONS. — Let

$$p = x_{21}(a)\,\tilde{h}\,x_{12}(b) \in C, \qquad t \in R.$$

We put

$$L(x_{21}(t))(p) = x_{21}(a+t)\,\tilde{h}\,x_{12}(b) \in C,$$
$$L(x_{12}(t))(p) = x_{21}(a(1+at)^{-1})\,\langle t, a\rangle\,\tilde{h}_{12}(1+at)\,\tilde{h}\,x_{12}(b+u) \in C,$$

where u is such that $e_{12}(t(1+at)^{-1})\,\pi(\tilde{h}) = \pi(\tilde{h})\,e_{12}(u)$. (So $\sigma(\tilde{h})^2\,u = t(1+at)^{-1}$). Of course $L(x_{12}(t))(p)$ is only defined when $e_{12}(t)\,\pi(p) \in \Omega$ or, equivalently, when

$1+at \in R^*$. We do not define $L(y)$ for arbitrary $y \in St(2, R)$. For $\tilde{h}' \in \tilde{H}$ we put $L(\tilde{h}')(p) = x_{21}(u)\tilde{h}'\tilde{h}x_{12}(b) \in C$, where p is as above and u is such that

$$\pi(\tilde{h}')e_{21}(a) = e_{21}(u)\pi(\tilde{h}').$$

Note that $\pi(L(y)(p)) = \pi(y)\pi(p)$ when $L(y)(p)$ is defined, where $y = x_{ij}(t)$ or $y = \tilde{h}' \in \tilde{H}$. Further we put

$$R(x_{12}(t))(p) = x_{21}(a)\tilde{h}x_{12}(b+t) \in C,$$

$$R(x_{21}(t))(p) = x_{21}(a+u)\tilde{h}\langle b, t\rangle \tilde{h}_{12}(1+bt)x_{12}(b(1+bt)^{-1}) \in C,$$

where u is such that $\pi(\tilde{h})e_{21}(t(1+bt)^{-1}) = e_{21}(u)\pi(\tilde{h})$ and p is as before. The domain of $R(x_{21}(t))$ consists of the $p \in C$ with $\pi(p)e_{21}(t) \in \Omega$. Finally we put $R(\tilde{h}')(p) = x_{21}(a)\tilde{h}\tilde{h}'x_{12}(u)$ where $\tilde{h}' \in \tilde{H}$ and u is chosen such that

$$e_{12}(b)\pi(\tilde{h}') = \pi(h')e_{12}(u).$$

Note that $\pi(R(y)(p)) = \pi(p)\pi(y)$ in all these cases.

5.11. Lemma (*cf.* 4.6). — *Let* $p = x_{21}(a)\tilde{h}_0 x_{12}(b) \in C$,

$$y = x_{12}(t) \quad \textit{or} \quad y = x_{21}(t) \quad \textit{or} \quad y = \tilde{h}' \in \tilde{H},$$

$$z = x_{12}(v) \quad \textit{or} \quad z = x_{21}(v) \quad \textit{or} \quad z = \tilde{h}'' \in \tilde{H}.$$

If $\pi(y)\pi(p)$, $\pi(p)\pi(z)$, $\pi(y)\pi(p)\pi(z)$ *are elements of* Ω *then* $L(y)R(z)(p) = R(z)L(y))(p)$.

Proof. — First note that the assumptions are such that both $L(y)R(z)(p)$ and $R(z)L(y)(p)$ are defined. As $\pi(L(y)R(z)(p)) = \pi(y)\pi(p)\pi(z) = \pi(R(z)L(y)(p))$ we only have to compare the $\tilde{H}$-components (*cf.* 4.6).

Say

$$R(z)(p) = x_{21}(\star)\tilde{h}_0 h_1 x_{12}(\star), \qquad L(y)R(z)(p) = x_{21}(\star)\tilde{h}_2\tilde{h}_0\tilde{h}_1 x_{12}(\star),$$

$$L(y)(p) = x_{21}(\star)\tilde{h}_3\tilde{h}_0 x_{12}(\star), \qquad R(z)L(y)(p) = x_{21}(\star)\tilde{h}_3\tilde{h}_0\tilde{h}_4 x_{12}(\star).$$

In most cases $\tilde{h}_1 = \tilde{h}_4$, $\tilde{h}_2 = \tilde{h}_3$ and the result follows. The only exception is $y = x_{12}(t)$, $z = x_{21}(v)$, where one has to make more detailed computations. One finds

$$L(y)(p) = x_{21}(\star)\langle t, a\rangle \tilde{h}_{12}(1+at)\tilde{h}_0 x_{12}(b+\sigma(\tilde{h}_0)^{-2}t(1+at)^{-1}),$$

$$R(z)L(y)(p) = x_{21}(\star)\langle t, a\rangle \tilde{h}_{12}(1+at)\tilde{h}_0$$
$$\times \langle b+\sigma(\tilde{h}_0)^{-2}t(1+at)^{-1}, v\rangle \tilde{h}_{12}(1+bv+v\sigma(\tilde{h})^{-2}t(1+at)^{-1})x_{12}(\star),$$

and similarly

$$L(y)R(z)(p) = x_{21}(\star)\langle t, a+v\sigma(\tilde{h}_0)^{-2}(1+bv)^{-1}\rangle$$
$$\tilde{h}(1+at+tv\sigma(\tilde{h})^{-2}(1+bv)^{-1})\tilde{h}_0\langle b, v\rangle \tilde{h}_{12}(1+bv)x_{12}(\star).$$

So we need to prove an identity in $\tilde{H}$. Write $\tilde{h}_0 = \tilde{h}_{12}(s)\,D$, where $D \in D(R)$, $s = \sigma(\tilde{h}_0)$. Put

$$A = 1 + bv + s^{-2}\,tv(1+at)^{-1},$$

$$B = 1 + at + s^{-2}\,tv(1+bv)^{-1}.$$

Then $A(1+at) = B(1+bv)$, which also follows by applying π to the formulas. As

$$\langle t,\, a + s^{-2}v(1+bv)^{-1}\rangle = \langle t, a\rangle \langle t,\, s^{-2}v(1+at)^{-1}(1+bv)^{-1}\rangle$$

and

$$\langle b + s^{-2}t(1+at)^{-1},\, v\rangle = \langle b, v\rangle \langle s^{-2}t(1+at)^{-1}(1+bv)^{-1},\, v\rangle$$

(note the symmetry) we have to show that

$$\tilde{h}_{12}(1+at)\,\tilde{h}_{12}(s)\,\tilde{h}_{12}(A)\,\langle s^{-2}t(1+at)^{-1}(1+bv)^{-1},\, v\rangle$$

equals

$$\langle t,\, s^{-2}v(1+at)^{-1}(1+bv)^{-1}\rangle\,\tilde{h}_{12}(B)\,\tilde{h}_{12}(s)\,\tilde{h}_{12}(1+bv).$$

Now the latter equals

$$\langle s^{-2}t(1+at)^{-1}(1+bv)^{-1},\, v\rangle$$
$$\times \{A(1+bv)^{-1},\, s^{-2}(1+at)^{-1}(1+bv)^{-1}\}\,\tilde{h}_{12}(B)\,\tilde{h}_{12}(s)\,\tilde{h}_{12}(1+bv),$$

so remains to show that $\{1+at, s\}\,\{(1+at)\,s, A\}$ equals

$$\{A(1+bv)^{-1},\, s^{-2}(1+at)^{-1}(1+bv)^{-1}\}\,\{B, s\}\,\{Bs,\, 1+bv\}.$$

Plug in $A = B(1+bv)(1+at)^{-1}$ and the result is clear from the multiplicative rules for the symbols $\{\star, \star\}$.

5.12. The above computation is a generalisation of the computation in 4.6, at least for the non-symplectic case. [Although SL (2, R) is of symplectic type we made a "non-symplectic" computation by factoring out the defining relations of D (R), thereby forcing relation 3.2 (*g*).] The above computation can also be interpreted in terms of gradual replacements inside "words", as in [16] (*cf.* the comment in 6.10).

5.13. Definition. — Let G be the set of partially defined maps f that can be written as $f = L(\tilde{h})\,L(x_{21}(a))\,L(x_{12}(b))\,L(x_{21}(c))$, with $\tilde{h} \in \tilde{H}$, $a, b, c \in R$. Our purpose is to provide G with a group structure. Define $\pi : G \to E(2, R)$ by

$$\pi(f) = \pi(\tilde{h})\,e_{21}(a)\,e_{12}(b)\,e_{21}(c),$$

where f is as above. This is the fourth map π (*see* 5.9 for other three). The value of $\pi(f)$ does not depend on the way f is written, because $\pi(f(p)) = \pi(f)\,\pi(p)$ for all p in the domain of f. [The domain of f consists of the $p = x_{21}(r)\,\tilde{h}'\,x_{12}(\star)$ with $\pi(f)\,\pi(p) \in \Omega$, hence it consists of the p with $\pi(f)\,e_{21}(r) \in \Omega$. There is at least one such r, by 1-fold stability.]

5.14. LEMMA (*cf.* 4.8 *or* [18], *Lemma* 12.8). — *Let* $p, q \in C$. *There is one and exactly one* $f \in G$ *with* $f(p) = q$. *Moreover, if* $g, h \in G$ *are such that* $g(h(p)) = q$ *then* $f(r) = g(h(r))$ *whenever the right hand side is defined.*

Proof. — Say $p = x_{21}(t)\tilde{h}' x_{12}(\star)$. It is easy to see that we can choose f of the form

$$L(\tilde{h})\,L(x_{21}(\star))\,L(x_{12}(\star))\,L(x_{21}(-t)) \quad \text{such that } f(p) = q.$$

Uniqueness of f will follow from the second statement of the lemma, upon taking $h = \mathrm{id}$. Thus let g, h be such that $g(h(p)) = q$. Clearly $\pi(f) = \pi(g)\pi(h)$, so $f(r)$ is defined whenever $g(h(r))$ is defined. If z is as in Lemma 5.11 then

$$g(h(R(z)(p))) = g(R(z)h(p)) = R(z)g(h(p)) = R(z)f(p) = f(R(z)(p))$$

provided that the left hand side is defined (use Lemma 5.11). Let V be the domain of $r \mapsto g(h(r))$ and $X = \{v \in V \mid f(v) = g(h(v))\}$. Then X is non-empty and what we just showed amounts to the rule "if $v \in X$ and $R(z)(v) \in V$ then $R(z)(v) \in X$". [Note that one needs $R(z)(v)$ to be defined.] We want to use something like proposition 2.6, as employed in 4.8. (The present situation is much simpler.) Note that

$$V = \{x_{21}(\tilde{a})h_0 x_{12}(b) \mid \pi(h)e_{21}(a) \text{ and } \pi(g)\pi(h)e_{21}(a) \text{ lie in } \Omega\}.$$

The above rule implies that the following are equivalent:

(i) $x_{21}(a)\tilde{h}_0 x_{12}(b) \in X$;

(ii) $x_{21}(a)\tilde{h}_0 x_{12}(0) \in X$;

(iii) $x_{21}(a)\tilde{h}_{12}(1) x_{12}(0) \in X$.

In particular, the fact that $p = x_{21}(t)\tilde{h}' x_{12}(\star) \in X$ implies that $x_{12}(t)h_{12}(1)x_{12}(0) \in X$. But then also $x_{12}(t+u)\tilde{h}_{12}(1)x_{12}(0)$ is an element of X, whenever it is one of V. We may conclude that $V = X$.

5.15. LEMMA. — *Let* $f, g, h \in G$. *There is* $p \in C$ *such that* $f(g(h(p)))$ *is defined.*

Proof. — Try $p = x_{21}(t)\tilde{h}_{12}(1)x_{12}(0)$ and use 3-fold stability.

5.16. DEFINITION. — For $f, g \in G$ let $f \star g$ be the unique element of G that extends the map $p \mapsto f(g(p))$. If $f, g, h \in G$ one has

$$(f \star (g \star h))(p) = f(g(h(p))) = ((f \star g) \star h)(p) \quad \text{for some } p \in C,$$

so that $\star$ is associative. (Use last two lemmas.) It is easy to see that the composition $\star$ makes G into a group. Put $y_{12}(t) = L(x_{12}(t))$ and $y_{21}(t) = L(x_{21}(t))$. Then $y_{ij}(t) \in G$.

5.17. COMMENT. — Only 2-fold stability is needed to define the composition $\star$. [Given f and g one needs $p \in C$ so that $f(g(p))$ is defined, in order to assert uniqueness of $f \star g$.] However, we do not see how to prove associativity in G without something like 3-fold stability. Of course one can impose additional relations on the $\langle a, b \rangle$ and thus force G to be associative. Such an approach has little appeal however, as it leads eventually to awful presentations for K_2. We know, at least for local rings, that in fact K_2 is given by the nice presentation from 5.2. Therefore we only use the nice relations and see where we get with them. It then seems necessary to use 3-fold stability and we are unable to recover the result of Maazen en Stienstra for the case of a local ring with a very small residue field. It seems that we could obtain, by the present method, some awful presentation for K_2 of 2-fold stable rings. In this presentation the various obstacles would be reflected which we avoid here by 3-fold stability. So it is not true that the simple relations of 5.2 come out naturally from this sort of proof. It is true, however, that the relations from 5.2 (and their consequences) are the only short relations encountered when computing in the chunks. (Two more chunks are to come, see next section.) We will not point out the further obstacles we need 3-fold stability for. They are easy enough to find. The trick is to avoid them.

5.18. LEMMA. — *$x_{ij}(t) \to y_{ij}(t)$ defines a homomorphism $\varphi : \mathrm{St}(2, \mathrm{R}) \to \mathrm{G}$ sending $\langle a, b \rangle_{12}$ to $\mathrm{L}(\langle a, b \rangle)$.*

Proof. — We have to show that the Steinberg relations hold between the $y_{ij}(t)$. By Lemma 5.14 identities in G can be checked by evaluating at some conveniently chosen p in C. It is therefore trivial to prove $y_{ij}(t+u) = y_{ij}(t)\, y_{ij}(u)$. When $t \in \mathrm{R}^*$ and $a \in \mathrm{R}$ we need to show that

$$y_{12}(t)\, y_{21}(-t^{-1})\, y_{12}(t)\, y_{12}(a)\, (y_{12}(t)\, y_{21}(-t^{-1})\, y_{12}(y))^{-1} = y_{21}(-t^{-2}\, a)$$

or that

$$y_{12}(t)\, y_{21}(-t^{-1})\, y_{12}(a)\, y_{21}(t^{-1})\, y_{12}(-t) = y_{21}(-t^{-2}\, a).$$

If it happens that $1+at^{-1} \in \mathrm{R}^*$, we test at the "origin" $p = x_{21}(0)\, \tilde{h}_{12}(1)\, x_{12}(0)$ and we see it boils down to proving

$$\langle t, -at^{-2}(1+at^{-1})^{-1} \rangle \{(1+at^{-1})^{-1}, 1+at^{-1}\} \langle a, t^{-1} \rangle = 1,$$

or

$$\langle -at^{-2}, -t \rangle \{-1, 1+at^{-1}\} \langle a, t^{-1} \rangle = 1,$$

or

$$\{1+at^{-1}, -t\} \{-1, 1+at^{-1}\} \{1+at^{-1}, t^{-1}\} = 1,$$

which is clear. For arbitrary $a \in \mathrm{R}$ we choose, by 2-fold stability, $u \in \mathrm{R}$ such that $1+ut^{-1}$ and $1+(a-u)\, t^{-1}$ are units. We can write $y_{12}(a) = y_{12}(u)\, y_{12}(a-u)$ and apply the previous computation to the factors. So the Steinberg relations hold indeed. Now if u, $v \in \mathrm{R}^*$ the element

$$w_{12}(u)\, w_{12}(v) = x_{12}(u)\, x_{21}(-u^{-1})\, x_{12}(u+v)\, x_{21}(-v^{-1})\, x_{12}(v)$$

goes to $\mathrm{L}(\langle u+v, -v^{-1} \rangle\, \tilde{h}_{12}(-uv^{-1}))$, as one sees by testing at the origin again. So $h_{12}(u)$ goes to $\mathrm{L}(\tilde{h}_{12}(u))$ (put $v=-1$) and $\langle a, b \rangle_{12}$ goes to $\mathrm{L}(\langle a, b \rangle)$. [Test at $x_{21}(0)\, \tilde{h}_{12}(1+ab)\, x_{12}(0)$.]

5.19. For $x \in \mathrm{K}_2(2, \mathrm{R})$ one has $\pi(\varphi(x)) = 1$, so $\varphi(x)$ is everywhere defined. Furthermore $\varphi(x)\,(x_{21}(0)\, \tilde{h}_{12}(1)\, x_{12}(0)) = x_{21}(0)\, \mathrm{D} x_{12}(0)$ for some $\mathrm{D} \in \mathrm{D}(\mathrm{R})$. Put $\tau(x) = \mathrm{D}$. Then τ is a homomorphism of the type required in the theorem. [Use $\mathrm{L}(\tau(x)) = \varphi(x)$.]

6. The rank 2 case

6.1. We want to do the same sort of thing as in the previous section, now for $K_2(3, R)$. The chunk C^* in which we can prove the necessary relations lies over the big cell Ω again. However, some relations are easier to prove in a bigger chunk C, in which we incorporate the results of the previous section. This bigger chunk was used before to prove injective stability for K_2 (*see* [11]). Its role is only secondary in the present approach. We use it to avoid a few lengthy computations of the type occurring in 6.9, by reducing to the previous section. We will not establish the analogue of Lemma 5.11 with points ranging over all of C. (Before 6.9 we have trouble to extrapolate that far out from our "general position" results and we also do not see how to carry through the general position computations in a range that would be more adequate for the study of C.) Once we have established (in 6.9) the analogue of 5.11 with the small chunk C^* as range, we can extrapolate along paths "all the way out to St (3, R)" (*cf.* the application of 2.6 in 4.8).

Now let us explain the connection with the problem of injective stability for K_2, to motivate the definition of C. (Here we also introduce some notations.) Let $St'(2, R)$ be the group which is obtained by factoring out the kernel of $\tau : K_2(2, R) \to D(R)$ from St (2, R) (*see* 5.3). In other words, $St'(2, R)$ is the group G which was constructed in the previous section. The image of $x_{ij}(a)$ in $St'(2, R)$ will be denoted by $x_{ij}(a)$ and the image of $h_{ij}(t)$ by $h_{ij}(t)$. [This convention is harmless as we won't use St (2, R) any more.] The image of $K_2(2, R)$ in $St'(2, R)$ will be identified with D (R) in the obvious way. It is known (*see* [16]) that $\langle a, b \rangle \mapsto \langle a, b \rangle_{12}$ defines a homomorphism $D(R) \to K_2(3, R)$, and also that, because our ring is 3-fold stable and hence certainly 1-fold stable, the usual map $K_2(2, R) \to K_2(3, R)$ is surjective (*see* [20] or [25] and also 6.23 below). Now $K_2(2, R) \to K_2(3, R)$ factors through $D(R) \to K_2(3,R)$ (use 5.3), so that the latter map is also surjective. Therefore $D(R) \simeq K_2(3, R)$ follows if we show that the natural map $St'(2, R) \to St(3, R)$ is injective. (Check that there is such a map.) This is a problem of the same type as the problem of injective stability for K_2, so that it is natural to introduce the chunck C suggested by Keith Dennis in that setting (*cf.* [11].)

6.2. THEOREM. — *Let* R *be* 3-*fold stable. The map* $\langle a, b \rangle \mapsto \langle a, b \rangle_{12}$ *defines an isomorphism* $D(R) \to K_2(3, R)$.

6.3. The proof of this theorem is given in the remainder of this section.

DEFINITION. — We define a copy $St''(2, R)$ of $St'(2, R)$ in which the generators are written as $x_{23}(t)$, $x_{32}(t)$ instead of $x_{12}(t)$, $x_{21}(t)$ respectively. So $x_{ij}(t) \mapsto x_{i+1,j+1}(t)$ defines an isomorphism $St'(2, R) \to St''(2, R)$. We write the image of $\langle a, b \rangle$ as $\langle a, b \rangle$ i. e. we also identify D (R) with a subgroup of $St''(2, R)$. We write the image of $w_{ij}(u)$ as $w_{i+1,j+1}(u)$ and the image of $h_{ij}(u)$ as $h_{i+1,j+1}(u)$. The *big chunck* C consists of equivalence classes $[x, M, y]$ of triples (x, M, y), with

$$x \in St'(2, R), \quad M = e_{31}(q)\, e_{32}(r)\, e_{12}(a)\, e_{13}(b) \in E(3, R), \qquad y \in St''(2, R).$$

Here (x, M, y) is called equivalent with (x', M', y'), notation $(x, M, y) \sim (x', M', y')$ if there are $t, u \in R$, $D \in D(R)$ such that $x = x' x_{12}(t) D$, $y = D^{-1} x_{32}(u) y'$, $M' = e_{12}(t) M e_{32}(u)$. Check that this is an equivalence relation. (One may also check that this is really the same type of chunck as in [11].) We put

$$\pi(x_{ij}(t)) = e_{ij}(t), \qquad \pi((x, M, y)) = \pi([x, M, y]) = \pi(x) M \pi(y).$$

Check that this defines homomorphisms $St'(2, R) \to E(3, R)$, $St''(2, R) \to E(3, R)$, and a map $C \to E(3, R)$.

6.4. Let us mimic the multiplication in St (3, R), by defining partially defined permutations of C (*cf.* 5.10). If $z \in St'(2, R)$ put $L(z)(x, M, y) = (zx, M, y)$ and $L(z)[x, M, y] = [zx, M, y]$, the class of $L(z)(x, M, y)$.

If $z = e_{31}(t) e_{32}(u)$ put

$$L(z)(x, M, y) = (x, \pi(x)^{-1} z \pi(x) M, y)$$

and

$$L(z)[x, M, y] = \text{class of } L(z)(x, M, y).$$

If $z = e_{12}(t) e_{13}(u)$ put

$$R(z)[x, M, y] = [x, M \pi(y) z \pi(y)^{-1}, y].$$

If $z \in St''(2, R)$ put

$$R(z)[x, M, y] = [x, M, yz].$$

It is easy to check that all these maps are well-defined and that

$$\pi(L(z)(p)) = \pi(z)\pi(p) \quad \text{or} \quad z\pi(p),$$

$$\pi(R(z)(p)) = \pi(p)\pi(z) \quad \text{or} \quad \pi(p) z.$$

Next let

$$z = e_{13}(\star) e_{23}(\star), \qquad M = e_{31}(q) e_{32}(r) e_{12}(a) e_{13}(b),$$

$$\pi(x)^{-1} z \pi(x) = e_{13}(t) e_{23}(u), \qquad 1 + tq \in R^*.$$

Then we put

$$\begin{aligned} L(z)(x, M, y) = (&x h_{12}(1+tq) \langle -t, -q \rangle x_{21}(uq(1+tq)),\\ &e_{31}(q) e_{12}((a-br)(1+tq)^{-1}) e_{13}(t - au + bru + b + bqt),\\ &x_{23}(u(1+tq)) h_{23}(1+tq) x_{32}(r) y). \end{aligned}$$

Clearly this formula is more complicated than the ones in rank 1, which is one reason to do some of the work in the rank 1 setting. One has $\pi(L(z)(x, M, y)) = z\pi(x, M, y)$ again. We will show in 6.5 that the class of $L(z)(x, M, y)$ only depends on the class of (x, M, y). But note that $L(z)(x', M', y')$ need not be defined for all representatives of $[x, M, y]$, because the analogue of the statement $1 + tq \in R^*$ may fail. We will denote the class of $L(z)(x, M, y)$ by $L(z)[x, M, y]$.

6.5. Let us now show that $L(z)$ is compatible with equivalence $[z = e_{13}(\star)\, e_{23}(\star)$ as above$]$. Assume (x, M, y) is equivalent with (x', M', y') and assume $L(z)$ is defined at both of them. Write $x' = x x_{12}(-f)\, D^{-1}$, $y' = x_{32}(-g)\, D y$,

$$M' = e_{31}(q)\, e_{32}(r+g-fq)\, e_{12}(a+f+bg)\, e_{13}(b) \qquad \text{with} \quad D \in D(R),$$

q, r, a, b as above. Then $\pi(x')^{-1} z\, \pi(x') = e_{13}(t+fu)\, e_{23}(u)$ and $1+(t+fu)\, q$ must be a unit. Say $A = 1+tq$ and $B = 1+(t+fu)\, q$. We find

$$L(z)(x', M', y') = (x x_{12}(-f) D^{-1} h_{12}(B) \langle -t-fu, -q \rangle x_{21}(uqB), \star,$$
$$x_{23}(uB)\, h_{23}(B)\, x_{32}(r+g-fq)\, x_{32}(-g)\, D y).$$

Note that $\pi(L(z)(x, M, y)) = z\, \pi([x, M, y]) = \pi(L(z)(x', M', y'))$. Therefore, to show that $L(z)(x', M', y')$ is equivalent with $L(z)(x, M, y)$, it suffices to show that

$$x h_{12}(A) \langle -t, -q \rangle x_{21}(uqA)$$
$$= x x_{12}(-f) D_1^{-1} h_{12}(B) \langle -t-fu, -q \rangle x_{21}(uqB)\, x_{12}(f A^{-1} B^{-1})$$

and

$$x_{23}(uA)\, h_{23}(A)\, x_{32}(r)\, y = x_{32}(fq A^{-1} B^{-1})\, x_{23}(uB)\, h_{23}(B)\, x_{32}(r-fq)\, D_1 y$$

hold simultaneously, for some $D_1 \in D(R)$. One finds that the following identities must hold:

$$h_{12}(A) \langle -t, -q \rangle = \langle -f, uqB^{-1} \rangle h_{12}(AB^{-1})\, h_{12}(B) \langle -t-fu, -q \rangle D_1^{-1}$$

and

$$h_{23}(B) \langle uB^{-1}, -fq \rangle h_{23}(AB^{-1})\, D_1 = h_{23}(A).$$

In other words we need

$$\langle -t, -q \rangle^{-1} \langle -f, uqB^{-1} \rangle \{AB^{-1}, B\} \langle -t-fu, -q \rangle = D_1$$
$$= \langle uB^{-1}, -fq \rangle^{-1} \{B, AB^{-1}\}^{-1},$$

or

$$\langle -f, uqB^{-1} \rangle \langle -t-fu, -q \rangle \langle uB^{-1}, -fq \rangle = \langle -t, -q \rangle.$$

Now $\langle uB^{-1}, -fq \rangle = \langle -uqB^{-1}, f \rangle \langle fuB^{-1}, -q \rangle$ so we are left with the true relation $\langle -t-fu, -q \rangle \langle fuB^{-1}, -q \rangle = \langle -t, -q \rangle$.

6.6. DEFINITION. — We define the *small chunk* C^* to consist of the elements $[h_{12}(v)\, x_{21}(u)\, D, M, x_{23}(s)\, h_{23}(t)]$ with

$$u, s \in R, \qquad t, v \in R^*, \qquad D \in D(R), \qquad M = e_{31}(\star)\, e_{32}(\star)\, e_{12}(\star)\, e_{13}(\star)$$

as usual. Note that an element of C^* has exactly one representative of the form $(h_{12}(\star)\, x_{21}(\star)\, D, M, x_{23}(\star)\, h_{23}(\star))$. We call this representative the normalized one. When $p \in C$, $\pi(p) = (a_{ij})$, then p is in the small chunk if and only if (a_{ij}) is in the big cell Ω, i.e. if and only if both a_{11} and the minor $a_{11}a_{22}-a_{12}a_{21}$ are units. Say $p=[x, M, y]$ with $\pi(x)=(b_{ij})$ and $\pi(y)=(c_{ij})$. Then

$$a_{11}=b_{11} \quad \text{and} \quad a_{11}a_{22}-a_{12}a_{21}=c_{22}.$$

So $p\in C^*$ if and only if both x and y lie over Ω.

6.7. DEFINITION. — Let $p\in C^*$, $p=[h_{12}(v)\,x_{21}(u)\,x_{12}(a), M, y]$. For $d\in R^*$ we put

$$\begin{aligned}&L(h_{23}(d))(p)\\&\quad=[h_{12}(v)\{v, d\}x_{21}(du)x_{12}(d^{-1}a), \pi(h_{23}(d))M\pi(h_{23}(d))^{-1}, h_{23}(d)y].\end{aligned}$$

We have $\pi(L(h_{23}(d))(p))=\pi(h_{23}(d))\,\pi(p)$, as usual. Check that $L(h_{23}(d))$ is well-defined. (Its domain is C^*, by definition.)

6.8. DEFINITION. — Let $p\in C^*$. Write

$$p=[x, e_{31}(q)e_{32}(r)e_{13}(b), x_{23}(s)h_{23}(t)].$$

(Note that p can indeed be written this way.)

Let $d\in R$ be such that $\pi(p)\,e_{21}(d)\in\Omega$. We put

$$R(e_{21}(d))(p)=[x\,x_{21}(td), e_{31}(q+rtd)e_{32}(r)e_{13}(b), x_{23}(s-btd)h_{23}(t)].$$

One has $\pi(R(e_{21}(d))(p))=\pi(p)\,e_{21}(d)$.

There is another way to describe $R(e_{21}(d))(p)$. Say

$$(h_{12}(v)x_{21}(u)D, e_{31}(q)e_{32}(r')e_{12}(a)e_{13}(b), x_{23}(s)h_{23}(t))$$

is the normalized representative of p. Then

$$\begin{aligned}&R(e_{21}(d))(p)\\&\quad=[h_{12}(v)x_{21}(u+td(1+atd)^{-1})\langle a, td\rangle h_{12}(1+atd)D,\\&\qquad e_{31}(q+qatd+r'td)e_{32}(r'(1+atd)^{-1})e_{12}(a(1+atd)^{-1})e_{13}(b),\\&\qquad\qquad x_{23}(s-btd)\,h_{23}(t)],\end{aligned}$$

where we use the condition $\pi(p)\,e_{21}(d)\in\Omega$ to conclude that $1+atd$ is a unit. Check the equivalence of the two descriptions of $R(e_{21}(d))$ and note that the last one (or the first one, for that matter) is automatically well-defined, because it starts from a unique representative of p. In the first description the condition $\pi(p)\,e_{21}(d)\in\Omega$ seems useless, but we insist on it because we do not want $R(e_{21}(d))(p)$ to be defined too often, as that would make the proof of the next proposition harder. [So the next proposition only sounds natural because we choose this definition of $R(e_{21}(d))$.]

6.9. So far we have defined maps $L(z)$, $R(w)$ in the following cases:

$$\begin{gathered}z\in St'(2, R),\qquad z=e_{31}(\star)e_{32}(\star),\qquad z=e_{13}(\star)e_{23}(\star),\\ z=h_{23}(\star),\qquad w\in St''(2, R),\qquad w=e_{21}(\star),\qquad w=e_{12}(\star)e_{13}(\star).\end{gathered}$$

We will leave it there.

PROPOSITION (*cf.* 5.11). — *Let* $p \in C$. *If* $L(z)R(w)(p)$ *and* $R(w)L(z)(p)$ *are both defined, then they are equal.*

Proof. — Assume both are defined. As there are four possibilities for z and three for w, there are twelve cases in total. As $\pi(L(z)R(w)(p)) = \pi(R(w)L(z)(p))$, by associativity of matrix multiplication, it suffices to show there are x, y such that $L(z)R(w)(p) = [x, M, y]$ and $R(w)L(z)(p) = [x, M', y]$ for some M, M'. (Then $M = M'$ will be automatic.) In eleven of the twelve cases it is easy. [Use the second definition of $R(e_{21}(\star))$ when $w = e_{21}(\star)$, $z = h_{23}(\star)$.] Remains the tedious case

$$z = e_{13}(f')e_{23}(g'), \qquad w = e_{21}(d).$$

Say

$$p = [h_{12}(v)x_{21}(u)D,\ e_{31}(q)e_{32}(r)e_{12}(a)e_{13}(b),\ x_{23}(s)h_{23}(t)].$$

[Recall that p must be in C^* in order that $R(e_{21}(d))(p)$ be defined. There are further restrictions on p.] Say

$$\pi(h_{12}(v)x_{21}(u))^{-1} z\, \pi(h_{12}(v)x_{21}(u)) = e_{13}(f)e_{23}(g).$$

We get

$$\begin{aligned} L(z)R(w)(p) = L(z)[&h_{12}(v)x_{21}(u+td(1+atd)^{-1})\langle a, td\rangle h_{12}(1+atd)D,\\ &e_{31}(q+qatd+rtd)e_{32}(r(1+atd)^{-1})e_{12}(\star)e_{13}(\star),\\ &x_{23}(\star)h_{23}(t)]\\ = [&h_{12}(v)x_{21}(\star)\langle a, td\rangle h_{12}(1+atd)D\,h_{12}(A)\\ &\langle -f(1+atd)^{-1}, -(q+qatd+rtd)\rangle x_{21}(\star), \star,\\ &x_{23}(g(1+atd)A - ftd\,A)h_{23}(A)x_{32}(r(1+atd)^{-1})x_{23}(\star)h_{23}(t)]\end{aligned}$$

where $A = 1+fq+frtd(1+atd)^{-1}$.

Now we know, from the fact that $R(w)L(z)(p)$ is defined, that the answer lies over Ω, so we get

$$\begin{aligned} &L(z)R(w)(p)\\ &= [h_{12}(v)\langle a, td\rangle h_{12}(1+atd)D\,h_{12}(A)\langle -f(1+atd)^{-1},\\ &\quad -(q+qatd+rtd)\rangle x_{21}(\star), \star, h_{23}(A)\langle g(1+atd)A^{-1} - ftd\,A^{-1},\\ &\quad r(1+atd)^{-1}\rangle h_{23}((1+fq+gr)A^{-1})x_{23}(\star)h_{23}(t)].\end{aligned}$$

Next we compute

$$\begin{aligned} &R(w)L(z)(p)\\ &= R(w)[h_{12}(v)x_{21}(u)D\,h_{12}(1+fq)\langle -f, -q\rangle x_{21}(\star),\\ &\quad e_{31}(q)e_{12}((a-br)(1+fq)^{-1})e_{13}(f-ag+brg+b+bqf),\\ &\quad x_{23}(g(1+fq))h_{23}(1+fq)x_{32}(r)x_{23}(\star)h_{23}(t)]\end{aligned}$$

$$
\begin{aligned}
&= R(w)[h_{12}(v)\,D\,h_{12}(1+fq)\langle -f,\ -q\rangle x_{21}(\star),\\
&\qquad e_{31}(q)\,e_{32}(r(1+fq)^{-1}(1+fq+gr)^{-1})\\
&\qquad\quad \times e_{12}((a+afq+fr)(1+fq)^{-1}(1+fq+gr)^{-1})\,e_{13}(\star),\\
&\qquad h_{23}(1+fq)\langle g(1+fq)^{-1},\ r\rangle h_{23}((1+fq+gr)(1+fq)^{-1})\,x_{23}(\star)\,h_{23}(t)]\\
&= [h_{12}(v)\,D\,D_1^{-1}\,h_{12}(1+fq)\langle -f,\ -q\rangle x_{21}(\star)\\
&\qquad \langle (a+afq+fr)(1+fq)^{-1}(1+fq+gr)^{-1},\ t(1+fq+gr)\,d\rangle h_{12}(B),_{]}\star,\\
&\qquad D_1\,x_{23}(\star)\,h_{23}(1+fq)\langle g(1+fq)^{-1},\ r\rangle\ h_{23}((1+fq+gr)(1+fq)^{-1})\,h_{23}(t)],
\end{aligned}
$$

where $B = 1+atd+frtd(1+fq)^{-1}$ and $D_1 \in D(R)$ is a superfluous element which symbolizes that $R(e_{21}(d))$ has been evaluated *via* the second description in 6.8.

Now both answers can easily be put into the form $[h_{12}(\star)\,x_{21}(\star)\,D',\ M,\ D''\,x_{23}(\star)\,h_{23}(\star)]$ and their images in Ω are equal. It follows easily (*cf.* 2.3) that they can only differ in the components D', D'', after we have put them into the indicated form. All in all we need to show that there is $D_2 \in D(R)$ with

$$
\begin{aligned}
&\langle a,\ td\rangle h_{12}(1+atd)\,D_2\,h_{12}(A)\langle -f(1+atd)^{-1},\ -(q+qatd+rtd)\rangle\\
&\quad = h_{12}(1+fq)\langle -f,\ -q\rangle\\
&\qquad \times\langle (a+afq+fr)(1+fq)^{-1}(1+fq+gr)^{-1},\ t(1+fq+gr)\,d\rangle h_{12}(B)
\end{aligned}
$$

and with

$$
\begin{aligned}
&h_{23}(A)\langle g(1+atd)A^{-1}-ftdA^{-1},\ r(1+atd)^{-1}\rangle h_{23}((1+fq+gr)A^{-1})\,D_2^{-1}\\
&\quad = h_{23}(1+fq)\langle g(1+fq)^{-1},\ r\rangle h_{23}((1+fq+gr)(1+fq)^{-1}).
\end{aligned}
$$

Just as in 6.5 we can eliminate D_2 and reduce to proving one identity in $D(R)$. This identity essentially reads

$$
\begin{aligned}
&\langle q,\ f\rangle^{-1}\{1+fq,\ B\}\langle (a+afq+fr)(1+fr)^{-1}(1+fq+gr)^{-1},\ td(1+fq+gr)\rangle\\
&\quad \times\langle a,\ td\rangle^{-1}\langle q+qatd+rtd,\ f(1+atd)^{-1}\rangle\{A,\ 1+atd\}\\
&= \{A,\ -(1+fq+gr)\}\langle g(1+atd)A^{-1}-ftdA^{-1},\ r(1+atd)^{-1}\rangle\\
&\quad \times\langle g(1+fq)^{-1},\ r\rangle^{-1}\{-(1+fq+gr),\ 1+fq\}.
\end{aligned}
$$

To simplify it, use (D 1) and the following six relations:

$$
\langle a,\ td\rangle = \langle -frA^{-1}(1+atd)^{-1},\ td\rangle\langle (a+afq+fr)(1+fq)^{-1},\ td\rangle;
$$

$$
\begin{aligned}
&\langle (a+afq+fr)(1+fq)^{-1}(1+fq+gr)^{-1},\ td(1+fq+gr)\rangle\\
&\quad = \langle (a+afq+fr)(1+fq)^{-1},\ td\rangle\{B,\ 1+fq+gr\};
\end{aligned}
$$

$$
\begin{aligned}
&\langle g(1+atd)A^{-1}-ftdA^{-1},\ r(1+atd)^{-1}\rangle\\
&\quad = \langle -ftdA^{-1},\ r(1+atd)^{-1}\rangle\langle g(1+atd)(1+fq)^{-1},\ r(1+atd)^{-1}\rangle;
\end{aligned}
$$

$$\langle g(1+atd)(1+fq)^{-1}, r(1+atd)^{-1}\rangle$$
$$= \langle g(1+fq)^{-1}, r\rangle \ \{(1+fq+gr)(1+fq)^{-1}, 1+atd\}^{-1};$$
$$\langle q, f\rangle\{1+fq, 1+atd\}^{-1}$$
$$= \langle -rtd\,A^{-1}, f(1+atd)^{-1}\rangle\langle q+qatd+rtd, f(1+atd)^{-1}\rangle$$

[because both sides equal $\langle q(1+atd), f(1+atd)^{-1}\rangle$];

$$\langle -td, fr\,A^{-1}(1+atd)^{-1}\rangle$$
$$= \langle -rt\,dA^{-1}, f(1+atd)^{-1}\rangle\langle -tdf\,A^{-1}, r(1+atd)^{-1}\rangle\{A^{-1}(1+fq), A^{-1}(1+atd)\}.$$

Then it reduces to proving $\{1+fq, B\}\{A^{-1}(1+fq), (1+atd)A^{-1}\}\{A, 1+atd\}$

$$\{B, 1+fq+gr\}\{1+fq, 1+atd\}^{-1}$$
$$= \{A, -(1+fq+gr)\}\{(1+fq+gr)(1+fq)^{-1}, 1+atd\}^{-1}\{-(1+fq+gr), 1+fq\}.$$

Plug in $B = A(1+atd)(1+fq)^{-1}$ and the rest is straightforward.

6.10. COMMENT. — It may seem rather mysterious that a computation like the one above works. Here is a plausible interpretation. Both

$$R(e_{21}(d))L(e_{13}(f')e_{23}(g'))(p) \quad \text{and} \quad L(e_{13}(f')e_{23}(g'))R(e_{21}(d))(p)$$

mimic a way to reduce

$$x_{13}(f')x_{23}(g')h_{12}(v)x_{21}(u)x_{31}(q)x_{32}(r)x_{12}(a)x_{13}(b)x_{23}(s)h_{23}(t)x_{21}(d)$$

in St (3, R) to the form

$$h_{12}(\star)x_{21}(\star)x_{31}(\star)x_{32}(\star)x_{12}(\star)x_{13}(\star)x_{23}(\star)h_{23}(\star)k,$$

with $k \in K_2(3, R)$. The two ways are quite different. That is why one gets such a complicated relation in terms of elements $\langle \star, \star\rangle_{ij}$ when comparing the results. However, there are many intermediate ways to reduce the original expression to the desired form. Comparing two ways that are closer to each other will result in simpler relations between the $\langle \star, \star\rangle_{ij}$. In the above situation there are apparently sufficiently many intermediate ways to bridge the total gap by means of many easy comparisons. We forced the existence of such intermediate ways by requiring that $1+atd$, $1+fq$, $1+fq+gr$, A are units. [These "general position" conditions are implicit in the assumption that both $R(w)L(z)(p)$ and $L(z)R(w)(p)$ are defined.] Note that this is exactly how similar units were used in [16]. (In *loc. cit.* the units are equal to 1 modulo the Jacobson radical.) In our case the four units are also needed to write down the desired identity in terms of elements of D (R). This is not typical. See for instance *loc. cit.* 3.6 VIII, where one uses one more unit (viz. $1+x_3z_1+x_2y_1-x_2y_3z_1$) to prove a relation in D (R) than one needs to state the relation, or to prove its analogue in $K_2(3, R)$.

6.11. LEMMA. — *Let*

$$p\in C, \qquad z_1, z_2\in St'(2, R), \qquad t, u\in R$$

such that $L(z_1) L(e_{13}(t) e_{23}(u)) L(z_2)(p)$ *is defined. Then*

$$L(z_1 z_2) L(\pi(z_2)^{-1} e_{13}(t) e_{23}(u) \pi(z_2))(p)$$

is also defined and its value is the same.

Proof. — Trivial.

6.12. LEMMA. — *Let* $p \in C^*$, t, u, f, $g \in R$ *such that both* $L(e_{13}(t) e_{23}(u))(p)$ *and* $L(e_{13}(t+f) e_{23}(u+g))(p)$ *are elements of* C^* *(and defined). Then*

$$L(e_{13}(t+f) e_{23}(u+g))(p) = L(e_{13}(f) e_{23}(g)) L(e_{13}(t) e_{23}(u))(p).$$

Proof. — By means of the previous lemma we can reduce to the case

$$p = [1, e_{31}(q) e_{32}(r) e_{12}(\star) e_{13}(\star), y].$$

As we may push $e_{32}(r)$ over to y we may further assume $r = 0$. Then

$$\begin{aligned} & L(e_{13}(f) e_{23}(g)) L(e_{13}(t) e_{23}(u))(p) \\ & = L(e_{13}(f) e_{23}(g)) [h_{12}(1+tq) \langle -t, -q \rangle x_{21}(\star), \\ & \qquad e_{31}(q) e_{12}(\star) e_{13}(\star), x_{23}(\star) h_{23}(1+tq) y] \\ & = [h_{12}(1+tq) \langle -t, -q \rangle h_{12}(1+fq(1+tq)^{-1}) \\ & \qquad \langle -f(1+tq)^{-1}, -q \rangle x_{21}(\star), \star, x_{23}(\star) h_{23}(1+fq(1+tq)^{-1}) h_{23}(1+tq) y] \\ & = [h_{12}(1+fq+tq) \{1+tq, 1+fq(1+tq)^{-1}\} \\ & \qquad \langle -f-t, -q \rangle x_{21}(\star), \star, x_{23}(\star) \{1+fq(1+tq)^{-1}, 1+tq\} h_{23}(1+fq+tq) y] \\ & = L(e_{13}(t+f) e_{23}(u+g))(p). \end{aligned}$$

(For the same reason as in the proof of proposition 6.9 we need not compute the coefficients of x_{21} and x_{23}).

6.13. DEFINITION. — For $A \in E(3, R)$ put

$$V(A) = \{g \in \Omega \mid A g \in \Omega\},$$
$$W(A) = \{p \in C^* \mid \pi(p) \in V(A)\}.$$

6.14. LEMMA. — *Let* A_1, A_2, $A_3 \in E(3, R)$. *Then* $V(A_1) \cap V(A_2) \cap V(A_3)$ *is non-empty.*

Proof. — Let $g = e_{21}(p) e_{31}(q) e_{32}(r)$. For each i there is a choice of p, q, r such that $(A g)_{11}$ is a unit. (Use 1-fold stability, *cf.* [24], proof of Th. 1.) By 3-fold stability we may use one and the same p. Fixing such p we can use one and the same q.

Fix it too. Now there is a value of r such that $A_i g \in \Omega$ for $i = 1, 2, 3$. (Use 3-fold stability and check unimodularity of the relevant pairs by passing to residue fields.)

6.15. DEFINITION. — Define left to consist of partially defined maps f from C^* to C^* such that:

(i) there is $A \in E(3, R)$ such that $W(A)$ is the domain of f and such that $\pi(f(p)) = A\,\pi(p)$ for $p \in W(A)$. We write $A = \pi(f)$. (Clearly A is unique.);

(ii) $f(R(w)(p)) = R(w) f(p)$ when both sides are defined. (Compare 6.9 and [11], 4.10.)

6.16. LEMMA. — *Let* $f = L(z)\,L(e_{13}(t)\,e_{23}(u))$, *where* $z \in St'(2, R)$, t, $u \in R$. *Let* g *be the restriction of* f *to* $W(\pi(z)\,e_{13}(t)\,e_{23}(u))$. *Then* $g \in$ *left*.

Proof. — First let us check that the domain of f is big enough to define g. If $p \in C^*$ and $\pi(z)\,e_{13}(t)\,e_{23}(u)\,\pi(p) \in \Omega$, then $e_{13}(t)\,e_{23}(u)\,\pi(p)$ is a matrix (c_{ij}) whose top left minor $c_{11}\,c_{22} - c_{12}\,c_{21}$ is invertible. Say $p = [x, M, y]$. There is, by 1-fold stability, an element $\lambda \in R$ such that

$$e_{12}(\lambda)\,\pi(x)^{-1}\,e_{13}(t)\,e_{23}(u)\,\pi(p) \in \Omega.$$

Now $L(e_{13}(t)\,e_{23}(u))\,(x x_{12}(-\lambda),\, e_{12}(\lambda)\,M,\, y)$ is defined, so $f(p)$ is defined (check this).

So much for the domain. Other things being obvious, we still have to prove that $g\,R(e_{21}(d))(p) = R(e_{21}(d))\,g(p)$ when both sides are defined ($d \in R$). Write $p = L(x')\,p'$ where p' has the form $[1,\, e_{31}(\star)\,e_{32}(\star)\,e_{13}(\star),\, h_{23}(\star)\,x_{23}(\star)]$. Now

$$g\,R(e_{21}(d))(p) = g\,L(x')\,R(e_{21}(d))(p')$$

by Proposition 6.9. Also, using 6.11, it is easy to see that $g\,L(x')$ is the restriction of some map $f' = L(zx')\,L(e_{13}(\star)\,e_{23}(\star))$. It suffices to show that

$$f'\,R(e_{21}(d))(p') = R(e_{21}(d))\,f'(p'),$$

so we may assume $p = p'$. Then there is $\lambda \in R$ such that

$$e_{12}(\lambda)\,\pi(p)\,e_{21}(d), \qquad e_{12}(\lambda)\,e_{13}(t)\,e_{23}(u)\,\pi(p),$$
$$e_{12}(\lambda)\,e_{13}(t)\,e_{23}(u)\,\pi(p)\,e_{21}(d)$$

are all in Ω. (Use 3-fold stability and compare with the beginning of this proof.) We get

$$R(e_{21}(d))\,g(p)$$
$$= R(e_{21}(d))\,L(z\,x_{12}(-\lambda))\,L(e_{12}(\lambda)\,e_{13}(t)\,e_{23}(u)\,e_{12}(-\lambda))\,L(x_{12}(\lambda))(p)$$

and in the last expression $R(e_{21}(d))$ can be moved three steps to the right, because proposition 6.9 applies three times (check this). The result follows.

6.17. PROPOSITION (*cf.* 4.8 *or* 5.14). — *Let* f, $g \in$ *left*, $p \in C^*$, *such that* $f(p) = g(p)$. *Then* $f = g$.

Proof. — Put $A = \pi(f)$. Clearly $A = \pi(g)$ too. Put

$$X = \{\, q \in W(A) \mid f(q) = g(q) \,\}.$$

Then X is invariant under the R (z) inside W (A), i. e. if $q \in X$, and $R(z)(q) \in W(A)$ then $R(z)(q) \in X$ (*cf.* proof of 5.14). To show that $X = W(A)$ it is therefore sufficient to show that any $q \in W(A)$ can be joined with p, inside W (A), by successive applications of maps R (z) (*cf.* 4.8). If $D \in D(R)$ then $D \in St''(2, R)$ and we have the map R (D) at our disposal. Therefore the problem of joining p with q is really only a problem about matrices. Thus let us join $\pi(p)$ with $\pi(q)$ inside V (A), by successive multiplications from the right. [We will use factors $e_{ij}(\star)$ with ij distinct from 31, as we may.] Write $\pi(p)$ as $M_1 M_2$ where M_1 is a lower triangular matrix, M_2 is an upper triangular matrix with ones on the diagonal. Instead of joining $\pi(p)$ with $\pi(q)$ inside V (A) we may join M_2 with $M_1^{-1}\pi(q)$ inside $V(AM_1)$. Therefore we may assume $M_1 = 1$. But then it is clear that M_2 may be joined with 1 inside V (A) by means of the $e_{ij}(\star)$ with $i < j$. We may thus assume that M_2 is also 1. Say $\pi(q) = M$. We can join M with a lower triangular matrix by means of the $e_{ij}(\star)$ with $i < j$. So say M is lower triangular, with $M_{11} = t$. By 2-fold stability there is $u \in R^*$ such that $M e_{21}(-ut) \in V(A)$. Then join M with $M e_{21}(-ut) e_{12}(t^{-1}u^{-1} - u^{-1}) e_{21}(u) e_{12}(-u^{-1} + tu^{-1})$ *via* the partial products. [Check that they are in V (A).] This reduces the situation to the case $M_{11} = 1$. Similarly we can achieve $M_{22} = 1$. [Or recall that we actually may multiply by diag $(1, v, v^{-1})$.] Now M is of the form $e_{21}(a) e_{31}(b) e_{32}(c)$. By 2-fold stability there is $\lambda \in R$ such that $A_{11} + (a+\lambda) A_{12} \in R^*$ and $M e_{21}(\lambda) \in V(A)$. Replacing M by $M e_{21}(\lambda)$ we may thus assume $A_{11} + a A_{12} \in R^*$. Now there is, by 3-fold stability, a unit μ such that $e_{21}(a) e_{31}(b) e_{32}(\mu) \in V(A)$, $\mu(A_{11} + a A_{12}) - b A_{12} \in R^*$. We can join M with $e_{21}(a) e_{31}(b) e_{32}(\mu)$ and then with

$$e_{21}(a) e_{31}(b) e_{32}(\mu) e_{21}(-b\mu^{-1}) = e_{21}(a - b\mu^{-1}) e_{32}(\mu).$$

Then we can join it with $e_{21}(a - b\mu^{-1})$ and finally with $1 = \pi(p)$.

6.18. DEFINITIONS. — We define G_L to consist of the $f \in$ left that are obtained by restricting a map $g = L(e_{31}(\star) e_{32}(\star)) L(h_{23}(\star)) L(z)$ to C^*, where $z \in St'(2, R)$ is such that $\pi(z)$ is lower triangular. [Check that any such g does indeed yield an element of left. Also note that z must have the form $h_{12}(\star) x_{21}(\star) D$ with $D \in D(R)$.] Similarly we define G_U to consist of the $f \in$ left that are obtained by restriction from a map $L(x_{12}(\star)) L(e_{13}(\star) e_{23}(\star))$ (*cf.* 6.16.)

6.19. LEMMA. — *Let $f \in$ left, $g \in G_L$. Then the composite maps fg and gf are both elements of left.*

Proof. — As g is everywhere defined on C^*, this is easy.

6.20. LEMMA. — *Let $f \in$ left, $g \in G_L$ such that $\pi(f)\pi(g) \in \Omega$. Then there are $h_1 \in G_L$, $h_2 \in G_U$ such that $fg = h_1 h_2$.*

Proof. — It is easy to see that there are $h_1 \in G_L$, $h_2 \in G_U$ such that $fg(p) = h_1 h_2(p)$ for $p = [1, 1, 1]$. Now apply proposition 6.17.

6.21. LEMMA. — *Let f, $g \in$ left. There is a unique $h \in$ left extending $p \mapsto fg(p)$.*

Proof. — Uniqueness follows from lemma 6.14 and proposition 6.17. Now assume f, g are obtained by restricting

$$L(x_{12}(a))L(e_{13}(b)e_{23}(c)), \qquad L(x_{12}(r))L(e_{13}(s)e_{23}(t))$$

respectively. Choose $h \in G_U$ such that $\pi(f)\pi(g) = \pi(h)$.

If $fg(p)$ is defined there is $\lambda \in R$ such that

$$\begin{aligned} &L(x_{12}(\lambda))fg(p)\\ &\quad = L(e_{13}(\star)e_{23}(\star))L(x_{12}(\lambda+a))g(p)\\ &\quad = L(e_{13}(\star)e_{23}(\star))L(x_{12}(\lambda+a+r))(p) = L(x_{12}(\lambda))h(p).\end{aligned}$$

[Use 6.11, 6.12 and 3-fold stability (*cf.* proof of 6.16).]

So h extends fg. Let us return to the general case.

Choose $k \in G_L$ such that $\pi(k) \in V(\pi(g)) \cap V(\pi(f)\pi(g))$, using 6.14. Then $gk = g_1 g_2$ and $fg_1 = f_1 f_2$ for some $f_1, g_1 \in G_L$, $f_2, g_2 \in G_U$.

Choose $k' \in G_L$ such that kk' is the identity, for instance with the help of proposition 6.17. By the above there is $h_2 \in G_U$ which extends $f_2 g_2$. Then $f_1 h_2 k'$ extends $f_1 f_2 g_2 k' = fg$.

6.22. Definition. — For $f, g \in$ left let $f \star g$ be the unique element of left that extends fg. Let G be the group with underlying set left and $\star$ as composition. (It is a group for the same reasons as in 5.16.) Put $y_{ij}(a) = L(e_{ij}(a))$ when ij is one of the pairs 13, 23, 31, 32. Put $y_{ij}(a) = L(x_{ij}(a))$ when ij equals 12 or 21.

6.23. It is easy to see that the y_{ij} satisfy the Steinberg relations, as we may test at $p = [1, 1, 1]$ (*cf.* proof of 5.18). So we have a homomorphism $\varphi : St(3, R) \to G$ and thus, as in 5.19, there is a homomorphism $\tau : K_2(3, R) \to D(R)$, defined by $\varphi(x)[1, 1, 1] = [\tau(x), 1, 1]$ or by $\varphi(x) = L(\tau(x))$. It sends $\langle a, b \rangle_{12}$ to $\langle a, b \rangle$ because of the way $St'(2, R)$ is built into C. Thus we have an inverse for the map $D(R) \to K_2(3, R)$ which sends $\langle a, b \rangle$ to $\langle a, b \rangle_{12}$. Theorem 6.2 follows, because we know that the $\langle a, b \rangle_{12}$ generate $K_2(3, R)$ (*see* 6.1). That the $\langle a, b \rangle_{12}$ generate $K_2(3, R)$ can also be seen in the fashion of section 4, as follows. Define $\sigma : C^* \to St(3, R)$ as suggested by the rules

$$x_{ij} \to x_{ij}, \qquad e_{ij} \to x_{ij}$$

(e. g. $[x_{12}(a), e_{31}(b), x_{32}(c)] \mapsto x_{12}(a)x_{31}(b)x_{32}(c)$). Then check that

$$\sigma(y_{ij}(t)(p)) = x_{ij}(t)\sigma(p).$$

For $f \in$ left show, as in 6.17, that there is a unique element of $St(3, R)$, denoted by $\sigma(f)$, such that $\sigma(f(p)) = \sigma(f)\sigma(p)$ whenever the left hand side is defined. Note that σ is the inverse of φ, so that φ is an isomorphism. The theorem follows again. From the proof of 6.21 one sees that $G = G_L G_U G_L$. This shows that there is a normal form in $St(3, R)$, similar to the one in 5.8. By proving this directly, as in 5.8, one can also see that the $\{u, v\}_{ij}$ generate $K_2(3, R)$, and it is standard that $\{u, v\}_{ij} = \{u, v\}_{12}$ in $K_2(3, R)$.

7. The stable case

7.1. THEOREM. — *Let* R *be* 3-*fold stable. The map* $\langle a, b \rangle \mapsto \langle a, b \rangle_{12}$ *defines an isomorphism* $D(R) \to K_2(m, R)$ *for* $m \geqq 3$. *In particular*, $K_2(R) \simeq D(R)$.

Proof. — The case $m = 3$ has been proved in the previous section. Remains to show that $K_2(n, R) \to K_2(n+1, R)$ is an isomorphism for $n \geqq 3$. But this follows from [23], Theorem 4.1, because a 3-fold stable ring is certainly 1-fold stable and therefore has the minimum possible stable range. (In the conventions of *loc. cit.*, the stable range is 1.)

7.2. For the sake of completeness we will show how to derive the isomorphism $K_2(n, R) \cong K_2(n+1, R)$ from our own work on injective stability. So Theorem 7.1 can be proved entirely by the chunck method. [When saying this we ignore the easy part of the theorem stating that the $\langle a, b \rangle_{12}$ generate $K_2(m, R)$.] Note that if R happens to be semi-local, Theorem 1 of [11] applies. We want to show that Theorem 4 of [11] (§ 3) applies for any 3-fold stable ring. We have to check the technical conditions. We restrict ourselves to the hardest one, leaving the remaining conditions to the reader. (One should argue as in 6.14.)

7.3. Let A, B, C, D be four 2 by 3 matrices, each obtained by taking the top two rows of an element of GL (3, R). Further let $a, b, c, d \in R^2$. Our objective is to show that there is $v \in R^3$ such that $a + Av$, $b + Bv$, $c + Cv$, $d + Dv$ are unimodular. [Recall that $w \in R^2$ is unimodular if and only if there is an R-linear map $f : R^2 \to R$ with $f(w) = 1$.] Clearly we may replace A, B, C, D by AU, BU, CU, DU for any $U \in GL(3, R)$. As A is part of an invertible matrix we can arrange that its first row is (1, 0, 0). Then we can multiply the four matrices by a suitable matrix $e_{21}(\lambda)\, e_{31}(\mu)$ such that B_{11}, C_{11}, D_{11} are also units. [Use 3-fold stability (*cf.* proof of 6.14).] We may further assume $A_{21} = 0$, because a, A may be replaced by $e_{21}(t)\, a$, $e_{21}(t)\, A$ with $t \in R$. Now choose $v \in R^3$ such that $a + Av$ has a unit in its second coordinate. Note that the first coordinate v_1 of v has no influence on the second coordinate of $a + Av$. Therefore we may change v_1, by 3-fold stability, so that $a + Av$, $b + Bv$, $c + Cv$, $d + Dv$ are unimodular.

7.4. REMARKS. — Actually the proof in [11] can be simplified a little for 3-fold stable rings. Instead of invoking the technical conditions $SR_n^p(c, u)$ one may use 3-fold stability directly. Our first proof of injective stability for 3-fold stable rings followed the line of section 2, 3, 4, 5. So instead of using the analogue of C from section 6 this original proof only used the analogue of C* from section 6. For higher dimensional maximal spectra one needs C however, which is why we introduced C in [11]. The proof in [11] (just as the original proof) can be adapted so that it works for 2-fold stable rings, but that still yields a result that is weaker than the one in [23]. In *loc. cit.* Suslin and Tulenbayev proved injective stability for K_2 in the form it had been conjecture ([7], Problem 4). To do this they had to unravel the Steinberg groups at hand. Our approach is quite different. Instead of trying to understand a given group better, we look for groups in which the part we already understand is big enough to extrapolate from. The advantage of our approach is that one is sure to get a theorem. Which theorem depends on how carefully the extrapolation is done. We have not been very careful for types different from type SL_n. (It is clear that one can do better than in sections 2, 3, 4.) Working with the crude method of open dense subsets (in a suitable topology) one should be able to prove stability results for symplectic K_2, say, of a noetherian ring of dimension d when there are only infinite residue fields. When some residue fields are finite, one has to be more specific about the size of the sets where statements are to be proved. By means of results like 5.14 and 6.17 one can presumably "recover ground" each time a statement is proved in too small a set. The smaller the residue fields, the harder it will be to carry the extrapolation through.

8. Comparing some presentations

8.1. PROPOSITION. — *Let* R *be* U-*irreducible* (*see* 1.3) *and* A *an abelian group. Let* $c : R^* \times R^* \to A$ *be a map satisfying*

$$c(x, y)\, c(xy, z) = c(x, yz)\, c(y, z),$$

$$c(1, 1) = 1,$$

$$c(x, y) = c(x^{-1}, y^{-1}) \quad \textit{when} \quad x, y, z \in R^*,$$

$$c(u, v) = c(u, (1-u)v) \quad \textit{when} \quad u, v, 1-u \in R^*.$$

Then c *also satisfies* $c(x, y) = c(x, -xy)$ *for* $x, y \in R^*$.

Proof. — We partly follow the proof of [17], 5.7.

Step 1. — Suppose x, y, $1-x$ are units. Then $c(x, y) = c(x, -xy)$ because

$$c(x, y) = c(x^{-1}, y^{-1}) = c(x^{-1}, (1-x^{-1})\, y^{-1}) = c(x, x(x-1)^{-1}\, y) = c(x, -xy).$$

Step 2. — If $u \in R^*$ then $c(1, u) = c(u, 1) = 1$ (easy).

Step 3. — Suppose $x, y, 1-x, 1-y, 1-xy$ are units. Then $c(x, y) = c(y^{-1}, x)$ because

$$\begin{aligned} 1 = c(xy, 1) = c(xy, -y^{-1}x^{-1}) &= c(x, -x^{-1})\, c(y, -y^{-1}x^{-1})\, c(x, y)^{-1} \\ &= c(x, 1)\, c(y, x^{-1})\, c(x, y)^{-1} = c(y^{-1}, x)\, c(x, y)^{-1}. \end{aligned}$$

Step 4. — Put $b(x, y) = c(x, y)\, c(y, x)^{-1}$ for $x, y \in R^*$.

Then b is bimultiplicative because

$$\begin{aligned} b(xy, z) &= c(xy, z)\, c(z, xy)^{-1} = c(y, z)\, c(x, yz)\, c(x, y)^{-1}\, c(z, xy)^{-1} \\ &= c(y, z)\, c(x, zy)\, c(zx, y)^{-1}\, c(z, x)^{-1} = c(y, z)\, c(x, z)\, c(z, y)^{-1}\, c(z, x)^{-1}. \end{aligned}$$

And

$$b(p, q) = b(q, p)^{-1}.$$

Step 5. — Suppose $x, y, 1-x, 1-y, 1-xy$ are units. Then

$$c(x, y^2) = b(x, y)$$

because

$$\begin{aligned} c(x, y^2)\,c(x, y)^{-1} &= c(xy, y)\,c(y, y)^{-1} = c(yx, -x^{-1})\,c(y, -1)^{-1} \\ &= c(y, x)^{-1}\,c(x, -x^{-1}) = c(y, x)^{-1}. \end{aligned}$$

Step 6. – Suppose x, y, $1-x$ are units. Then $c(y, x) = c(y(1-x), x)$, because

$$\begin{aligned} c(y, x)\,c(y(1-x), x)^{-1} &= c(x, y)\,c(x, y(1-x))^{-1},\ b(x, y)^{-1}\,b(x, y(1-x)) \\ &= b(x, 1-x) = c(1-x, x)^{-1} = 1. \end{aligned}$$

Therefore, if $c'(x, y)$ is defined by $c'(x, y) = c(y, x)$, then c' has the same properties as c. For instance, if x, y, $1-y$ are units then $c(x, y) = c(-xy, y)$.

Step 7. – Suppose x, y, $1-x$ are units. Then $c(x, y^2) = b(x, y)$ because there is, by U-irreducibility, an element $t \in R$ such that

$$\begin{aligned} c(x, y^2) &= c(x, t^2 y^2 t^{-2}) = c(xt^2, y^2 t^{-2})\,c(x, t^2)\,c(t^2, y^2 t^{-2})^{-1} \\ &= b(xt^2, yt^{-1})\,b(x, t)\,b(t^2, yt^{-1})^{-1} = b(x, y). \end{aligned}$$

(One has to choose t such that t, $t-y$, $1-xt^2$, $1-txy$, $1-t^2$, $1-xt$, $1-yt$ are units.)

Step 8. – Suppose x, y are units. Choose $t \in R$ such that t, t^2-x, $t^2-x^2 y$, $1-t^2$, $1+t^2 y$ are units. Then

$$\begin{aligned} c(x, -xy) &= c(t^2 xt^{-2}, -xy) = c(t^2, -x^2 t^{-2} y)\,c(xt^{-2}, -xy)\,c(t^2, xt^{-2})^{-1} \\ &= c(t^2, -x^2 t^{-2} y)\,c(xt^{-2}, t^2 y)\,c(t^2, xt^{-2})^{-1} \\ &= b(t, -x^2 t^{-2} y)\,c(x, y)\,c(xt^{-2}, t^2)\,c(t^2, y)^{-1}\,b(t, xt^{-2})^{-1} \\ &= b(t, -xy)\,c(x, y)\,b(xt^{-2}, t)\,c(t^2, -t^2 y)^{-1} \\ &= b(t, -xy)\,c(x, y)\,b(t, xt^{-2})^{-1}\,b(t, -t^2 y)^{-1} = c(x, y). \end{aligned}$$

8.2. Remark. – Clearly the above proof also works for a local ring whose residue field contains at least ten elements.

8.3. Definition. – Let R be a commutative ring. We call US (R) the group generated by symbols $\{x, y\}$, where $x, y \in R^*$, subject to the following relations and their consequences:

(i) US (R) is abelian;

(ii) $\{x, yz\} = \{x, y\}\{x, z\}$ for $x, y, z \in R^*$;

(iii) $\{xy, z\} = \{x, z\}\{y, z\}$ for $x, y, z \in R^*$;

(iv) $\{x, 1-x\} = 1$ if $x, 1-x \in R^*$.

8.4 Theorem. – *Let* R *be* 5-*fold stable. Then* $\tau(\{x, y\}) = \langle (x-1)y^{-1}, y\rangle$ *defines an isomorphism* US (R) $\to$ D (R).

Proof. – We compute in US (R).

Step 1. – If x, $1-x$ are units, then $\{x, -x\} = \{x, 1\} = 1$ (*see* step 1 of previous proof).

Step 2. — If y, z, $1-y$, $1-z$, $1-yz$ are units, then $\{y, z\}\{z, y\} = 1$ because

$$\{y, z\}\{z, y\} = \{y, -yz\}\{z, -yz\} = \{yz, -yz\} = 1.$$

Step 3. — If y, z, $1-y$ are units, then $\{y, z\}\{z, y\} = 1$ because

$$\{y, z\}\{z, y\} = \{y, zt\}\{zt, y\} = 1,$$

where t is chosen such that t, $1-t$, $1-ty$, $1-zt$, $1-zty$ are units.

Step 4. — If y, z are units, then $\{y, z\}\{z, y\} = 1$ because

$$\{y, z\}\{z, y\} = \{yt, z\}\{z, yt\} = 1$$

when t, $1-t$, $1-ty$ are units.

Step 5. — If y is a unit, then $\{y, -y\} = 1$ because

$$\{t, -t\}\{y, -y\} = \{yt, -yt\} = 1$$

when t, $1-t$, $1-yt$ are units.

Put

$$\langle a, b\rangle(x) = \{1+ax, x\}\{(1+ab)(1+ax)^{-1}, (b-x)(1+ax)^{-1}\},$$

when $1+ab$, x, $1+ax$, $b-x$ are units.

Step 6. — If a, $1+ab$, x, $1+ax$, $b-x$ are units, then $\langle a, b\rangle(x) = \{-a, 1+ab\}$ because

$$\{(1+ab)(1+ax)^{-1}, a(x-b)(1+ax)^{-1}\} = 1 \quad \text{and} \quad \{1+ax, -ax\} = 1$$

(both by the same rule).

Step 7. — If b, $1+ab$, x, $1+ax$, $b-x$ are units, then $\langle a, b\rangle(x) = \{1+ab, b\}$ because

$$\{xb^{-1}, (b-x)b^{-1}\} = 1 \quad \text{and} \quad \{-b(1+ax)(x-b)^{-1}, x(1+ab)(x-b)^{-1}\} = 1$$

(as above).

It follows that $\langle a, b\rangle(x)$ doesn't depend on x when $1+ab$, a are units or when $1+ab$, b are units. We therefore often write $\langle a, b\rangle$ for $\langle a, b\rangle(x)$ in these cases. [The idea is that this $\langle a, b\rangle$ will serve as $\tau^{-1}(\langle\langle a, b\rangle\rangle)$.] Note that

$$\langle a, b\rangle(x) = \langle a, x\rangle\langle a, (b-x)(1+ax)^{-1}\rangle$$

when the left hand side is defined.

Step 8. — If $1+ab$, x, $1+ax$, $b-x$, $1+ac$, $b+c+abc-x$, c are units, then $\langle a, b\rangle(x)\langle a, c\rangle = \langle a, b+c+abc\rangle(x)$ because

$$\begin{aligned}&\langle a, b+c+abc\rangle(x)\langle a, x\rangle^{-1}\\ &\quad = \langle a, (b+c+abc-x)(1+ax)^{-1}\rangle((b-x)(1+ax)^{-1}) = \langle a, (b-x)(1+ax)^{-1}\rangle\langle a, c\rangle.\end{aligned}$$

Step 9. — If $1+ab$, x, $1+ax$, $b-x$, y, $b-y$, $1+ay$, $y-x$ are units, then

$$\langle a, b\rangle(x) = \langle a, b\rangle(y)$$

because

$$\langle a, b\rangle(x) = \langle a, y\rangle(x)\langle a, (b-y)(1+ay)^{-1}\rangle = \langle a, b\rangle(y).$$

Step 10. — If $1+ab, x, 1+ax, b-x, y, b-y, 1+ay$ are units, then $\langle a, b\rangle(x) = \langle a, b\rangle(y)$ because

$$\langle a, b\rangle(x) = \langle a, b\rangle(t) = \langle a, b\rangle(y)$$

when $t, b-t, 1+at, t-x, t-y$ are units.

So $\langle a, b\rangle(x) = \langle a, b\rangle(y)$ whenever both sides are defined. Therefore we also write $\langle a, b\rangle$ for $\langle a, b\rangle(x)$. Note that $\langle a, b\rangle$ is defined [in US (R)] exactly when $1+ab \in R^*$.

Step 11. — $\langle a, b\rangle\langle a, c\rangle = \langle a, b+c+abc\rangle$ when $1+ab$, $1+ac$ are units, because

$$\begin{aligned}&\langle a, b+c+abc\rangle(x)\langle a, x\rangle^{-1}\\ &\quad= \langle a, (b-x+c+abc)(1+ax)^{-1}\rangle(y) = \langle a, c\rangle(y)\langle a, (b-x)(1+ax)^{-1}\rangle\end{aligned}$$

when $x, 1+ax, b-x+c+abc, b-x, y, 1+ay, c-y, (b-x+c+abc)(1+ax)^{-1}-y$ are units.

Step 12. — If x, $1+abx$ are units, then $\langle a, bx\rangle = \langle ax, b\rangle\langle ab, x\rangle$ because

$$\langle a, bx\rangle(xy)\langle ab, x\rangle^{-1}(\langle ax, b\rangle(y))^{-1} = 1$$

when $y, b-y, 1+axy$ are units.

Step 13. — If $x, y, x+y, 1+axb, 1+axb+ayb$ are units, then

$$\langle ax+ay, b\rangle = \langle ax, b\rangle\langle ay(1+axb)^{-1}, b\rangle$$

because

$$\begin{aligned}&\langle ax, b\rangle\langle ay(1+axb)^{-1}, b\rangle = \langle a, xb\rangle\langle a, yb(1+axb)^{-1}\rangle\langle ab, x\rangle^{-1}\\ &\quad\times\langle ab, y(1+axb)^{-1}\rangle^{-1} = \langle a, xb+yb\rangle\langle ab, x+y\rangle^{-1} = \langle ax+ay, b\rangle.\end{aligned}$$

Step 14. — If $1+a, 1+b+ab, 1+ab$ are units, then $\langle a, b\rangle = \langle -b, -a\rangle^{-1}$ because

$$\begin{aligned}\langle a,b\rangle &= \langle a, b\rangle\langle a, 1\rangle = \langle a, 1+b+ab\rangle = \{(1+a)(1+ab), 1+b+ab\}\\ &= \langle b(1+ab)^{-1}, (1+a)(1+ab)\rangle^{-1} = \langle b(1+ab)^{-1}, -a\rangle\langle b(1+ab)^{-1}, 1\rangle^{-1}\\ &= \langle -b, a(1+ab)^{-1}\rangle\langle ab, -(1+ab)^{-1}\rangle^{-1}\\ &= \langle -b, -a\rangle^{-1}\langle -b, 0\rangle\{(1+ab)^{-1}, -(1+ab)^{-1}\}^{-1} = \langle -b, -a\rangle^{-1}.\end{aligned}$$

Step 15. — If $1+a$, $1+ab$ are units, then $\langle a, b\rangle = \langle -b, -a\rangle^{-1}$ because

$$\begin{aligned}\langle a, b\rangle &= \langle a, xb\rangle\langle a, (1-x)b(1+axb)^{-1}\rangle\\ &= \langle -xb, -a\rangle^{-1}\langle -(1-x)b(1+axb)^{-1}, -a\rangle^{-1} = \langle -b, -a\rangle^{-1}\end{aligned}$$

when $1+axb$, $1+xb+axb$, $1+ab+b(1-x)$, x, $1-x$ are units.

Step 16. — If $1+ab$ is a unit, then $\langle a, b\rangle = \langle -b, -a\rangle^{-1}$ because the previous computation is also valid when $1+axb$, $1-xb$, $1+ab+(1-x)(-b-ab)$, x, $1-x$ are units. (So we iterate step 15.)

Step 17. — If $1+abc$ is a unit, then $\langle a, bc \rangle = \langle ab, c \rangle \langle ac, b \rangle$ because

$$\begin{aligned}\langle a, bc \rangle &= \langle a, xc \rangle \langle a, (b-x)c(1+axc)^{-1} \rangle \\ &= \langle ax, c \rangle \langle ac, x \rangle \langle a(b-x)(1+axc)^{-1}, c \rangle \langle ac, (b-x)(1+axc)^{-1} \rangle \\ &= \langle -c, -ax \rangle^{-1} \langle -c, -a(b-x)(1+axc)^{-1} \rangle^{-1} \langle ac, b \rangle \\ &= \langle -c, -ab \rangle^{-1} \langle ac, b \rangle\end{aligned}$$

when x, $b-x$, $1+axc$ are units.

It follows from the above that $\langle a, b \rangle \mapsto \langle a, b \rangle$ defines a homomorphism $\sigma : \mathrm{D}(\mathrm{R}) \to \mathrm{US}(\mathrm{R})$. Also, it follows from Lemma 5.6 that $\tau(\{x, y\}) = \langle (x-1)y^{-1}, y \rangle$ does at least define a homomorphism $\mathrm{US}(\mathrm{R}) \to \mathrm{D}(\mathrm{R})$. Clearly $\sigma\tau = \mathrm{id}$ and it is also easy to see that $\tau\sigma = \mathrm{id}$.

8.5. COROLLARY. — *Let* R *be* 5*-fold stable. Then* $\mathrm{K}_2(\mathrm{R})$ *is isomorphic with* $\mathrm{US}(\Phi, \mathrm{R})$.

Proof. — As R is also 3-fold stable we have $\mathrm{K}_2(\mathrm{R}) \simeq \mathrm{D}(\mathrm{R})$, by 7.1.

8.6. *Example* (R. K. Dennis). — Let $\mathrm{R} = \mathbf{F}_4 \times \mathbf{F}_4$. So R has 16 elements and R is 3-fold stable. Let ζ be a primitive third root of unity in $\mathbf{F}_4$. Put

$$c((\zeta^a, \zeta^b), (\zeta^c, \zeta^d)) = \zeta^{ac-bd}.$$

It is easy to check that c induces a homomorphism $\mathrm{US}(\mathrm{R}) \to \mathbf{F}_4^*$. However, c does not satisfy the rule $c(x, -x) = 1$, because $c(x, -x) = c(x, x) = \zeta$ for $x = (\zeta, 1)$. It follows that the analogues of proposition 8.1 and theorem 8.4 do not hold under the weaker condition of 3-fold stability (*see* Lemma 5.6).

8.7. *Remarks.* — We do not know whether 4-fold stability suffices for theorem 8.4. We even do not know whether 4-fold stability suffices when one weakens the theorem by adding the relation $\{x, -x\} = 1$ to the list in 8.3. Anyway, it is known that something like D (R) is needed for a local ring with a very small residue field. [Say $\mathrm{R} = \mathbf{Z}_p$ where p is the prime ideal $2\,\mathbf{Z}$. Then we have in $\mathrm{K}_2(\mathrm{R})$ the relation $\{27, 13\} = 1$, because $\{27, 13\} = \langle 2, 13 \rangle = \langle 2, 1 \rangle^3$. But in a group like US (R) such information is lost, because 8.3 (iv) never applies.]

9. Power norm residue symbols

9.1. Let Φ_n denote the nth cyclotomic polynomial, i. e. the minimum polynomial over $\mathbf{Q}$, with leading coefficient 1, for a primitive nth root of unity ω. It is well-known that $\Phi_n(\mathrm{X}) \in \mathbf{Z}[\mathrm{X}]$ and that $\Phi_n(\mathrm{X}) = \prod_{\substack{0<i<n \\ \mathrm{g.c.d.}(i,n)=1}} (\mathrm{X}-\omega^i)$.

If k is a field and α an element of k, then the following statements are equivalent:

(i) $\Phi_n(\alpha) = 0$ and n is invertible in k;

(ii) α is a primitive nth root of unity in k.

9.2. DEFINITION. – Let α be an element of the commutative ring R. We say that α is a *primitive nth root of unity in* R if $\Phi_n(\alpha) = 0$ and n is invertible in R. Equivalently, α is a primitive nth root of unity in R if the image of α in R/m is a primitive nth root of unity for all maximal ideals m of R.

9.3. If $\varphi : R \to S$ is a homomorphism of rings and ω is a primitive nth root of unity in R, then $\varphi(\omega)$ is one in S. In particular this applies when R is a field and ω a primitive nth root of unity in R, in the usual sense.

9.4. Let ω be a primitive nth root of unity in R. If $a, b \in R^*$ we define, as in [18] (§ 15), an associative algebra $A_\omega(a, b)$ (with identity 1) which is generated by elements x and y subject to the relations $x^n = a\,1$, $y^n = b\,1$, $yx = \omega\, xy$. As an R-module $A_\omega(a, b)$ is free of rank n^2.

Consider the special case $a = c^n$, $c \in R^*$. Define the ring homomorphism $f : R[x] \to R \times \ldots \times R$ by

$$f(x) = (c, \omega c, \ldots, \omega^{n-1} c),$$

$$f(r) = (r, r, \ldots, r) \qquad \text{for} \quad r \in R.$$

If R is a field, then f is injective and, as the dimensions are equal, f is even an isomorphism. If R is a local ring, then f is an isomorphism because of the Nakayama lemma. For general R we still get an isomorphism because the question is local. We gave such an elaborate argument because it exemplifies how to generalize the arguments of [18] (§ 15). In particular, we find n orthogonal idempotents e_i in $R[x]$ and isomorphisms $A_\omega(c^n, b) \to \mathrm{Hom}_R(A\,e_i, A\,e_i)$. Note that the $y^j e_i$, $1 \leqq i \leqq n$, $0 \leqq j \leqq n-1$ form a basis of $A_\omega(c^n, b)$, so that the left ideals $A\,e_i$ are free R-modules.

From the special case just considered we see that in general $A_\omega(a, b)$ is an Azumaya algebra, split by $S = R[T]/(T^n - a)$. Let Br (R) denote the Brauer group of R (*see* [15]). The following is standard.

9.5. PROPOSITION. – *Let* ω *be a primitive nth root of unity in* R:

(i) *there is a homomorphism* $c_\omega : \mathrm{US}(R) \to \mathrm{Br}(R)$ *which sends* $\{a, b\}$ *to the class of* $A_\omega(a, b)$. *We call* c_ω *an nth power norm residue symbol;*

(ii) *if* R *is semi-local, a symbol* $\{a, b\}$ *is in the kernel of* c_ω *if and only if a is a norm from the extension* $R[T]/(T^n - b)$ *of* R.

Proof. – *See* Milnor's discussion and adapt it in the fashion indicated above.

9.6. *Remark.* – One can be more general. Let G be the cyclic group of order n, say with generator σ. Then $A_\omega(a, b)$ is a crossed-product algebra $\Delta(f; S; G)$ in the sense of [2], p. 404. Here $S = R[y]$ and

$$f(\sigma^i, \sigma^j) = 1 \qquad \text{if} \quad i+j < n,$$

$$f(\sigma^i, \sigma^j) = a \qquad \text{if} \quad i+j \geqq n \qquad (0 \leqq i < n, 0 \leqq j < n).$$

It is easy to see that a is a norm from S if and only if f is a coboundary. Thus (ii) is a special case of [2] (Th. A 15) or [4] (Cor. 5.5).

9.7. THEOREM. — *Let* R *be semi-local or* 5*-fold stable,* ω *a primitive nth root of unity in* R. *There is an nth power norm residue symbol* $c_\omega : K_2(R) \to Br(R)$, *sending* $\{u, v\}_{12}$ *to the class of* $A_\omega(u, v)$, *for* $u, v \in R^*$.

9.8. REMARK. — Assuming $n > 1$ the residue fields of R have at least three elements, so we know by [21] that $K_2(R)$ is generated by the $\{u, v\}_{12}$ [*cf.* 5.3, 5.4 (1), (2)].

PROOF OF THE THEOREM. — When R is 5-fold stable we have $K_2(R) \cong US(R)$ and the result is an obvious consequence of 9.5. Remains the case that R is semi-local with at least one small residue field. (Here "small" means less than 6 elements. Let N be the set of integers m for which the rule $\{u, v\}_{12} \mapsto$ (mth power of the class of $A_\omega(u,v)$) defines a homomorphism $K_2(R) \to Br(R)$. Clearly N is a subgroup of **Z**. We have to show $N = \mathbf{Z}$. For $m = n$ we get the trivial homomorphism (*cf.* 9.4), so $n \in N$. As R has a "small" residue field, n is 2 or 3 or 4. (We have a primitive nth root of unity.) Say $p_1, \ldots, p_k$ are the characteristics of the residue fields of R. Put $m = p_1 p_2 \ldots p_k + n^2$, $S = R[T]/(T^m - T^{n^2} + 1)$. Then $R \to S$ is a finite etale extension of constant rank m, so by [14] there is a norm or corestriction map $Br(S) \to Br(R)$ such that the composition $Br(R) \to Br(S) \to Br(R)$ sends a class to its mth power. Now S is 5-fold stable, as it is a semi-local ring without "small" residue fields. So we have a composite homomorphism $K_2(R) \to K_2(S) \to Br(S) \to Br(R)$ which sends $\{u, v\}_{12}$ to the mth power of the class of $A_\omega(u, v)$. Therefore $m \in N$ and, as m is prime to n, we get $N = \mathbf{Z}$.

9.9. COROLLARY. — *Let* R *be semi-local, with primitive nth root of unity* ω. *There is an nth power norm residue homomorphism* $D(R) \to Br(R)$, *sending* $\langle (u-1)v^{-1}, v \rangle$ *to the class of* $A_\omega(u, v)$, *for* $u, v \in R^*$.

9.10. REMARK. — Let R be a noetherian domain which is smooth over a fied, ω a primitive n-th root of unity in R. Let F be the field of fractions of R. We have a commutative diagram

$$\begin{array}{ccc} K_2(R) & \to & Br(R) \\ \downarrow & & \downarrow \\ K_2(F) & \underset{c_\omega}{\to} & Br(F) \end{array}$$

For, by Hoobler (to appear), the intersection of the images of the $Br(R_{\mathscr{P}})$ in $Br(F)$ is isomorphic with $Br(R)$. (Here $\mathscr{P}$ runs through the maximal spectrum of R.) And the composite map $K_2(R) \to K_2(F) \to Br(F)$ factors through $K_2(R_{\mathscr{P}}) \to Br(R_{\mathscr{P}})$.

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