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INDUCED REPRESENTATIONS OF REDUCTIVE $\mathfrak{p}$-ADIC GROUPS. I

BY I. N. BERNSTEIN AND A. V. ZELEVINSKY

Introduction

Let G be a reductive group over a non-archimedean local field F. It follows from works of Harish-Chandra and Jacquet that there are two main problems in the studying of irreducible representations of the group G :

(*a*) The description of cuspidal irreducible representations of the group G.

(*b*) The studying of induced representations. More precisely, let P be a parabolic subgroup in G, M its Levi subgroup, ρ a cuspidal irreducible representation of M. Let π be the representation of G induced by ρ in a standard way (such representations will be called the induced ones). The problem is to study the conditions of irreducibility of π, the decomposition of π in irreducible components and the connections between different induced representations.

We will deal only with the second problem.

Let us formulate the main results of this paper. Let P and Q be parabolic subgroups in G, M and N their Levi subgroups, ρ and ρ' irreducible cuspidal representations of M and N; let π and π' be the corresponding induced representations.

In paragraph 2 we prove the equivalence of the following conditions (Theorem 2.9):

(i) Pairs (M, ρ) and (N, ρ') are conjugate by some element of G.

(ii) $\mathrm{Hom}(\pi, \pi') \neq 0$.

(iii) π and π' have the same families of composition factors.

(iv) π and π' have a common composition factor.

One can prove the equivalence (i)-(iii) using the theory of intertwiningo perators (*see* [11], [14] and [15]). We use another method based on the study of the functor r adjoint to the functor of inducing. For the representations of principal series this method was used by Casselman in [6]. Such a method allows us to get some information about subquotients of π and π'; e. g. using it we prove the implication (iv) $\Rightarrow$ (i). Furthermore, we obtain an estimate for the length of π which depends only on G and M (Th. 2.8). It refines the results of Howe [12].

In case $G = G_n = GL(n, F)$, we obtain more precise results. The basic method used here is that of the restriction of induced representations to the subgroup P_n [this subgroup consists of the matrices with the last row $(0, 0, \ldots, 0, 1)$]. This is a method of Gelfand and Kajdan ([8], [9]).

In paragraph 3 we study in details representations of the group P_n. It turns out that there is a one-to-one correspondence between irreducible representations of P_n and such of the family of groups G_k, $0 \leqq k < n$ (Cor. 3.5).

In paragraph 4 we describe the decomposition of the restriction to P_n of any induced representation of the group G_n. Using this decomposition we get the main results of the paper namely:

(*a*) Criterion of irreducibility of induced representations of the group G_n (Th. 4.2). If $\rho_1, \ldots, \rho_k$ are irreducible cuspidal representations of the groups $G_{n_1}, \ldots, G_{n_k}$ and $\pi = \rho_1 \times \ldots \times \rho_k$ is the corresponding induced representation of the group G_n, $n = n_1 + \ldots + n_k$, then π is reducible iff for some i, j $n_i = n_j$ and $\rho_j = |\det|.\rho_i$.

(*b*) The existence of the Kirillov model for any irreducible non-degenerate representation of G_n (Conjecture of Gelfand-Kajdan [8]), Theorem 4.9.

Note that in Theorem 4.2 we prove only sufficient conditions of irreducibility. The necessity of these conditions will be proved in Part II of this paper.

Our proof of the Gelfand-Kajdan conjecture is based on Theorem 4.11, which describes when an induced representation has a degenerate P_n-subrepresentation. This theorem is of an independent interest.

Few words about methods. We use the theory of algebraic (= smooth in the sense of Harish-Chandra) representations of locally compact totally disconnected groups (we call them *l*-groups), *see* [1]. Our basic tools are the functors of inducing $I_{U,\theta}$, $i_{U,\theta}$ and the adjoint functor of "localisation" $r_{U,\theta}$, which connect the representations of different *l*-groups. Some properties of these functors and the needed information about representations of *l*-groups are collected in paragraph 1.

Proofs of the results of paragraphs 2, 4 are based on the geometrical Lemmas 2.12 and 4.13. Using these lemmas we obtain all our results by purely category-theoretical arguments dealing only with connections between some functors. The geometrical Lemmas describe the composition of the functors $r_{V,\psi}$ and $i_{U,\theta}$ in some special situations. They are in fact the particular cases of Theorem 5.2 which describes such a composition in a very general case. We prove this theorem in paragraph 5 and in paragraphs 6, 7 deduce from it the geometrical Lemmas.

The results of this paper are announced in [2].

Recently we have received the paper by W. Casselman „Introduction to the theory of admissible representations of p-adic reductive groups" which contains the proof of almost all results paragraph 2 and paragraph 6 of our paper and some more. Casselman's results are based on the ideas of Harish-Chandra and Jacquet, and so are our results. We are very obliged to W. Casselman for his paper.

1. Preliminaries

In this section we shall introduce some basic notions and notations. They will be used throughout the whole paper. All proofs may be found in [1].

1.1. A Hausdorff topological group G is called an *l*-group, if any neighbourhood of the identity contains an open compact subgroup.

Example. – A $\mathfrak{p}$-adic Lie group is an l-group.

1.2. If π is a representation of an l-group G on a complex vector space E, we briefly write $\pi = (\pi, G, E)$; the space E is called a G-module. No topology on E is considered.

A representation (π, G, E) is called algebraic [1] if the stabilizer stab ξ of any vector $\xi \in E$ is an open subgroup of G. If (π, G, E) is a representation let E_{alg} be the subspace of all vectors $\xi \in E$ with an open stabilizer; the representation $\pi_{alg} = (\pi|_{E_{alg}}, G, E_{alg})$ is said to be the algebraic part of π.

Let G be an l-group. Denote by Alg G the abelian category of algebraic representations of G. Let Irr G be the subcategory of Alg G, consisting of irreducible representations and let $\overline{\text{Irr}}$ G be the set of equivalence classes of representations from Irr G.

If $\pi \in$ Alg G, we call subquotient of π any representation of the form π_2/π_1, where $\pi_1 \subset \pi_2 \subset \pi$. Let JH (π) denote the subset of $\overline{\text{Irr}}$ G, consisting of equivalence classes of irreducible subquotients of π. Let $l(\pi)$ denote the length of π. If $l(\pi) < \infty$, we denote by $\text{JH}^0(\pi)$ the set of irreducible quotients of Jordan-Hölder series of π [each element $\omega \in$ JH (π) is contained in $\text{JH}^0(\pi)$ with some multiplicity].

1.3. A representation $(\pi, G, E) \in$ Alg G is called admissible, if the subspace of K-invariant vectors $\xi \in E$ is finite-dimensional for any open subgroup $K \subset G$.

1.4. Let $(\pi, G, E) \in$ Alg G and E* the space of all linear forms on E. We define the representation (π^*, G, E^*) by

$$\langle \pi^*(g)\xi^*, \xi \rangle = \langle \xi^*, \pi(g^{-1})\xi \rangle, \qquad g \in G, \quad \xi^* \in E^*, \quad \xi \in E.$$

Define the contragredient representation $(\tilde{\pi}, G, \tilde{E})$ to π by $\tilde{\pi} = (\pi^*)_{alg}$, $\tilde{E} = (E^*)_{alg}$. If π is admissible, so is $\tilde{\pi}$, and $\tilde{\tilde{\pi}} = \pi$.

1.5. We call character of an l-group G any locally constant homomorphism $\theta : G \to \mathbf{C}^*$. If $(\pi, G, E) \in$ Alg G, then the representation $(\theta\pi, G, E) \in$ Alg G is defined by $(\theta\pi)(g) = \theta(g).\pi(g)$. We obtain the functor θ : Alg G $\to$ Alg G. Note that $\widetilde{\theta\pi} = \theta^{-1}\tilde{\pi}$.

1.6. Let G_1, G_2 be l-groups and $G = G_1 \times G_2$. Using tensor products, one obtains the bifunctor

$$\otimes : \quad \text{Alg}\, G_1 \times \text{Alg}\, G_2 \to \text{Alg}\, G \quad ((\rho_1, \rho_2) \to \rho_1 \otimes \rho_2).$$

If $\rho_i \in \text{Alg}\, G_i$ $(i = 1, 2)$ are admissible and irreducible, so is $\rho_1 \otimes \rho_2 \in$ Alg G; conversely any admissible $\rho \in$ Irr G has such a form, and the equivalence classes of ρ_i are determined by ρ. Note, that in this case $\widetilde{\rho_1 \otimes \rho_2} = \tilde{\rho}_1 \otimes \tilde{\rho}_2$.

1.7. Let M, N be l-groups and $\sigma : M \to N$ an isomorphism. Define the functor σ : Alg M $\to$ Alg N. If $(\rho, M, L) \in$ Alg M then the representation $(\sigma\rho, N, L) \in$ Alg N is defined by

$$(\sigma\rho)(n)\xi = \rho(\sigma^{-1} n)\xi, \qquad n \in N, \ \xi \in L.$$

If η is an inner automorphism of M, then the functors σ, $\sigma \circ \eta$: Alg M $\to$ Alg N are isomorphic.

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(1) It is smooth in the sense of Harish-Chandra.

Let σ be an automorphism of an l-group U. Denote by $\mathrm{mod}_{\mathrm{U}}\,\sigma$ the (topological) module of σ. It is defined by the formula

$$\int_{\mathrm{U}} f(\sigma^{-1}u)\,d\mu(u) = \mathrm{mod}_{\mathrm{U}}\,\sigma.\int_{\mathrm{U}} f(u)\,d\mu(u),$$

where μ is a Haar measure on U. If U is a closed subgroup of an l-group G and an element $g \in \mathrm{G}$ normalises U, then denote by $\mathrm{mod}_{\mathrm{U}}(g)$ the module of the automorphism $u \to gug^{-1}$; $\mathrm{mod}_{\mathrm{U}}$ is a character of the normaliser of U in G. The character $\Delta_{\mathrm{G}} = \mathrm{mod}_{\mathrm{G}}^{-1}$ is said to be the module of the group G.

If U′ is a closed subgroup of an l-group U, and $\Delta_{\mathrm{U}} = 1$, $\Delta_{\mathrm{U}'} = 1$, then there exists an invariant measure on the quotient space $\mathrm{U}'\diagdown\mathrm{U}$. Let σ be an automorphism of U, such that $\sigma(\mathrm{U}') = \mathrm{U}'$. One can define the module $\mathrm{mod}_{\mathrm{U}'\diagdown\mathrm{U}}\,\sigma$. The Fubini theorem implies that $\mathrm{mod}_{\mathrm{U}'\diagdown\mathrm{U}} = \mathrm{mod}_{\mathrm{U}}/\mathrm{mod}_{\mathrm{U}'}$.

All characters of the form $\mathrm{mod}_{\mathrm{U}}$ are positive. So we shall frequently use the following simple statement: any positive character of a compact l-group equals 1.

1.8. Now we describe some functors which play the fundamental role in this paper.

Let G be an l-group, M, U closed subgroups, such that M normalises U, $\mathrm{M} \cap \mathrm{U} = \{e\}$ and the subgroup $\mathrm{P} = \mathrm{MU} \subset \mathrm{G}$ is closed; let θ be a character of U normalised by M. In such a situation we define the functors

$$\mathrm{I}_{\mathrm{U},\theta},\qquad i_{\mathrm{U},\theta}:\quad \mathrm{Alg}\,\mathrm{M} \to \mathrm{Alg}\,\mathrm{G},$$

$$r_{\mathrm{U},\theta}:\quad \mathrm{Alg}\,\mathrm{G} \to \mathrm{Alg}\,\mathrm{M}.$$

(*a*) Let $(\rho, \mathrm{M}, \mathrm{L}) \in \mathrm{Alg}\,\mathrm{M}$. Denote by I (L) the space of functions $f : \mathrm{G} \to \mathrm{L}$, satisfying the following conditions :

(1) $$f(umg) = \theta(u)\,\mathrm{mod}_{\mathrm{U}}^{1/2}(m)\,\rho(m)(f(g)),\quad u \in \mathrm{U},\ m \in \mathrm{M},\quad g \in \mathrm{G}.$$

(2) There exists an open subgroup $\mathrm{K}_f \subset \mathrm{G}$ such that

$$f(gk) = f(g)\qquad \text{for}\quad g \in \mathrm{G},\quad k \in \mathrm{K}_f.$$

Define the representation $(\delta, \mathrm{G}, \mathrm{I}(\mathrm{L})) \in \mathrm{Alg}\,\mathrm{G}$ by $(\delta(g)f)(g') = f(g'g)$. We call δ an induced representation and denote it by $\mathrm{I}_{\mathrm{U},\theta}(\rho)$ (or, more complete, $\mathrm{I}_{\mathrm{U},\theta}(\mathrm{G}, \mathrm{M}, \rho)$).

Denote by i (L) the subspace of I (L), consisting of all functions compactly supported modulo the subgroup P = MU. The restriction of δ on the space i (L) is called compactly induced and is denoted by $i_{\mathrm{U},\theta}(\rho)$ (or $i_{\mathrm{U},\theta}(\mathrm{G}, \mathrm{M}, \rho)$).

(*b*) Let $(\pi, \mathrm{G}, \mathrm{E}) \in \mathrm{Alg}\,\mathrm{G}$. Denote by $\mathrm{E}(\mathrm{U}, \theta) \subset \mathrm{E}$ the subspace, spanned by the vectors of the form

$$\pi(u)\xi - \theta(u)\xi,\qquad u \in \mathrm{U},\qquad \xi \in \mathrm{E}.$$

The quotient space E/E (U, θ) is called the θ-localisation of the space E and is denoted by $r_{\mathrm{U},\theta}(\mathrm{E})$. Define the representation $(\delta, \mathrm{M}, r_{\mathrm{U},\theta}(\mathrm{E}))$ by

$$\delta(m)(\xi + \mathrm{E}(\mathrm{U}, \theta)) = \mathrm{mod}_{\mathrm{U}}^{-1/2}(m).(\pi(m)\xi + \mathrm{E}(\mathrm{U}, \theta)),\qquad m \in \mathrm{M},\qquad \xi \in \mathrm{E};$$

it is easily verified that δ is well defined. Call the representation δ the θ-localisation of π and denote it by $r_{U,\theta}(\pi)$ (or $r_{U,\theta}(M, G, \pi)$).

If $U = \{ e \}$, then the functors I, i turn into the ordinary inducing (without any factor) and r turns into the restriction of the representations. If $MU = G$, then the representation $I_{U,\theta}(\rho)$ equals $i_{U,\theta}(\rho)$ and it acts on the same space as ρ.

1.9. Describe the properties of functors I, i and r. We say that an l-group U is a limit of compact subgroups if for any compact $K \subset U$ there exists a compact subgroup $U' \subset U$, containing K; in particular, in this case $\Delta_U = 1$.

PROPOSITION. — (*a*) *The functors* $I_{U,\theta}$, $i_{U,\theta}$ *are exact. If* U *is a limit of by compact subgroups, then* $r_{U,\theta}$ *is exact.*

(*b*) *The functor* $r_{U,\theta}$ *is left adjoint to* $I_{U,\theta}$, *i. e. for any* $\rho \in \operatorname{Alg} M$, $\pi \in \operatorname{Alg} G$ *there is a natural isomorphism*

$$\operatorname{Hom}(r_{U,\theta}(\pi), \rho) = \operatorname{Hom}(\pi, I_{U,\theta}(\rho)).$$

(*c*) *Let* N, V *be subgroups of* M *and* θ' *be a character of* V *such that the functors*

$$I_{V,\theta'}, \quad i_{V,\theta'}; \quad \operatorname{Alg} N \to \operatorname{Alg} M \quad \text{and} \quad r_{V,\theta'}: \operatorname{Alg} M \to \operatorname{Alg} N$$

are well defined. Define the character θ^0 *of the group* $U^0 = UV$ *by* $\theta^0(uv) = \theta(u)\,\theta'(v)$ *Then:*

$$i_{U,\theta} \circ i_{V,\theta'} = i_{U^0,\theta^0}, \qquad I_{U,\theta} \circ I_{V,\theta'} = I_{U^0,\theta^0}, \qquad r_{V,\theta'} \circ r_{U,\theta} = r_{U^0,\theta^0}.$$

(*d*) *If* $\Delta_U = 1$ *then there is a natural isomorphism*

$$\widetilde{\Delta_G\, i_{U,\theta}(\rho)} = I_{U,\theta^{-1}}(\widetilde{\Delta_M \rho}), \qquad \rho \in \operatorname{Alg} M.$$

(*e*) *Suppose that* G *is compact modulo* $P = MU$. *Then the functors* $I_{U,\theta}$ *and* $i_{U,\theta}$ *coincide and carry admissible representations into admissible ones.*

(*f*) *If* χ *is a character of* G *then:*

$$\chi \circ i_{U,\theta} = i_{U,\chi\theta} \circ \chi, \qquad \chi \circ I_{U,\theta} = I_{U,\chi\theta} \circ \chi, \qquad \chi \circ r_{U,\theta} = r_{U,\chi\theta} \circ \chi.$$

(*g*) *The functors* $i_{U,\theta}$ *and* $r_{U,\theta}$ *commute with inductive limits; if* H *is an l-group and* $\tau \in \operatorname{Alg} H$ *then* $i_{U,\theta}$ *and* $r_{U,\theta}$ *commute with the functors* $\bigstar \mapsto \bigstar \otimes \tau$ *and* $\bigstar \mapsto \tau \otimes \bigstar$.

Parts (*f*) and (*g*) can be directly verified. The other assertions are proved in ([1], chap. I); since the definitions in [1] don't include the factor $\operatorname{mod}_U^{1/2}$, in (*b*), (*c*) and (*d*) one has to verify that all such factors are compatible.

1.10. If $\mathscr{A}$ and $\mathscr{B}$ are abelian categories, then additive functors from $\mathscr{A}$ to $\mathscr{B}$ form an abelian category. We will freely use such notions as an exact sequence of functors, a subfunctor a. s. o. They all can be understood "locally with respect to $\mathscr{A}$", e. g. to set a subfunctor Φ of the functor F one has to choose the subobject $\Phi(\rho) \subset F(\rho)$ for any $\rho \in \mathscr{A}$, such that for any morphism $\varphi : \rho \to \tau$ the morphism $F(\varphi)$ carries $\Phi(\rho)$ into $\Phi(\tau)$.

1.11. Let $\mathscr{A}$ be an abelian category and $C_1, \ldots, C_k \in \mathscr{A}$. We say that the object $D \in \mathscr{A}$ is glued from $C_1, \ldots, C_k$, if there is a filtration $0 = D_0 \subset D_1 \subset \ldots \subset D_k = D$ in D, such that the set of quotients $\{ D_i/D_{i-1} \}$ is isomorphic after a permutation to the set $\{ C_i \}$.

2. The induced representations of reductive groups

2.1. Let F be a locally compact nonarchimedean field. From now on by an algebraic F-group we mean the group of F-points of some algebraic group, defined over F. In a natural locally compact topology such groups are l-groups.

Let G be a connected (in an algebraic sense) reductive F-group. Fix from now on a minimal parabolic subgroup $P_0 \subset G$ and a maximal split torus $A_0 \subset P_0$.

Let P be a parabolic subgroup, containing P_0, U the unipotent radical of P. There exists a unique Levi subgroup in P containing A_0; denote it by M (it is a connected reductive F-group). It is known that P normalises U and has the Levi decomposition $P = MU$, $M \cap U = \{ e \}$. A group M, which can be obtained by such a construction, is called a standard subgroup in G (the notation is $M < G$) and a triple (P, M, U) is called a parabolic triple. Note that P and U are determined by M, since $P = P_0 M$.

In any standard subgroup $M < G$ we fix the minimal parabolic subgroup $P_0 \cap M$ and the maximal split torus A_0. It is clear that $N < M$ implies $N < G$.

2.2. *Example*. Let $G = G_n = GL(n, F)$, P_0 be the group of upper triangular matrices and A_0 the group of diagonal matrices. The standard subgroups of G are numerated by (ordered) partitions of n: to each partition $\alpha = (n_1, \ldots, n_r)$ there corresponds the subgroup $G_\alpha = G_{n_1} \times \ldots \times G_{n_r}$, embedded into G as the subgroup of cellular-diagonal matrices. Furthermore, $G_\beta < G_\alpha$ iff β is a subpartition of α.

2.3. Let $M < G$ and (P, M, U) be the corresponding parabolic triple. Define the functors

$$i_{G,M}\colon \ \mathrm{Alg}\,M \to \mathrm{Alg}\,G \qquad \text{and} \qquad r_{M,G}\colon \ \mathrm{Alg}\,G \to \mathrm{Alg}\,M,$$

by $i_{G,M} = i_{U,1}$, $r_{M,G} = r_{U,1}$ (*see* 1.8).

PROPOSITION. — (*a*) *The functors* $i_{G,M}$ *and* $r_{M,G}$ *are exact.*

(*b*) *The functor* $r_{M,G}$ *is left adjoint to* $i_{G,M}$.

(*c*) *If* $N < M < G$ *then:*

$$i_{G,M} \circ i_{M,N} = i_{G,N}, \qquad r_{N,M} \circ r_{M,G} = r_{N,G}.$$

(*d*) $\widetilde{i_{G,M}(\rho)} = i_{G,M}(\tilde{\rho})$, $\rho \in \mathrm{Alg}\,M$.

(*e*) *The functors* $i_{G,M}$ *and* $r_{M,G}$ *carry admissible representations into admissibles ones.*

Parts (*a*)-(*d*) and the first part of (*e*) follow from the corresponding points of the proposition 1.9. We have only to use the following statements: U is a limit of compact subgroups, G is compact modulo P and G is unimodular. It was proved by Jacquet that $r_{M,G}$ carries admissible representations into admissible ones. (*See* [7] and [1] for the case $G = GL(n, F)$).

2.4. A representation $\pi \in \mathrm{Alg}\,G$ is called quasicuspidal if $r_{M,G}(\pi) = 0$ for any standard subgroup $M \lneqq G$. It follows from 2.3 (*a*), that in this case all subquotients of π are quasicuspidal.

A quasicuspidal admissible representation is called cuspidal.

THEOREM. — *(a) Any representation $\pi \in \mathrm{Alg}\,G$ decomposes into $\pi = \pi_c \oplus \pi_c^{\perp}$, where π_c is quasicuspidal and $\pi_c^{\perp}$ has no non-zero quasicuspidal subquotients.*

(b) If $\pi \in \mathrm{Alg}\,G$ is admissible and $\omega \in \mathrm{JH}(\pi)$ is cuspidal, then π has a submodule (and a factormodule), equivalent to ω.

(c) If π is cuspidal then so is $\tilde{\pi}$.

(d) If $M \lneq G$, $\rho \in \mathrm{Alg}\,M$ and $\pi = i_{G,M}(\rho)$ then $\pi_c = 0$.

Parts (a)-(c) in the case $G = GL(n, F)$ are proved in ([1], chap. II); in the general case proofs are similar. Part (d) follows from 2.3 (b), since

$$\mathrm{Hom}(\pi_c, \pi) = \mathrm{Hom}(r_{M,G}(\pi_c), \rho) = 0.$$

2.5. THEOREM. — *Let $\omega \in \mathrm{Irr}\,G$. Then there exists a subgroup $M < G$ and a cuspidal representation $\rho \in \mathrm{Irr}\,M$ such that ω can be embedded into $i_{G,M}(\rho)$; in particular, ω is admissible.*

The proof can be found in [10]; for $G = GL(n, F)$ *see* [1, chap. II].

2.6. The main purpose of this section is to clear up the relations between the representations of G, induced from different standard subgroups. For this we study the action of the Weyl group on standard subgroups.

Let $W = W_G = N_G(A_0)/Z_G(A_0)$ be the Weyl group of the group G. For any $w \in W$ choose a representative $\bar{w} \in N_G(A_0)$ and define the map $w : G \to G$ by $w(g) = \bar{w}.g.\bar{w}^{-1}$. If $M < G$ then $M \supset Z_G(A_0)$ hence for any $w \in W_M \subset W_G$ its representative $\bar{w}$ belongs to M.

Let M and N be standard subgroups of G. We set

$$W(M, N) = \{ w \in W \mid w(M) = N \};$$

it is clear that $W_N.W(M, N).W_M = W(M, N)$. The subgroups M and N are called associated (the notation $M \sim N$) if $W(M, N) \neq \varnothing$.

Any element $w \in W(M, N)$ determines the functor $w : \mathrm{Alg}\,M \to \mathrm{Alg}\,N$ (*see* 1.7). The representations $\rho \in \mathrm{Alg}\,M$, $\rho' \in \mathrm{Alg}\,N$ are called associated, if $\rho' \approx w(\rho)$ for some $w \in W(M, N)$ (the notation $\rho \sim \rho'$).

2.7. *Example.* — Let $G = G_n$, $\alpha = (n_1, \ldots, n_r)$ and $\beta = (n'_1, \ldots, n'_s)$ be partitions of n, $M = G_\alpha < G$, $N = G_\beta < G$. Then the condition $M \sim N$ means that $r = s$ and the family $(n_1, \ldots, n_r)$ is a permutation of $(n'_1, \ldots, n'_s)$. Such permutations correspond to elements of $W(M, N)/W_M$.

Let

$$\rho_i \in \mathrm{Irr}\,G_{n_i}, \qquad \rho'_i \in \mathrm{Irr}\,G_{n'_i}, \qquad \rho = \otimes \rho_i \in \mathrm{Irr}\,M, \qquad \rho' = \otimes \rho'_i \in \mathrm{Irr}\,N.$$

Then $\rho \sim \rho'$ iff the sets $(\rho_1, \ldots, \rho_r)$ and $(\rho'_1, \ldots, \rho'_r)$ are equal up to a permutation.

2.8. Now we state main theorems about induced representations.

Let M be a standard subgroup of G, $\rho \in \mathrm{Irr}\,M$ be a cuspidal representation and $\pi = i_{G,M}(\rho)$. Set

$$W(M, \bigstar) = \bigcup_{N < G} W(M, N) = \{ w \in W \mid w(M) < G \},$$

and denote by $l(\mathrm{M})$ the cardinality of the set $\mathrm{W}(\mathrm{M}, \bigstar)/\mathrm{W}_{\mathrm{M}}$. For example, if $\mathrm{M} = \mathrm{G}_{n_1} \times \ldots \times \mathrm{G}_{n_r} < \mathrm{G} = \mathrm{G}_n$, then $l(\mathrm{M}) = r!$

THEOREM. — *The length $l(\pi)$ of the representation π is finite; moreover $l(\pi) \leqq l(\mathrm{M})$.*

2.9. In conditions of 2.8 let N be a standard subgroup of G, $\rho' \in \operatorname{Irr} \mathrm{N}$ a cuspidal representation and $\pi' = i_{\mathrm{G},\mathrm{N}}(\rho')$.

THEOREM. — (*a*) *The following conditions are equivalent:*

(i) $\mathrm{M} \sim \mathrm{N}$ *and* $\rho \sim \rho'$;

(ii) $\operatorname{Hom}(\pi, \pi') \neq 0$.

(iii) $\mathrm{JH}^0(\pi) = \mathrm{JH}^0(\pi')$;

(iv) $\mathrm{JH}(\pi) \cap \mathrm{JH}(\pi') \neq \emptyset$.

(*b*) *Set*

$$\mathrm{W}(\rho, \rho') = \{ w \in \mathrm{W}(\mathrm{M}, \mathrm{N}) \mid w(\rho) \approx \rho' \}.$$

Then :

$$\dim \operatorname{Hom}(\pi, \pi') \leqq |\mathrm{W}(\rho, \rho')/\mathrm{W}_{\mathrm{M}}|.$$

2.10. *Remark.* — If in theorems 2.8 and 2.9 we give up the assumption that ρ and ρ' are cuspidal, then the estimate $l(\mathrm{M}) \leqq |\mathrm{W}|$ in 2.8 and the implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) in 2.9 remain valid, while the estimate $l(\pi) \leqq l(\mathrm{M})$ in 2.8 and implications (ii) $\Rightarrow$ (i), (iv) $\Rightarrow$ (i) in 2.9 may fail.

2.11. Proofs of theorems 2.8 and 2.9 are based on the following two lemmas, which will be proved in paragraph 6.

Let M, N be standard subgroups in G. Set

$$\mathrm{W}^{\mathrm{M},\mathrm{N}} = \{ w \in \mathrm{W} \mid w(\mathrm{M} \cap \mathrm{P}_0) \subset \mathrm{P}_0,\ w^{-1}(\mathrm{N} \cap \mathrm{P}_0) \subset \mathrm{P}_0 \}.$$

LEMMA. — (*a*) *In each double coset* $\mathrm{W}_{\mathrm{N}} \backslash \mathrm{W}/\mathrm{W}_{\mathrm{M}}$ *there exists a unique element of* $\mathrm{W}^{\mathrm{M},\mathrm{N}}$.

(*b*) *If* $w \in \mathrm{W}^{\mathrm{M},\mathrm{N}}$ *then* $\mathrm{M} \cap w^{-1}(\mathrm{N}) < \mathrm{M}$ *and* $w(\mathrm{M}) \cap \mathrm{N} < \mathrm{N}$.

2.12. Let $\mathrm{M}, \mathrm{N} < \mathrm{G}$. The following lemma describes the composition of functors $r_{\mathrm{N},\mathrm{G}}$ and $i_{\mathrm{G},\mathrm{M}}$; it plays the main role in proofs of theorems 2.8 and 2.9.

For any $w \in \mathrm{W}^{\mathrm{M},\mathrm{N}}$ define the functor

$$\mathrm{F}_w : \ \operatorname{Alg} \mathrm{M} \to \operatorname{Alg} \mathrm{N} \qquad \text{by} \quad \mathrm{F}_w = i_{\mathrm{N},\mathrm{N}'} \circ w \circ r_{\mathrm{M}',\mathrm{M}},$$

where $\mathrm{M}' = \mathrm{M} \cap w^{-1}(\mathrm{N})$, $\mathrm{N}' = w(\mathrm{M}) \cap \mathrm{N}$ ($\mathrm{M}' < \mathrm{M}$ and $\mathrm{N}' < \mathrm{N}$ according to 2.11).

GEOMETRICAL LEMMA. — *The functor* $\mathrm{F} = r_{\mathrm{G},\mathrm{N}} \circ i_{\mathrm{G},\mathrm{M}} : \operatorname{Alg} \mathrm{M} \to \operatorname{Alg} \mathrm{N}$ *is glued from functors* F_w, $w \in \mathrm{W}^{\mathrm{M},\mathrm{N}}$ (*see* 1.10, 1.11).

In other words, there exists a numeration $w_1, \ldots, w_k$ of elements of $\mathrm{W}^{\mathrm{M},\mathrm{N}}$ satisfying the following condition: for any $\rho \in \operatorname{Alg} \mathrm{M}$ $\mathrm{F}(\rho)$ has a filtration $0 = \tau_0 \subset \tau_1 \subset \ldots \subset \tau_k = \mathrm{F}(\rho)$ and a system of isomorphisms $\mathrm{C}_i : \tau_i/\tau_{i-1} \to \mathrm{F}_{w_i}(\rho)$, functorially depending on ρ.

2.13. COROLLARY. — *Let* $M, N < G$, $\rho \in \operatorname{Alg} M$ *be quasicuspidal, and* $\tau = r_{N,G} \circ i_{G,M}(\rho)$. *Then :*

(*a*) *If* N *has no standard subgroups associated to* M *in* G, *then* $\tau = 0$.

(*b*) *If* M *is not associated to* N, *then* τ *has no non-zero quasicuspidal subquotients.*

(*c*) *If* $M \sim N$, *then* τ *is glued from representations* $w(\rho)$ *where* $w \in W(M, N)/W_M$*; in particular,* τ *is quasicuspidal.*

Proof. — Part (*a*) follows directly from 2.12, since $r_{M',M}(\rho) = 0$ for any $M' \nleqq M$; to prove (*c*) one has to note that

$$W(M, N) \cap W^{M,N} = W_N \backslash W(M, N)/W_M = W(M, N)/W_M.$$

Part (*b*) follows from 2.4 (*d*), (*a*).

2.14. *Proof of the Theorem* 2.8. — (1) First of all we shall find some restrictive conditions on subquotients of π. If $\pi_0 \in \operatorname{Alg} G$, $\pi_0 \neq 0$, then in accordance to 2.3 (*c*) there exists such a subgroup $L < G$, that $r_{L,G}(\pi_0)$ is quasicuspidal and non-zero.

We claim that if π_0 is a subquotient of π, then $L \sim M$; moreover, if $\omega \in JH(r_{L,G}(\pi_0))$, then $\omega \sim \rho$. Indeed, exactness of the functor $r_{L,G}$ implies that $\omega \in JH(r_{L,G}(\pi))$ and, since ω is cuspidal, our statement follows from 2.13 (*b*), (*c*).

(2) Now we prove the Theorem 2.8. Define the function l' on Alg G by

$$l'(\tau) = \sum_{L \sim M} l(r_{L,G}(\tau)),$$

where l is the length of a representation. It follows from the exactness of $r_{L,G}$, that l' is additive, i. e. if $\pi_1 \subset \pi_2$, then $l'(\pi_1) + l'(\pi_2/\pi_1) = l'(\pi_2)$. According to (1) $l'(\pi_0) > 0$ for any non-zero subquotient π_0 of π; hence, for such subquotients $l(\pi_0) \leqq l'(\pi_0)$. In particular, $l(\pi) \leqq l'(\pi) = \sum_{L \sim M} |W(M, L)/W_M| = l(M)$ [*see* 2.13 (*c*)].

2.15. *Proof of the Theorem* 2.9. — (1) Implications (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) are trivial. Implication (iv) $\Rightarrow$ (i) follows from the step (1) in 2.14, since any element of $JH(\pi)$ allows us to construct a pair (L, ω) associated to (M, ρ).

(2) (i) $\Rightarrow$ (ii). According to 2.3 (*b*)

$$\operatorname{Hom}(\pi, \pi') = \operatorname{Hom}(r_{N,G} \circ i_{G,M}(\rho), \rho') = \operatorname{Hom}(F(\rho), \rho').$$

If $\rho \sim \rho'$, then $\rho' \in JH(F(\rho))$, so 2.4 (*b*) implies that $F(\rho)$ has a factormodule, isomorphic to ρ'. Hence, $\operatorname{Hom}(\pi, \pi') = \operatorname{Hom}(F(\rho), \rho') \neq 0$.

(3) Note that $\dim \operatorname{Hom}(\pi, \pi') = \dim \operatorname{Hom}(F(\rho), \rho')$ is no more than the multiplicity of ρ' in $JH^0(F(\rho))$. But according to 2.13 (*c*) this multiplicity equals $|W(\rho, \rho')/W_M|$. This implies Part (*b*) of the Theorem 2.9.

2.16. The implication (i) $\Rightarrow$ (iii) in 2.9 is proved in two steps.

(1) Suppose that $l(M) = 2$.

Fix non-zero morphisms $A : \pi \to \pi'$, $A' : \pi \to \pi$ (*see* 2.15). We have $l'(\pi) = l'(\pi') = 2$ (*see* 2.14). If π is irreducible, then A is an embedding and, since

$$l'(\pi'/A\pi) = l'(\pi') - l'(\pi) = 0,$$

A is an isomorphism. So we may assume that π and π' have proper submodules π_0 and π'_0. It is clear that $l'(\pi_0) = l'(\pi/\pi_0) = l'(\pi_0) = l'(\pi'/\pi_0) = 1$; in particular all these four representations are irreducible.

In what follows we do not consider the case $M = N$, $\rho \approx \rho'$ when $\pi \approx \pi'$. Then $l'(\pi_0) = 1$ and $\mathrm{Hom}(\pi_0, \pi) = \mathrm{Hom}(r_{M,G}(\pi_0), \rho) \neq 0$ imply that

$$\mathrm{Hom}(\pi_0, \pi') = \mathrm{Hom}(r_{N,G}(\pi_0), \rho') = 0, \qquad \text{so} \quad A(\pi_0) = 0.$$

In particular it follows that π_0 is the unique proper submodule in π (otherwise $A(\pi) = 0$). Analogously $A'(\pi'_0) = 0$ and π'_0 is the unique proper submodule in π'.

Hence, $\pi/\pi_0 \approx \pi'_0$ and $\pi'/\pi'_0 \approx \pi_0$, so

$$JH^0(\pi) = JH^0(\pi') = \{\pi_0, \pi'_0\}.$$

(2) Let $M, N < G$, $w \in W(M, N)$. We call the map $w : M \to N$ elementary, if there exists a subgroup $L < G$ such that $M, N < L$, $w \in W_L$ and $l(M) = 2$ inside L. It follows from the preceding step and 2.3 (*c*), (*a*) that in this case the condition (iii) of 2.9 is valid for $\rho' = w(\rho)$. Thus the implication (i) $\Rightarrow$ (iii) follows from the following Lemma.

2.17. LEMMA. — *Let* $M, N < G$, $N = w(M)$. *Then there exists a chain of standard subgroups* $N_0 = M, N_1, N_2, \ldots, N_k = N$ *of* G *and elementary maps* $w_i : N_{i-1} \to N_i$ *such that* $w = w_k \circ w_{k-1} \circ \ldots \circ w_1$.

This Lemma will be proved in paragraph 6.

In the case when $G = GL(n, F)$ and M, N are the groups of cellular-diagonal matrices an elementary map is just a transposition of two neighbour cells and the lemma means that any permutation of cells is a composition of such transpositions.

3. Representations of the group P_n

3.1. Later on we shall study the representations of the group $G = G_n = GL(n, F)$. Our main method is to study the restriction of representations of G_n to the subgroup $P = P_n \subset G_n$; P is by definition the subgroup of matrices with the last row $(0, 0, \ldots, 0, 1)$. In this section we classify irreducible representations of the group P.

From now on assume that the group G_{n-1} (and also P_{n-1}) is embedded into P_n in a standard way; denote by $V = V_n$ the unipotent radical of P_n:

$$V = \{(g_{ij}) \in P \mid g_{ij} = \delta_{ij} \quad \text{for} \quad j < n\}.$$

It is clear that G_{n-1} normalises V, $G_{n-1} \cap V = \{e\}$ and $G_{n-1}.V = P$.

Fix a non-trivial additive character ψ of the field F and define the character θ of the group V_n by $\theta((v_{ij})) = \psi(v_{n-1,n})$. It is clear that P_{n-1} normalises θ.

3.2. The main role in our study of representations of the group P_n is played by the functors

$$\Psi^- : \ \mathrm{Alg}\, P_n \to \mathrm{Alg}\, G_{n-1}, \qquad \Psi^+ : \ \mathrm{Alg}\, G_{n-1} \to \mathrm{Alg}\, P_n,$$

$$\Phi^- : \ \mathrm{Alg}\, P_n \to \mathrm{Alg}\, P_{n-1}, \qquad \Phi^+, \hat{\Phi}^+ : \ \mathrm{Alg}\, P_{n-1} \to \mathrm{Alg}\, P_n,$$

defined by

$$\Psi^- = r_{V,1}, \qquad \Phi^- = r_{V,\theta}, \qquad \Psi^+ = i_{V,1}, \qquad \Phi^+ = i_{V,\theta}, \qquad \hat{\Phi}^+ = I_{V,\theta},$$

(*see* 1.8). Note that these functors differ from the ones. defined in [1] by the "twisting" with the character $\mathrm{mod}_V^{1/2}$.

Now we describe a "multiplication table" of these functors.

PROPOSITION. — (*a*) *All the functors* Ψ^-, Ψ^+, Φ^-, Φ^+, $\hat{\Phi}^+$ *are exact.*

(*b*) Ψ^- *is left adjoint to* Ψ^+ *and* Φ^+ *is left adjoint to* Φ^-.

(*c*) Φ^- *is left adjoint to* $\hat{\Phi}^+$.

(*d*) $\Phi^- \circ \Psi^+ = 0$, $\Psi^- \circ \Phi^+ = 0$.

(*e*) *Let*

$$i: \ \Phi^+ \Phi^- \to \mathrm{Id}, \qquad i': \ \mathrm{Id} \to \Phi^- \Phi^+, \qquad j: \ \mathrm{Id} \to \Psi^+ \Psi^-$$
$$\text{and} \qquad j': \ \Psi^- \Psi^+ \to \mathrm{Id},$$

be the adjunction maps from (*b*) (*see* [13]). *Then* i' *and* j' *are isomorphisms, and* i *and* j *form the exact sequence*

$$0 \to \Phi^+ \Phi^- \to \mathrm{Id} \to \Psi^+ \Psi^- \to 0.$$

(*f*) *Let us consider the morphisms of functors*

$$\mathrm{Id} \xrightarrow{i'} \Phi^- \Phi^+ \xrightarrow{k} \Phi^- \hat{\Phi}^+ \xrightarrow{l} \mathrm{Id},$$

where i' *is defined in* (*e*), l *is an adjunction map from* (*c*) *and* k *is the morphism, induced by the natural embedding* $\Phi^+ \to \hat{\Phi}^+$. *Then* i', k, l *are isomorphisms and* $l \circ k \circ i'$ *is an identical morphism.*

Proof. — Parts (*a*), (*c*) and the first part of (*b*) follow from 1.9. The second part of (*b*), (*d*) and (*e*) are proved in ([1], § 5).

In the proof of the proposition 5.12 in [1] we have shown that the morphism $l \circ k$ is an isomorphism. Moreover, the morphism i' by definition equals $(l \circ k)^{-1}$, hence $l \circ k \circ i' = \mathrm{id}$. Since Φ^+ is left adjoint to Φ^- and Φ^- is left adjoint to $\hat{\Phi}^+$, we conclude that $\Phi^- \circ \Phi^+$ is left adjoint to $\Phi^- \circ \hat{\Phi}^+$ and the morphism $k \circ i'$: $\mathrm{id} \to \Phi^- \hat{\Phi}^+$ corresponds to the morphism $l \circ k$: $\Phi^- \Phi^+ \to \mathrm{id}$. Since $l \circ k$ is isomorphism so is $k \circ i'$. Hence l and k are isomorphisms.

3.3. *Remarks.* — (*a*) It follows from 3.2 that Φ^+ establishes an equivalence of the category $\mathrm{Alg}\, P_{n-1}$, with the complete subcategory $\mathrm{Im}\, \Phi^+$ in $\mathrm{Alg}\, P_n$, consisting of representations π, such that $\Psi^- (\pi) = 0$. Since the functor Ψ^- is exact, $\mathrm{Im}\, \Phi^+$ is closed with respect to subobjects. It follows that for any $\tau \in \mathrm{Alg}\, P_{n-1}$ the functor Φ^+ determines an isomorphism of the lattice of submodules of τ with the lattice of submodules of $\Phi^+ (\tau)$ (analogously for Ψ^+). In particular, Φ^+ and Ψ^+ carry irreducible representations into irreducible ones.

(*b*) It follows from 3.2 (*d*), (*e*) that for $\tau \in \operatorname{Alg} P_n$ $\Phi^-(\tau) = 0$ iff $\tau|_V$ is a trivial representation. In particular it follows from 3.2 (*f*) that for any $\rho \in \operatorname{Alg} P_{n-1}$ the group V acts trivially on $\hat{\Phi}^+(\rho)/\Phi^+(\rho)$. Using the explicit constructions from ([1], § 5), one can describe the appropriate representation of the group G_{n-1}.

3.4 Proposition. — (*a*) *If* $\rho \in \operatorname{Alg} G_{n-1}$ *then* $\widetilde{\Delta_P \Psi^+(\rho)} \approx \Psi^+(\tilde{\rho})$.

(*b*) *If* $\tau \in \operatorname{Alg} P_{n-1}$ *then* $\widetilde{\Delta_P \Phi^+(\tau)} \approx \hat{\Phi}^+(\widetilde{\Delta_P \tau})$.

Proof. — Part (*a*) and the equality $\widetilde{\Delta_P \Phi^+(\tau)} \approx \bar{\Phi}^+(\widetilde{\Delta_P \tau})$ follow from 1.9 (*d*), where $\bar{\Phi}^+ = I_{V, \theta^{-1}}$ differs from $\hat{\Phi}^+$ by replacement θ to θ^{-1}. So we have only to prove that functors $\bar{\Phi}^+$ and $\hat{\Phi}^+$ are isomorphic.

Let $\bar{h} = (h_{ij}) \in P$ be the diagonal matrix with $h_{ii} = -1$ $(i < n)$, $h_{nn} = 1$; let h be the inner automorphism, corresponding to $\bar{h}$. It is clear, that $\bar{\Phi}^+ = h \circ \hat{\Phi}^+ \circ h^{-1}$. Since the functor $h : \operatorname{Alg} P_{n-1} \to \operatorname{Alg} P_{n-1}$ is identical, and $h : \operatorname{Alg} P_n \to \operatorname{Alg} P_n$ is isomorphic to identical one, $\bar{\Phi}^+ \approx \hat{\Phi}^+$.

3.5. Let $\tau \in \operatorname{Alg} P_n$. We define the representations $\tau^{(k)} \in \operatorname{Alg} G_{n-k}$ $(k = 1, \ldots, n)$ by (²) $\tau^{(k)} = \Psi^- \circ (\Phi^-)^{k-1}(\tau)$. We call $\tau^{(k)}$ the k-th derivative of τ. If $\tau^{(k)} \neq 0$ and $\tau^{(m)} = 0$ for $m > k$, then $\tau^{(k)}$ is called the highest derivative of τ.

It follows from the proposition 3.2, that τ is glued from the representations $(\Phi^+)^{k-1} \circ \Psi^+(\tau^{(k)})$, i. e. there exists a natural filtration $0 \subset \tau_n \subset \ldots \subset \tau_1 = \tau$, such that

$$\tau_k/\tau_{k+1} = (\Phi^+)^{k-1} \circ \Psi^+(\tau^{(k)}) \qquad (\text{here } \tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}(\tau))$$

In particular, if $\tau \in \operatorname{Irr} P_n$, then exactly one of the representations $\tau^{(k)}$ is non-zero, and we obtain the following.

Corollary. — *Any representation* $\tau \in \operatorname{Irr} P_n$ *is equivalent to a representation of the form* $(\Phi^+)^{k-1} \Psi^+(\rho)$, *where* $1 \leqq k \leqq n$ *and* $\rho \in \operatorname{Irr} G_{n-k}$; *moreover the number* k *and the class of equivalence of* ρ *are uniquely determined by* τ.

3.6. All irreducible representations of reductive groups are admissible, hence the contragredient representations are also irreducible. This doesn't remain true for the group P: if $\tau \in \operatorname{Irr} P$, then τ is not admissible as a rule and JH $(\tilde{\tau})$ has the cardinality of a continuum. It appears, however, that $\tilde{\tau}$ has a certain structure. It is convenient to describe this structure in terms of pairings of representations of the group P.

Definition. — Let G be an l-group, χ be a character of G, and

$$(\pi_i, G, E_i) \in \operatorname{Alg} G \; (i = 1, 2).$$

By a χ-pairing of π_1 and π_2 we mean a bilinear form B on $E_1 \times E_2$, satisfying the condition

$$\chi(g) . B(\pi_1(g)\xi_1, \pi_2(g)\xi_2) = B(\xi_1, \xi_2), \qquad g \in G, \qquad \xi_1 \in E_1, \qquad \xi_2 \in E_2.$$

(²) By $(\Phi^-)^m$ we mean the composition of functors $\operatorname{Alg} P_n \xrightarrow{\Phi^-} \operatorname{Alg} P_{n-1} \xrightarrow{\Phi^-} \ldots \xrightarrow{\Phi^-} \operatorname{Alg} P_{n-m}$.

It is clear, that the space of all χ-pairings of π_1 and π_2 is isomorphic to

$$\operatorname{Hom}(\pi_1, \widetilde{\chi\pi_2}) = \operatorname{Hom}(\pi_2, \widetilde{\chi\pi_1}).$$

We say, that B is non-degenerate w. r. t. π_1, if the corresponding morphism $\pi_1 \to \widetilde{\chi\pi_2}$ is an embedding; the non-degeneracy of B w. r. t. π_2 is defined analogously.

Usually we shall be interested in Δ-pairings, where $\Delta = \Delta_G$ is the module of G. The space of such pairings is denoted by $\operatorname{Bil}(\pi_1, \pi_2)$.

3.7. PROPOSITION. — *Let* $\rho, \rho' \in \operatorname{Alg} G_{n-1}$, $\tau, \tau' \in \operatorname{Alg} P_{n-1}$. *Then*

(*a*) $$\operatorname{Bil}(\Psi^+ \rho, \Psi^+ \rho') \approx \operatorname{Bil}(\rho, \rho').$$

(*b*) $$\operatorname{Bil}(\Phi^+ \tau, \Phi^+ \tau') \approx \operatorname{Bil}(\tau, \tau').$$

(*c*) $$\operatorname{Bil}(\Psi^+ \rho, \Phi^+ \tau) = 0.$$

Moreover, isomorphisms in (*a*) *and* (*b*) *preserve the non-degeneracy of pairings.*

Proof. — (*b*) We have

$$\begin{aligned} &\operatorname{Bil}(\Phi^+ \tau, \Phi^+ \tau') \\ &\quad = \operatorname{Hom}(\Phi^+ \tau', \widetilde{\Delta_P \Phi^+ \tau}) = \operatorname{Hom}(\Phi^+ \tau', \hat{\Phi}^+ (\widetilde{\Delta_P \tau})) \\ &\quad \approx \operatorname{Hom}(\Phi^- \Phi^+ \tau', \widetilde{\Delta_P \tau}) = \operatorname{Hom}(\tau', \widetilde{\Delta_P \tau}) = \operatorname{Bil}(\tau, \tau'), \end{aligned}$$

[*see* 3.4 (*b*), 3.2 (*c*), (*e*)]. The preserving of non-degeneracy means that the isomorphism

$$\operatorname{Hom}(\Phi^+ \tau', \hat{\Phi}^+ (\widetilde{\Delta_P \tau})) \approx \operatorname{Hom}(\Phi^- \Phi^+ \tau', \widetilde{\Delta_P \tau})$$

carries embeddings into embeddings. It follows from 3.3 (*a*) and the fact that this isomorphism functorially depends on τ'.

Parts (*a*) and (*c*) can be proved analogously (but simpler).

3.8. PROPOSITION. — *Let* $\tau, \pi \in \operatorname{Alg} P_n$. *Suppose that there exists a non-degenerate w. r. t.* τ Δ*-pairing* B *of* τ *and* π. *Let* $\tau^{(k)}$ *be the highest derivative of* τ (*see* 3.5). *Then there exists an* 1*-pairing of* $\tau^{(k)}$ *and* $\pi^{(k)}$, *which is non-degenerate w. r. t.* $\tau^{(k)}$.

Proof. — If $k = 1$, then $\tau = \Psi^+(\tau^{(1)})$. According to 3.7 (*c*) B is trivial on $\Phi^+ \Phi^- (\pi)$, i. e. it determines a Δ-pairing of τ and $\Psi^+ \Psi^- (\pi) = \Psi^+ (\pi^{(1)})$; so our statement follows from 3.7 (*a*). Let $k > 1$. Set $\tau_1 = \Phi^+ \Phi^- (\tau)$, $\pi_1 = \Phi^+ \Phi^- (\pi)$. We claim that B induces a non-degenerate w. r. t. τ_1 pairing of τ_1 and π_1. This is the corollary of the fact, that, according to 3.7 (*c*) and 3.3 (*a*), any submodule of τ_1 has only zero Δ-pairing with $\pi/\pi_1 = \Psi^+ \Psi^- (\pi)$. Set $\tau' = \Phi^- (\tau)$, $\pi' = \Phi^- (\pi)$; then 3.7 (*b*) implies that there exists a non-degenerate w. r. t. τ' Δ-pairing of τ' and π'. Since $\tau^{(k)} = (\tau')^{(k-1)}$, $\pi^{(k)} = (\pi')^{(k-1)}$, induction up to k proves our statement.

3.9. COROLLARY. — *Let* $\pi = (\Phi^+)^{k-1}\,\Psi^+(\rho)$, *where* $\rho \in \operatorname{Irr} G_{n-k}$, *and* $\bar{\pi} = \widetilde{\Delta_P\,\pi}$. *Then* $\bar{\pi}$ *has an irreducible submodule* $\bar{\pi}_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}(\bar{\pi}) \subset \bar{\pi}$ (*see* 3.5), *isomorphic to* $(\Phi^+)^{k-1}\,\Psi^+(\rho)$ *and any non-zero submodule of* $\bar{\pi}$ *contains* $\bar{\pi}_k$.

To prove the statement it is sufficient to apply 3.8 to representations τ and π, where τ is an arbitrary submodule in $\bar{\pi}$.

4. The restriction of induced representations to the group P

4.1. Now we begin to study induced representations of the group $G = G_n$. Fix the subgroups P_0 and A_0 in G as in 2.2. Let $\beta = (n_1, \ldots, n_r)$ be a partition of the number n and $\rho_i \in \operatorname{Alg} G_{n_i}$ $(i = 1, \ldots, r)$.

Denote by $\rho_1 \times \ldots \times \rho_r \in \operatorname{Alg} G$ the representation $i_{G,M}(\rho_1 \otimes \ldots \otimes \rho_r)$, where $M = G_{n_1} \times \ldots \times G_{n_r}$ is a standard subgroup of G, corresponding to β (*see* 2.2); $\rho_1 \times \ldots \times \rho_r$ is called the product of $\rho_1, \ldots, \rho_r$.

It is convenient to reformulate in these terms the results of paragraph 2. Let $\omega \in \operatorname{Irr} G$. Then there exists a partition $\beta = (n_1, \ldots, n_r)$ of n and cuspidal representations $\rho_i \in \operatorname{Irr} G_{n_i}$, such that $\omega \in \operatorname{JH}(\rho_1 \times \ldots \times \rho_r)$. The set $(\rho_1, \ldots, \rho_r)$ is determined by ω uniquely up to a permutation; we call it the support of ω (the notation is $(\rho_1, \ldots, \rho_r) = \operatorname{supp}\omega$). One can choose such an ordering $(\rho_{i_1}, \ldots, \rho_{i_r})$ in $\operatorname{supp}\omega$, that ω can be embedded into $\rho_{i_1} \times \ldots \times \rho_{i_r}$ (*see* 2.5, 2.7 and 2.9).

4.2. Further we shall denote by ν the character of G_n defined by $\nu(g) = |\det g|$, where $|\ |$ is a standard norm of the field F. It is easy to prove that $\Delta_P = \nu^{-1}|_P$.

THEOREM (Criterion of irreducibility). — *Let* $\rho_i \in \operatorname{Irr} G_{n_i}$ *be cuspidal representations* $(i=1, \ldots, r)$. *Suppose that* $\rho_j \not\approx \nu\rho_i$ *for any* i, j. *Then the representation* $\pi = \rho_1 \times \ldots \times \rho_r$ *is irreducible.*

Remarks. — (1) Of course the condition $\rho_j \approx \nu\rho_i$ means in particular that $n_j = n_i$. If $m > 0$, and $\rho \in \operatorname{Irr} G_m$ then $\rho \not\approx \nu\rho$ since ρ and $\nu\rho$ differ, when restricted to the center of the group G_m.

(2) In fact, the inverse theorem is true too; if $\rho_j \approx \nu\rho_i$ for some i, j, then the representation $\rho_1 \times \ldots \times \rho_r$ is reducible. It will be proved in Part II of this paper.

4.3. We shall prove the Theorem 4.2 by studying the restriction of representation from G to P. If $\pi \in \operatorname{Alg} G$, denote by $\pi|_P$ the restriction of π to P. If $\pi \in \operatorname{Alg} G_n$ then derivatives $\pi^{(k)} \in \operatorname{Alg} G_{n-k}$ $(k=0, 1, \ldots, n)$ are defined by $\pi^{(0)} = \pi$, $\pi^{(k)} = (\pi|_P)^{(k)}$ $(k=1, \ldots, n)$ (*see* 3.5). If $\pi^{(k)} \neq 0$ and $\pi^{(m)} = 0$ for $m > k$, then we call $\pi^{(k)}$ the highest derivative of π.

4.4. The following theorem by I. M. Gelfand and D. A. Kajdan, describes the derivatives of cuspidal representations.

THEOREM. — *Let* $\pi \in \operatorname{Alg} G_n$ *be quasicuspidal. Then* $\pi^{(k)} = 0$ *for* $0 < k < n$. *If* π *is cuspidal and irreducible, then* $\pi^{(n)} = 1$ *is a one-dimensional representation.*

For the proof, *see* ([1], chap. III). Note that the inverse assertion is true too: if $\pi^{(k)} = 0$ when $0 < k < n$, then π is quasicuspidal; moreover if in this case $\pi^{(n)} = 1$, then π is irreducible.

4.5. Now we describe the derivatives of a product of two representations.

LEMMA. — *Let*

$$\rho \in \mathrm{Alg}\, G_m, \qquad \omega \in \mathrm{Alg}\, G_l, \qquad \pi = \rho \times \omega \in \mathrm{Alg}\, G_{m+l}.$$

Then for each k the representation $\pi^{(k)}$ is glued from $\rho^{(i)} \times \omega^{(k-i)}$, where $i = 0, 1, \ldots, k$.

A more transparent formulation can be given in terms of a representation ring of groups G_n. Let $\mathcal{R}_n$ be the Grothendieck group of the category of algebraic G_n-modules of a finite length and $\mathcal{R} = \oplus \mathcal{R}_n$ $(n = 0, 1, \ldots)$. For any algebraic G_n-module π of a finite length we denote by the same symbol π its image in $\mathcal{R}_n$, so in $\mathcal{R}$ we have $\pi = \Sigma\omega$, $\omega \in \mathrm{JH}^0(\pi)$. The multiplication $(\pi_1, \pi_2) \to \pi_1 \times \pi_2$ turns $\mathcal{R}$ into a graded ring; we call it the representation ring of the groups G_n. If $\pi \in \mathrm{Alg}\, G_n$ has a finite length, set

$$\mathcal{D}\pi = \Sigma \pi^{(k)} \in \mathcal{R} \, (k = 0, 1, \ldots, n).$$

Let us extend the map $\pi \mapsto \mathcal{D}\pi$ to the **Z**-linear operator $\mathcal{D} : \mathcal{R} \to \mathcal{R}$. Then our lemma implies, that $\mathcal{D}$ is a homomorphism of rings.

Lemma 4.5 is proved in 4.14.

4.6. COROLLARY. — *If $\pi = \rho_1 \times \ldots \times \rho_r$, then $\mathcal{D}\pi = \mathcal{D}\rho_1 \times \ldots \mathcal{D}\rho_r$.*

4.7. The representation $\tau \in \mathrm{Alg}\, G_n$ (or $\tau \in \mathrm{Alg}\, P_n$) is called non-degenerate (resp. degenerate), if $\tau^{(n)} \neq 0$ (resp. $\tau^{(n)} = 0$). We deduce the Theorem 4.2 from the following

LEMMA. — *Let $\rho_i \in \mathrm{Irr}\, G_{n_i}$ be cuspidal $(i = 1, \ldots, r)$ and $\pi = \rho_1 \times \ldots \times \rho_r$. Then:*

(*a*) *If $\sigma \in \mathrm{JH}(\pi^{(m)})$ $(m = 0, 1, \ldots, n)$, then* $\mathrm{supp}\,\sigma \subset (\rho_1, \ldots, \rho_r)$ (*see* 4.1).

(*b*) *If ω is a non-zero* P*-submodule of π, $\omega^{(k)}$ the highest derivative of ω and σ an irreducible submodule of $\omega^{(k)}$ then* $\nu.\mathrm{supp}\,\sigma \subset (\rho_1, \ldots, \rho_r)$.

Let us show how Theorem 4.2 follows from this Lemma. It follows from 4.4 and 4.6, that $\pi^{(n)} = 1$. Since $\pi^{(n)} = \Sigma\omega^{(n)}$ $(\omega \in \mathrm{JH}^0(\pi))$ exactly one of the elements of $\mathrm{JH}^0(\pi)$ is non-degenerate. Hence if π is reducible then there exists a degenerate subquotient ω in $\mathrm{JH}^0(\pi)$. Permuting $\rho_1, \ldots, \rho_r$ in a certain way, we can suppose, that $\omega \subset \pi$ (*see* 4.1); the conditions of the Theorem 4.2 remain valid. Let $\omega^{(k)}$ be the highest derivative of ω and σ an irreducible submodule of $\omega^{(k)}$. Since ω is degenerate, $k < n$; hence $\mathrm{supp}\,\sigma \neq \emptyset$. Let $\rho \in \mathrm{supp}\,\sigma$. Then Lemma 4.7 implies that $\rho \in (\rho_1, \ldots, \rho_r)$ and $\nu\rho \in (\rho_1, \ldots, \rho_r)$. This contradicts the condition of the Theorem 4.2.

4.8. *Proof of the lemma* 4.7. — Part (*a*) follows immediately from 4.4 and 4.6.

(*b*) Set $\bar{\pi} = \nu\tilde{\pi}$; according to 2.3 (*d*) and 1.9 (*f*),

$$\bar{\pi} = \bar{\rho}_1 \times \ldots \times \bar{\rho}_r, \qquad \text{where} \quad \bar{\rho}_i = \nu\tilde{\rho}_i.$$

The natural 1-pairing of π and $\tilde{\pi}$ induces a ν^{-1}-pairing of π and $\bar{\pi}$. Restricting to the group P, we obtain the non-degenerate Δ-pairing B of $\pi\,|_P$ and $\bar{\pi}\,|_P$; restricting B to ω, we see that there exists a non-degenerate w. r. t. ω Δ-pairing of ω and $\bar{\pi}\,|_P$. It follows from 3.8 that there exists a non-degenerate w. r. t. $\omega^{(k)}$ 1-pairing of $\omega^{(k)}$ and $\bar{\pi}^{(k)}$. From the existence of non-zero pairing of σ and $\overline{\pi}^{(k)}$ it follows that $\tilde{\sigma} \in \mathrm{JH}(\overline{\pi}^{(k)})$ (since $\tilde{\sigma}$ is irreducible) According to 2.4 (*c*) and 1.9 (*f*) the representations $\overline{\rho}_i$ are cuspidal. Hence (*a*) implies that $\operatorname{supp} \tilde{\sigma} \subset (\overline{\rho}_1, \ldots, \overline{\rho}_r)$. It obviously follows, that

$$\operatorname{supp}. \sigma \subset (\tilde{\overline{\rho}}_1, \ldots, \tilde{\overline{\rho}}_r) = (\nu^{-1} \rho_1, \ldots, \nu^{-1} \rho_r)$$

so the Lemma is proved.

4.9. The remaining part of this section is devoted to the proof of the following.

THEOREM. — *Let $\omega \in \operatorname{Irr} G_n$ be non-degenerate. Then ω has no non-zero degenerate* P-*submodules.*

4.10. *Remark.* — Let $\omega \in \operatorname{Alg} G_n$ and $\omega^{(n)} = 1$. Then the condition, that ω has no non-zero degenerate P-submodules means that ω has a Kirillov model (*see* [1], chap. III). So the Theorem 4.9 means, that any non-degenerate irreducible representation of the group G_n has a Kirillov model (it is the conjecture by Gelfand-Kajdan [8]).

4.11. THEOREM. — *Let $\rho_i \in \operatorname{Irr} G_{n_i}$ be cuspidal ($i = 1, \ldots, r$). Suppose that $\rho_j \not\approx \nu\rho_i$ for any i, j such that $i < j$. Then the representation $\pi = \rho_1 \times \ldots \times \rho_r$ has no non-zero degenerate* P-*submodules.*

Let us deduce the Theorem 4.9 from this Theorem. Let $\operatorname{supp} \omega = (\rho_1, \ldots, \rho_r)$. We order the ρ_i so that $\rho_j \not\approx \nu\rho_i$ when $i < j$, and set $\pi = \rho_1 \times \ldots \times \rho_r$. Let σ be an irreducible submodule of π. Theorem 4.11 implies that σ has no non-zero degenerate P-submodules; in particular σ is non-degenerate. Since only one element of $\mathrm{JH}^0(\pi)$ is non-degenerate (*see* 4.7), $\omega \approx \sigma$ and the Theorem 4.9 is done.

Remark. — The conditions of Theorem 4.11 are necessary and sufficient. In fact, suppose $\rho_j \approx \nu\rho_i$ for some $i < j$ we'll show that π has a non-zero degenerate G-submodule. If one permutes two factors ρ_k and ρ_{k+1} in $\rho_1 \times \ldots \times \rho_r$ with $\rho_{k+1} \not\approx \nu^{\pm 1} \rho_k$ then due to 2.9 and Theorem 4.2 the product comes into the isomorphic one. So, one may assume $j = i+1$.

In virtue of Remark 2 to Theorem 4.2 $\rho_i \times \nu\rho_i$ is reducible hence it containes a irreducible degenerate subquotient ω. Due to Theorem 4.11 ω could not be embedded in $\nu\rho_i \times \rho_i$, so it could be embedded in $\rho_i \times \nu\rho_i = \rho_i \times \rho_{i+1}$. Hence, π containes a degenerate submodule $\rho_1 \times \ldots \times \rho_{i-1} \times \omega \times \rho_{i+2} \times \ldots \times \rho_r$.

4.12. For the proof of the Theorem 4.11, we define the multiplication of representations of the groups G_k and P_m. Let k, m be integers and $n = k+m$. We define the product functor $\operatorname{Alg} M_1 \times \operatorname{Alg} M_2 \xrightarrow{\times} \operatorname{Alg} G$ $((\rho, \tau) \mapsto \rho \times \tau)$ in the following three situations:

(I) $\quad M_1 = G_k, \quad M_2 = G_m, \quad G = G_n;$

(II) $\quad M_1 = G_k, \quad M_2 = P_m, \quad G = P_n;$

(III) $\quad M_1 = P_m, \quad M_2 = G_k, \quad G = P_n.$

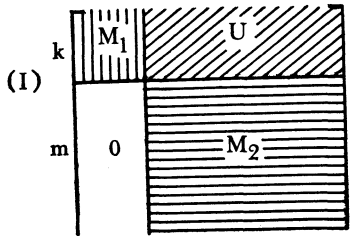
The scheme of the definition in all three cases is the following:

(1) The product factors in the form

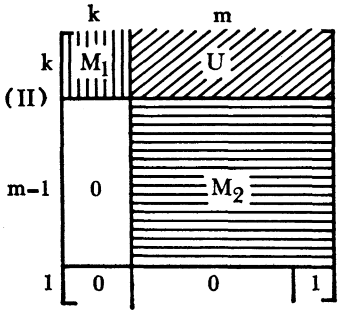
$$\operatorname{Alg} M_1 \times \operatorname{Alg} M_2 \xrightarrow{\otimes} \operatorname{Alg} M \xrightarrow{i} \operatorname{Alg} G, \qquad (M = M_1 \times M_2).$$

(2) The group M is embedded in a certain way into G. The functor i has the form $i = i_{U,1} \circ \varepsilon$, where ε is a character of M, and U is a subgroup of G.

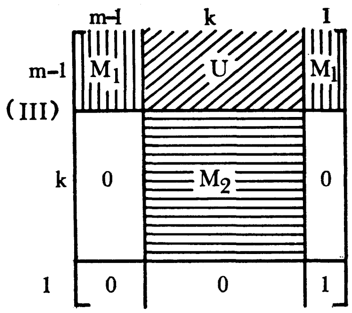
Let us show, how $M_1 \times M_2$ is embedded into G and what are U and ε in our three cases:



$\varepsilon = 1$



$\varepsilon = 1$



$\varepsilon(m_1, m_2) = \nu(m_2)^{-1/2}$

More precisely, introduce some notations:

(*a*) If $1 \leqq k \leqq l \leqq n$, then

$$G^{k,l} = \{ g = (g_{ij}) \in G_n \mid g_{ij} = \delta_{ij}, \text{ outside the square } k \leqq i,j \leqq l \};$$

it is clear that $G^{k,l} \simeq G_{l-k+1}$.

(*b*) If $1 \leqq k < n$, then :

$$U^k = \{ g = (g_{ij}) \in G_n \mid g_{ij} = \delta_{ij}, \quad \text{for} \quad i > k, \quad \text{and for} \quad j \leqq k \}.$$

Then M_1, M_2 and U are as follows :

(I) $\quad M_1 = G^{1,k}, \quad M_2 = G^{k+1,n}, \quad U = U^k.$

(II) $\quad M_1 = G^{1,k}, \quad M_2 = G^{k+1,n} \cap P_n, \quad U = U^k.$

(III) $\quad M_1 = G^{1,m-1}.(U^{m-1} \cap U^{n-1}), \quad M_2 = G^{m,n-1}, \quad U = U^{m-1} \cap G^{1,n-1}.$

Note that the definition (I) coincides with the one given in 4.1.

4.13. We describe how the functors Ψ^-, Ψ^+, Φ^-, Φ^+ and the restriction to P act on the products.

PROPOSITION. — *Let* $\rho \in \text{Alg } G_k$, $\sigma \in \text{Alg } G_m$, $\tau \in \text{Alg } P_m$.

(*a*) *In* Alg P_{k+m} *there exists an exact sequence*

$$0 \to (\rho|_P) \times \sigma \to (\rho \times \sigma)|_P \to \rho \times (\sigma|_P) \to 0.$$

(*b*) *If* Ω *is one of functors* Ψ^-, Ψ^+, Φ^-, Φ^+, *then* $\rho \times \Omega(\tau) \simeq \Omega(\rho \times \tau)$.

(*c*) $\Psi^-(\tau \times \rho) \simeq \Psi^-(\tau) \times \rho$ *and there exists an exact sequence*

$$0 \to \Phi^-(\tau) \times \rho \to \Phi^-(\tau \times \rho) \to \Psi^-(\tau) \times (\rho|_P) \to 0.$$

(*d*) *Suppose that* $k > 0$. *Then for any non-zero* P-*submodule* $\omega \subset \tau \times \rho$ *we have* $\Phi^-(\omega) \neq 0$.

All morphisms in (*a*), (*b*), (*c*) *are functorial. We assume that* $P_0 = \emptyset$ *and* $\rho|_{P_0} = 0$.

This proposition will be proved in paragraph 7.

4.14. COROLLARY. — (*a*) *If* $i \geqq 1$, *then* $(\rho \times \tau)^{(i)} = \rho \times \tau^{(i)}$.

(*b*) *If* $i \geqq 1$ *then* $(\tau \times \rho)^{(i)}$ *is glued from* $\tau^{(j)} \times \rho^{(i-j)}$ ($j = 1, 2, \ldots, i$).

(*c*) *If* $i \geqq 0$, *then* $(\rho \times \sigma)^{(i)}$ *is glued from* $\rho^{(j)} \times \sigma^{(i-j)}$ ($j = 0, 1, \ldots, i$).

Proof. — Part (*a*) follows from 4.13 (*b*); (*b*) from (*a*) and 4.13 (*c*); (*c*) from (*a*), (*b*) and 4.13 (*a*).

Note, that (*c*) coincides with 4.5.

4.15. *Proof of Theorem* 4.11. — We use induction over r. If $r = 1$, our statement follows from 4.4. Let $r > 1$; then $\pi = \rho_1 \times \pi^0$ where $\pi^0 = \rho_2 \times \ldots \times \rho_r$. Suppose that π has a degenerate P-submodule $\omega \neq 0$; one can assume that ω is irreducible. It follows from 4.13 (*a*) that either $\omega \subset \sigma = (\rho_1|_P) \times \pi^0$, or $\omega \subset \pi/\sigma = \rho_1 \times (\pi^0|_P)$. Consider the two cases.

CASE 1. — $\omega \subset \sigma = (\rho_1 |_P) \times \pi^0$. It follows from 4.13 (*c*) and 4.4 that

$$(\Phi^-)^i(\sigma) = (\Phi^-)^i(\rho_1 |_P) \times \pi^0 (i < n_1); \qquad (\Phi^-)^{n_1}(\sigma) = \pi^0 |_P,$$

and

$$\sigma^{(i)} = \rho_1^{(i)} \times \pi^0 = 0 (i < n_1).$$

So $\omega^{(i)} = 0$ for $i < n_1$, and 3.5 implies that $(\Phi^-)^{n_1-1}(\omega) \neq 0$. Since

$$(\Phi^-)^{n_1-1}(\omega) \subset (\Phi^-)^{n_1-1}(\sigma) = (\Phi^-)^{n_1-1}(\rho_1 |_P) \times \pi^0,$$

it follows from 4.13 (*d*) that $(\Phi^-)^{n_1}(\omega) \neq 0$.

Hence $\omega' = (\Phi^-)^{n_1}(\omega)$ is a non-zero degenerate submodule in $(\Phi^-)^{n_1}(\sigma) = \pi^0 |_P$; it contradicts the induction hypothesis.

CASE 2. — $\omega \subset \pi/\sigma = \rho_1 \times (\pi^0 |_P)$. Let $\omega^{(k)}$ be the highest derivative of ω and ω' be an irreducible submodule of $\omega^{(k)}$. Then $\omega' \subset (\pi/\sigma)^{(k)} = \rho_1 \times (\pi^0)^{(k)}$ [*see* 4.14 (*a*)]. Hence, $\rho_1 \in \operatorname{supp} \omega'$. According to Lemma 4.7 (*b*) there exists j such that $\rho_j \simeq \nu\rho_1$. It contradicts the condition of Theorem 4.11.

Theorem 4.11 is proved.

5. Composition of functors r and i

In this section we prove one general theorem about the composition of functors r and i (Th. 5.2).

5.1. Let G be an l-group, P, M, U and Q, N, V be closed subgroups, θ be a character of U and ψ be a character of V. Suppose that

(1) $MU = P$, $NV = Q$, $M \cap U = N \cap V = \{ e \}$, M normalises U and θ, N normalises V and ψ.

According to 1.8, there are defined functors

$$i_{U,\theta}: \ \operatorname{Alg} M \to \operatorname{Alg} G \qquad \text{and} \qquad r_{V,\psi}: \ \operatorname{Alg} G \to \operatorname{Alg} N.$$

We want to compute the functor

$$F = r_{V,\psi} \circ i_{U,\theta}: \ \operatorname{Alg} M \to \operatorname{Alg} N.$$

It requires some complementary conditions. Suppose that

(2) The group G is countable in infinity, and U, V are limits of compact subgroups (*see* 1.9).

Consider the space $X = P \backslash G$ with its quotient-topology and the action δ of G on X defined by

$$\delta(g)(Ph) = Phg^{-1} (g, h \in G, Ph \in X).$$

Suppose that

(3) The subgroup Q has a finite number of orbits on X. According to ([1], 1.5), one can choose a numbering $Z_1, \ldots, Z_k$ of the Q-orbits on X such that all sets

$$Y_1 = Z_1, \qquad Y_2 = Z_1 \cup Z_2, \ \ldots, \ Y_k = Z_1 \cup \ldots \cup Z_k = X$$

are open in X. In particular, all Q-orbits on X are locally closed.

Fix a Q-orbit $Z \subset X$. Choose $\bar{w} \in G$ such that $P\bar{w}^{-1} \in Z$ and denote by w the corresponding inner automorphism of $G : w(g) = \bar{w}g\bar{w}^{-1}$. Call a subgroup $H \subset G$ decomposable with respect to the pair (M, U), if $H \cap (MU) = (H \cap M).(H \cap U)$. Suppose that

(4) The groups $w(P)$, $w(M)$ and $w(U)$ are decomposable with respect to (N, V); the groups $w^{-1}(Q)$, $w^{-1}(N)$ and $w^{-1}(V)$ are decomposable with respect to (M, U).

If the conditions (1)-(4) hold, we define the functor $\Phi_Z : \mathrm{Alg}\, M \to \mathrm{Alg}\, N$. Consider the condition

(★) The characters $w(\theta)$ and ψ coincide when restricted to the subgroup $w(U) \cap V$.

If (★) does not hold, set $\Phi_Z = 0$. If (★) holds then define the functor Φ_Z in the following way.

Set

$$M' = M \cap w^{-1}(N), \qquad N' = w(M') = w(M) \cap N,$$

$$V' = M \cap w^{-1}(V), \qquad \psi' = w^{-1}(\psi)|_{V'}, \qquad U' = N \cap w(U), \qquad \theta' = w(\theta)|_{U'}.$$

It is clear that the following functors are defined

$$r_{V', \psi'} : \ \mathrm{Alg}\, M \to \mathrm{Alg}\, M',$$

$$w : \ \mathrm{Alg}\, M' \to \mathrm{Alg}\, N', \qquad i_{U', \theta'} : \ \mathrm{Alg}\, N' \to \mathrm{Alg}\, N,$$

(*see* 1.7, 1.8). Let $\varepsilon_1 = \mathrm{mod}_U^{1/2}.\mathrm{mod}_{U \cap w^{-1}(Q)}^{-1/2}$ be a character of M',

$$\varepsilon_2 = \mathrm{mod}_V^{1/2}.\mathrm{mod}_{V \cap w(P)}^{-1/2}$$

be a character of N' and $\varepsilon = \varepsilon_1 . w^{-1}(\varepsilon_2)$ be a character of M'. We define Φ_Z by

$$\Phi_Z = i_{U, \theta'} \circ w \circ \varepsilon \circ r_{V', \psi'} : \quad \mathrm{Alg}\, M \to \mathrm{Alg}\, N$$

(here ε is considered as a functor, *see* 1.5). In a more symmetric form

$$\Phi_Z = i_{U', \theta'} \circ \varepsilon_2 \circ w \circ \varepsilon_1 \circ r_{V', \psi'},$$

5.2. THEOREM. — *Under the conditions* (1)-(4) *from* 5.1 *the functor* $F = r_{V, \psi} \circ i_{U, \theta}$: $\mathrm{Alg}\, M \to \mathrm{Alg}\, N$ *is glued from the functors* Φ_Z *where* Z *runs through all* Q-*orbits on* X. *More precisely, if orbits* $Z_1, \ldots, Z_k$ *are numerated so that all sets* $Y_i = Z_1 \cup \ldots \cup Z_i$ $(i = 1, \ldots, k)$ *are open in* X *then there exists a filtration* $0 = F_0 \subset F_1 \subset \ldots \subset F_k = F$ *such that* $F_i/F_{i-1} \simeq \Phi_{Z_i}$.

The remaining part of this Section is devoted to the proof of this Theorem.

5.3. By an l-space we mean a Hausdorff topological space X such that compact open subsets form a base of the topology of X. Denote by $C^\infty(X)$ the ring of all locally constant complex-valued functions on X and by $S(X)$ the subring of all functions with a compact support.

In the situation of 5.1 the space $X = P\backslash G$ is an l-space (*see* [1], 1.4). Let Y be a Q-invariant open subset of X. We shall define the subfunctor $F_Y \subset F$. Let

$$(\rho, M, L) \in \operatorname{Alg} M.$$

The representation $i_{U,\theta}(\rho)$ acts on the space $i(L)$ (*see* 1.8). Denote by $i_Y(L) \subset i(L)$ the subspace consisting of functions which are equal to 0 outside the set

$$PY = \{g \in G \mid Pg \in Y\}.$$

Let τ and τ_Y be the representations of the group Q on the spaces $i(L)$ and $i_Y(L)$. Put $F_Y(\rho) = r_{V,\psi}(\tau_Y) \in \operatorname{Alg} N$. Since $r_{V,\psi}$ is exact (*see* 1.9), $F_Y(\rho) \subset F(\rho) = r_{V,\psi}(\tau)$, hence F_Y is a subfunctor of F.

PROPOSITION.

$$F_{Y \cap Y'} = F_Y \cap F_{Y'}, \qquad F_{Y \cup Y'} = F_Y + F_{Y'}, \qquad F_\emptyset = 0, \qquad F_X = F.$$

Proof. — Since $r_{V,\psi}$ is exact, it is sufficient to prove similar formulae for τ_Y. The only non-trivial one is the equality $\tau_{Y\cap Y'} = \tau_Y \cap \tau_{Y'}$, due to the fact that for any compact set $K \subset Y \cap Y'$ there exist $\varphi \in S(Y)$, $\varphi' \in S(Y')$ such that $(\varphi + \varphi')|_K = 1$ (*see* [1], 1.3).

5.4. For any Q-invariant locally closed set $Z \subset X$ we define the functor

$$F_Z : \quad \operatorname{Alg} M \to \operatorname{Alg} N.$$

For this choose a Q-invariant open $Y \subset X$ such that $Y \cap Z = \emptyset$ and $Y \cup Z$ is open in X (one can take $Y = X \backslash \overline{Z}$), and put $F_Z = F_{Y \cup Z}/F_Y$. It follows from 5.3 that all such F_Z constructed by different Y, are canonically isomorphic.

Let $Z_1, \ldots, Z_k$ be Q-orbits on X, numerated as in the Theorem 5.2. Then by definitions F has the filtration $0 \subset F_{Y_1} \subset F_{Y_2} \subset \ldots \subset F_{Y_k} = F$ and $F_{Y_i}/F_{Y_{i-1}} = F_{Z_i}$. Hence, to prove the Theorem 5.2 we only have to prove that $F_Z \simeq \Phi_Z$ for any Q-orbit $Z \subset X$.

5.5. *Remarks* — (*a*) From now on we fix a Q-orbit $Z \subset X$ and begin to prove that $F_Z \simeq \Phi_Z$. The condition (3) from 5.1 is not necessary for this; we need only conditions (1), (2) and (4) (for our Z).

(*b*) The isomorphism $A : F_Z \xrightarrow{\sim} \Phi_Z$, which will be constructed, is not canonical. It depends on the choise of a Haar measure μ on the quotient-space $V \cap w(P)\backslash V$. We give the explicit expression for A.

Let

$$(\rho, M, L) \in \operatorname{Alg} M, \qquad L^+ = r_{V',\psi'}(L) \qquad \text{and} \qquad p : \ L \to L^+$$

be the canonical projection. Let $i(\mathrm{L})$ be the space of the representation $i_{\mathrm{U},\theta}(\rho)$ (*see* 1.8). Consider the subspaces

$$\mathrm{E} = \{ f \in i(\mathrm{L}) \mid f(\mathrm{P}\overline{\mathrm{Z}} \setminus \mathrm{PZ}) = 0 \},$$
$$\mathrm{E}' = \{ f \in i(\mathrm{L}) \mid f(\mathrm{PZ}) = 0 \} \quad \text{in } i(\mathrm{L}).$$

By definition, $\mathrm{F_Z}(\rho)$ acts on the space $r_{\mathrm{V},\psi}(\mathrm{E}/\mathrm{E}')$ and $\Phi_{\mathrm{Z}}(\rho)$ acts on the space $i(\mathrm{L}^+)$. So to define $\mathrm{A} : \mathrm{F_Z} \to \Phi_{\mathrm{Z}}$ we have to construct an operator $\overline{\mathrm{A}} : \mathrm{E} \to i(\mathrm{L}^+)$ such that

$$\overline{\mathrm{A}}(\mathrm{E}') = 0 \qquad \text{and} \qquad \overline{\mathrm{A}}(\mathrm{E}(\mathrm{V}, \psi)) = 0.$$

Define $\overline{\mathrm{A}}$ by

$$\overline{\mathrm{A}} f(n) = \int \psi^{-1}(v)\, p(f(\overline{w}^{-1} v n))\, d\mu(v), \qquad f \in \mathrm{E}, \qquad n \in \mathrm{N}, \quad v \in (\mathrm{V} \cap w(\mathrm{P})) \setminus \mathrm{V}.$$

One can easily verify that if the condition (★) from 5.1 holds then $\overline{\mathrm{A}}$ is well defined and determines a morphism $\mathrm{A} : \mathrm{F_Z}(\rho) \to \Phi_{\mathrm{Z}}(\rho)$.

5.6. Let us make in 5.1 the following replacements:

$$\hat{\mathrm{P}} = w(\mathrm{P}), \qquad \hat{\mathrm{M}} = w(\mathrm{M}), \qquad \hat{\mathrm{U}} = w(\mathrm{U}), \qquad \hat{\theta} = w(\theta), \qquad \hat{\overline{w}} = e.$$

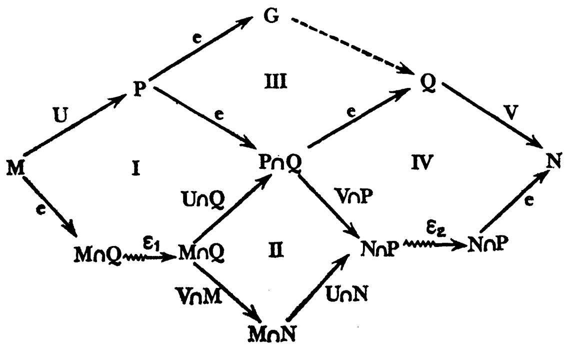
It is clear that

$$\mathrm{F}_{\hat{\mathrm{Z}}}(w\rho) = \mathrm{F_Z}(\rho), \qquad \Phi_{\hat{\mathrm{Z}}}(w\rho) = \Phi_{\mathrm{Z}}(\rho) \quad \text{for all } \rho \in \mathrm{Alg\,M}.$$

Hence further on we can assume that $\overline{w} = e$ (so w is an identical automorphism). We have

$$\mathrm{M}' = \mathrm{N}' = \mathrm{M} \cap \mathrm{N}, \qquad \mathrm{U}' = \mathrm{U} \cap \mathrm{N}, \qquad \mathrm{V}' = \mathrm{V} \cap \mathrm{M}, \qquad \theta' = \theta|_{\mathrm{U}'}, \qquad \psi' = |_{\mathrm{V}'}.$$

5.7. Consider the diagram



Here points correspond to categories and arrows to functors in the following way. A group H in the diagram means the category Alg H, an arrow $\overset{\mathrm{H}}{\nearrow}$ means the functor $i_{\mathrm{H},\theta}$, an arrow $\searrow^{\mathrm{H}}$ means the functor $r_{\mathrm{H},\psi}$ and an arrow $\overset{\varepsilon}{\rightsquigarrow}$ means the functor ε (*see* 1.5). Note that the arrow $\mathrm{G} \dashrightarrow \mathrm{Q}$ in the diagram does not correspond to any functor; but there is determined the functor corresponding to the composition $\mathrm{P} \to \mathrm{G} \dashrightarrow \mathrm{Q}$.

By definition the composition of functors along the highest path of the diagram is $\mathrm{F_Z}$; if the condition ($\bigstar$) from 5.1 holds, then the composition along the lowest path is Φ_{Z}. So it suffices to prove that the diagram above is commutative if ($\bigstar$) holds and its highest path is 0 otherwise. For this we shall check this statement for parts I, II, III, IV of our diagram. It is clear that each of these parts is a particular case of the whole diagram. So it suffices to prove the equality $\mathrm{F_Z} = \Phi_{\mathrm{Z}}$ in the following four cases:

$$\text{I. } \mathrm{P}=\mathrm{G},\quad \mathrm{V}=\{e\};\qquad \text{II. } \mathrm{P}=\mathrm{G}=\mathrm{Q};$$

$$\text{III. } \mathrm{U}=\mathrm{V}=\{e\};\qquad \text{IV. } \mathrm{U}=\{e\},\quad \mathrm{Q}=\mathrm{G}.$$

Note that in cases I, II, IV, $\mathrm{PQ} = \mathrm{G}$ hence $\mathrm{Z} = \mathrm{X}$. So in these cases we shall write F and Φ instead of $\mathrm{F_Z}$ and Φ_{Z}.

5.8. Case I. — $\mathrm{P} = \mathrm{G}$, $\mathrm{V} = \{e\}$. Let $(\rho, \mathrm{M}, \mathrm{L}) \in \mathrm{Alg}\,\mathrm{M}$, $\pi = \mathrm{F}(\rho)$, $\sigma = \Phi(\rho)$. By definitions π and σ act on the same space L, and we have

$$\pi(u)=\sigma(u)=\theta(u).1 \qquad \text{for}\quad u\in \mathrm{U}\cap\mathrm{N},$$

$$\sigma(m)=\varepsilon_1(m).\mathrm{mod}^{1/2}_{\mathrm{U}\cap\mathrm{N}}(m)\,\rho(m)=\mathrm{mod}^{1/2}_{\mathrm{U}}(m)\,\rho(m)=\pi(m)\qquad\text{for}\quad m\in\mathrm{M}'.$$

Hence $\pi = \sigma$.

5.9. Case II. — $\mathrm{P} = \mathrm{G} = \mathrm{Q}$. In notations of 5.8 the representation π acts on the space

$$r_{\mathrm{V},\psi}(\mathrm{L})=r_{\mathrm{V}',\psi}(r_{\mathrm{U}\cap\mathrm{V},\psi}(\mathrm{L}))$$

[*see* 1.8, 1.9 (*c*)]. If $\theta\,|_{\mathrm{U}\cap\mathrm{V}} \neq \psi\,|_{\mathrm{U}\cap\mathrm{V}}$ [it means that 5.1 ($\bigstar$) does not hold] then $r_{\mathrm{U}\cap\mathrm{V},\psi}(\mathrm{L}) = 0$ hence $\pi = 0$. Suppose that $\theta\,|_{\mathrm{U}\cap\mathrm{V}} = \psi\,|_{\mathrm{U}\cap\mathrm{V}}$. Then π and σ act on the same space $r_{\mathrm{V}',\psi}(\mathrm{L}) = \mathrm{L}/\mathrm{L}(\mathrm{V}', \psi)$ (*see* 1.8). For $u \in \mathrm{U}\cap\mathrm{N}$ we have

$$\pi(u)=\sigma(u)=\theta(u).1$$

and for $m \in \mathrm{M}' = \mathrm{M}\cap\mathrm{N}$, $\xi' = \xi \bmod \mathrm{L}(\mathrm{V}', \psi)$ we have

$$\pi(m)\,\xi'=\big[(\mathrm{mod}^{1/2}_{\mathrm{U}}.\mathrm{mod}^{-1/2}_{\mathrm{V}})(m).\rho(m)\,\xi\big]\bmod \mathrm{L}(\mathrm{V}',\psi),$$

$$\sigma(m)\,\xi'=\big[(\mathrm{mod}^{-1/2}_{\mathrm{V}'}.\mathrm{mod}^{1/2}_{\mathrm{U}'})(m).\rho(m)\,\xi\big]\bmod \mathrm{L}(\mathrm{V}',\psi).$$

Since $\mathrm{mod_U} = \mathrm{mod_{U'}}.\mathrm{mod_{U\cap V}}$, $\mathrm{mod_V} = \mathrm{mod_{V'}}.\mathrm{mod_{U\cap V}}$, it follows that $\pi = \sigma$.

5.10. In cases III, IV we use the notion of an *l*-sheaf, defined in [1]. Let us collect the basic definitions and results about *l*-sheaves (in a form somewhat different from [1], chap.I).

Definition. — Let X be an *l*-space (*see* 5.3), $\mathscr{C}$ the constant sheaf with the fiber **C** on X (meaning that the space $\Gamma(\mathscr{C}, \mathrm{Y})$ of sections is $\mathrm{C}^\infty(\mathrm{Y})$ for any open $\mathrm{Y}\subset\mathrm{X}$). By an *l*-sheaf on X we mean an arbitrary sheaf of modules over the sheaf of rings $\mathscr{C}$. Denote by Sh (X) the category of *l*-sheaves on X.

If $\mathscr{F} \in \mathrm{Sh}(X)$ denote by $\mathscr{F}(X)$ the space of sections of $\mathscr{F}$ over X and by $\mathscr{F}_c(X)$ the subspace of sections with a compact support. It is clear that $\mathscr{F}(X)$ and $\mathscr{F}_c(X)$ are modules over the ring $C^\infty(X)$ and hence over $S(X)$ (*see* 5.3).

PROPOSITION (*see* [1], 1.14). — *The functor* $\mathscr{F} \mapsto \mathscr{F}_c(X)$ *is an equivalence of the category* $\mathrm{Sh}(X)$ *with the category of all* $S(X)$ *-modules* M *satisfying the condition* $S(X).M = M$.

5.11. Let $q : Y \to X$ be a continuous map of *l*-spaces, $\mathscr{F} \in \mathrm{Sh}(X)$. Define the *l*-sheaf $q^*\mathscr{F}$ on Y as corresponding to the $S(Y)$ -module $S(Y) \otimes_{S(X)} \mathscr{F}_c(X)$. If q is an embedding of a locally closed subset $Y \subset X$ into X then we write $\mathrm{res}_Y(\mathscr{F})$ instead of $q^*\mathscr{F}$ and $\mathscr{F}_c(Y)$ instead of $(\mathrm{res}_Y(\mathscr{F}))_c(Y)$.

If $Y \subset X$ is open we have natural maps $\mathscr{F}_c(Y) \to \mathscr{F}_c(X)$ (extension by 0) and $\mathscr{F}_c(X) \to \mathscr{F}_c(X \setminus Y)$ (restriction).

PROPOSITION (*see* [1], 1.16). — *The sequence* $0 \to \mathscr{F}_c(Y) \to \mathscr{F}_c(X) \to \mathscr{F}_c(X \setminus Y) \to 0$ *is exact.*

5.12. Let $x \in X$. For any open compact neighbourhood Y of x the map $f \mapsto \chi_Y . f$ is a projection of $\mathscr{F}_c(X)$ into the subspace $\mathscr{F}_c(Y) \subset \mathscr{F}_c(X)$. In particular the fiber $\mathscr{F}_x$ equals $\varinjlim \mathscr{F}_c(Y) = \varinjlim \chi_Y . \mathscr{F}_c(X)$ where the inductive limit is taken over all compact open neighbourhoods $Y \ni x$. It follows that $\mathscr{F}_x$ is canonically isomorphic to $\mathscr{F}_c(\{x\})$.

5.13. Let X, Y be *l-spaces*, $\mathscr{F} \in \mathrm{Sh}(X)$, $\mathscr{E} \in \mathrm{Sh}(Y)$. By an isomorphism of $(X, \mathscr{F})$ with $(Y, \mathscr{E})$ we mean a pair consisting of a homeomorphism $\gamma : X \to Y$ and an isomorphism $\mathscr{F}$ with $\gamma^*(\mathscr{E})$.

We call action of an *l*-group G on a pair $(X, \mathscr{F})$ a homomorphism

$$\gamma : \quad G \to \mathrm{Aut}(X, \mathscr{F})$$

such that the action of G on X is continuous and the representation of G on $\mathscr{F}_c(X)$ is an algebraic one. Fix a continuous action γ_0 of G on X. Let us define the category $\mathrm{Sh}(X, G)$ of G-sheaves on X. An object of $\mathrm{Sh}(X, G)$ is an *l*-sheaf $\mathscr{F} \in \mathrm{Sh}(X)$ with an action γ of G on $(X, \mathscr{F})$ such that the restriction of γ on X is γ_0. By morphisms in $\mathrm{Sh}(X, G)$ we mean G-equivariant morphisms of sheaves on X.

For example $\mathrm{Sh}(X, \{e\}) = \mathrm{Sh}(X)$, $\mathrm{Sh}(\{x\}, G) = \mathrm{Alg}\, G$.

The correspondence $\mathscr{F} \mapsto \mathscr{F}_c(X)$ determines the functor

$$\mathrm{Sec} : \quad \mathrm{Sh}(X, G) \to \mathrm{Alg}\, G.$$

If Q is a closed subgroup of G and Z is a locally closed Q-invariant subset of X then the correspondence $\mathscr{F} \to \mathrm{res}_Z(\mathscr{F})$ determines the functor $\mathrm{res} : \mathrm{Sh}(X, G) \to \mathrm{Sh}(Z, Q)$. In particular if $Z = \{x\}$ is a point then $\mathrm{res}\, \mathscr{F} \in \mathrm{Sh}(\{x\}, Q) = \mathrm{Alg}\, Q$ is the representation of Q on the fiber $\mathscr{F}_x$ of $\mathscr{F}$.

5.14. The most important case for us is when the action γ_0 of G on X is transitive. If the group G is countable at infinity then X is homeomorphic to the quotient-space $P \setminus G$ where P is a stabilizer of some point $x \in X$ (see [1], 1.5). In this case we shall define the functor $\mathrm{ind} : \mathrm{Alg}\, P \to \mathrm{Sh}(X, G)$ which is inverse to the functor

$$\mathrm{res} : \quad \mathrm{Sh}(X, G) \to \mathrm{Sh}(\{x\}, P) = \mathrm{Alg}\, P.$$

Let $(\rho, P, L) \in \operatorname{Alg} P$. Consider the space $i(L)$ of the representation $i(\rho) = i_{\{e\}, 1}(\rho)$ (*see* 1.8). It has a natural structure of an $S(X)$-module and so according to 5.10 determines the l-sheaf $\operatorname{ind}(\rho)$ on X. The representation $i(\rho)$ and the action γ_0 of G on X determine a structure of a G-sheaf on $\operatorname{ind}(\rho)$. We have by definitions

$$\operatorname{Sec} \circ \operatorname{ind} = i_{\{e\}, 1} : \quad \operatorname{Alg} P \to \operatorname{Alg} G.$$

PROPOSITION (*see* [1], 2.23). – *If* $X = P \backslash G$ *then the functors*

$$\operatorname{res} : \quad \operatorname{Sh}(X, G) \to \operatorname{Alg} P \qquad \text{and} \qquad \operatorname{ind} : \operatorname{Alg} P \to \operatorname{Sh}(X, G),$$

are inverse to each other and determine an equivalence of categories (*i. e.* $\operatorname{res} \circ \operatorname{ind} \simeq \operatorname{Id}$, $\operatorname{ind} \circ \operatorname{res} \simeq \operatorname{Id}$).

5.15. Let us return to our Theorem. Let us describe the functor F_Z in terms of l-sheaves. For any Q-invariant locally closed subset $Z \subset X$ the functor F_Z decomposes into a composition of functors

$$\operatorname{Alg} M \xrightarrow{i_{U,\theta}} \operatorname{Alg} P \xrightarrow{\operatorname{ind}} \operatorname{Sh}(X, G) \xrightarrow{\operatorname{res}} \operatorname{Sh}(Z, Q) \xrightarrow{\operatorname{Sec}} \operatorname{Alg} Q \xrightarrow{r_{V,\psi}} \operatorname{Alg} N$$

[for open Z it is a definition of F_Z (*see* 5.3, 5.4) and for others follows from 5.4, 5.11 and the exactness of $r_{V,\psi}$]. Now consider the following.

CASE III. – $U = V = \{e\}$, i. e. $M = P$, $N = Q$. It is clear that $Z \simeq P \cap Q \backslash Q$ (here Z is the Q-orbit of the point $\bar{e} = Pe \in X$). Let $\rho \in \operatorname{Alg} P$ and $i'(\rho) \in \operatorname{Sh}(Z, Q)$ be the restriction of the sheaf $\operatorname{ind}(\rho)$ to Z. It suffices to prove that $i'(\rho)$ is isomorphic to the sheaf

$$\operatorname{ind}(\rho') \in \operatorname{Sh}(Z, Q), \qquad \text{where} \quad \rho' = r_{\{e\}, 1}(P \cap Q, P, \rho).$$

It follows from 5.13 since for both sheaves the representation of the group $P \cap Q$ on the fiber over the point $\bar{e}$ equals ρ.

5.16. CASE IV. – $U = \{e\}$, $G = Q$. Divide this case into two cases IV_1 and IV_2 using the diagram

$$\begin{tikzcd}
 & & Q=NV \arrow[ddrr, "V"] & & \\
 & (M\cap N).V \arrow[ur, "e"] \arrow[ddr] \arrow[rr, phantom, "\mathrm{IV}_1"] & & & \\
M \arrow[ur, "e"] \arrow[dr, "V\cap M"'] \arrow[r, phantom, "\mathrm{IV}_2"] & & & & N \\
 & M\cap N \arrow[r, squiggly, "\varepsilon_2"] & M\cap N \arrow[urr, "e"'] & &
\end{tikzcd}$$

CASE IV_1. – $U = \{e\}$, $G = Q$, $V \subset M = P$. Let

$$(\rho, M, L) \in \operatorname{Alg} M, \qquad \mathscr{F} = \operatorname{ind}(\rho) \in \operatorname{Sh}(X, Q).$$

Since in this case the group V acts trivially on $X = M\backslash G$. the space $r_{V,\psi}(\mathcal{F}_c(X))$ becomes an $S(X)$-module and so, according to 5.10, it corresponds to some l-sheaf $\mathcal{E}$ on X. The representation $F(\rho)$ of the group N on this space determines on the pair $(X, \mathcal{E})$ the structure of an N-sheaf. Since N transitively acts on X, using 5.14 we have only to check that the representation of the group $M \cap N$ on the fiber $\mathcal{E}_{\bar{e}}$ of the sheaf $\mathcal{E}$ equals $r_{V,\psi}(\rho)$. In other words we have to prove that $\mathcal{E}_{\bar{e}} = r_{V,\psi}(\mathcal{F}_{\bar{e}})$. Since the functor $r_{V,\Psi}$ commutes with inductive limits [*see* 1.9 (*g*)], using 5.11 we have

$$\begin{aligned}\mathcal{E}_{\bar{e}} &= \varinjlim \chi_Y . \mathcal{E}_c(X)\\ &= \varinjlim \chi_Y r_{V,\psi}(\mathcal{F}_c(X)) = r_{V,\psi}(\varinjlim \chi_Y \mathcal{F}_c(X)) = r_{V,\psi}(\mathcal{F}_{\bar{e}}).\end{aligned}$$

5.17. CASE IV_2. — $U = \{e\}$, $G = Q$, $N \subset M$. In this case

$$X = NV'\backslash NV \simeq V'\backslash V, \qquad \text{where} \quad V' = V \cap M.$$

Choose a Haar measure μ on $V'\backslash V$ (*see* 1.7). Let

$$(\rho, M, L) \in \operatorname{Alg} M, \qquad L^+ = r_{V',\psi}(L) \qquad \text{and} \qquad p:\ L \to L^+,$$

be a natural projection. Let $i(L)$ be the space of the representation $\tau = i_{\{e\},1}(\rho)$. Define the map $\bar{A} : i(L) \to L^+$ by

$$\bar{A}f = \int_{V'\backslash V} \psi^{-1}(v)\, p(f(v))\, d\mu(v).$$

It is easily seen that this definition is correct and that

$$\bar{A}(\tau(v, f) = \psi(v)\bar{A}(f) \qquad \text{for} \quad v \in V,$$

i. e. $\bar{A}$ determines the map $A : r_{V,\psi}(i(L)) \to L^+$. We prove that $A \in \operatorname{Hom}(\pi, \sigma)$, where

$$\pi = r_{V,\psi}(\tau) = F(\rho), \qquad \sigma = \varepsilon_2 . r_{V',\psi}(\rho) = \Phi(\rho).$$

We have

$$\begin{aligned}A(\pi(n)\bar{f}) &= \operatorname{mod}_V^{-1/2}(n)\bar{A}(\pi(n) f)\\ &= \operatorname{mod}_V^{-1/2}(n) . \int_{V'\backslash V} \psi^{-1}(v)\, p(f(vn))\, d\mu(v) \qquad (n \in N).\end{aligned}$$

Use in this integral the replacement $v = n^{-1}vn$; according to 1.7 it equals

$$\operatorname{mod}_{V'\backslash V}(n) . \int_{V'\backslash V} \psi^{-1}(\tilde{v})\, p(f(n\tilde{v}))\, d\mu(\tilde{v}).$$

Since

$$f(n\tilde{v}) = \rho(n) f(\tilde{v}), \qquad p(f(n\tilde{v})) = \operatorname{mod}_{V'}^{1/2}(n) . \varepsilon_2^{-1}(n) . \sigma(n)(p(f(\tilde{v}))),$$

substituting this expression into the latter integral, we obtain that $A\pi(n) = \sigma(n) A$.

Therefore we have constructed a morphism of functors $A : F \to \Phi$. We have only to check that A is an isomorphism. We can suppose that $N = \{e\}$, $M = V'$; replacing ρ by $\psi^{-1}\rho$, we can suppose that $\psi = 1$.

Assume first that $\rho = i_{\{e\},1}(V', \{e\}, 1)$ is the regular representation of V' on the space S (V'). Then $\tau = i_{\{e\},1}(\rho)$ is the regular representation of the group V [*see* 1.9 (*c*)]. The uniqueness of a Haar measure on V and V' implies that the spaces $r_{V',1}(S(V'))$ and $r_{V,1}(S(V))$, are one-dimensional; it is clear that in this case $A \neq 0$. Hence if ρ is the regular representation then A is an isomorphism. According to 1.9 (*g*) A is an isomorphism for any free representation ρ (it means that ρ is a direct sum of regular representations). It is easy to prove that any representation $\rho \in \operatorname{Alg} V'$ is a quotient of some free representation hence ρ has a free resolution. Therefore exactness of F and Φ implies that $A : F(\rho) \to \Phi(\rho)$ is an isomorphism for any ρ.

Theorem 5.2 is proved.

6. Proof of lemmas of paragraph 2

6.1. Let us formulate some statements about subgroups of the reductive group G. All of them are proved in [4].

Let G, P_0, A_0 be the groups defined in 2.1. Denote by Λ the lattice of rational characters $\lambda : A_0 \to F^*$; we write the group operation in Λ additively. Let $\Sigma' \in \Lambda$ be the set of roots of G relative to A_0 and Σ the reduced part of Σ' (Σ consists of non-divisible roots of Σ'). To each $\gamma \in \Sigma$ there corresponds the subgroup $U_\gamma \subset G$; it is the maximal unipotent subgroup, normalised by A^0, in which A_0 has weights γ and 2γ. The Weyl group $W = N_G(A_0)/Z_G(A_0)$ (*see* 2.6) acts on A_0, hence on Λ. This action allows us to identify W with the Weyl group of the system Σ (*see* [4], 5.3). Put

$$\Sigma^+ = \{\gamma \in \Sigma \mid U_\gamma \subset P_0\} -$$

it is a system if positive roots, corresponding to P_0; denote by Π the corresponding set of simple roots. We shall sometimes write $\gamma > 0$ ($\gamma < 0$) instead of $\gamma \in \Sigma^+$ ($\gamma \in -\Sigma^+$).

Let S be a subset of Σ. We call S closed if $(S+S) \cap \Sigma \subset S$ and convex if S is an intersection of Σ with some convex cone in $\Lambda \otimes \mathbf{Q}$. A closed subset $S \subset \Sigma$ is called symmetric if $S = -S$; in this case S is a root system and we denote by $W_S \subset W$ its Weyl group. A closed subset $S \subset \Sigma$ is called unipotent if $S \subset w(\Sigma^+)$ for some $w \in W$.

For any closed $S \subset \Sigma$ denote by G (S) the algebraic subgroup in G, generated by subgroups $Z_G(A_0)$ and U_γ, $\gamma \in S$. If S is unipotent denote by U (S) the algebraic subgroup in G, generated by all U_γ, $\gamma \in S$. Using the results of [4], one can prove that G (S) (resp. U (S)) is generated by $Z_G(A_0)$ and U_γ (resp. by U_γ) as an abstract group.

PROPOSITION. (*See* [4], 3.22). — *Let* S, T *be closed subsets of* Σ.

(*a*) *If* S *and* T *are convex then* $G(S) \cap G(T) = G(S \cap T)$ (3).

(*b*) *If* T *is unipotent then* $G(S) \cap U(T) = U(S \cap T)$.

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(3) In ([4], 3.22) there is required only the convexity of $S \cap T$. But in this form the statement is not true. Counterexample: G is a split group of the type G_2, $S = \{\pm\alpha, \pm(\alpha + 2\beta)\}$, $T = \{\pm\beta, \pm(2\alpha + 3\beta)\}$, where α is a long simple root and β is a short one. The proof in [4] is based on the false statement $W_S \cap W_T = W_{S \cap T}$. For convex S and T it is true (and follows from [16], Append., (36)) therefore in this case the proof from [4] is correct.

6.2. We call a closed subset $\mathscr{P} \subset \Sigma$ parabolic if $\mathscr{P} \supset \Sigma^+$. In this case $\mathscr{M} = \mathscr{P} \cap (-\mathscr{P})$ is called a standard subset of Σ (the notation is $\mathscr{M} < \Sigma$) and the triple $(\mathscr{P}, \mathscr{M}, \mathscr{U} = \mathscr{P} \setminus \mathscr{M})$ is called a parabolic triple.

PROPOSITION (*see* [4], 5.12-5.18). — *Let* Γ *be a subset of* Π. *Denote by* $\mathscr{P}$, $\mathscr{M}$ *the closed subsets generated by* $\Sigma^+ \cup (-\Gamma)$ *and* $\Gamma \cup (-\Gamma)$ *respectively. Then* $(\mathscr{P}, \mathscr{M}, \mathscr{U} = \mathscr{P} \setminus \mathscr{M})$ *is a parabolic triple and* $(P = G(\mathscr{P}), M = G(\mathscr{M}), U = U(\mathscr{U}))$ *is a parabolic triple of subgroups in* G (*see* 2.1). *Any parabolic triple* $(\mathscr{P}, \mathscr{M}, \mathscr{U})$ *and parabolic triple of subgroups* (P, M, U) *have such a form; moreover* Γ *is uniquely determined by* $(\mathscr{P}, \mathscr{M}, \mathscr{U})$ *and by* (P, M, U).

It follows, in particular, that in a parabolic triple $(\mathscr{P}, \mathscr{M}, \mathscr{U})$ *the subsets* $\mathscr{P}$, $\mathscr{M}$ *and* $\mathscr{U}$ *are convex.*

6.3. We begin to prove our Lemmas. Denote by $\mathscr{P}$, $\mathscr{M}$, $\mathscr{U}$, $\mathscr{Q}$, $\mathscr{N}$, $\mathscr{V}$, the subsets of Σ, corresponding to the groups, P, M, U, Q, N, V; put

$$\mathscr{M}^+ = \mathscr{M} \cap \Sigma^+, \qquad \mathscr{N}^+ = \mathscr{N} \cap \Sigma^+.$$

It is clear that

$$W_M = W_{\mathscr{M}}, \qquad W_N = W_{\mathscr{N}}.$$

Furthermore it follows from 6.1-6.2 that

$$W^{M, N} = W^{\mathscr{M}, \mathscr{N}} = \{ w \in W \mid w(\mathscr{M}^+) \subset \Sigma^+, \; w^{-1}(\mathscr{N}^+) \subset \Sigma^+ \}.$$

Let $w \in W^{\mathscr{M}, \mathscr{N}}$. Then $\mathscr{M} \cap w^{-1}(\mathscr{Q}) \supset \mathscr{M}^+$ is a parabolic subset of $\mathscr{M}$; therefore

$$(\mathscr{M} \cap w^{-1}(\mathscr{Q}), \; \mathscr{M} \cap w^{-1}(\mathscr{N}), \; \mathscr{M} \cap w^{-1}(\mathscr{V}))$$

is a parabolic triple in $\mathscr{M}$. It follows from 6.1-6.2 that

$$(M \cap w^{-1}(Q), \; M \cap w^{-1}(N), \; M \cap w^{-1}(V))$$

is a parabolic triple of subgroups in M. Similarly $(N \cap w(P), N \cap w(M,) \; N \cap w(U))$ is a parabolic triple in N. In particular, it proves 2.11 (*b*).

We shall often use the following statements (*see* [16; Append., I, II]) :

(1) If $w \in W_{\mathscr{M}}$ then $w(\Sigma^+ \setminus \mathscr{M}^+) \subset \Sigma^+ \setminus \mathscr{M}^+$.

(2) The length $l(w)$ of an element $w \in W$ is equal to a number of roots $\gamma > 0$ such that $w(\gamma) < 0$.

(3) If $w \in W$ and $\gamma \in \Pi$ then the conditions $w(\gamma) > 0$ and $l(w \sigma_\gamma) > l(w)$ are equivalent (here σ_γ is the reflection corresponding to the root γ).

It follows from (3) that $W^{\mathscr{M}, \mathscr{N}} = \{ w \in W \mid l(w \sigma_\gamma) > l(w)$ for any $\gamma \in \Pi \cap \mathscr{M}$, $l(w^{-1} \sigma_\gamma) > l(w^{-1})$ for any $\gamma \in \Pi \cap \mathscr{N} \}$. Therefore the statement 2.11 (*a*) follows from [5; chap. IV, § 1, exer. 3].

6.4. *Proof of the Lemma* 2.12. — Use the Bruhat decomposition. It implies that the map $W \to G$ $(w \mapsto \bar{w}^{-1})$ determines a bijection $W_N \setminus W / W_M \xrightarrow{\sim} P \setminus G / Q$ (*see* [4], 5.15, 5.20). In particular for any Q-orbit $Z \subset X = P \setminus G$ there exists a unique point of the form $P \bar{w}^{-1}$ $(w \in W^{M, N})$, belonging to Z (*see* 2.11); denote the orbit $Z \ni P \bar{w}^{-1}$ by $Z(w)$.

For the computation of the functor F from 2.12 use the theorem 5.2. Conditions (1)-(4) from 5.1 follow directly from 6.1 and the condition (★) holds since $\theta = 1$, $\psi = 1$. We conclude that F is glued from the functors $\Phi_{Z(w)}$, $w \in W^{M, N}$. It follows easily from 6.3 that $\Phi_{Z(w)}$ differs from the functor F_w describing in 2.12 by only a character ε. Therefore we have only to prove that $\varepsilon = 1$.

It is easily seen that for any automorphism σ of an F-group H we have

$$\operatorname{mod}_H(\sigma) = \operatorname{mod}_{\mathscr{H}}(\sigma).$$

where $\mathscr{H}$ is the Lie algebra of H (it follows e. g. from [4], 3.11).

Therefore if $\tilde{F}$ is a finite extension of the field F and $\tilde{H} = H(\tilde{F})$ is a group of $\tilde{F}$-points of H then $\operatorname{mod}_{\tilde{H}}(\sigma) = \operatorname{mod}_H^k(\sigma)$ where $k = [\tilde{F} : F]$. Extending in such a way all our groups (G, M, U etc.) we replace ε by $\tilde{\varepsilon} = \varepsilon^k$; since ε is positive, it suffices to prove that $\tilde{\varepsilon} = 1$. One can choose $\tilde{F}$ such that G splits over $\tilde{F}$. It follows that in proving the equality $\varepsilon = 1$ we may assume that G is split. In particular assume that A_0 is a maximal torus in G hence $Z_G(A_0) = A_0$.

Since ε is positive, $\varepsilon|_{U(\mathscr{M}'^+)} = 1$ (*see* 1.7). Therefore if follows from the Bruhat decomposition

$$M' = U(\mathscr{M}'^+).N_{M'}(A_0).U(\mathscr{M}'^+)$$

and from the finiteness of $N_{M'}(A_c)/A_0$ that ε is determined by its restriction to A_0. Since A_0 is a maximal torus, ıor any unipotent subset $S \subset \Sigma$ we have $\operatorname{mod}_{U(S)}|_{A_0} = \operatorname{mod}_S$ where

$$\operatorname{mod}_S(a) = \prod_{\gamma \in S} |\gamma(a)|, \qquad a \in A_0,$$

(*see* [4], 3.11). It follows that $\varepsilon_1^2 = \operatorname{mod}_U . \operatorname{mod}_{U \cap w^{-1}(Q)}^{V-1} = \operatorname{mod}_S$ where

$$S = \mathscr{U} \setminus (\mathscr{U} \cap w^{-1}(\mathscr{Q})) = \mathscr{U} \cap (\Sigma \setminus w^{-1}(\mathscr{Q})) = \mathscr{U} \cap w^{-1}(-\mathscr{V}).$$

Similarly

$$\varepsilon_2^2 = \operatorname{mod}_{w(-\mathscr{U}) \cap \mathscr{V}} = \operatorname{mod}_{-w(S)},$$

hence

$$\varepsilon^2 = \varepsilon_1^2 . w^{-1}(\varepsilon_2^2) = \operatorname{mod}_S . \operatorname{mod}_{-S} = 1.$$

6.5. *Proof of the lemma* 2.17. — (1) Put

$$\overline{W}(\mathscr{M}, \mathscr{N}) = W(\mathscr{M}, \mathscr{N}) \cap W^{\mathscr{M}, \mathscr{N}};$$

$\overline{W}(\mathscr{M}, \mathscr{N})$ is a system of representatives of double cosets $W_{\mathscr{N}} \setminus W(\mathscr{M}, \mathscr{N})/W_{\mathscr{M}}$. It is clear that $w \in \overline{W}(\mathscr{M}, \mathscr{N})$ iff $w(\Pi \cap \mathscr{M}) = \Pi \cap \mathscr{N}$. For any standard subset $\mathscr{L} < \Sigma$ denote by $s_{\mathscr{L}} \in W_{\mathscr{L}}$ such an element that

$$s_{\mathscr{L}}(\mathscr{L}^+) = -\mathscr{L}^+$$

(hence $s_{\mathscr{L}}(\Pi \cap \mathscr{L}) = -(\Pi \cap \mathscr{L})$; the existence and uniqueness of $s_{\mathscr{L}}$ follow from [16], Append. (24)).

(2) Let $\mathcal{M}<\Sigma$, $\Gamma=\Pi\cap\mathcal{M}$, $w\in W$ be such an element that $w(\Gamma)>0$ and $w'=w.w.s_{\mathcal{M}}$. We prove that

(a) $l(w')=l(w)+|\mathcal{M}^+|$

(b) If $w(\Gamma)\subset\Pi$ then for any $\gamma\in\Sigma\setminus\mathcal{M}$ the roots $w(\gamma)$ and $w'(\gamma)$ have the same sign.

Since $s_{\mathcal{M}}(\Sigma^+\setminus\mathcal{M}^+)=\Sigma^+\setminus\mathcal{M}^+$, we have

$$|\{\gamma\in\Sigma^+\setminus\mathcal{M}^+\,|\,w(\gamma)<0\}|=|\{\gamma\in\Sigma^+\setminus\mathcal{M}^+\,|\,w'(\gamma)<0\}|.$$

Since $w(\mathcal{M}^+)>0$, $w'(\mathcal{M}^+)<0$, $l(w')=l(w)+|\mathcal{M}^+|$ [*see* 6.3 (2)]. Now let $w(\Gamma)\subset\Pi$. Then $\mathcal{N}=w(\mathcal{M})<\Sigma$ and it is easily verified that

$$w\,W_{\mathcal{M}}.w^{-1}=W_{\mathcal{N}}\qquad\text{and}\qquad w\,s_{\mathcal{M}}w^{-1}=s_{\mathcal{N}}.$$

If $\gamma\in\Sigma\setminus\mathcal{M}$, then $w(\gamma)\in\Sigma\setminus\mathcal{N}$, hence $w'(\gamma)=ws_{\mathcal{M}}(\gamma)=s_{\mathcal{N}}w(\gamma)$ has the same sign as γ (*see* 6.3. [1]).

(3) Let $\mathcal{M}<\mathcal{L}<\Sigma$, $\Gamma=\Pi\cap\mathcal{M}$. Suppose that $\Gamma'=\mathcal{L}\cap\Pi$ has the form $\Gamma'=\Gamma\cup\{\alpha\}$. Then there are precisely two elements $w\in W_{\mathcal{L}}$ such that $w(\Gamma)\subset\Pi$. Actually, if $w(\alpha)>0$ then $w(\Gamma')>0$ hence $w=1$. If $w(\alpha)<0$ and $w'=w\,s_{\mathcal{M}}$ then (2) implies that $w'(\Gamma')<0$ hence $w'=s_{\mathcal{L}}$ and $w=s_{\mathcal{L}}s_{\mathcal{M}}^{-1}=s_{\mathcal{L}}.s_{\mathcal{M}}$. Therefore in this case the element $\hat{w}=s_{\mathcal{L}}s_{\mathcal{M}}$ determines an elementary (in the sense of 2.16) map $\hat{w}:\mathcal{M}\to\hat{w}(\mathcal{M})$.

(4) We want to prove the Lemma 2.17. In terms of subsets in Σ it states that for any $\mathcal{M}$, $\mathcal{N}<\Sigma$, $w(\mathcal{M})=\mathcal{N}$ the element w is a composition of elements $\hat{w}$ such as constructed in (3). One may assume that $w\in\overline{W}(\mathcal{M},\mathcal{N})$, i. e. $w(\Gamma)\subset\Pi$, where $\Gamma=\mathcal{M}\cap\Pi$. Use the induction on $l(w)$. Let $\alpha\in\Pi\setminus\Gamma$ be such a root that $w(\alpha)<0$ (if such α does not exist then $w=1$). Consider the standard subset $\mathcal{L}$ generated by $\Gamma'=\Gamma\cup\{\alpha\}$ and put

$$\hat{w}=s_{\mathcal{L}}s_{\mathcal{M}},\qquad \mathcal{M}'=\hat{w}(\mathcal{M})\qquad\text{and}\qquad w'=w.\hat{w}^{-1}.$$

Then

$$\mathcal{M}'<\Sigma,\qquad w'\in\overline{W}(\mathcal{M}',\mathcal{N})\qquad\text{and}\qquad \hat{w}:\ \mathcal{M}\to\mathcal{M}'$$

is an elementary map according to (3). So it suffices to prove that $l(w')<l(w)$.

Put $w''=w.s_{\mathcal{M}}$; then $w'=w''s_{\mathcal{L}}$. According to (2), $l(w'')=l(w)+|\mathcal{M}^+|$ and $w''(\Gamma')<0$. Therefore

$$w'(\Gamma')=w''(s_{\mathcal{L}}(\Gamma'))>0,$$

hence (2) implies that

$$l(w')=l(w'')-|\mathcal{L}^+|=l(w)+|\mathcal{M}^+|-|\mathcal{L}^+|<l(w).$$

Lemma 2.17 follows.

7. Proof of the proposition 4.13

We shall use the notations of 4.12-4.13.

7.1. *Proof of* 4.13 (*a*). — Let $M = G_k \times G_m$ be embedded into $G = G_n$ as in 4.12. (I). By definition $(\rho \times \sigma)|_P = F(\rho \otimes \sigma)$, where the functor F is defined as in 5.1 in the following situation:

$$U = U^k, \qquad \theta = 1, \qquad N = P_n, \qquad V = \{e\}.$$

To compute F we apply the Theorem 5.2. Conditions (1), (2) and (★) from 5.1 hold trivially. It is easily seen that there are two Q-orbits on $X = P\backslash G$: the closed orbit Z of the point $P e \in X$ and the open orbit Y of the point $P w^{-1} \in X$, where w is the matrix of the cyclic permutation $(k \mapsto n \mapsto n-1 \mapsto \ldots \mapsto k+1 \mapsto k)$; it follows e. g. from the Bruhat decomposition. Condition (4) from 5.1 can be checked directly or by using 6.1. The character ε from 5.1 is computed as in 6.5. After all it turns out that

$$\Phi_Y(\rho \otimes \sigma) = \rho|_P \times \sigma, \qquad \Phi_Z(\rho \otimes \sigma) = \rho \times \sigma|_P$$

and 4.13 (*a*) follows.

7.2. *Proof of* 4.13 (*b*), (*c*). — If Ω is one of the functors Φ^+ and Ψ^+ then 4.13 (*b*) follows directly from 1.9 (*c*). In the cases $\Omega = \Phi^-$, $\Omega = \Psi^-$ in 4.13 (*b*) and for the proof of 4.13 (*c*) one has to use the theorem 5.2. We leave the details to the reader and describe only the situations and orbits.

STATEMENT 4.13 (*b*). — We have $G = P_n$; M, U are defined in 4.12. (II).

In the case $\Omega = \Psi^-$ we have

$$N = G_{n-1}, \qquad V = V_n, \qquad Q = G,$$

hence there is only one Q-orbit.

In the case $\Omega = \Phi^-$ we have $N = P_{n-1}$, $V = V_n$. There are two orbits—the closed orbit Z of the point $P e \in X$ and the open orbit Y of the point $P w_0^{-1}$ where w_0 is the matrix corresponding to the cyclic permutation $(k \mapsto n-1 \mapsto n-2 \to \ldots \to k)$. Note that $\Phi_Y = 0$ since for Y the condition (★) from 5.1 does not hold; therefore $F = \Phi_Z$.

STATEMENT 4.13 (*c*). — $G = P_n$; M, U are defined in 4.12. (III).

For Ψ^-, we have $N = G_{n-1}$, $V = V_n$; there is only one orbit.

For Φ^-, $N = P_{n-1}$, $V = V_n$; there are two orbits — the closed one of the point $P e$ and the open one of the point $P w_0^{-1}$, where w_0 corresponds to the permutation

$$((m-1) \mapsto n-1 \mapsto n-2 \mapsto \ldots \mapsto m-1).$$

7.3. *Proof of* 4.13 (*d*). — If $\Phi^-(\omega) = 0$ then the restriction of ω to V_n is trivial [*see* 3.3]. Let f be a non-zero element of ω. By the definition of $\tau \times \rho$, f is a vector function on the group P_n [*see* 4.12. (III) and 1.8). Denote by T its support:

$$T = \{g \in P_n \mid f(g) \neq 0\}.$$

One may assume that $e \in T$. Since V_n acts trivially, $V_n \subset T$. Furthermore by the definition of $i_{U,1}$ the set T is compact modulo MU, where M and U are defined in 4.12. (III). But it is easily seen that any set $K \supset V_n$ cannot be compact modulo MU. Actually, it follows from the fact that $MU\backslash MUV_n \simeq F^k$ in not compact. We obtain the contradiction which proves 4.13 (*d*).

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