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ON A LOCAL PROPERTY OF GOOD FRAMES IN THOM-BOARDMAN SINGULARITIES

BY CARLOS S. SUBI

0. Introduction

In [2] Mather defines good frames associated to an ideal in a power series ring. In [5] Mount and Villamayor study the sets $\Sigma(I; i_1, \ldots, i_r)$ of, roughly speaking, all points having the same Thom-Boardman singularity.

The main result of this paper is to prove that the "analitic" construction of Mather is valid locally, i. e., there is an open set in $\Sigma(I; i_1, \ldots, i_r)$ where the given frame is a good frame.

1. The schemas $\Sigma(I; i_1, i_2, \ldots, i_r)$

The reference for this section is [5] (Section 3).

Suppose that M is a finitely generated module over a ring R.

We shall denote $f_j(M)$ the jth Fitting invariant of M (*see* [1] or [3]). If $0 \to K \to F \to M \to 0$ is an exact sequence of R-modules where F is free with a basis $f_1, \ldots, f_n$ and if K is generated by the elements $\sum_j a_{ij} f_j$, then the rth Fitting invariant of M is the ideal of R generated by the $(n-r)\times(n-r)$ subdeterminants of the matrix (a_{ij}). Set k be an algebraically closed field of characteristic zero. Set $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates. If B is a k-algebra, $D(B/k)$ will denote the B-module of k-differentials of B and $d_B : B \to D(B/k)$ the canonical k-derivation.

If $I \subset A$ is an ideal, $D([A/I]/k)$ is an A/I-module of finite type that we shall denote D. The canonical derivation $d_{A/I} : A/I \to D$ will be noted d.

DEFINITION 1.1. — If j is a nonnegative integer we set:

(i) $$z_j(D) = f_{j-1}(D);$$

(ii) $$D(j) = (A/I)/z_j(D) \otimes_{A/I} \frac{D}{(dz_j(D))},$$

where $(dz_j(D))$ stands for the sub-A/I-module of D generated by dZ, $Z \in z_j(D)$.

If $i_1, i_2, \ldots, i_r$ is a sequence of integers, then we define a sequence of ideals and modules as follows:

(iii) $$z(i_1) = z_{i_1}(D);$$

(iv) $$D(i_1) = \frac{A/I}{z_{i_1}(D)} \otimes_{A/I} \frac{D}{(dz_{i_1}(D))};$$

(v) if $D(i_1, i_2, \ldots, i_{r-1})$ and $z(i_1, i_2, \ldots, i_{r-1})$ have been defined, then

$$z(i_1, i_2, \ldots, i_r) = z(i_1, i_2, \ldots, i_{r-1}) + z_{i_r}(D(i_1, i_2, \ldots, i_{r-1}))$$

and

$$D(i_1, i_2, \ldots, i_r) = \frac{A/I}{z(i_1, i_2, \ldots, i_r)} \otimes_{A/I} \frac{D(i_1, i_2, \ldots, i_{r-1})}{(dz(i_1, i_2, \ldots, i_r))}.$$

DEFINITION 1.2. — If A, I, D, d are as in Definition 1.1 and $i_1, i_2, \ldots, i_r$ is a sequence of non negative integers, we define a subchema of Spec (A/I) which we shall denote by $\Sigma(I; i_1, i_2, \ldots, i_r)$ as follows:

(i) the support of $\Sigma(I; i_1, i_2, \ldots, i_r)$ in the underlying topological space, consists of all $p \in \text{Spec}(A/I)$ such that $p \supset z(i_1, \ldots, i_r)$ and such that $p \not\supset z(i_1, \ldots, i_j + 1)$ for $1 \leqq j \leqq r$.

(ii) the structure sheaf of $\Sigma(I; i_1, i_2, \ldots, i_r)$ is the sheaf of rings induced on $\Sigma(I; i_1, i_2, \ldots, i_r)$ by the ring (i. e. A/I-algebra) $(A/I)/z(i_1, i_2, \ldots, i_r)$.

2. φ^* frames

In order to present the central result of Section 7, we must introduce a lemma due to Mount and Villamayor. For this purpose we shall give some definitions that make clear the meaning of the lemma. All the definitions and results of this section are in [6].

As before, k is a field of characteristic zero; $P_k(n) = k[[x_1, \ldots, X_n]]$, the power series ring in n indeterminates; $r_k(n) = (x_1, \ldots, x_n)$, the maximal ideal of $P_k(n)$; $P_k(n/m) = P_k(n)/r_k(n)^{m+1}$, the ring of truncated power series and $r_k(n/m)$ its radical.

$\rho(m', m) : P_k(n/m) \to P_k(n/m')$ if $m' < m$, and $\rho(m) : P_k(n) \to P_k(n/m)$ will be the canonical projections.

DEFINITION 2.1. — If $Z_1, \ldots, Z_n \in P_k(n/m)$ are such that $P_k(n/m) = k[Z_1, \ldots, Z_n]$ (the k-subalgebra generated by the Z_i) then the ordered set $\{Z_1, \ldots, Z_n\}$ will be called a frame for $P_k(n/m)$.

If $Z_1, \ldots, Z_r$ are elements of $P_k(n/m)$ so that $\{Z_1, \ldots, Z_r, Z_{r_1}, \ldots, Z_n\}$ is a frame for $P_k(n/m)$, then $\{Z_1, \ldots, Z_r\}$ will be called a partial frame for $P_k(n/m)$.

DEFINITION 2.2. — Suppose that $I \subset r_k(n/m)$ is an ideal of $P_k(n/m)$.

We shall define by induction the concept of a $\varphi^* - \lambda$ pair $(Z_1, \ldots, Z_{\lambda(m)}; \lambda(1), \ldots, \lambda(m))$ for I where $Z_1, \ldots, Z_{\lambda(m)}$ is a partial frame for $P_k(n/m)$ and $0 \leqq \lambda(1) \leqq \ldots \leqq \lambda(m)$ is a sequence of integers.

The sequence $\lambda(1), \ldots, \lambda(m)$ will be called the λ-sequence of I.

The definition will be by induction on m.

Case $m = 1$. — If $I = (0)$, then a pair $(\Phi; 0)$ where Φ is the empty frame and 0 is the sequence $\lambda(1) = 0$ is the only $\varphi^* - \lambda$ pair.

If $I \neq (0)$, then a pair $(Z_1, \ldots, Z_j; j)$, where $Z_1, \ldots, Z_j$ is a k-basis for I and $j = \lambda(1)$ is the λ-sequence, is a $\varphi^* - \lambda$ pair.

Case $m > 1$. — Assume that the concept of $\varphi^* - \lambda$ pair has been defined for each ideal $I \subset r_k(n/m-1)$. If $I = (0)$, then the only $\varphi^* - \lambda$ pairs are $(\Phi; 0, \ldots, 0)$ where Φ is the empty partial frame and $\lambda(1) = 0 = \ldots = 0 = \lambda(m)$ is the λ-sequence. If $I \neq (0)$, then a pair $(Z_1, \ldots, Z_{\lambda(m)}; \lambda(1), \ldots, \lambda(m))$, $n \geqq \lambda(m)$, is said to be a $\varphi^* - \lambda$ pair for I iff:

(i) the pair $(\rho(m-1, m) Z_1, \ldots, \rho(m-1, m) Z_{\lambda(m-1)}; \lambda(1), \ldots, \lambda(m-1))$ is a $\varphi^* - \lambda$ pair for $\rho(m-1, m) I$;

(ii) $I \subset (Z_1, \ldots, Z_{\lambda(1)}) + (Z_1, \ldots, Z_{\lambda(2)})^2 + \ldots + (Z_1, \ldots, Z_{\lambda(m)})^m$;

(iii) $\lambda(m)$ is the smallest integer among pair $(Z_1, \ldots, Z_{\lambda(m)} : \lambda(1), \ldots, \lambda(m)$ satisfying conditions (i) and (ii).

THEOREM 2.3. — *If* I *is an ideal of* $P_k(n/m)$ *contained in* $r_k(n/m)$, *then there is a* $\varphi^* - \lambda$ *pair for i. Furthermore if* $(Z_1, \ldots, Z_{\lambda(m)}; \lambda(1), \ldots, \lambda(m))$ *and* $(w_1, \ldots, w_{\sigma(m)}; \sigma(1), \ldots, \sigma(m))$ *are two* $\varphi^* - \lambda$ *pairs for* I, *then* $\lambda(i) = \sigma(i)$, $1 \leqq i \leqq m$.

For the proof *see* [6].

DEFINITION 2.4. — If $I \subset r_k(n/m)$ is an ideal of $P_k(n/m)$, and if $(x_1, \ldots, x_{\lambda(m)}; \lambda(1), \ldots, \lambda(m))$ is a $\varphi^* - \lambda$ pair for I, then $x_1, \ldots, x_{\lambda(m)}$ will be called a φ^*-frame for I and the integers $(n-\lambda(1), \ldots, n-\lambda(m)) = (\varphi(1), \ldots, \varphi(m))$ will be called the φ-invariants of I.

We shall denote by $\theta(1)$ the first integer such that $\lambda(\theta(1)) \neq 0$, and by $\theta(j)$ the first integer such that $\lambda(\theta(j)) > \lambda(\theta(j-1))$.

Remark. — If σ is a k-automorphism of $P_k(n/m)$, then it is clear that $\sigma(I)$ and I have the same λ-sequence.

Furthermore if $x_1, \ldots, x_{\lambda\theta(t)}$ is a φ^*-frame for I, then $\sigma(x_1), \ldots, \sigma(x_{\lambda\theta(t)})$ is a φ^*-frame for $\sigma(I)$.

DEFINITION 2.5. — Suppose now that $I \subset r_k(n)$ is an ideal in $P_k(n)$. We shall say that I has φ-sequence $(\varphi(1), \ldots, \varphi(m), \ldots)$ if for each $m \geqq 1$ the sequence $(\varphi(1), \ldots, \varphi(m))$ is the φ-sequence for $\rho(m) I \subset P_k(n/m)$. Similarly for the λ-sequence of I.

It is clear that each ideal contained in the radical of $P_k(n)$ has a uniquely determined sequence. Note also that for an ideal I in the radical of $P_k(n)$ the φ-sequence for I is decreasing. Because $\varphi(i)$ are bounded below by zero, the φ-sequence for I must be constant for sufficiently large values of i. It is thus clear that for sufficiently large m, the jump sequences for the $\rho(m) I$ are all the same.

DEFINITION 2.6. — Suppose that $I \subset r_k(n)$ is an ideal of $P_k(n)$.

If for each $m \geqq M$, $\varphi(m) = \varphi(m+1) = \ldots$ then we shall call the jump sequence for $\rho(M) I$ in $P_k(n/m)$ the jump sequence for I.

Furthermore we shall call a partial frame $Z_{\lambda(1)}, \ldots, Z_{\lambda(M)}$ of $P_k(n)$ a φ^*-frame for I if $\rho(m) Z_1, \ldots, \rho(m) Z_{\lambda(m)}$ is a φ^*-frame for $\rho(m) I$ for each $m \geqq M$.

The following lemma is fundamental in what follows.

LEMMA 2.7. — *Suppose that* $I \subset r_k(n)^2$ *is an ideal in* $P_k(n)$ *with* λ*-sequence* $\lambda(1), \lambda(2), \ldots, \lambda(m), \ldots$ *and jump numbers* $\theta(1), \ldots, \theta(t)$.

There exists a φ^*-frame $x_1, \ldots, x_{\mu(t)}$ (where $\mu(t) = \lambda\theta(t)$) for I which is part of a frame $x_1, \ldots, x_n$ for $P(n)$ such that the following conditions are satisfied for all $1 \leqq c \leqq t$.

$(D_c - 1)$: there are generators $f_1, \ldots, f_s$ for I such that $f_i = \sum_{d=2}^{\infty} F_{id}$, where F_{id} is a homogeneous polynomial of degree d in $x_1, \ldots, x_n$.

$(D_c - 2)$: for each jump number $\theta(b)$ write

$$F_{i\theta(b)} = F^*_{i\theta(b)} + F^{**}_{i\theta(b)}$$

where $F^*_{i\theta(b)} \in (x_1, \ldots, x_{\mu(b)})^{\theta(b)}$ and where $F^*_{i\theta(b)}$ contains no monomial in

$$(x_1, \ldots, x_{\mu(1)})^{\theta(1)} + \ldots + (x_1, \ldots, x_{\mu(b-1)})^{\theta(b-1)}.$$

If $b \leqq c$, there is a set of row indices

$$S_b = \{(1, m^{(1)}_{\theta(b)}(1)), \ldots, (1, m^{(1)}_{\theta(b)}(i_1)), \ldots, (s, m^{(s)}_{\theta(b)}(1)), \ldots, (s, m^{(s)}_{\theta(b)}(i_s))\},$$
$$i_1 + i_2 + \ldots + i_s = \mu(b) - \mu(b-1),$$

such that the submatrix of the polarization matrix of the forms $F^*_{1\theta(b)}, \ldots, F^*_{s\theta(b)}$ with column indices $x_{\mu(b-1)+1}, \ldots, x_{\mu(b)}$ and with row indices the entries in S_b, is invertible.

Note. — If no row index for f_1 occurs in S_b we set $(i, m^{(i)}_{\theta(b)}(.)) = (i, 0)$.

$(D_c - 3)$: if M is a monomial of degree $\theta(d)$ for $1 \leqq d \leqq t$ and M occurs with nonzero coefficient in $F_{i\theta(d)}$ such that M is divisible by $m^{(i)}_{\theta(b)}(v)$ for some $b \leqq c$, then $M \in (x_1, \ldots, x_{\mu(1)})^{\theta(1)} + \ldots + (x_1, \ldots, x_{\mu(d-1)})^{\theta(d-1)}$.

In view of this latter result, we shall give the following.

DEFINITION 2.8. — Suppose $I \subset r_k(n)^2$ is an ideal of $P_k(n)$.

If I has λ-sequence $\lambda(1), \ldots, \lambda(j), \ldots$ and jump numbers $\theta(1), \ldots, \theta(t)$, then a φ^*-frame for I and a set of generators $f_1, \ldots, f_s$ for I which satisfy $(D_c - 1), \ldots, (D_c - 3)$ for all $1 \leqq c \leqq t$ will be called a prepared φ^*-frame for I and a prepared set of generators for I with respect to the frame.

An element $(j, m) \in S_i$ will be called a distinguished row index for the prepared generators, and the sets $S_1, \ldots, S_t$ will be called the sets of distinguished indices for the prepared φ^*-frame $x_1, \ldots, x_{\mu(t)}$ and prepared generators $f_1, \ldots, f_s$.

The idea of the proof of the Lemma 2.7 is the following:

Start with a frame $\{y_1, \ldots, y_n\}$ for $P_k(n)$ and a system $f_1, \ldots, f_s$ of generators for I, written in terms of the y_i.

From the polarization matrix of the forms of minimal degree of the f_i (may be that some form is zero), is obtained the first set of distinguished indices.

Then, if $m^{(i)}_{\theta(1)}(v)$ is one of the distinguished monomials, every monomial multiple of $m^{(i)}_{\theta(1)}(v)$ is "cleared" from $F^*_{i\theta(2)}, \ldots, F^*_{i\theta(t)}$ by means of automorphisms σ of $P_k(n)$ which are the identity on $y_j, j > \mu(1)$, obtaining that in $\sigma(I)$ holds D_1-1, D_1-2, D_1-3, with respect to the frame $\{y_1, \ldots, y_n\}$.

This process is iterated and, after a finite number of stages, it is obtained an automorphism σ of $P_k(n)$ (of a very particular form) such that conditions D_c-1, D_c-2, D_c-3 are satisfied, $1 \leqq c \leqq t$, by the ideal $\sigma(I)$, generators $\sigma(f_1), \ldots, \sigma(f_s)$ and the frame $\{y_1, \ldots, y_n\}$.

The φ^*-frame required is the $\sigma^{-1}(y_1), \ldots, \sigma^{-1}(y_{\mu(t)})$.

3. Goods frames for jacobian chains

In this section we follow [2].

Following J. Mather we may define:

DEFINITION 3.1. – Set $A = k[x_1, \ldots, x_n]$ as in Section 1.

Set $m \subset A$ be a maximal ideal and $I \subset m$ an ideal.

We define $\operatorname{rank}_m(I) = \dim_k \bar{I}$, where $\bar{I} = \operatorname{Im}(\Pi/I)$ and $\Pi : m \to m/m^2$ is the canonical projection. It is easy to see that $\bar{I} \simeq (I+m^2)/m^2$ as k-vector spaces.

When no confusion can arrive we will put rank (I) for $\operatorname{rank}_m(I)$.

DEFINITION 3.2. – Let $m \subset A$ be a maximal ideal.

A frame of m will be any ordered set $\{Z_1, \ldots, Z_n\} \subset m$ such that $(Z_1, \ldots, Z_n) = m$ A, where $(Z_1, \ldots, Z_n)$ stands for the ideal generated by the Z_i.

DEFINITION 3.3. – Let $I \subset m \subset A$ as in Definition 3.1.

$\delta_m I = I + I'$, where I' is the ideal of A generated by the determinants of dimension $(\operatorname{rank}_m I+1)$ of the Jacobian matrix of I; i. e. the matrix $\partial f_\alpha/\partial x_i$, $f_\alpha \in I$, $i = 1, 2, \ldots, n$.

We put $\delta^k I = \delta(\delta^{k-1} I)$ and $\delta^0 I = I$ (where $\delta = \delta_m$).

It is clear that $\delta^{k-1} I \subseteq \delta^k I \subseteq m$, $k \geqq 1$. The chain $\delta^i_m I$ will be called the Jacobian chain of I in m.

DEFINITION 3.4. – Suppose $I \subset m \subset A$ as in Definition 3.1.

We shall say that the ordered set $\{y_1, \ldots, y_n\}$ is a good frame of m for the chain $\{\delta^{i-1}_m(I), i = 1, 2, \ldots\}$ if $\{y_1, \ldots, y_n\}$ is a frame of m and if $\{y_1, \ldots, y_{s_i}\} \subset \delta^{i-1}_m(I)$, $i = 1, 2, \ldots$ where $s_i = \operatorname{rank}_m(\delta^{i-1} I)$.

THEOREM 3.5. – *If $\{y_1, \ldots, y_n\}$ is a good frame for the Jacobian chain of* I *in m and if $s_i = \operatorname{rank}(\delta^{i-1} I)$, $i = 1, 2, \ldots$, then*

(i) $$I \subset (y_1, \ldots, y_{s_1}) + (y_1, \ldots, y_{s_2})^2 + \ldots + (y_1, \ldots, y_{s_k})^k + m^{k+1};$$

(ii) *if* $\{Z_1, \ldots, Z_n\}$ *is another frame of* m *and holds an inclusion of the form*

$$I \subset (Z_1, \ldots, Z_{t_1}) + (Z_1, \ldots, Z_{t_2})^2 + \ldots + (Z_1, \ldots, Z_{t_k})^k + m^{k+1}$$

then $(s_1, \ldots, s_k) \leqq (t_1, \ldots, t_k)$ *in the lexicographic order.*

Proof [2].

4. Relation between φ*—frames and good frames

In this section δ will denote δ_m.

LEMMA 4.1. – *Let* $I = (f_1, \ldots, f_s) \subset r_k(n)$ *be an ideal of* $P_k(n)$ *with* λ*-sequence* $\lambda(1)$, $\lambda(2)$, ... *and jump numbers* $\theta(1), \ldots, \theta(t)$.

Let $\{y_1, \ldots, y_n\}$ *be a good frame of* $r_k(n)$ *for the Jacobian chain of* I.

Then there exists $\mu(t)$ *polynomials* $P_1, \ldots, P_{\mu(t)}$ *(with coefficients in* k*) depending on* $\mu(t)$ *indeterminates and a prepared* φ^**-frame for* I *and the generators* $f_1, \ldots, f_s$ *such that*

$$(\bigstar) \qquad y_j = P_j(x_1, \ldots, x_{\mu(t)}), \qquad j = 1, 2, \ldots, \mu(t).$$

Furthermore: the Jacobian matrix of the system ($\bigstar$) *is a unit in* P (n).

Proof. – Being $\{y_1, \ldots, y_{\mu(t)}\}$ a φ^*-frame for I (after Theorem 3.5) we have

$$I \subset (y_1, \ldots, y_{\mu(1)})^{\theta(1)} + \ldots + (y_1, \ldots, y_{\mu(t)})^{\theta(t)}.$$

Then if $f_i = \sum f_{id}$ $(i = 1, 2, \ldots, s)$ with f_{id} the homogeneous component of degree d (in terms of the y_i) we have that for $1 \leqq i \leqq s$:

$$\begin{aligned}
&f_{i\theta(1)}, \ldots, f_{i\theta(2)-1} \in (y_1, \ldots, y_{\mu(1)})^{\theta(1)} \\
&f_{i\theta(2)}, \ldots, f_{i\theta(3)-1} \in (y_1, \ldots, y_{\mu(1)})^{\theta(1)} + (y_1, \ldots, y_{\mu(2)})^{\theta(2)} \\
&\vdots \\
&f_{i\theta(t-1)}, \ldots, f_{i\theta(t)-1} \in (y_1, \ldots, y_{\mu(1)})^{\theta(1)} + \ldots + (y_1, \ldots, y_{-1})^{\theta(t-1)} \\
&f_{id} \in (y_1, \ldots, y_{\mu(1)})^{\theta(1)} + \ldots + (y_1, \ldots, y_{\mu(t)})^{\theta(t)},\ d \geqq \theta(t),
\end{aligned}$$

by computing degrees and the fact that I is contained in an ideal, not only homogeneous, but generated by monomials.

We will put: $f_{ia} = f_{ia}^* + f_{ia}^{**}$ where

(i) if $\theta(b) < a < \theta(b+1)$ (i. e. a is not a jump number) then $f_{ia}^* = 0$ and $f_{ia}^{**} = f_{ia}$.

(ii) if $a = \theta(b)$, then $f_{ia}^{**} \in (y_1, \ldots, y_{\mu(1)})^{\theta(1)} + \ldots + (y_1, \ldots, y_{\mu(b-1)})^{\theta(b-1)}$ and $f_{ia}^* \in (y_1, \ldots, y_{\mu(b)})^{\theta(b)}$, where f_{ia}^* consists of the linear combination of all monomials of f_{ia} which are not in $(y_1, \ldots, y_{\mu(1)})^{\theta(1)} + \ldots + (y_1, \ldots, y_{\mu(b-1)})^{\theta(b-1)}$.

So, in particular, in $f_{i\theta(b)}^*$ appear only the variables $y_1, \ldots, y_{\mu(b)}$ $(i = 1, 2, \ldots, s)$.

From the proof of Lemma 2.7 applied to this situation, we can conclude that the monomials that appear in the set S_b are monomials (of degree $\theta(b)-1$) in $y_1, \ldots, y_{\mu(b)}$, $b = 1, 2, \ldots, t$.

If in $f^*_{i\theta(b)}$ appears the monomial M multiple of $m^{(i)}_{\theta(c)}(j)$ (supposing $(i, m^{(i)}_{\theta(c)}(j)) \in S_c$, for some $c < b$) then $M = m^{(i)}_{\theta(c)}(j)$. Q, where Q is a monomial in $y_{\mu(c)1}, \ldots, y_{\mu(b)}$, as M was a monomial of $f^*_{i\theta(b)}$ and if in Q appear a variable of indice $\leqq \mu(c)$ then $M \in (y_1, \ldots, y_{\mu(c)})^{\theta(c)}$, which contradicts the definition of $f^*_{i\theta(b)}$.

This monomial is "cleaned" by means of an automorphism σ of $P_k(n)$ of the form:

$$\begin{cases} \sigma(y_i) = y_i + a_i Q, & \mu(c-1) < i \leqq \mu(c) \quad \text{where } a_i \in k \text{ convenient.} \\ \sigma(y_i) = y_i, & \text{in other case.} \end{cases}$$

After a finite number of stages we obtain a $\sigma \in \operatorname{Aut}(P_k(n))$, composition of all the ones used in the process of elimination of multiples of the distinguished monomials in the components $f^*_{i\theta(b)}$, such that $\sigma(I)$ and $\sigma(f_1), \ldots, \sigma(f_s)$ are prepared with respect to $y_1, \ldots, y_{\mu(t)}$ and

$$(\bigstar\bigstar) \qquad \begin{cases} \sigma(y_j) = y_j + \pi_j(y_{\mu(b+1)-1}, \ldots, y_{\mu(t)}), & \text{if } \mu(b) < j \leqq \mu(b+1), \\ \pi_j \text{ polynomials } 0 \leqq b \leqq t-1 & [\text{Put } \mu(0) = 0]. \\ \sigma(y_i) = y_j, & j \geqq \mu(t). \end{cases}$$

As noted before, putting $x_j = \sigma^{-1}(y_j)$, $j = 1, \ldots, \mu(t)$, I and $f_1, \ldots, f_s$ became prepared with respect to the φ^*-frame $\{x_1, \ldots, x_{\mu(t)}\}$ and we have $y_j = P_j(x_1, \ldots, x_{\mu(t)})$, $j = 1, 2, \ldots, \mu(t)$.

It is also clear from $(\bigstar\bigstar)$ that the Jacobian matrix

$$\frac{\partial y_1, \ldots, y_{\mu(t)}}{\partial x_1, \ldots, x_{\mu(t)}}$$

is a unit in $P_k(n)$, and the proof is complete.

COROLLARY 4.2. — *Under the same assumptions that in lemma* 4.1, *we have*

$$\{x_1, \ldots, x_{\mu(t)}\} \subset \delta^{\theta(t)-1} I.$$

Proof. — It follows from the following facts:

(i) $\{y_1, \ldots, y_{\mu(t)}\} \subset \delta^{\theta(t)-1} I$, by definition of a good frame;

(ii) The Jacobian matrix of $(\bigstar)$ is invertible and so it is possible to obtain the x_j as power series in $y_1, \ldots, y_{\mu(t)}$;

(iii) Every ideal of $P_k(n)$ is closed under the operation of limits (in the m-adic topology) as $P_k(n)$ is a Zariski ring.

5. Jacobian chains and the schemas $\Sigma(I; i_1, i_2, \ldots)$

As always $A = k[x_1, \ldots, x_n]$, with k algebraically closed field of characteristic zero.

Let $I \subset A$ be an ideal, say $I = (f_1, \ldots, f_k)$.

Let $\lambda : A \to A/I$ be the canonical mapping.

Consider the following sequence of A/I-modules induced by λ:

$$I/I^2 \underset{\alpha}{\rightarrow} A/I \underset{A}{\otimes} D(A/k) \underset{d\lambda}{\rightarrow} D([A/I]/k) \rightarrow 0 \quad (\bigstar)$$

where

$$\alpha(\bar{f}) = 1 \otimes d_A f, \qquad \text{if} \quad \bar{f} \in I/I^2, \qquad f \in I$$

and

$$d\lambda(1 \otimes d_A f) = d_{A/I}(\lambda f).$$

($\bigstar$) is and exact sequence in every localization at a maximal ideal of A/I (*see* [7]). Then it is an exact sequence of A/I-modules.

It is well known that $D(A/k)$ is a free A-module with basis $d_A x_1, \ldots, d_A x_n$.

Furthermore: $\operatorname{Im}(\alpha)$ is the A/I-submodule of $A/I \underset{A}{\otimes} D(A/k)$ generated by $1 \otimes d_A f_1$, $i = 1, 2, \ldots, h$.

The matrix of relations of $D([A/I]/k)$ is then $\lambda(\partial f_i/\partial x_j)$, as we have:

$$1 \otimes d_A f_i = \sum_{j=1}^{n} \lambda\left(\frac{\partial f_i}{\partial x_j}\right)(1 \otimes d_A x_j), \qquad i = 1, 2, \ldots, h,$$

i. e. it is the Jacobian matrix of I, mod I.

If $m \supset I$ is a maximal ideal of A, localizing ($\bigstar$) we σ et the following exact sequence of A_m/IA_m-modules:

$$I/I^2 \underset{\alpha}{\rightarrow} A_m/I \underset{A_m}{\otimes} D(A_m/k) \underset{d\lambda}{\rightarrow} D([A_m/I]/k) \rightarrow 0 \quad (\bigstar\bigstar)$$

where I denotes IA_m.

Since k is the residual field of A and A_m/I, if we apply $k \underset{A_m/I}{\otimes}$ to ($\bigstar\bigstar$) we get following exact sequence of k-vector spaces:

$$k \underset{A_m/I}{\otimes} I/I^2 \underset{1\otimes\alpha}{\longrightarrow} m/m^2 \rightarrow k \underset{A_m/I}{\otimes} D([A/I]/k) \rightarrow 0,$$

because $k \underset{A_m}{\otimes} D(A_m/k) \simeq m/m^2$ (*see* [7] for example).

Note that $\operatorname{Im}(1 \otimes \alpha) = \bar{I} = \operatorname{Im}(\Pi/I)$, where $\Pi : m \rightarrow m/m^2$.

This shows that:

$$\operatorname{rank}_m I + r\,k_m(D([A/I]/k)) = n \quad (\bigstar\bigstar\bigstar)$$

where $rk_m(M)$ denotes the rank of the module $A_m \otimes M$ over the local ring A_m, i. e. $\dim_k (k \underset{A}{\otimes} M)$.

If $F_j \subset A/I$ are the Fitting invariants of the A/I-module $D([A/I]/k)$, from the definition of the schemas $\Sigma(I; i_1, i_2, \ldots)$ one can see that if $m \in \Sigma(I; i_1, i_2, \ldots)$:

$$i_1 = \max\{j : F_j \subset m/I\} + 1,$$

and so

$$i_1 = r\,k_m(\mathrm{D}([\mathrm{A}/\mathrm{I}]/k)).$$

Then if $s_1 = \mathrm{rank}_m\,\mathrm{I}$, we obtain from (★★★) the equality

$$s_1 + i_1 = n \quad (★★★★)$$

After (★★★★) it is easy to show that the following proposition holds:

PROPOSITION 5.1:

$$\lambda^{-1}(\mathrm{F}_{i_1-1}) = \delta_m \mathrm{I}.$$

Repeating the arguments to the situation $\mathrm{A} \xrightarrow{\lambda} \mathrm{A}/\delta_m \mathrm{I}$ we get:

(1) $$s_2 + i_2 = n,$$

where

$$s_2 = \mathrm{rank}_m(\delta_m \mathrm{I}), \qquad i_2 = r\,k_m(\mathrm{D}([\mathrm{A}/\delta_m \mathrm{I}]/k),$$

and

(2) $$\lambda^{-1}(\mathrm{F}_{i_2-1}) = \delta_m^2 \mathrm{I},$$

where F_{i_2-1} is the (i_2-1)th Fitting invariant of the $\mathrm{A}/\delta_m\,\mathrm{I}$-module $\mathrm{D}([\mathrm{A}/\delta_m\,\mathrm{I}]/k)$.

More generally, we can state the following:

PROPOSITION 5.2. — *If* $m \in \Sigma(\mathrm{I}; i_1, i_2, \ldots)$ *then for every* $\nu \geqq 1$ *we have:*

(1) $$s_\nu + i_\nu = n, \qquad s_\nu = \mathrm{rank}_m(\delta_m^{\nu-1}\mathrm{I}), \qquad i_\nu = rg_m(\mathrm{D}([\mathrm{A}/\delta_m^{\nu-1}\mathrm{I}]/k));$$

(2) $$\delta_m^\nu \mathrm{I} = \lambda^{-1}(\mathrm{F}_{i_\nu-1}),$$

where $\mathrm{F}_{i_\nu-1}$ *denotes the* $(i_\nu-1)$th *Fitting invariant of the* $\mathrm{A}/\delta_m^{\nu-1}\,\mathrm{I}$-*module* $\mathrm{D}([\mathrm{A}/\delta_m^{\nu-1}\,\mathrm{I}]/k)$ *and* $\lambda : \mathrm{A} \to \mathrm{A}/\delta^\nu\,\mathrm{I}$ *is the canonical map.*

COROLLARY 5.3. — *If* m, m' *are maximal ideals of* A *containing* I *and* $m, m' \in \Sigma(\mathrm{I}; i_1, i_2, \ldots)$ *then* $\delta_m^k(\mathrm{I}) = \delta_{m'}^k(\mathrm{I})$, $k \geqq 0$.

COROLLARY 5.4. — *If* $m \in \Sigma(\mathrm{I}; i_1, i_2, \ldots)$ *then the Jacobian chain* $\{\delta_m^k \mathrm{I}\}$ *of* I *is in every other maximal point* m' *of* $\Sigma(\mathrm{I}; i_1, i_2, \ldots)$.

6. A lemma of globalization

LEMMA 6.1. — *Let* m *be a maximal point of* Spec (A) *such that* $m \in \Sigma(\mathrm{I}; i_1, i_2, \ldots)$.

Let $\{y_1, y_2, \ldots, y_n\}$ *be a good frame of* m *for the Jacobian chain of* I. *Suppose that the chain stops at* $\delta_m^{k-1}\,\mathrm{I}$ *and that* $s_i = \mathrm{rank}_m(\delta_m^{i-1}\,\mathrm{I})$.

Then there exists an open neighbourhood V *of* m *in* $\Sigma(\mathrm{I}; i_1, i_2, \ldots)$ *such that* $\{y_1, \ldots, y_{s_k}\}$ *is part of a good frame of* m' *for the Jacobian chain of* I, *for every* $m' \in \mathrm{V}$.

Proof. — By corollary 5.4 $\{y_1, \ldots, y_{s_k}\} \subset m'$, for every $m' \in \Sigma(\mathrm{I}; i_1, i_2, \ldots)$.

It is clear that $\{y_1, \ldots, y_{s_k}\}$ will be part of a frame of m' if and only if $\{\bar{y}_1, \ldots, \bar{y}_{s_k}\}$ are k-independent in m'/m'^2.

If $\mathrm{P}' = (\alpha'_1, \ldots, \alpha'_n) \in k^n$ represents the maximal ideal m', the coordinates of the $\bar{y}_i$ in m'/m'^2 will be:

$$(\bigstar) \qquad \left(\frac{\partial y_i}{\partial x_1}(p'), \ldots, \frac{\partial y_i}{\partial x_n}(p')\right),$$

in the basis $x'_i = (x_i - \alpha'_i)$.

Therefore k-independence is given by the non-vanishing of a certain determinant associated to a s_k-minor of the matrix that has $(\bigstar)$ as rows.

This condition determinates the required open set.

Finally, by Corollary 5.3, $\delta_m \mathrm{I} = \delta_{m'} \mathrm{I}$ and so, if $\{y_1, \ldots, y_{s_k}\}$ is part of a frame of m' then it is part of a good frame of m' for the Jacobian chain of I.

7. Central results

THEOREM 7.1. — *As before* $\mathrm{A} = k[x_1, \ldots, x_n]$.

Let $\mathrm{I} = (f_1, \ldots, f_s) \subset \mathrm{A}$ *be an ideal, and let* $m \subset \mathrm{A}$ *be a maximal ideal such that* $m \in \Sigma(\mathrm{I}; i_1, i_2, \ldots)$.

Then there exists an open neighbourhood V *of* m *in* $\Sigma(\mathrm{I}; i_1, i_2, \ldots)$ *and a* φ^**-frame* $\{x_1, \ldots, x_{\mu(t)}\}$ *for* $\mathrm{I}\hat{\mathrm{A}}_m$ *that prepares* $\mathrm{I}\hat{\mathrm{A}}_m$ *and the generators* $f_1, \ldots, f_s$ *such that* $\{x_1, \ldots, x_{\mu(t)}\}$ *is a* φ^**-frame for* $\mathrm{I}\hat{\mathrm{A}}_{m'}$ *that prepares* $\mathrm{I}\hat{\mathrm{A}}_{m'}$ *and the generators* $f_1, \ldots, f_s$ *for every other maximal* $m' \in \mathrm{V}$.

Note. — When $m \subset \mathrm{A}$ is a maximal ideal, $\hat{\mathrm{A}}_m$ denotes the completion of A in the $m\,\mathrm{A}_m$-adic topology.

Proof. — Let $\{y_1, \ldots, y_n\}$ be a good frame of m for the Jacobian chain of I.

By Lemma 6.1, there exists an open set v (containing m) of $\Sigma(\mathrm{I}; i_1, \ldots)$ where $\{y_1, \ldots, y_{\mu(t)}\}$ is part of a good frame of m', $m' \in v$.

By Corollary 5.4, $\{y_1, \ldots, y_{\mu(t)}\} \subset m'$, $m' \in \Sigma(\mathrm{I}; i_1, i_2, \ldots)$.

By Lemma 4.1, there exists a φ^*-frame $\{x_1, \ldots, x_{\mu(t)}\}$ in $\hat{\mathrm{A}}_m$ that prepares $\mathrm{I}\hat{\mathrm{A}}_m$ and $f_1, \ldots, f_s$, and $\mu(t)$ polynomials P_j such that

$$(\bigstar) \qquad y_j = \mathrm{P}_j(x_1, \ldots, x_{\mu(t)}), \qquad j = 1, 2, \ldots, \mu(t).$$

Let $j(x_1, \ldots, x_{\mu(t)})$ be the Jacobian of the system $(\bigstar)$.

It's clear that $j(x_1, \ldots, x_{\mu(t)}) \in \mathrm{A}$.

Let $m' \in \Sigma(\mathrm{I}; i_1, i_2, \ldots)$ be such that $j(x_1, \ldots, x_{\mu(t)}) \notin m'$.

As the "preparation" of $\mathrm{I}\hat{\mathrm{A}}_{m'}$ can be carried out in a similar way as it was done in Lemma 4.1 for $\mathrm{I}\hat{\mathrm{A}}_{m'}$ (starting in both cases from the partial frame $\{y_1, \ldots, y_{\mu(t)}\}$), let us see that the process of "cleaning" in $\hat{\mathrm{A}}_{m'}$ is identic to that of $\hat{\mathrm{A}}_m$.

For this purpose it suffices to observe that:

(i) $f'_{i\theta(b)}$ has the same expression in $\hat{A}_{m'}$, as it depends only on the variables $y_1, \ldots, y_{\mu(t)}$.

(ii) Although the other components will vary respect to some other frame of $\hat{A}_{m'}$, say $\{y_1, \ldots, y_{\mu(t)}, y'_{\mu(t)1}, \ldots, y'_n\}$, the same relations will hold, as those that existed in $\hat{A}_m$, for in the chosen frame are $y_1, \ldots, y_{\mu(t)}$.

More explicitly if $f_i = \sum G_{ia}$, G_{ia} component of degree a in the variables $\{y_1, \ldots, y_{\mu(t)}, y'_{\mu(t)1}, \ldots, y'_n\}$ then:

(*a*) $$G^*_{i\theta(b)} = F^*_{i\theta(b)}, \qquad b = 1, 2, \ldots, t;$$

(*b*) $$G^{**}_{i\theta(b)} \in (y_1, \ldots, y_{\mu(1)})^{\theta(1)} + \ldots + (y_1, \ldots, y_{\mu(b-1)})^{\theta(b-1)};$$

(*c*) $$G_{ia} \in (y_1, \ldots, y_{\mu(1)})^{\theta(1)} + \ldots + (y_1, \ldots, y_{\mu(b-1)})^{\theta(b-1)}, \qquad \text{if} \quad \theta(b) < a < \theta(b+1),$$

then: the polarization matrix, the distinguished rows and the automorphisms of the process in $A_{m'}$, can be chosen in the same way to that done in $\hat{A}_m$.

In particular, a frame $\{x'_1, \ldots, x'_{\mu(t)}\}$ that prepares $I\hat{A}_{m'}$ and the generators $f_1, \ldots, f_s$ and the same polynomials P_j are obtained such that $y_j = P_j(x'_1, \ldots, x'_{\mu(t)})$.

Then $x_i = x'_i$, $i = 1, 2, \ldots, \mu(t)$ and this holds at every $m' \in V$ such that $j(x_1, \ldots, x_{\mu(t)}) \notin m'$, clearly this conditions determines the open set and the theorem follows.

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