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NON-TRIVIAL CHARACTERISTIC INVARIANTS OF HOMOGENEOUS FOLIATED BUNDLES

BY FRANZ W. KAMBER (*) AND PHILIPPE TONDEUR (*)

A Monsieur Henri Cartan
pour son soixante-dixième anniversaire.

ABSTRACT. — The authors have given a construction of characteristic classes for foliated bundles which generalizes the Chern-Weil construction for the characteristic classes of principal bundles. In this paper these classes are evaluated for the particular case of locally homogeneous foliated bundles. The generalized characteristic homomorphism is then the composition of three maps. The first of these maps is associated to the representation defining a foliated vector bundle from a foliated principal bundle. The second map is expressible purely in terms of relative Lie algebra and Weil algebra cohomology. The third map can be interpreted as the characteristic homomorphism of a flat bundle. The computation of the characteristic homomorphism reduces then to purely algebraic problems which can be solved with the methods discussed in earlier papers. Many non-trivial realizations of secondary characteristic invariants in various geometric contexts are given.

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1. Introduction

The authors have given in [22] to [25] a construction of characteristic classes for foliated bundles which generalizes the Chern-Weil construction for the characteristic classes of principal bundles. A foliated bundle is a principal bundle $P \to M$ with a foliation on the base space M and a partial connection on P (defined along the leaves of the foliation only) which has zero curvature. In other words: the horizontal spaces of the partial connection

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define a foliation of P projecting onto the foliation of M. This concept allows the simultaneous discussion of characteristic invariants for ordinary bundles, bundles with an infinitesimal or global group action, flat bundles and normal bundles of foliations.

This particular point of view concerning secondary characteristic invariants of foliations ([24], [25]) is one of the various independently discovered approaches to this topic by Godbillon-Vey [12], Bott-Haefliger ([5], [7], [15]), Bernstein-Rosenfeld [2], Malgrange and the authors. An account of the developments leading to the authors point of view can be found in the introduction to [25] and in the lecture notes [25′].

In this paper we evaluate the generalized characteristic homomorphism for the particular case of locally homogeneous foliated bundles. The computation reduces then to purely algebraic problems which can be solved with the methods discussed in [23]. Some results of this paper are contained in the announcement [27]. The methods here developed lead to many other applications which wil be presented in later papers.

The authors wish to emphasize how much this work on secondary invariants builds on the classical work of Koszul ([29], [30]) and Cartan-Weil [8]). That subject is presented in the forthcoming book by Greub-Halperin-Vanstone [14] (Vol. III).

The outline of the paper is as follows. In section 2 we review the concept of foliated bundle, give several examples and describe in particular locally homogeneous foliated bundles given by a Lie group $\overline{G}$, subgroups $H \subset G \subset \overline{G}$, with H closed in $\overline{G}$ and a discrete subgroup $\Gamma \subset \overline{G}$ as in (2.14). An important example is the normal bundle of the foliation of a group $\overline{G}$ by the cosets of a subgroup G. Invariant adapted connections are characterized algebraically by (2.6) and (2.8).

In section 3 we review first our construction of generalized characteristic classes [Theorems (3.4) and (3.5)]. Then we give an evaluation principle for the generalized characteristic homomorphism Δ_* for the case of locally homogeneous foliated bundles [Theorems (3.7) and (3.11)]. For the corresponding associated foliated vectorbundles the construction is detailed in Theorem (3.7′). According to these results, Δ_* is the composition of three maps. The first map is associated to the representation of (G, H) which defines a foliated vectorbundle in terms of a foliated principal bundle. The second map is expressible purely in terms of relative Lie algebra and Weil algebra cohomology. The third map is the characteristic homomorphism of a flat bundle. The rest of the paper proceeds according to this decomposition. First we study in section 4 the map Δ_* for flat bundles. Then we compute Δ_* for certain foliated principal and vector bundles in section 5 and 6. Finally we discuss in section 7 interrelations between these results. Typical applications of the computations carried out are the following results.

THEOREM 6.49. — *Let* $Q_{U(r)}$ *be the normal bundle of the foliation of* $SU(r+1)$ *by the left cosets of the unitary group* $U(r)$ *with quotient the complex projective space* $P^r\mathbf{C}$. *The image of the generalized characteristic homomorphism* $\Delta_*(Q_{U(r)})$ *in*

$$H^+(SU(r+1)) \cong \Lambda^+(\bar{y}_2, \ldots, \bar{y}_{r+1})$$

is the ideal generated by the primitive element $\bar{y}_{r+1}$, *the suspension of the top-dimensional Chern polynomial* $\bar{c}_{r+1} \in I^{2r+2}(SU(r+1))$.

This implies that $\dim \operatorname{im} \Delta_*^+ (Q_{U(r)}) = 2^{r-1}$, whereas $\dim H(SU(r+1)) = 2^r$. It further shows the abundant existence of non-trivial linearly independent secondary invariants in dimensions greater than $2r+1$.

THEOREM 6.52. — *Let* $Q_{SO(2r)}$ *be the normal bundle of the foliation of* $SO(2r+1)$ *by the left cosets of the orthogonal group* $SO(2r)$ *with quotient the sphere* S^{2r}. *The image of* $\Delta_*(Q_{SO(2r)})$ *in* $H(SO(2r+1)) \cong \Lambda(\bar{y}_1, \ldots, \bar{y}_r)$ *is the direct sum*

$$\operatorname{Id}(\bar{y}_r) \oplus \Lambda(\bar{y}_{[r/2]+1}, \ldots, \bar{y}_{r-1})$$

of the ideal generated by the suspension $\bar{y}_r$ *of the top-dimensional Pontrjagin polynomial* $\bar{p}_r \in I^{4r}(SO(2r+1))$, *and the exterior algebra generated by the primitive elements* $\bar{y}_{[r/2]+1}, \ldots, \bar{y}_{r-1}$.

Note that the foliations in both these examples are Riemannian, and the generalized characteristic homomorphism

$$H(W(\mathfrak{gl}(2r), SO(2r))_r) \to H_{DR}(M)$$

is certainly trivial. However e. g. in the last example the bundle $Q_{SO(2r)}$ with its canonical trivialization gives according to our general construction rise to a generalized characteristic homomorphism

$$\Delta_*: \quad H(W(\mathfrak{so}(2r))_r) \to H_{DR}(SO(2r+1))$$

which according to the result just stated is highly non-trivial. Moreover one obtains non-trivial realizations of rigid secondary characteristic invariants in this case.

We wish to emphasize that the functorial properties of the generalized characteristic homomorphism are immediate consequences of the general construction in [25], whereas the evaluation for the cases here discussed is efficiently done through the principles embodied in Theorems (3.7), (3.7′) and (3.11).

In section 4 the map Δ_* is evaluated for flat bundles. Non-trivial characteristic classes for flat bundles [Theorems (4.7), (4.17), (4.18)] have already occured in our earlier work ([20], [24], [25]). A further application is the following. Let G be compact and H closed in G. An H-bundle $P' \to M$ with trivial G-extension P is characterized by a homotopy class $f: M \to G/H$. The generalized characteristic homomorphism Δ_* of P is then essentially $f^*: H(G/H) \to H(M)$ [*see* (4.13) and (4.14)]. This leads e. g. to secondary invariants for real structures on a trivial complex vectorbundle [Theorem (4.15)]. The lowest dimensional of these invariants corresponds to the Maslov class entering in quantization conditions [1] [Proposition (4.16)]. We discuss briefly the dual non-compact situation of flat $GL(m)$-bundles with an $O(m)$-reduction, and prove the statements (4.17)-(4.18) given without proof in [25]. Another application is to $SO(2m-1)$-bundles with a trivial $SO(2m)$-extension, for which the suspension $\sigma(e)$ of the Pfaffian polynomial $e \in I^{2m}(SO(2m))$ gives a secondary characteristic invariant on the odd-dimensional sphere S^{2m-1}. The result of the explicit computation of this invariant is (4.20). A geometric interpretation is given in (4.21) for Riemannnian immersions in $\mathbf{R}^{2m}$ with codimension 1 in terms of the normal degree of the immersion.

The generalized characteristic homomorphism is computed explicitly for certain symmetric pairs $(\overline{G}, G)$ in section 6. In section 5 we first prove general results for reductive pairs. The framework for these computations is Theorem (5.6). This result expresses the secondary characteristic homomorphism $\Delta(\theta)$ of the foliated G-bundle $P = \Gamma\backslash\overline{G} \times G \to \Gamma\backslash\overline{G}$ in terms of the primary characteristic homomorphism $h(\theta)$ of the (ordinary) G-bundle $G \to \overline{G} \to \overline{G}/G$. For this purpose the relevant complexes have to be replaced by the cohomologically equivalent complexes according to the algorithm described in [23]. The relevant material is reviewed in the appendix to this paper. More details on this algorithm are to appear in [28] (*see* also [25′]). In top dimension a necessary and sufficient condition for the surjectivity of $\Delta(P)_*$ is the surjectivity of the primary characteristic homomorphism $h(\theta)_*$ [Theorem (5.10)]. This condition is satisfied if the Lie algebras $\overline{\mathfrak{g}}$ and $\mathfrak{g}$ are of equal rank [Corollary (5.10′).] This leads to non-trivial secondary characteristic numbers. In the remainder of section 5 we prove and apply a formula for the suspension $\overline{\sigma}(\Phi) \in H(\overline{\mathfrak{g}})$ of an invariant polynomial $\Phi \in I(\overline{G})$ in terms of the restriction $i^* \Phi \in I(G)$ and the $\mathfrak{g}$-DG-algebra structure of $\Lambda\,\overline{\mathfrak{g}}^*$ [*see* Corollary (5.25) (i)]. This technical result is crucial for the computations in section 6 (and also of interest in other contexts). It can be used to relate the cycles in the image of the characteristic homomorphism $\Delta(\theta)_*$ to the primitive elements of $H(\overline{\mathfrak{g}})$. Thereby one obtains an estimate on the dimension of the image of $\Delta(\theta)_*$. Theorem (6.28) and Corollary (6.30) are established in this way. A complete determination of $\Delta(\theta)_*$ for certain symmetric pairs is the following main result of section 6.

THEOREM 6.40. — *Let* $(\overline{G}, G)$ *be a symmetric pair of equal rank* r *and satisfying conditions* (6.21) *and* (6.22). *Then for the generalized characteristic homomorphism* $\Delta(\theta)_*$ *of the left coset foliation of* $\overline{G}$ *by* G *we have*

$$\operatorname{im}\Delta(\theta)_* = \operatorname{Id}(\overline{y}_j) + \Lambda(y_{t+1}, \ldots, y_r) \subset H(\overline{\mathfrak{g}}).$$

The primitive class $\overline{y}_j$ *of* $\overline{\mathfrak{g}}$ *is the suspension of the distinguished generator* $\overline{c}_j$ *of* $I(\overline{G})$ *in* (6.22), *and the primitive classes* y_i *of* $\mathfrak{g}$ $(i = t+1, \ldots, r)$ *are the suspensions of the generators* c_i *of* $I(G)$ *satisfying* $\deg c_i > 2q' = \dim \overline{\mathfrak{g}}/\mathfrak{g}$.

The ideal $\operatorname{Id}(\overline{y}_j)$ generated by the element $\overline{y}_j$ has already dimension 2^{r-1}, since $H(\overline{\mathfrak{g}}) = \Lambda(\overline{y}_1, \ldots, \overline{y}_r)$ has dimension 2^r. This produces linearly independent secondary invariants in dimensions greater than $2q'+1$.

In the statement above $\Delta(\theta)_*$ is more precisely the characteristic homomorphism of the foliated G-bundle $\overline{G} \times G \to \overline{G}$, to which the normal bundle of the foliation of $\overline{G}$ by G is associated. We refer to the text for a detailed explanation of the occuring terms. Conditions (6.21) (6.22) are satisfied for the symmetric pairs

$$\overline{G}, G) = (SU(r+1), U(r)) \quad \text{and} \quad (\overline{G}, G) = (SO(2r+1), SO(2r)).$$

As applications of Theorem (6.40) we obtain Theorems (6.49) and (6.52) already stated in this introduction.

In section 7 we finally show how for certain non-compact pairs $(\overline{G}, G)$ the invariants discussed are interrelated. The main result is the factorization of the characteristic homomorphism on the cochain level given in diagram (7.9), which together with the injectivity result in Theorem (4.7) gives a Lie algebraic description of secondary characteristic classes. This applies in particular for hermitian symmetric spaces and flag manifolds. The results announced in [27] for this situation will be presented elsewhere. Applications to linear independence results for certain cohomology classes of $B\Gamma_q$ are announced in [25'], p. 179-185 and will be proved in a forthioming paper.

The appendix has been described before. It reviews the cohomological methods of [23] which are needed in the present paper.

2. Foliated bundles

The concept of a foliated bundle involves three independent data: a principal G-bundle $\pi : P \to M$, a foliation L on the base space M, and a flat partial connection on P with respect to L ([25], [25']).

We review this concept for the case of smooth manifolds considered in this paper A foliation on M is given by a involutive subbundle L of the tangent bundle T_M. The dimension q of the quotient bundle $Q = T_M/L$ is the codimension of the foliation. The sections of the dual bundle Q^* are the 1-forms annihilating the vectorfields tangent to the foliation. A flat partial connection in the G-bundle P with respect to the foliation L on M is a G-equivariant foliation ω_0 on P projecting onto L, and such that the tangentspace to the foliation at each point $u \in P$ has only the zero vectorspace in common with the tangent space to the fiber at u. A flat partial connection in P can be described by a connection ω in P whose horizontal space at $u \in P$ contains the tangentspace to the foliation of P. Such a connection is called adapted to the foliation of P. The components of the curvature Ω of an adapted connection ω in P are then contained in the ideal generated in the de Rham complex $\Omega^{\cdot}(P)$ of P by the 1-forms which are the sections of $\pi^* Q^* \subset \Omega^1(P)$. A flat partial connection ω_0 in P with respect to L (or mod Q^*) is completely characterized by a class of connections (called adapted) with the curvature property above and such that the differences are tensorial 1-forms on P of type Ad, where all components are sections of $\pi^* Q^*$. A foliated bundle is then a triple (P, Q^*, ω_0).

There is a natural pull back $f^*(P, Q^*, \omega_0)$ for any map $f : M' \to M$. It suffices to define $Q'^* = f^* Q^*$. Then the pullback bundle $P' = f^* P$ inherits a flat connection ω'_0 mod Q'^* in a canonical way. For Q'^* to be the bundle of annihilating 1-forms of a (non-singular) foliation on M', we have to require that f is transversal to the foliation on M. The codimension of the pullback foliation on M' then equals the codimension of the foliation on M.

Let P_0, P_1 be foliated bundles on M. P_0, P_1 are integrably homotopic, if there exists a foliated bundle P on $M \times [0, 1]$ such that $P_i \cong j_i{}^*P$ for $i = 0, 1$ (as foliated bundles), where $j_\tau : M \to M \times \{ \tau \}$ for $\tau \in [0, 1]$.

For $Q^* = \Omega^1_M$ ($L = 0$) the concept of a foliated bundle is simply a bundle P equipped with a connection. For $Q^* = 0$ ($L = T_M$) it is a bundle P equipped with a flat connection. If P is the GL (n)-frame bundle of a vectorbundle $E \to M$, the covariant derivative defined by a connection ω in P is given locally by

$$(2.1) \qquad \nabla_\xi s_j = \sum_{i=1}^{n} s^* \omega_{ij}(\xi) s_i$$

for a vectorfield ξ on M and a local trivializing section $s = (s_1, \ldots, s_n)$ of P. ω_{ij} is the matrix of ω with respect to the canonical basis in $\mathfrak{g}\,l(n)$. If ω is an adapted connection in P, the covariant derivative defined in (2.1) is well-defined only for vectorfields ξ annihilated by forms in Q^*. It turns E into a foliated vectorbundle (*see* [21]-[22]). An example is the bundle Q^* itself with Bott connection defined by $\nabla_\xi \omega = i(\xi) d\omega$ for ω belonging to Q^* and ξ belonging to L.

We recall the notion of a basic connection ([25], (7.8), (7.9)], which will be used later on. Let ω be an adapted connection in the foliated bundle $P \to M$. Then

$$(2.2) \qquad \omega \text{ L-basic} \Leftrightarrow \Theta(\tilde{\xi})\omega = i(\tilde{\xi})\Omega = 0 \quad \text{for vectorfields } \xi \text{ in L}.$$

On the RHS $\tilde{\xi}$ denotes the canonical lift of ξ to a "horizontal" vectorfield in P. The geometric significance is that the flow of $\tilde{\xi}$ leaves ω invariant. A typical example is the case of a submersion $f : M \to M'$ and the foliation $T(f)$ defined by the fibers of f. Any connection ω' in the frame bundle $P' \to M'$ of M' pulls back to a $T(f)$-basic connection in $P = f^* P'$, the frame bundle of the normal bundle $T_M/T(f)$.

We turn now to the description of homogeneous and locally homogeneous foliated bundles. Consider a Lie group $\bar{G}$ with Lie subgroup $G \subset \bar{G}$ and corresponding Lie algebras $\mathfrak{g} \subset \bar{\mathfrak{g}}$. On the G-bundle

$$(2.3) \qquad \bar{G} \times G \to \bar{G}$$

the G-orbits of the diagonal G-action on $\bar{G} \times G$ defined by $(\bar{g}, g) g' = (\bar{g}g', g'^{-1} g)$ for $\bar{g} \in \bar{G}$, $g \in G$, $g' \in G$ lift the G-orbits on $\bar{G}$ (the left cosets) to $\bar{G} \times G$. Thus the bundle (2.3) has a canonical foliated bundle structure.

Next consider a Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. We assume that H is closed in $\bar{G}$, hence closed in G. Since the foliation just described on $\bar{G} \times G$ is G-invariant, the G-bundle

$$(2.4) \qquad P = \bar{G} \times_H G \to \bar{G}/H$$

inherits a canonical foliated bundle structure. A foliation L_G on the homogeneous base space $\bar{G}/H$ is induced by the orbits (left cosets) of the (right) G-action on $\bar{G}$. H acts on the G-orbits of the diagonal action on $\bar{G} \times G$ and the foliation L_G on $\bar{G}/H$ lifts to P, thus defining a partial flat connection in P.

A left $\bar{G}$-invariant connection ω on P adapted to this canonical foliated structure is given by a $\mathfrak{g}$-valued 1-form in the de Rham complexes

$$^{\bar{G}}\Gamma(\Omega_P) \subset \Gamma(\Omega_P) \subset \Gamma(\Omega_{\bar{G} \times G})$$

satisfying the conditions (on $\bar{G}\times G$):

$$(2.5)\qquad \begin{cases} \text{(i) } R_g^*\omega = \mathrm{Ad}(g^{-1})\omega & \text{for } g\in G;\\ \text{(ii) } \bar{L}_{\bar{g}}^*\omega = \omega & \text{for } \bar{g}\in\bar{G};\\ \text{(iii) } \omega(\eta^*) = (0, \eta_L) = \eta & \text{for } \eta\in\mathfrak{g};\\ \text{(iv) } \omega(\bar{\xi}_L, -\xi_R) = 0 & \text{for } \xi\in\mathfrak{g};\\ \text{(v) } L_h^*\omega = \bar{R}_h^*\omega & \text{for } h\in H. \end{cases}$$

Here we have used the notations $\bar{R}$, R for the canonical right action of $\bar{G}$, G on $\bar{G}\times G$ and $\bar{L}$, L for the corresponding canonical left actions. Further $\bar{\xi}_R = (\bar{\xi}_R, 0)$ denotes the right invariant vectorfield on $\bar{G}\times G$ corresponding to $\xi\in\bar{\mathfrak{g}}$, and $\bar{\xi}_L = (\bar{\xi}_L, 0)$ the corresponding left invariant vectorfield. Similarly $\eta_R = (0, \eta_R)$ and $\eta_L = (0, \eta_L)$ resp. denote the right and left invariant vectorfields an $\bar{G}\times G$ corresponding to $\eta\in\mathfrak{g}$. For $\eta\in\mathfrak{g}$ the infinitesimal transformation of $R_{\exp t\eta}$ is $\eta^* = (0, \eta_L)$ on $\bar{G}\times G$. Conditions (i), (iii) express then that ω is a connection form, whose $\bar{G}$-invariance is given by (ii). (iv) expresses that ω vanishes on the orbits of the diagonal G-action on $\bar{G}\times G$, whereas (v) expresses the invariance under the diagonal H-action (so that ω is in fact defined on $\bar{G}\times_H G$). To express these conditions more algebraically, we need the following

(2.6) LEMMA. – *There is a bijection between left $\bar{G}$-invariant connections ω on $\bar{G}\times G$ and linear maps $\theta : \bar{\mathfrak{g}} \to \mathfrak{g}$ determined by the formula*

$$(2.7)\qquad \omega_{(\bar{g}, g)}(\bar{\xi}_L, \eta_R) = \mathrm{Ad}(g^{-1})\circ[\theta(\xi)+\eta]$$

for $\xi\in\bar{\mathfrak{g}}$, $\eta\in\mathfrak{g}$.

In view of the left $\bar{G}$-invariance (2.5) (ii) and right $\bar{G}$-covariance (2.5) (i) it is clear that θ determines ω completely.

(2.8) LEMMA. – *For a left $\bar{G}$-invariant connection ω on $\bar{G}\times G$ characterized by $\theta : \bar{\mathfrak{g}}\to\mathfrak{g}$ the following are equivalent:*

(i) ω *is adapted to the foliation defined by the diagonal* G-*action;*

(ii) $\omega(\bar{\xi}_L, -\xi_R) = 0$ *for* $\xi\in\mathfrak{g}$;

(iii) $\theta\,|\,\mathfrak{g} = \mathrm{id}$.

Proof. – (i) ⇔ (ii) is clear, since the diagonal G-action is given by $\bar{R}_g\circ L_{g^{-1}}$. (ii) ⇔ (iii) follows from (2.7) which reads

$$\omega_{(\bar{g}, g)}(\bar{\xi}_L, -\xi_R) = \mathrm{Ad}(g^{-1})\circ[\theta(\xi)-\xi] \qquad \text{for } \xi\in\mathfrak{g}. \quad \blacksquare$$

Note that the vector field $(\bar{\xi}_L, -\xi_R)$ on $\bar{G}\times G$ is the infinitesimal transformation of the flow of $\exp t\,\xi$ under the diagonal G-action.

(2.9) LEMMA. — *Let* ω *be as above. The following conditions are equivalent:*

(i) $L_h^* \omega = \overline{R}_h^* \omega$ *for* $h \in H$*;*

(ii) θ *is equivariant for* $\mathrm{Ad}_{\bar{\mathfrak{g}}}(H)$ *and* $\mathrm{Ad}_{\mathfrak{g}}(H)$.

We omit the trivial verification. These conditions make ω a form on $\overline{G} \times_H G$. The preceding observations give a purely Lie algebraic characterization of $\overline{G}$-invariant adapted connections on P, which is summarized as follows.

(2.10) PROPOSITION. — *Let* P *be the canonically foliated homogeneous bundle* (2.4). *There is a bijection between left* $\overline{G}$*-invariant and adapted connections* ω *on* P *and* H*-equivariant splittings* $\theta : \bar{\mathfrak{g}} \to \mathfrak{g}$ *of the exact sequence* $0 \to \mathfrak{g} \to \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\mathfrak{g} \to 0$, *where* ω *and* θ *are related by* (2.7).

For a connection ω on P the $\mathfrak{g}$-valued curvature form $\Omega = d\omega + (1/2)[\omega, \omega]$ is horizontal, i. e. $i(\eta^*)\Omega = 0$, $\eta \in \mathfrak{g}$. A direct calculation shows that

$$\Omega_{(\bar{g}, g)}((\bar{\xi}_L, \eta_R), (\bar{\xi}'_L, \eta'_R)) = \mathrm{Ad}(g^{-1})(K(\theta)(\xi, \xi')), \tag{2.11}$$

where $\xi, \xi' \in \bar{\mathfrak{g}}$, $\eta, \eta' \in \mathfrak{g}$ and $K(\theta) : \Lambda^2 \bar{\mathfrak{g}} \to \mathfrak{g}$ denotes the $\mathfrak{g}$-valued algebraic curvature form of θ given by

$$\alpha K(\theta) \equiv d_\Lambda(\alpha\theta) + \frac{1}{2}\alpha[\theta, \theta], \qquad \alpha \in \mathfrak{g}^*.$$

d_Λ is the Chevalley-Eilenberg differential in $\Lambda\, \mathfrak{g}^*$.

(2.12) LEMMA. — *Let* ω *be a left* $\overline{G}$*-invariant adapted connection on* P *characterized by an* H*-equivariant splitting* $\theta : \bar{\mathfrak{g}} \to \mathfrak{g}$ *of* $0 \to \mathfrak{g} \to \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\mathfrak{g} \to 0$. *The following conditions are equivalent:*

(i) ω *is* L_G*-basic;*

(ii) $i(\bar{\xi}_L, -\xi_R)\Omega = 0$, $\xi \in \mathfrak{g}$*;*

(iii) $i(\xi)(\alpha K(\theta)) = 0$, $\xi \in \mathfrak{g}$; $\alpha \in \mathfrak{g}^*$*;*

(iv) θ *is* $\mathfrak{g}$*-equivariant.*

Proof. — (i) $\Leftrightarrow$ (ii) follows from (2.2) and (2.8) (ii). (ii) $\Leftrightarrow$ (iii) follows from (2.11). (iii) $\Leftrightarrow$ (iv): For $\xi \in \mathfrak{g}$, $\alpha \in \mathfrak{g}^*$, $\eta \in \bar{\mathfrak{g}}$ we have by (2.8) (iii):

$$i(\xi)(\alpha K(\theta))(\eta) = \alpha(K(\theta))(\xi, \eta) = -\alpha\theta[\xi, \eta] + \alpha[\theta(\xi), \theta(\eta)] = \alpha(-\theta[\xi, \eta] + [\xi, \theta(\eta)]).$$

Hence $i(\xi)(\alpha K(\theta)) = 0$, $\xi \in \mathfrak{g}$, $\alpha \in \mathfrak{g}^*$ is equivalent to $\theta(\mathrm{ad}_{\bar{\mathfrak{g}}}(\xi)\eta) = \mathrm{ad}_{\mathfrak{g}}(\xi)\theta(\eta)$, $\xi \in \mathfrak{g}$, $\eta \in \bar{\mathfrak{g}}$, i. e. to the $\mathfrak{g}$-equivariance of θ. ∎

(2.13) LEMMA. — *Assume* $G \subset \overline{G}$ *to be closed. Then* P *admits an* L_G*-basic adapted connection.*

Proof. – The bundle P for the case H = G becomes $P' = \overline{G} \times_G G = \overline{G} \to \overline{G}/G$, the canonical G-bundle on $\overline{G}/G$. There is an obvious G-bundle map $(\bar{f}, f)$:

$$\begin{array}{ccc} P = \overline{G} \times_H G & \xrightarrow{\bar{f}} & P' = \overline{G} \\ \downarrow & & \downarrow \\ \overline{G}/H & \xrightarrow{f} & \overline{G}/G \end{array}$$

and $P \cong f^* P'$. The foliation on $\overline{G}/H$ is given by the fibers of $f : L_G = T(f)$. Any connection ω' in P' lifts to an L_G-basic adapted connection $\omega = f^* \omega'$ in P. ■

Finally we consider a discrete subgroup $\Gamma \subset \overline{G}$ acting properly discontinuously and without fixed points on $\overline{G}/H$, so that $M = \Gamma \backslash \overline{G}/H$ is a manifold. Dividing the G-bundle (2.4) by the action of Γ, we obtain the locally homogeneous G-bundle

$$P = (\Gamma \backslash \overline{G}) \times_H G \to M = \Gamma \backslash \overline{G}\, H. \tag{2.14}$$

Since the foliation considered before on the bundle (2.4) is $\overline{G}$-invariant, it is inherited by the bundle (2.14). Left $\overline{G}$-invariant connections on (2.4) pass to (2.14), so that the preceding discussion of adapted connections applies equally well to this locally homogeneous foliated bundle.

Consider the exact sequence of H-modules

$$0 \to \mathfrak{g}/\mathfrak{h} \to \bar{\mathfrak{g}}/\mathfrak{h} \to \bar{\mathfrak{g}}/\mathfrak{g} \to 0$$

for which $\theta : \bar{\mathfrak{g}} \to \mathfrak{g}$ characterizing an invariant and adapted connection gives an H-module splitting. In the exact sequence of vectorbundles

$$\begin{array}{ccccccccc} 0 \to & \Gamma \backslash \overline{G} \times_H \mathfrak{g}/\mathfrak{h} & \to & \Gamma \backslash \overline{G} \times_H \bar{\mathfrak{g}}/\mathfrak{h} & \to & \Gamma \backslash \overline{G} \times_H \bar{\mathfrak{g}}/\mathfrak{g} & \to 0 \\ & \| & & \| & & \| & \\ 0 \longrightarrow & L_G & \longrightarrow & T_M & \longrightarrow & Q_G & \longrightarrow 0 \end{array} \tag{2.15}$$

the last term is associated to P and hence $Q_G = P \times_G \bar{\mathfrak{g}}/\mathfrak{g}$. It is the normal bundle $Q_G = T_M/L_G$ of the foliation and is canonically foliated. In fact the foliation induced from P coincides with the Bott connection in Q_G.

3. **Generalized characteristic classes**

For the convenience of the reader we recall briefly our construction of generalized characteristic classes of foliated bundles as first announced in [22] to [24] and detailed in [25]. For the sake of simplicity, we review our construction as it applies in the category of smooth manifolds (*see* [25′]). For this purpose consider the Weil homomorphism

$$k(\omega) : \quad W(\mathfrak{g}) \to \Gamma(P, \Omega_P^{\bullet})$$

defined by a global connection ω in a smooth principal G-bundle $G \to P \to M$. $W(\mathfrak{g})$ denotes the Weil algebra [8] of the Lie algebra $\mathfrak{g}$ of G, Ω_P the sheaf of differential forms on P with global sections the de Rham complex $\Gamma(P, \Omega_P^{\cdot})$ of P. On the subalgebra of G-invariant polynomials $I(G) \subset W(\mathfrak{g})$ the map $k(\omega)$ induces the Chern-Weil homomorphism assigning to $\Phi \in I(G)$ the de Rham cohomology class $[k(\omega)\Phi] \in H_{DR}(M)$. Let now P be a foliated bundle. The fundamental observation concerning the Weil homomorphism of a foliated bundle is the following. For a connection ω in P adapted to the partial flat connection ω_0 the Weil homomorphism $k(\omega)$ is filtration preserving for the canonical filtration F^{2p} of $W(\mathfrak{g})$ (*see* below) and the filtration F^p of the de Rham complex $\Gamma(P, \Omega_P^{\cdot})$ defined by the foliation in the base space M. The latter filtration breaks off above the codimension q of the foliation in M. This leads to an induced homomorphism

$$k(\omega): \quad W(\mathfrak{g})_q \to \Gamma(P, \Omega_P^{\cdot})$$

on the truncated Weil algebra $W(\mathfrak{g})_q = W(\mathfrak{g})/F^{2(q+1)}W(\mathfrak{g})$. This is a generalization of Bott's vanishing theorem in [4]. Whereas the Weil algebra is contractible, the cohomology of $W(\mathfrak{g})_q$ is non-trivial and thus leads to generalized characteristic invariants in the de Rham cohomology $H_{DR}(P)$. This construction was motivated by the Chern-Simons construction on invariant polynomials in [9], [10]. Additional devices lead to invariants in $H_{DR}(M)$.

For our construction one needs the further data of a closed subgroup $H \subset G$ and an H-reduction P′ of the G-bundle Pgiven by a section s of $P/H \xrightarrow{\hat{\pi}} M : P' = s^* P$. $W(\mathfrak{g}, H)$ denotes the algebra of H-basic elements in the Weil algebra $W(\mathfrak{g})$. These are the elements in $W(\mathfrak{g})$ killed by the operators $i(x)$, $x \in \mathfrak{h}$ and invariant under the H-action induced from the canonical G-action on $W(\mathfrak{g})$. The canonical filtration of $W(\mathfrak{g})$ by

$$(3.1) \qquad F^{2p}W(\mathfrak{g}) = S^p(\mathfrak{g}^*).W(\mathfrak{g}), \qquad F^{2p-1}W = F^{2p}W$$

induces a filtration on $W(\mathfrak{g}, H)$. Denote,

$$(3.2) \qquad W(\mathfrak{g}, H)_k = W(\mathfrak{g}, H)/F^{2(k+1)}W(\mathfrak{g}, H) \qquad \text{for} \quad k \geqq 0.$$

The construction sketched above establishes then the existence of the generalized characteristic homomorphism Δ_* in Theorem (3.4) below. Δ_* is defined as the composition $\Delta_* = s^* \circ k_*$, where the homomorphism

$$(3.3) \qquad k_*: \quad H(W(\mathfrak{g}, H)_q) \to H_{DR}(P/H)$$

is induced by the generalized Weil homomorphism restricted to H-basic elements and does not depend on the choice of the adapted connection ω (*see* [23] to [25] and [25′] for more details.) The construction of Δ_* will be particularized below for the case of locally homogeneous foliated bundles, so that the present account can be read without consultation of the papers giving the theory in the general case. We have then the following result.

(3.4) THEOREM [25]. — *Let* $P \to M$ *be a foliated* G-*bundle with an* H-*reduction*, $H \subset G$ *a closed subgroup with finitely many connected components.*

(i) *There is a canonical multiplicative homomorphism*

$$\Delta_* : \quad H(W(\mathfrak{g}, H)_q) \to H_{DR}(M),$$

where q *is the codimension of the given foliation on* M. Δ_* *is the generalized characteristic homomorphism of the bundle* P *equipped with the foliation* ω_0 *and the* H-*reduction* s.

(ii) Δ_* *is functorial under pullbacks:*

$$f^* \Delta_*(P, \omega_0, s) = \Delta_*(f^* P, f^* \omega_0, f^* s).$$

(iii) Δ_* *is invariant under integrable homotopies.*

This construction contains beside the usual (primary) characteristic classes of P secondary invariants, which therefore are naturally explained in the framework of the Chern-Weil theory. For the particular foliated bundle obtained from the normal bundle of a non-singular foliation or from a Haefliger Γ_q-cocycle, the relationship of these invariants with the invariants of Godbillon-Vey [12] and Bott-Haefliger ([7], [15]) has been explained in [25]. The computation of the universal generalized characteristic invariants has been carried out in [23]. The method is explained in the appendix to this paper.

With the definitions given in section 2 it is e. g. clear that the generalized characteristic homomorphism is invariant under integrable homotopies. Namely is P is a foliated bundle on $M \times [0, 1]$ providing an integrable homotopy between the foliated bundles P_0, P_1 on M, then the generalized characteristic homomorphism

$$\Delta(P)_* : \quad H(W(\mathfrak{g}, H)_q) \to H_{DR}(M \times I)$$

of P composed with $j_\tau^* : H_{DR}(M \times I) \to H_{DR}(M)$ is independent of τ, where

$$j_\tau : \quad M \to M \times \{\tau\} \subset M \times [0, 1].$$

But $\Delta(P_i)_* = j_i^* \circ \Delta(P)_*$ for $i = 0, 1$ and hence $\Delta(P_0)_* = \Delta(P_1)_*$.

The characteristic homomorphism Δ_* is also contravariant functorial for homomorphisms $(G', H') \to (G, H)$ in an obvious sense. This fact is used in the proof of the next result, which is based on the construction of the complex $A(W(\mathfrak{g})_q, H)$ for the cohomology $H(W(\mathfrak{g}, H)_q)$ in [23] (*see* appendix to this paper).

(3.5) THEOREM [25]. — *Let* P *be a foliated bundle as in Theorem* 3.4 *and* $P' = s^* P$ *the* H-*reduction of* P *given by a section* s *of* $P/H \to M$.

(i) *There is a split exact sequence of algebras*

$$(3.6) \qquad 0 \to H(K_q) \to H(W(\mathfrak{g}, H)_q) \overset{\times}{\underset{\curvearrowleft}{\to}} I(G)_q \otimes_{I(G)} I(H) \to 0$$

and the composition $\Delta_* \circ \varkappa$ *is induced by the characteristic homomorphism* $\mathrm{I}(\mathrm{H}) \to \mathrm{H}_{\mathrm{DR}}(\mathrm{M})$ *of* P'.

(ii) *If the foliation of* P *is induced by a foliation of* P′, *then* $\Delta_* \,|\, \mathrm{H}(\mathrm{K}_q^\cdot) = 0$.

The classes in $\Delta_* \mathrm{H}(\mathrm{K}_q^\cdot)$ are secondary invariants. Their appearance is by (ii) due to the incompatibility of the foliation of P with the H-reduction P′.

It is clear from the construction of Δ_* that it depends only on the homotopy class of $s : \mathrm{M} \to \mathrm{P}/\mathrm{H}$. If the subgroup $\mathrm{H} \subset \mathrm{G}$ contains a maximal compact subgroup K of G, then an H-reduction of P exists and Δ_* does not depend on the choice of the H-reduction. To see this, it suffices to show that G/H is contractible, since the sections of $\mathrm{P}/\mathrm{H} \to \mathrm{M}$ are classified by maps $\mathrm{M} \to \mathrm{G}/\mathrm{H}$. But this follows from the fibration $\mathrm{H}/\mathrm{K} \to \mathrm{G}/\mathrm{K} \to \mathrm{G}/\mathrm{H}$, since both H/K and G/K are contractible. In general Δ_* does depend on s and this dependence is in fact of great interest.

We describe now the construction of Δ_* in the case of locally homogeneous foliated bundles. We have the following result.

(3.7) THEOREM. — *Let* $\mathrm{H} \subset \mathrm{G} \subset \overline{\mathrm{G}}$ *be Lie groups,* H *closed in* $\overline{\mathrm{G}}$ *with finitely many connected components, and* $\Gamma \subset \overline{\mathrm{G}}$ *a discrete subgroup acting properly discontinuously and without fixed points on* $\overline{\mathrm{G}}/\mathrm{H}$.

The canonical G-*foliation* L_{G} *of* $\mathrm{M} = \Gamma\backslash\overline{\mathrm{G}}/\mathrm{H}$ *has codimension* $q = \dim \bar{\mathfrak{g}}/\mathfrak{g}$. *The* G-*bundle* $\mathrm{P} = (\Gamma\backslash\overline{\mathrm{G}}) \times_{\mathrm{H}} \mathrm{G} \to \mathrm{M}$ *is canonically foliated. Let* ω *be a locally* $\overline{\mathrm{G}}$-*invariant adapted connection on* P, *characterized by an* H-*equivariant splitting* θ *of the exact sequence*

$$0 \to \mathfrak{g} \underset{\theta}{\rightleftarrows} \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\mathfrak{g} \to 0. \tag{3.8}$$

Then the generalized characteristic homomorphism $\Delta(\omega)$ *of* P *on the cochain level factorizes as follows*

$$\begin{array}{ccc} \mathrm{W}(\mathfrak{g}, \mathrm{H})_q & \xrightarrow{\Delta(\theta)_{\mathrm{H}}} & \Lambda^\cdot(\bar{\mathfrak{g}}/\mathfrak{h})^{*\mathrm{H}} \\ & {}_{\Delta(\omega)}\searrow & \downarrow \gamma \\ & & \Gamma(\mathrm{M}, \Omega_{\mathrm{M}}^\cdot) \end{array} \tag{3.9}$$

where γ *is the canonical inclusion and* $\Delta(\theta)_{\mathrm{H}}$ *is induced by the* H-DG-*homomorphism*

$$\Delta(\theta) : \quad \mathrm{W}(\mathfrak{g}) \to \Lambda\bar{\mathfrak{g}}^*$$

which is completely determined by

$$\begin{cases} \Delta(\theta)\alpha = \alpha\theta & \text{for } \alpha \in \Lambda^1 \mathfrak{g}^*, \\ \Delta(\theta)\tilde{\alpha} = \alpha \mathrm{K}(\theta) = d_\Lambda \alpha\theta + \dfrac{1}{2}\alpha[\theta, \theta] & \text{for } \tilde{\alpha} \in \mathrm{S}^1 \mathfrak{g}^*. \end{cases} \tag{3.10}$$

Note that the induced maps on the cohomology level do not depend of the choice of ω, resp. θ. Under a stronger assumption on θ we have the following result.

(3.11) THEOREM. — *Let the situation be as in Theorem* (3.7). *Assume that* θ *is a* G-*equivariant splitting of the exact sequence* (3.8). *Then there exists an* L_G-*basic and locally* $\overline{G}$-*invariant adapted connection* ω *on* P, *and* $\Delta(\omega)$ *on the cochain level factorizes as follows*

(3.12)
$$\begin{tikzcd} W(\mathfrak{g}, H)_{[q/2]} \arrow[r, "\Delta(\theta)"] \arrow[dr, "\Delta(\omega)"'] & \Lambda^{\cdot}(\bar{\mathfrak{g}}/\mathfrak{h})^{*H} \arrow[d, hook, "\gamma"] \\ & \Gamma(M, \Omega_M^{\cdot}) \end{tikzcd}$$

where γ, $\Delta(\theta)$ *are defined as in Theorem* (3.7).

For any G-module V let P (V) be the associated foliated vectorbundle. Assume more specifically that

$$(3.13)\qquad \rho : \ (G, H) \to (GL(V), L(V)) \qquad [\text{resp. } \rho : (G, H) \to (GL(V), O(V))],$$

where L (V) denotes the linear maps of V with determinant ± 1, and O (V) the orthogonal group with respect to an Euclidean metric in V. Then ρ induces a canonical homomorphism

$$3.14)\quad \rho^* : \ W(\mathfrak{gl}(V), L(V)) \to W(\mathfrak{g}, H) \qquad [\text{resp. } \rho^* : W(\mathfrak{gl}(V), O(V)) \to W(\mathfrak{g}, H)].$$

The generalized characteristic homomorphism of P (V) is then obtained by composing the previous homomorphisms with ρ^*. Specifically we obtain e. g. for the case of L (V) the following result in cohomology.

(3.7)′ THEOREM. — *Let the situation be as in Theorem* (3.7) *and* V *a* G-*module such that* $\rho : (G, H) \to (GL(V), L(V))$.

The generalized characteristic homomorphism $\Delta_*(P(V))$ *of the locally homogeneous foliated vectorbundle* P (V) *factorizes as in the commutative diagram*

(3.9)′
$$\begin{tikzcd} H^{\cdot}(W(\mathfrak{g}, H)_q) \arrow[r, "\Delta(\theta)_*"] \arrow[d, "\rho^*"] & H^{\cdot}(\Lambda(\bar{\mathfrak{g}}/\mathfrak{h})^{*H}) \arrow[d, "\gamma^*"] \\ H^{\cdot}(W(\mathfrak{gl}(V), L(V))_q) \arrow[r, "\Delta_*(P(V))"] & H_{DR}^{\cdot}(M) \end{tikzcd}$$

A similar result for Δ_* holds in case the particular situation of Theorem (3.11) is realized. The statement is analogeous, where q has now to be replaced by $[q/2]$.

For the proof of these results we need the concept of a commutative G−DG-algebra $E^{\cdot}$ with respect to a Lie group G (*see* [25′], 3.12). The corresponding Lie algebra concept is defined in the appendix. The only difference is that the $\mathfrak{g}$-action θ is the differential of a G-action. For a subgroup $H \subset G$ the H-invariant elements are denoted E^H, the H-basic elements by $E_{H^{\cdot}}$, e. g. $W(\mathfrak{g})_H \equiv W(\mathfrak{g}, H)$.

It is useful to recall the universal property of the Weil algebra in the following slightly more general form.

(3.15) LEMMA. — *Let $E^{\bullet}$ be a commutative* G-DG-*algebra and* $H \subset G$ *a subgroup. Let* $\nu : \mathfrak{g}^* \to E^1$ *be a linear map commuting with the operators* $i(x)$, $x \in \mathfrak{h}$ *and the* H-*actions on* $\mathfrak{g}^*$ *and* $E^{\bullet}$. *Then there exists a unique* H-DG-*algebra homomorphism* $k(\nu) : W(\mathfrak{g}) \to E$ *such that the following diagram is commutative*

$$\begin{tikzcd} W(\mathfrak{g}) \arrow[r, "k(\nu)"] & E \\ \Lambda \mathfrak{g}^* \arrow[u, hook] \arrow[ur, "\Lambda\nu"'] & \end{tikzcd}$$

Proof. — $k(\nu)$ is defined in $\Lambda \mathfrak{g}^*$ by multiplicative extension of ν and on $S\,\mathfrak{g}^*$ by multiplicative extension of $k(\nu)\,\tilde{\alpha} = d\omega(\alpha) - \omega(d'\alpha)$. By direct verification one shows that $k(\nu)$ is a DG-homomorphism, and it commutes with $i(x)$, $x \in \mathfrak{h}$ and the H-actions by the assumptions on ν. ■

Let now ω be a locally $\overline{G}$-invariant adapted connection on P and $s : M \to P/H$ the canonical cross-section $\overline{G} \to \overline{G} \times (e)$ of $\overline{G} \times G$. To evaluate $\Delta(\omega) = s^* \circ k(\omega)$ it is as in section 2 convenient to perform all calculations on $\overline{G} \times G$ under observation of the correct equivariance properties. Then by left $\overline{G}$-invariance $\Delta(\omega)$ factorizes through left $\overline{G}$-invariant forms as follows:

$$\Delta(\omega) : \quad W(\mathfrak{g}) \xrightarrow{k(\omega)} {}^{\overline{G}}\Gamma(\Omega^{\bullet}_{\overline{G}\times G}) \xrightarrow{s^*} {}^{\overline{G}}\Gamma(\Omega_{\overline{G}}) = \Lambda \bar{\mathfrak{g}}^* \overset{\gamma}{\hookrightarrow} \Gamma(\Omega^{\bullet}_{\overline{G}}).$$

Let $\theta : \bar{\mathfrak{g}} \to \mathfrak{g}$ be the H-equivariant splitting of $0 \to \mathfrak{g} \to \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\mathfrak{g} \to 0$ characterizing ω by Lemma 2.6.

(3.16) LEMMA. — *With the notation above, we have*

$$(3.17) \qquad \Delta(\omega)\,\alpha = \gamma(\alpha\theta) \qquad \textit{for} \quad \alpha \in \Lambda^1 \mathfrak{g}^*;$$

$$(3.18) \qquad \Delta(\omega)\,\tilde{\alpha} = \gamma(\alpha\,K(\theta)) \qquad \textit{for} \quad \tilde{\alpha} \in S^1 \mathfrak{g}^*.$$

Here $\Delta(\omega)\,\tilde{\alpha}$ *is defined as usual by the curvature:*

$$\Delta(\omega)\,\tilde{\alpha} = d\omega(\alpha) - \omega(d'\alpha) = \alpha\,\Omega,$$

where α *denotes the element in* $\Lambda^1 \mathfrak{g}^* = \mathfrak{g}^*$ *corresponding to* $\tilde{\alpha} \in S^1 \mathfrak{g}^* = \mathfrak{g}^*$.

Proof. — (3.17) and (3.18) follow directly from (2.7) and (2.11). ■

From Lemma 3.15 it follows that there exists a unique homomorphism of H-DG-algebras $\Delta(\theta) : W(\mathfrak{g}) \to \Lambda \bar{\mathfrak{g}}^*$ satisfying $\Delta(\theta)\,\alpha = \alpha\theta$, $\Delta(\theta)\,\tilde{\alpha} = \alpha\,K(\theta)$, $\alpha \in \Lambda^1 \mathfrak{g}^*$, $\tilde{\alpha} \in S^1 \mathfrak{g}^*$. Lemma 3.16 and the universal property of $W(\mathfrak{g})$ imply the equation $\Delta(\omega) = \gamma \circ \Delta(\theta)$. ■

On H-basic elements we have then the following induced factorization of $\Delta(\omega)$:

(3.19)
$$\begin{tikzcd} W(\mathfrak{g}, H) \arrow[r, "\Delta(\theta)_H"] \arrow[dr, "\Delta(\omega)"'] & (\Lambda \bar{\mathfrak{g}}^*)_H \arrow[d, hook, "\gamma"] \\ & \Gamma(\Omega_{\bar{G}/H}) \end{tikzcd}$$

With the definition

(3.20)
$$F^p(\mathfrak{g})\Lambda \bar{\mathfrak{g}}^* = \Lambda^p \mathfrak{m}^* . \Lambda \bar{\mathfrak{g}}^*$$

for $\mathfrak{m} = \ker \theta \subset \bar{\mathfrak{g}}$ we have the following result for $\Delta(\theta)$.

(3.21) LEMMA.
$$\Delta(\theta) : \ F^{2p}(\mathfrak{g}) W(\mathfrak{g}) \to F^p(\mathfrak{g}) \Lambda \bar{\mathfrak{g}}^* \qquad \textit{for} \quad p \geqq 0.$$

Proof. — Since both filtrations and $\Delta(\Theta)$ are multiplicative, it suffices to show that

$$\Delta(\theta) : \ F^2(\mathfrak{g}) W(\mathfrak{g}) = S^1 \mathfrak{g}^* . W(\mathfrak{g}) \to F^1(\mathfrak{g}) \Lambda \bar{\mathfrak{g}}^* = \mathfrak{m}^* . \Lambda \bar{\mathfrak{g}}^*.$$

We have in fact

$$(\Delta(\theta)\tilde{\alpha})(\xi, \eta) = \alpha(K(\theta))(\xi, \eta) = \alpha([\theta(\xi), \theta(\eta)] - \theta[\xi, \eta])$$

for $\xi, \eta \in \bar{\mathfrak{g}}$. But $\theta \,|\, \mathfrak{g} = \mathrm{id}$, so that $(\Delta(\theta)\tilde{\alpha})(\xi, \eta) = 0$ for $\xi, \eta \in \mathfrak{g}$, i. e. $\Delta(\theta)\tilde{\alpha} \in \mathfrak{m}^* . \Lambda \bar{\mathfrak{g}}^*$, which proves the desired result. ∎

Since for $q = \dim \bar{\mathfrak{g}}/\mathfrak{g} = \dim \mathfrak{m}$ we have clearly $F^{q+1}(\mathfrak{g}) \Lambda \bar{\mathfrak{g}}^* = 0$, the preceding facts complete the proof of Theorem (3.7).

To prove Theorem (3.11), it suffices to observe in addition the following fact.

(3.22) LEMMA. — *Assume* $\theta : \bar{\mathfrak{g}} \to \mathfrak{g}$ *to be* G-*equivariant. Then*
$$\Delta(\theta) : \ F^{2p}(\mathfrak{g}) W(\mathfrak{g}) \to F^{2p}(\mathfrak{g}) \Lambda \bar{\mathfrak{g}}^*, \qquad p \geqq 0.$$

Proof. — Again it suffices to verify this for $p = 1$. But for $\xi \in \mathfrak{g}$ and $\eta \in \bar{\mathfrak{g}}$ we have since $\theta \,|\, \mathfrak{g} = \mathrm{id}$
$$\begin{aligned}(\Delta(\theta)\tilde{\alpha})(\xi, \eta) &= \alpha([\xi, \theta(\eta)] - \theta[\xi, \eta]) \\ &= \alpha(\mathrm{ad}_{\mathfrak{g}}(\xi)\theta(\eta) - \theta(\mathrm{ad}_{\bar{\mathfrak{g}}}(\xi)\eta)) = 0,\end{aligned}$$

since θ is G-equivariant. Therefore indeed $\Delta(\theta)\tilde{\alpha} \in F^2(\mathfrak{g}) \Lambda^2 \bar{\mathfrak{g}}^* = \Lambda^2 \mathfrak{m}^*$. ∎

4. Flat bundles

The simplest case of the constructions in the preceding sections occurs for $G = \bar{G}$ and therefore $q = 0$. Recall that $H \subset G$ is a closed subgroup with finitely many connected components, and $\Gamma \subset G$ a discrete subgroup, operating properly discontinuously and without fixed points on G/H, so that $M = \Gamma \backslash G/H$ is a manifold. The foliation of M consists of one single leaf equal to M. The bundle

(4.1)
$$P = \Gamma \backslash G \times_H G \cong G/H \times_\Gamma G \to M = \Gamma \backslash GH$$

is flat. Clearly

$$W(\mathfrak{g}, H)_0 \cong (W(\mathfrak{g})/F^2 W(\mathfrak{g}))_H \cong (\Lambda \mathfrak{g}^*)_H.$$

Since $\Delta(\theta) =$ identity, the generalized characteristic homomorphism of P reduces to the canonical homomorphism

$$(4.2) \qquad \gamma_* : \ H^{\cdot}(\mathfrak{g}, H) \to H^{\cdot}_{DR}(M)$$

on the relative Lie algebra cohomology $H^{\cdot}(\mathfrak{g}, H)$, induced by the canonical inclusion

$$(4.3) \qquad \gamma : \ \Lambda^{\cdot}(\mathfrak{g}/\mathfrak{h})^{*H} \cong (\Lambda^{\cdot} \mathfrak{g}^*)_H \subsetneq \Gamma(M, \Omega^{\cdot}_M).$$

The following fact is well-known.

(4.4) LEMMA. — *If* G *is compact and connected, then* γ_* *is an isomorphism.*

The realization of $H_{DR}(G/H)$ by G-invariant cohomology classes establishes the isomorphism $H(\mathfrak{g}, H) \cong H_{DR}(G/H)$. The result follows, since the finite group $\Gamma \subset G$ acts trivially on $H_{DR}(G/H)$. The following is an immediate consequence.

(4.5) THEOREM. — *Let* H *be a closed subgroup of a compact connected Lie group* G *and* Γ *a finite subgroup of* G *acting without fixed points on* G/H. *The generalized characteristic homomorphism*

$$\Delta_* : \ H(W(\mathfrak{g}, H)_0) \cong H(\mathfrak{g}, H) \xrightarrow{\gamma^*} H_{DR}(M)$$

of the flat bundle

$$P = \Gamma\backslash G \times_H G \to M = \Gamma\backslash G/H$$

is an isomorphism.

For $\Gamma = \{ e \}$ the map $\gamma : (\Lambda^{\cdot} \mathfrak{g}^*)_H \subsetneq \Gamma(G/H, \Omega^{\cdot}_{G/H})$ is the inclusion of left-invariant forms into $\Gamma(G, \Omega^{\cdot}_G)$, restricted to H-basic elements. Thus e. g. for a compact connected group G the Chevalley-Eilenberg cohomology isomorphism $H(\mathfrak{g}) \to H(G)$ induced by the canonical inclusion of left-invariant forms $\Lambda^{\cdot} \mathfrak{g}^* \subsetneq \Gamma(G, \Omega^{\cdot}_G)$ can be interpreted as the generalized characteristic homomorphism of the trivial bundle $G \times G \to G$. The foliation is given by the diagonal action of G on $G \times G$, which is non-compatible with the trivialization, and therefore gives rise to secondary invariants by the principle embodied in Theorem 3.5.

Assume in particular G/H to be a symmetric space. It is then easy to check that the differential on $(\Lambda \mathfrak{g}^*)_H$ is zero, so that in fact $\gamma_* : \ (\Lambda \mathfrak{g}^*)_H \cong H(G/H)$. An application is to the symmetric space defined by the $G \times G$-action on G *via* $(g, g').g'' = gg'' g'^{-1}$. In this case the diagonal ΔG plays the role of the subgroup H and by the remark above

$$\gamma_* : \ \Lambda(\mathfrak{g} \times \mathfrak{g})^*_{\Delta\mathfrak{g}} \cong H(G).$$

The left-hand side represents now the biinvariant forms on G with trivial differential and hence is isomorphic to H(G). This is the interpretation of the generalized characteristic homomorphism in this case.

Before giving applications of Theorem (4.5), we turn to the case of a not necessarily compact G. We need the following fact (*see* e. g. [20], Lemma (4.21)).

(4.6) LEMMA. — *Assume that* $M = \Gamma \backslash G/H$ *is a compact orientable manifold, and that* $H(\mathfrak{g}, H)$ *satisfies Poincaré-duality with respect to a non-zero* $\mu \in \Lambda^n (\mathfrak{g}/\mathfrak{h})^{*H}$, $n = \dim \mathfrak{g}/\mathfrak{h}$. *Then* γ_* *is injective.*

Proof. — μ defines a G-invariant nowhere zero n-form on G/H, which is *a fortiori* Γ-invariant and induces hence a non-trivial cohomology class of degree n on the compact manifold M. It follows that γ_* is an isomorphism in dimension n.

Let now $x \neq 0 \in H^i(\mathfrak{g}, H)$. By Poincaré-duality there exists $y \in H^{n-i}(\mathfrak{g}, H)$ such that $x.y = \mu \in H^n(\mathfrak{g}, H)$. Since γ_* is multiplicative and $\gamma_*(x.y) \neq 0$, it follows that $\gamma_*(x) \neq 0$. ■

Note that for compact H the isotropy representation of H in $\mathfrak{g}/\mathfrak{h}$ is unimodular, so that the existence of a non-zero $\mu \in \Lambda^n (\mathfrak{g}/\mathfrak{h})^{*H}$ is then always guaranteed.

For the flat bundles considered in [20] (4.14) this leads to the following result ([24], [25], 8.6).

(4.7) THEOREM. — *Let* G *be connected semi-simple with finite center and containing no compact factor,* $K \subset G$ *a maximal compact subgroup and* $\Gamma \subset G$ *a discrete uniform and torsion-free subgroup. Then the generalized characteristic homomorphism*

$$\Delta_* : \quad H(W(\mathfrak{g}, K)_0) \cong H(g, K) \xrightarrow{\gamma^*} H_{DR}(M)$$

of the flat bundle

$$P = \Gamma \backslash G \times_K G \to M = \Gamma \backslash G/K$$

is injective.

Proof. — The existence of a discrete uniform torsion free subgroup $\Gamma \subset G$ is proved in [3]. The result follows from 4.6. ■

It is useful to recall at this place how this homomorphism fits into the commutative diagram of [20] (4.18) or [25] (8.6) :

(4.8)
$$\begin{tikzcd} & H(B_G) \arrow[r, "\cong"] \arrow[d, "B\alpha^*"'] & H(B_K) \arrow[d, "\varkappa"] \\ H(\Gamma) \cong & H(M) & H(\mathfrak{g}, K) \arrow[l, "\gamma^*"'] \end{tikzcd}$$

where $\varkappa$ is the characteristic homomorphism of the bundle $K \to G \to G/K$ with values in the invariant forms on G/K, and $B\alpha^*$ is induced by the classifying map $B\alpha : M \to B_G$ of P. Here $B\alpha$ is induced by the holonomy homomorphism $\alpha : \Gamma \to G$.

This is a special case of the construction of Δ_* for a flat G-bundle $P \to M$ with H-reduction in section 4.3 of [25]. The classifying map g of P factorizes as $g : M \xrightarrow{\zeta} B_\Gamma \xrightarrow{B\alpha} B_G$, where ζ classifies the universal covering $\tilde{M} \to M$, Γ is the fundamental group of M and $\alpha : \Gamma \to G$ the holonomy homomorphism of the flat connection in P [20]. If $g' : M \to B_H$

denotes the classifying map of the H-reduction, then $\Delta_*(\mathrm{P})$ appears more generally in the commutative diagram

$$(4.9)\qquad \begin{tikzcd}
\mathrm{H}(\mathrm{B_G}) \arrow[r] \arrow[d, "\mathrm{B}(\alpha)^*"] \arrow[dd, bend right=60, "g^*"'] & \mathrm{H}(\mathrm{B_H}) \arrow[ddl, "g'^*"'] & \mathrm{I}(\mathrm{H}) \arrow[l, "\nu"'] \arrow[dd, "\mathrm{H}"] \\
\mathrm{H}(\mathrm{B}_\Gamma) \arrow[d, "\varkappa^*"] & & \\
\mathrm{H}(\mathrm{M}) & & \mathrm{H}(\mathfrak{g},\mathrm{H}) \arrow[ll, "\Delta_*(\mathrm{P})"']
\end{tikzcd}$$

where $\varkappa$ is again the characteristic homomorphism of the pair (G, H) with values in the invariant forms of G/H (for this we assume the existence of an H-invariant split of $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$), and $\nu : \mathrm{I}(\mathrm{H}) \to \mathrm{H}(\mathrm{B_H})$ the Chern-Weil homomorphism of the universal H-bundle over $\mathrm{B_H}$. The commutativity of the right hand side is the content of Theorem (3.5), (i). For H compact, ν is an isomorphism. But this diagram holds in full generality. In the case discussed above the manifold $\mathrm{M} = \Gamma\backslash\mathrm{G/K}$ is a B_Γ and the classifying map $\mathrm{B}_\Gamma \to \mathrm{B_G}$ is realized as $\mathrm{B}\,\alpha$, $\alpha : \pi_1(\mathrm{M}) = \Gamma \to \mathrm{G}$ the holonomy homomorphism.

Recall now that for a reductive pair (G, H) satisfying condition (C) of the appendix

$$(4.10)\qquad \mathrm{H}(\mathfrak{g}, \mathrm{H}) \cong \Lambda\hat{\mathrm{P}} \otimes \mathrm{I}(\mathrm{H})/\mathrm{I}^+(\mathrm{G}).\mathrm{I}(\mathrm{H}),$$

where $\mathrm{I}^+(\mathrm{G}).\mathrm{I}(\mathrm{H})$ denotes the ideal in $\mathrm{I}(\mathrm{H})$ generated by the image of the restriction $\mathrm{I}(\mathrm{G}) \to \mathrm{I}(\mathrm{H})$ (we use the notations of the appendix). $\Delta_*(\mathrm{P})$ restricted to $\mathrm{I}(\mathrm{H})/\mathrm{I}^+(\mathrm{G}).\mathrm{I}(\mathrm{H})$ is induced by the characteristic homomorphism $\mathrm{I}(\mathrm{H}) \to \mathrm{H}_{\mathrm{DR}}(\mathrm{M})$ of the H-reduction of P. Thus by (4.9) the classes of main interest obtained by $\Delta_*(\mathrm{P})$ are those coming from $\Lambda\hat{\mathrm{P}}$. Moreover a linear basis of $\Lambda\hat{\mathrm{P}}$ leads to linearly independent cohomology classes in $\mathrm{H}(\mathfrak{g}, \mathrm{H})$.

As subspace of the primitive elements $\mathrm{P}_\mathfrak{g}$ of $\mathfrak{g}$, all elements of $\hat{\mathrm{P}}$ are suspensions σ of invariant polynomials $\Phi \in \mathrm{I}(\mathrm{G})$. If condition (CS) of the appendix holds, $\hat{\mathrm{P}}$ is spanned by the suspensions $\sigma\Phi$ of $\Phi \in \ker(\mathrm{I}(\mathrm{G}) \to \mathrm{I}(\mathrm{H}))$. It is therefore useful to have an explicite formula for the suspension of invariant polynomials. Recall that by Hopf-Samelson $\mathrm{H}(\mathfrak{g}) \cong (\Lambda\,\mathfrak{g}^*)^\mathfrak{g} \cong \Lambda\,\mathrm{P}_\mathfrak{g}$ [29]. To give a formula for $\sigma\Phi$ let $\mathrm{X}_1, \ldots, \mathrm{X}_m$ be a basis of $\mathfrak{g}$ with dual basis X_j^* of $\mathfrak{g}^* = \Lambda^1\mathfrak{g}^*$ and corresponding basis $\tilde{\mathrm{X}}_j^*$ of $\mathrm{S}^1\mathfrak{g}^*$. Then for an invariant polynomial of degree p :

$$\Phi = \sum_{j_1 \ldots j_p} a_{j_1 \ldots j_p} \tilde{\mathrm{X}}_{j_1}^* \ldots \tilde{\mathrm{X}}_{j_p}^* \in \mathrm{I}^{2p}(\mathrm{G})$$

with symmetric coefficients $a_{j_1 \ldots j_p}$ we have ([28], [14]) :

$$(4.11)\qquad \sigma\Phi = \frac{(p-1)!\,p!}{(2p-1)!} \sum_{j_1 \ldots j_p} a_{j_1 \ldots j_p} d_\Lambda \mathrm{X}_{j_1}^* \wedge \ldots \wedge d_\Lambda \mathrm{X}_{j_{p-1}}^* \wedge \mathrm{X}_{j_p}^*$$

with $d_\Lambda \mathrm{X}_{j_k}^* \in \Lambda^2\mathfrak{g}^*$. The following fact is useful.

(4.12) LEMMA. — *Let* $\Phi \in I^{2p}(G)$ *restrict to zero on* H. *Then* $\sigma\Phi \in (\Lambda \mathfrak{g}^*)_H$, i. e. $\sigma\Phi$ *is* H-*basic.*

This is proved in section 5 (Corollary 5.26). For H-basic forms (4.11) simplifies then considerably. We return to this in an example at the end of this section.

We turn now to applications of Theorem (4.5) (G and H compact). For $\Gamma = \{e\}$ note that P is the G-extension of the H-bundle $G \to G/H$. Let more generally $P' \to M$ be an H-bundle with trivial G-extension $P = P' \times_H G$. The fibration $G/H \to B_H \to B_G$ shows that P′ together with a trivialization of P is characterized by a homotopy class $f : M \to G/H$. Diagram (4.9) is in this case induced by the commutative diagram of space maps

(4.13)
$$\begin{tikzcd} B_G & & B_H \arrow[ll] \\ M \arrow[u, "g\simeq 0"] \arrow[urr, "g'"] \arrow[rr, "f"'] & & G/H \arrow[u] \end{tikzcd}$$

where g classifies P, g' classifies P′. The generalized characteristic homomorphism Δ_* (P) of the trivial G-bundle P with the H-reduction P′ factorizes then as follows

(4.14)
$$\Delta_* : \quad H(\mathfrak{g}, H) \underset{\cong}{\xrightarrow{\gamma_*}} H_{DR}(G/H) \xrightarrow{f_*} H_{DR}(M)$$

and Δ_* depends only on the homotopy class of $f : M \to G/H$.

A first application of this construction is to the case $G = U(m)$ and $H = O(m)$. It is well-known that

$$H(\mathfrak{u}(m), O(m)) \cong \Lambda(y_1, y_3, \ldots, y_{m'}),$$

where $m' = 2[(m+1)/2]-1$ is the largest odd integer $\leqq m$. The generators y_i of degree $2i-1$ are primitive elements transgressing to the odd Chern classes $c_i \in I(U(m))$. With the definition

$$c(A) = \sum c_j(A)\, t^j = \det\left(\mathrm{Id} - \frac{t}{2\pi i} A\right) \qquad \text{for} \quad A \in \mathfrak{u}(m)$$

this gives a normalization of the y_i and e. g.

$$y_1 = \frac{i}{2\pi} \operatorname{trace} \in (\Lambda^1 \mathfrak{u}(m)^*)_{O(m)}.$$

(4.15) THEOREM. — *Let* $P' \to M$ *be an* O (m)-*bundle with a trivial* U (m)-*extension* (*trivial complex bundles with real structures*). *There are well-defined secondary characteristic invariants*

$$\Delta_*(y_i) \in H_{DR}^{2i-1}(M) \quad \text{for } i = 1, 3, \ldots, m', \qquad m' = 2\left[\frac{m+1}{2}\right] - 1.$$

These invariants are according to Theorem 3.5 obstructions to the triviality of the real structure on a trivial complex vectorbundle.

To give an interpretation of $\Delta_*(y_1)$, we refer now to the cohomology class introduced by Maslov and which intervenes in quantization conditions (*see* the discussion by Arnold in [1]).

(4.16) PROPOSITION. — *Let* $P' \to M$ *be an* $O(m)$*-bundle with a trivial* $U(m)$*-extension* P, ω *the connection form of the trivial connection in* P. *Then* $\Delta_*(y_1)$ *is represented by a closed* 1*-form* $\Delta(\omega)(y_1)$ *on* M. *The Maslov class of* P' *is the characteristic class*

$$-2\,\Delta_*(y_1) \in H^1_{DR}(M)$$

and

$$-2\int_\gamma \Delta(\omega)(y_1) = \deg(\det^2 \circ f(\gamma)) \qquad \text{for} \quad \gamma \in \pi_1(M).$$

Proof. — The map $\det : U(m) \to S^1$ squared factorizes through $O(m)$ and defines $\det^2 : U(m)/O(m) \to S^1$. The RHS is then the degree of the map

$$S^1 \xrightarrow{f(\gamma)} U(m)/O(m) \xrightarrow{\det^2} S^1,$$

where $f : M \to U(m)/O(m)$ classifies P' with its trivialized $U(m)$-extension.

This is proved by observing that for the $O(m)$-reduction $\bar{s} : P' \to P$ given by $s : M \to P/O(m)$, we can represent $\Delta(\omega)(y_1)$ as the $O(m)$-basic 1-form $(i/2\pi)$ $\bar{s}^*$ (trace ω) on P'. It suffices to check the formula for the critical example

$$O(m) \to U(m) \to U(m)/O(m).$$

Observe further that for a lift $\tilde{\gamma}$ to $U(m)$ of $\gamma \in \pi_1(U(m)/O(m))$ clearly

$$\int_{\tilde{\gamma}} \bar{s}^* \text{ trace } \omega = \int_\gamma s^* \text{ trace } \omega.$$

Since $\pi_1(U(m)/O(m)) \cong \mathbf{Z}$, it suffices to verify that

$$-\frac{i}{\pi}\int_\gamma \bar{s}^* \text{trace}\, \omega = \deg(\det^2 \circ \gamma)$$

for a single path $\gamma : [0, 2\pi] \to U(m)$ which maps into a non-trivial loop in $\pi_1(U(m)/O(m))$. For the path $\gamma(\tau) = e^{i/2 \cdot t}$ in $U(m)$ it is then easily verified that the value of both terms is m, which completes the proof. ∎

At this place we digress to discuss the non-compact version of these classes. Let P be a flat $GL(m)$-bundle. It has an $O(m)$-reduction and the generalized characteristic homomorphism does not depend on the choice of the $O(m)$-reduction, since there is only one homotopy class of sections of $P/O(m) \to M$. Δ_* is a well-defined homomorphism

$$\Delta_* : \quad H(\mathfrak{gl}(m), O(m)) \to H^{\bullet}_{DR}(M).$$

Again $H(\mathfrak{gl}(m), O(m)) = \Lambda(y_1, y_3, \ldots, y_{m'})$, $m' = 2[m+1/2]-1$, where the primitive elements y_{2i-1} $(i = 1, \ldots, [(m+1)/2])$ are now transgressing to the Chern classes $c_{2i-1} \in I(GL(m))$.

Therefore

(4.17) THEOREM [25] (Theorem 4.5). — *Let* $P \to M$ *be a flat* $GL(m)$*-bundle. There are well-defined secondary characteristic invariants* $\Delta_*(y_i) \in H_{DR}^{2i-1}(M)$ *for* $i = 1, 3, \ldots, m'$. *If* P *is a flat* $O(m)$*-bundle, these invariants are zero.*

These classes are closely related to the invariants defined by Reinhart [32] and Goldman [13] on a leaf of a foliated bundle. The foliated normal bundle restricted to a leaf is flat, so that it carries the invariants described above.

Since the flat bundle P is completely characterized by the holonomy representation $\alpha : \pi_1(M) \to GL(m)$, it is interesting to determine the invariants $\Delta_*(y_i)$ from α. For the invariant $\Delta_*(y_1) \in H_{DR}^1(M)$ this is done by the following formula of [25] (Theorem 4.5).

(4.18) PROPOSITION. — *Let* P *be a flat* $GL(m)$*-bundle with connection form* ω. *Then* $\Delta_*(y_1)$ *is represented by a closed* 1*-form* $\Delta(\omega)(y_1)$ *on* M *and*

$$\int_\gamma \Delta(\omega)(y_1) = -\frac{1}{2\pi}\log|\det\alpha(\gamma)| \qquad \text{for} \quad \gamma \in \pi_1(M).$$

Note that by this formula $\Delta_*(y_1)$ is visibly not invariant under deformations, whereas this is the case for all invariants $\Delta_*(y_i)$, $i > 1$. Proposition 4.18 is proved by observing that for the $O(m)$-reduction $\bar{s} : P' \to P$ given by $s : M \to P/O(m)$, we can represent $\Delta(\omega)(y_1)$ as the $O(m)$-basic 1-form $\bar{s}^*$ trace ω on P'. Note that the normalization in (4.18) depends on the normalization for c_1 resp. y_1, which is here taken as $y_1 = 1/2\pi.\text{trace}$.

From (4.18) it follows that $\Delta_*(y_1)$ is non-zero if and only if the holonomy representation does not map the $(m \times m)$-matrices with determinant ± 1. In the following situation we obtain a non-trivial realization of this invariant.

(4.19) PROPOSITION. — *Let* M^m *be a compact affine hyperbolic manifold. Then* $\Delta_*(y_1)$ *is a non-trivial cohomology class.*

The hyperbolicity of the affine structure means that the universal covering of M^m is affinely isomorphic to an open convex subset of R^m containing no complete line (these are non-complete affine manifolds). According to Koszul [31], the hyperbolicity condition is equivalent to the existence of a closed 1-form α with positive definite covariant derivative $\nabla\alpha$ (symmetric since the connection is torsion-free). The 1-form $\Delta(\omega)(y_1)$ can be identified with α. From this it is now easy to conclude that $\Delta(\omega)(y_1)$ is not exact. Because if it were, the metric $\nabla\alpha$ would be the Hessian of a function $f : M \to \mathbf{R}$ in each critical point, which is absurd if M is compact. ■

We turn to a further application of the preceding constructions. Let $G = SO(2m)$ and $H = SO(2m-1)$, so that $G/H = S^{2m-1}$. For an $SO(2m-1)$-bundle $P' \to M$ with trivial $SO(2m)$-extension P there is a unique homotopy class $f : M \to S^{2m-1}$. Then

$$\ker(I(SO(2m)) \to I(SO\ 2m-1))$$

is generated by the (normalized) Pfaffian polynomial $e \in I^{2m}(SO(2m))$ which is defined as follows. Let $x_1, \ldots, x_{2m}$ be an orthonormal basis of $\mathbf{R}^{2m}$ with dual basis $x_1^*, \ldots, x_{2m}^*$ and define

$$\omega(A) = \sum_{i<j} A_{ij} x_i^* \wedge x_j^* \in \Lambda^2 \mathbf{R}^{2m*}$$

for $A = (A_{ij}) \in \mathfrak{so}(2m)$. $\omega(A)$ is $SO(2m)$-invariant and hence $\omega^m(A)$ an $SO(2m)$-invariant volume element on $\mathbf{R}^{2m}$. As such it is a non-zero multiple of $\mu = x_1^* \wedge \ldots \wedge x_{2m}^*$, and therefore $\omega^m(A) = (-1)^m . (2\pi)^m . m!\ e(A).\mu$ defines $e \in I^{2m}(SO(2m))$. The explicite formula for e in terms of a basis of $\mathfrak{so}(2m)^*$ is

$$e = \frac{(-1)^m}{(2\pi)^m m!} \sum_{\substack{i_k < j_k \\ k=1, \ldots, m}} \varepsilon_\sigma \tilde{X}^*_{i_1 j_1} \ldots \tilde{X}^*_{i_m j_m},$$

where $\varepsilon_\sigma = \operatorname{sign} \sigma$, $\sigma(1 \ldots 2m) = (i_1 j_1 \ldots i_m j_m)$. X_{ij} $(i < j)$ is the basis element of $\mathfrak{so}(2m)$ with a 1 in the i-th row and j-th column, -1 in the j-th row and i-th column and zero everywhere else. X_{ij}^* denotes the dual basis of $\mathfrak{so}(2m)^*$. Let

$$\mathfrak{so}(2m) = \mathfrak{so}(2m-1) \oplus \mathfrak{m}$$

be the symmetric space decomposition with $\mathfrak{m}$ spanned by the matrices

$$Z_\alpha = X_{\alpha 2m} \ (\alpha = 1, \ldots, 2m-1).$$

Note that

$$\sigma(e) \in (\Lambda^{2m-1} \mathfrak{so}(2m)^*)_{SO(2m-1)} \cong (\Lambda^{2m-1} \mathfrak{m}^*)^{SO(2m-1)}$$

is a multiple of the volume form

$$\mu = Z_1^* \wedge \ldots \wedge Z_{2m-1}^* \in (\Lambda^{2m-1} \mathfrak{m}^*)^{SO(2m-1)}.$$

(4.20) Proposition. — *Let the notation be as above. Then* $\sigma(e) = \alpha.\mu$ *with*

$$\alpha = \frac{-(m-1)!}{(2\pi)^m 2^{m-1} (2m-1)}.$$

Proof. — This follows by a direct computation using (4.11) and the fact that the volume of S^{2m-1} with respect to the invariant form μ is $(2/(m-1)!)\pi^m$. ∎

To give a geometric interpretation to the invariant $\sigma(e)$, consider the following situation.

(4.21) Theorem. — *Let* $h : M^{2m-1} \to \mathbf{R}^{2m}$ *be an isometric immersion of the compact oriented Riemannian manifold* M. *The tangent frame bundle* $SO(2m-1) \to P' \to M$ *has a trivial* $SO(2m)$*-extension* $P \to M$. *The generalized characteristic homomorphism* $\Delta_* = \Delta_*(P)$ *applied to* $\sigma(e)$ *given a top-dimensional cohomology class such that*

$$N(h) = -2^{2(m-1)}.(2m-1)\Delta_* \sigma(e)[M],$$

where $N(h)$ *is the normal degree of* h.

Proof. – The SO $(2m-1)$-bundle P′ with trivial SO $(2m)$-extension P is characterized by a map $f: M^{2m-1} \to S^{2m1}$ which (up to homotopy) is precisely the Gauss map g_h of h. Thus with the previous notations $\Delta_* = g_h^* \circ \gamma_*$ and in top degree $2m-1$,

$$\Delta_* = \deg(g_h).\gamma_* \equiv N(h).\gamma_*.$$

This establishes the functoriality of $\Delta_* \sigma(e)$ in the sense that

$$\Delta_* \sigma(e)[M^{2m-1}] = N(h)\cdot\Delta_* \sigma(e)[S^{2m-1}],$$

and it suffices to verify that

$$\Delta_* \sigma(e)[S^{2m-1}] = \int_{S^{2m-1}} \gamma_* \sigma(e) = \frac{-1}{2^{2(m-1)}(2m-1)}.$$

But with α as in proposition (4.20) :

$$\int_{S^{2m-1}} \gamma_* \sigma(e) = \alpha \int_{S^{2m-1}} \mu$$

and

$$\int_{S^{2m-1}} \mu = \frac{2}{(m-1)!}\pi^m,$$

which proves the desired result. ∎

It is of interest to contrast this last result with the situation when an even-dimensional compact manifold M^{2m} is immersed in R^{2m+1}. For such an immersion $h: M^{2m} \to \mathbf{R}^{2m+1}$ by Hopf ([18], [19]) the normal degree is given by $N(h) = (1/2)\chi(M)$, where $\chi(M)$ is the Euler number of M. In the framework of our discussion this formula follows from: (*a*) the functoriality

$$\Delta_* e[M] = N(h)\cdot\Delta_* e[S^{2m}]$$

of the primary invariant $\Delta_* e$ [e the Pfaffian in $I^{2m}(SO(2m))$], and (*b*) the evaluation

$$\Delta_* e[S^{2m}] = \int_{S^{2m}} \gamma_* e = 2$$

of this invariant on the sphere S^{2m}.

In contrast to the primary nature of the Euler number (its definition is independent of the immersion), our secondary invariant $\Delta_* \sigma(e)$ needs for its definition on M^{2m-1} beside the Riemannian structure a trivialization of $\tau_M \oplus \varepsilon_1$, where τ_M is the tangent bundle of M and ε_1 a trivial line bundle on S^{2m-1}. It is in fact an invariant defined for Riemannian π-manifolds M^{2m-1} and allows to test the Riemannian immersability of M^{2m-1} in R^{2m}. Note that Theorem 4.21 holds also for an isometric immersion $h: M^{2m-1} \to N^{2m}$ into a Riemannian parallelizable manifold N^{2m} [the Gauss map g_h and the normal degree $N(h) = \deg(g_h)$ are then defined].

5. Reductive pairs

In the next section the generalized characteristic homomorphism is evaluated for certain symmetric pairs. In this section we prove for reductive pairs general results needed below, some in slightly more general form in view of other applications. *See* also the outline of this section in the introduction.

Let $(\overline{G}, G)$ be a reductive pair with G connected. θ denotes a G-equivariant splitting of the exact sequence

$$(5.1) \qquad 0 \to \mathfrak{g} \overset{\theta}{\leftrightarrows} \overline{\mathfrak{g}} \to \overline{\mathfrak{g}}/\mathfrak{g} \to 0.$$

We assume further that

$$(5.2) \qquad \dim \overline{\mathfrak{g}}/\mathfrak{g} = 2q'.$$

This is always true in the case of greatest interest, where rank $\mathfrak{g}$ equals rank $\overline{\mathfrak{g}}$ [Lemma (5.11)]. For the sake of simplicity we let $H = \{e\}$. If $\Gamma \subset \overline{G}$ denotes a discrete uniform subgroup, the foliated bundle under discussion is

$$(5.3) \qquad P = \Gamma \backslash \overline{G} \times G \to M = \Gamma \backslash \overline{G}.$$

By Theorem (3.11) the crucial map to evaluate is the cohomology map $\Delta(\theta)_*$ induced by the G-DG-homomorphism

$$(5.4) \qquad \Delta(\theta) : \quad W(\mathfrak{g})_{q'} \to \Lambda \overline{\mathfrak{g}}^*$$

determined by $\theta : \overline{\mathfrak{g}} \to \mathfrak{g}$ (note that the truncation index is q').

Let as in section 3 :

$$(5.5) \qquad \overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{m}, \qquad \mathfrak{m} = \ker \theta,$$

so that $\Lambda \overline{\mathfrak{g}}^* = \Lambda \mathfrak{m}^* \otimes \Lambda \mathfrak{g}^*$. The map (5.4) is then induced from the G-DG-homomorphism $\Delta(\theta) : W(\mathfrak{g}) \to \Lambda \overline{\mathfrak{g}}^*$. Since $\Delta(\theta)$ preserves filtrations by (3.22), it maps $F^{2(q'+1)}(\mathfrak{g}) W(\mathfrak{g})$ into zero and hence induces the map 5.4 on the quotient

$$W(\mathfrak{g})_{q'} = W(\mathfrak{g})/F^{2(q'+1)} W(\mathfrak{g}).$$

For the computation of $\Delta(\theta)_*$ we replace the complexes in (5.4) by the corresponding cohomologically equivalent complexes according to the algorithm of [23], which is reviewed in the appendix. We have then the following result, which is the framework for the explicite computations done below.

(5.6) THEOREM. — *Let* $(\overline{G}, G)$ *be a reductive pair and* $\theta : \overline{\mathfrak{g}} \to \mathfrak{g}$ *a* G-*equivariant splitting of* (5.1). *There is a commutative diagram of filtration preserving* G-DG-*homomorphisms*

$$(5.7) \qquad \begin{array}{ccc} W(\mathfrak{g})_{q'} & \xrightarrow{\Delta(\theta)} & \Lambda \overline{\mathfrak{g}}^* \\ \varphi \uparrow \simeq & & \varphi \uparrow \simeq \\ \Lambda P_{\mathfrak{g}} \otimes I(G)_{q'} & \xrightarrow{\mathrm{id} \otimes h(\theta)} & \Lambda P_{\mathfrak{g}} \otimes (\Lambda \mathfrak{m}^*)^G \end{array}$$

where $h(\theta) = \Delta(\theta)_G : I(G)_{q'} \to (\Lambda\, \mathfrak{m}^*)^G$ *is induced by the (ordinary) characteristic homomorphism of* $(\overline{G}, G)$ *with values in the invariant forms* $(\Lambda\, \mathfrak{m}^*)^G$, *and where the vertical maps are cohomology isomorphisms.*

Here we have used the identification $(\Lambda\, \mathfrak{m}^*)^G \cong (\Lambda\, \bar{\mathfrak{g}}^*)_G$. If G is closed in $\overline{G}$, $h(\theta)$ is the characteristic homomorphism of the G-bundle $\overline{G} \to \overline{G}/G$ with values in the invariant forms. But the facts above hold regardless of this property. The result follows from the appendix. Note that φ restricted to $I(G)_{q'}$ and $(\Lambda\, \mathfrak{m}^*)^G$ is the canonical inclusion. On the primitive elements $P_{\mathfrak{g}}$ the definition of φ *via* the difference homomorphism λ^1 is explained in the appendix.

For compact connected $\overline{G}$ and closed $G \subset \overline{G}$ we have

$$H((\Lambda\, \mathfrak{m}^*)^G) \cong H(\bar{\mathfrak{g}}, G) \cong H(\overline{G}/G).$$

For $\Phi \in I^{2p}(G)$ the well-known formula

$$h(\theta)(\Phi) = \Phi(K(\theta) \wedge \ldots \wedge K(\theta))$$

reduces in view of $K(\theta) = -\theta[\ ,\ ]$ (since $[\theta, \theta] = 0$ on $\mathfrak{m}$) to

$$h(\theta)\Phi = (-1)^p \Phi(\underbrace{\theta[\ ,\ ] \wedge \ldots \wedge \theta[\ ,\ ]}_{p \text{ factors}}) \in (\Lambda^{2p}\, \mathfrak{m}^*)^G. \tag{5.8}$$

Note that for a symmetric pair in particular this simplifies in view of $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{g}$ further to

$$h(\theta)\Phi = (-1)^p \Phi(\underbrace{[\ ,\ ] \wedge \ldots \wedge [\ ,\ ]}_{p \text{ factors}}). \tag{5.9}$$

The main effect of the preceding Theorem is the expression of the secondary characteristic homomorphism $\Delta(\theta)$ of the foliated bundle (5.3) by the primary characteristic homomorphism $h(\theta)$ of the pair $(\overline{G}, G)$. This is made clear for the corresponding characteristic numbers by the following result.

(5.10) THEOREM. — *Let* $\overline{G}$ *be compact,* G *a connected subgroup* and $\Gamma = \{e\}$, *or* $\overline{G}$ *connected semi-simple non-compact,* G *a compact subgroup and* $\Gamma \subset \overline{G}$ *a discrete, uniform and torsionfree subgroup. Then the generalized characteristic homomorphism of the foliated bundle*

$$P = \Gamma \backslash \overline{G} \times G \to M = \Gamma \backslash \overline{G}$$

in degree $n = \dim \bar{\mathfrak{g}}$:

$$\Delta(P)_* : \ H^n(W(\mathfrak{g})_{q'}) \xrightarrow{\Delta(\theta)_*} H^n(\bar{\mathfrak{g}}) \underset{\cong}{\xrightarrow{\gamma_*}} H^n(M) \cong \mathbf{R}$$

is onto if and only if the characteristic homomorphism of $(\overline{G}, G)$ *with values in the invariant forms in degree* $2q' = \dim \bar{\mathfrak{g}}/\mathfrak{g}$:

$$h(\theta)_* : \ I^{2q'}(G) \to H^{2q'}(\bar{\mathfrak{g}}, G) \cong H^{2q'}((\Lambda\, \mathfrak{m}^*)^G) \cong \mathbf{R}$$

is onto.

When $\overline{\mathrm{G}}$ and G are compact, $h(\theta)_*$ is the characteristic homomorphism of the G-bundle $\overline{\mathrm{G}} \to \overline{\mathrm{G}}/\mathrm{G}$. In the case where $\overline{\mathrm{G}}$ is non-compact, $h(\theta)_*$ takes values in the cohomology of the G-invariant forms on $\overline{\mathrm{G}}/\mathrm{G}$. Note that in both cases γ_* in degree u is injective, hence an isomorphism.

In particular we have the following consequence of Theorem (5.10).

(5.10)′ COROLLARY. — *If* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is a pair of equal rank, then* $h(\theta)_*$ *is surjective* [*see* (5.11)]. *Therefore* $\Delta(\mathrm{P})_*$ *in degree* $n = \dim \bar{\mathfrak{g}}$ *is onto.*

We collect for reference a few facts needed for reductive and symmetric pairs.

(5.11) LEMMA. — *Let* $(\overline{\mathrm{G}}, \mathrm{G})$ *be a reductive pair of Lie groups with* G *connected. Then the following facts hold.*

(i) *The exact sequence* $0 \to \mathfrak{g} \to \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\mathfrak{g} = \mathfrak{m} \to 0$ *is a sequence of unimodular* G-*modules, and hence* $(\Lambda^q \mathfrak{m}^*)^{\mathrm{G}} \cong \mathrm{R}$ *for* $q = \dim \mathfrak{m}$.

(ii) *The differential* $d : (\Lambda^{q-1} \mathfrak{m}^*)^{\mathrm{G}} \to \Lambda^q \mathfrak{m}^*$ *is zero, and hence*

$$\mathrm{H}^q(\bar{\mathfrak{g}}, \mathrm{G}) \equiv \mathrm{H}^q(\Lambda\, \mathfrak{m}^*)^{\mathrm{G}}) = \Lambda^q \mathfrak{m}^* \cong \mathrm{R}.$$

(iii) *If* $\bar{\mathfrak{g}}$ *and* $\mathfrak{g}$ *are of equal rank, then* $h(\theta)_* : \mathrm{I}(\mathrm{G}) \to \mathrm{H}(\bar{\mathfrak{g}}, \mathrm{G})$ *is onto, and hence* $q = 2q'$ *is even.*

(iv) *If* $(\overline{\mathrm{G}}, \mathrm{G})$ *is a symmetric pair, then the differential in* $(\Lambda\, \mathfrak{m}^*)^{\mathrm{G}}$ *is zero and hence* $\mathrm{H}(\bar{\mathfrak{g}}, \mathrm{G}) \cong (\Lambda\, \mathfrak{m}^*)^{\mathrm{G}}$.

(v) *If* $(\overline{\mathrm{G}}, \mathrm{G})$ *is a symmetric pair with* $\mathfrak{g}$ *and* $\bar{\mathfrak{g}}$ *of equal rank, then* $h(\theta) : \mathrm{I}(\mathrm{G})_{q'} \to (\Lambda\, \mathfrak{m}^*)^{\mathrm{G}}$ *is onto.*

Proof. — (i) The adjoint representations of G in $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ are semi-simple, trivial on the center, hence unimodular. Therefore $\mathfrak{m}$ is also a unimodular G-module. The existence of a G-invariant volume on $\mathfrak{m}$ shows that $(\Lambda^q \mathfrak{m}^*)^{\mathrm{G}} \cong \mathrm{R}$.

(ii) Choose a basis $x_1, \ldots, x_m$ of $\mathfrak{g}$ and a basis $y_1 = x_{m+1}, \ldots, y_q = x_{m+q}$ of $\mathfrak{m}$. Then $\alpha \in (\Lambda^{q-1} \mathfrak{m}^*)^{\mathrm{G}}$ can be written

$$\alpha = \sum_{i=1}^{q} a_i\, y_1^* \wedge \ldots \wedge \hat{y}_i^* \wedge \ldots \wedge y_q^* \qquad \text{with} \quad \Theta(x_j)\alpha = 0,$$

$j = 1, \ldots, m$. We need the following expressions valid in $\Lambda\, \bar{\mathfrak{g}}^* = \Lambda\, \mathfrak{m}^* \otimes \Lambda\, \mathfrak{g}^*$ modulo $\Lambda\, \mathfrak{m}^* \otimes \Lambda\, \mathfrak{g}^{*+}$:

$$\begin{aligned}
&\Theta(y_k)(y_1^* \wedge \ldots \wedge \hat{y}_i^* \wedge \ldots \wedge y_q^*)\\
&\quad = -\sum_{j \neq i} \sum_{l=1}^{q} \bar{c}_{kl}^{j}\, y_1^* \wedge \ldots \wedge \underset{j\text{-th place}}{y_l^*} \wedge \ldots \wedge \hat{y}_i^* \wedge \ldots \wedge y_q^*\\
&\quad = -\sum_{j \neq i} \bar{c}_{kj}^{j}\, y_1^* \wedge \ldots \wedge \hat{y}_i^* \wedge \ldots \wedge y_q^* - \sum_{j \neq i} (-1)^{i+j}\, \bar{c}_{ki}^{j}\, y_1^* \wedge \ldots \wedge \hat{y}_j^* \wedge \ldots \wedge y_q^*,
\end{aligned}$$

where $\Theta(y_k)\, y_j^* = -\sum_{l=1}^{q} \bar{c}_{kl}^{j}\, y_l^*$. Then we have

$$d\alpha = \frac{1}{2} \sum_{k=1}^{q} y_k^* \wedge \Theta(y_k)\alpha = \frac{1}{2} \sum_{k,i=1}^{q} a_i\, y_k^* \wedge \Theta(y_k)(y_1^* \wedge \ldots \wedge \hat{y}_i^* \wedge \ldots \wedge y_q^*) = \varkappa . y_1^* \wedge \ldots \wedge y_q^*,$$

where

$$\varkappa = \frac{1}{2} \sum_{i=1}^{q} \sum_{j \neq i} (-1)^{i+1} a_i (\bar{c}^j_{ij} + \bar{c}^j_{ji}).$$

But $\varkappa = 0$ in view of the skew-symmetry of the $\bar{c}'$ s, which up to a shift of indices are the structure constants of $\bar{\mathfrak{g}}$. Therefore $d\,\alpha = 0$.

(iii) For the equal rank case $h(\theta)_* : \mathrm{I}(\mathrm{G}) \to \mathrm{H}(\bar{\mathfrak{g}}, \mathrm{G})$ is onto by [14] (vol. III) (*see* also the beginning of section 6). Since I (G) is evenly graded, so is $\mathrm{H}(\bar{\mathfrak{g}}, \mathrm{G})$. But $\mathrm{H}^q(\bar{\mathfrak{g}}, \mathrm{G})$ is non-trivial by (ii). Thus q must be even.

(iv) This is well-known, and follows from the compatibility of the differential with the involutive symmetry, whose differential at the identity has $\mathfrak{m}$ as eigenspace corresponding to the eigenvalue -1.

(v) Follows from (iii) and (iv). ∎

For the proof of Theorem (5.10) we need the following result.

(5.12) LEMMA. — *Let* $\Phi \in \mathrm{I}^{2q'}(\mathrm{G})$ *and* $y_1, \ldots, y_r$ *a basis of the primitive space* $\mathrm{P}_{\mathfrak{g}}$ (r = rank of $\mathfrak{g}$). *Then for* $1 \leqq i_1 < \ldots < i_s \leqq r$ *we have*

$$\varphi(y_{i_1} \wedge \ldots \wedge y_{i_s} \otimes \Phi) = y_{i_1} \ldots y_{i_s} \Phi \in \mathrm{W}(\mathfrak{g})_{q'}.$$

Proof. — By (A.5) (*see* appendix) we have for the suspension $\sigma\, c = y$ of $c \in \mathrm{I}^+(\mathrm{G})$ the universal transgression $\mathrm{T}\,c \in \mathrm{W}(\mathfrak{g})$:

$$\mathrm{T}\,c \equiv \sigma\, c = y \bmod \mathrm{F}^2\,\mathrm{W}(\mathfrak{g}), \tag{5.13}$$

since the projection π has as kernel precisely $\mathrm{F}^2\,\mathrm{W}(\mathfrak{g})$. By definition of φ:

$$\varphi(y_{i_1} \wedge \ldots \wedge y_{i_s} \otimes \Phi) = \mathrm{T}\,c_{i_1} \ldots \mathrm{T}\,c_{i_s} \cdot \Phi \in \mathrm{F}^{2q'}\,\mathrm{W}(\mathfrak{g})$$

But (5.13) implies that

$$\mathrm{T}\,c_{i_1} \ldots \mathrm{T}\,c_{i_s} . \Phi \equiv \sigma\, c_{i_1} \ldots \sigma\, c_{i_s} . \Phi = y_{i_1} \ldots y_{i_s} . \Phi \bmod \mathrm{F}^{2(q'+1)}\,\mathrm{W}(\mathfrak{g})$$

which proves the desired result. ∎

Proof of Theorem 5.10. — The monomials $y_{i_1} \wedge \ldots \wedge y_{i_s} \otimes \Phi$, $\Phi \in \mathrm{I}^{2q'}(\mathrm{G})$ are cocycles in $\wedge\, \mathrm{P}_{\mathfrak{g}} \otimes \mathrm{I}(\mathrm{G})_{q'}$, and it is convenient to identify them with their images under φ in $\mathrm{W}(\mathfrak{g})_{q'}$. Using (5.7) we have then in particular in degree $n = \dim \bar{\mathfrak{g}}$:

$$\Delta(\theta)(y_1 \ldots y_r . \Phi) = y_1 \ldots y_r . h(\theta)\,\Phi. \tag{5.14}$$

Therefore $\Delta(\theta)_*$ in degree n is non-zero if and only if $h(\theta)$ in degree $2\,q'$ is non-zero, which proves Theorem (5.10). ∎

(5.14) together with (5.8) gives the explicit formula

$$\Delta(\theta)(y_1 \ldots y_r . \Phi) = (-1)^{q'} y_1 \ldots y_r . \Phi(\underbrace{\theta[\ ,\ ] \wedge \ldots \wedge \theta[\ ,\ ]}_{q' \text{ factors}}).$$

In the case of a symmetric pair, θ inside Φ is redundant, since the bracket $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{g}$ and $\theta/\mathfrak{g} = \mathrm{id}$.

If G is compact and $f : \overline{G} \to \overline{G}/G$, then the primary invariant $h(\theta)\,\Phi$ and the secondary invariant $\Delta(\theta)(y_1 \ldots y_r . \Phi)$ are related via integration f_* over the fiber G by the formula

$$f_*(\Delta(\theta)(y_1 \ldots y_r . \Phi)) = \int_G (y_1 \ldots y_r) . h(\theta)\,\Phi \in (\Lambda^{2q'} \mathfrak{m}^*)^G,$$

where $\displaystyle\int_G y_1 \ldots y_r$ is a non-zero constant.

For the computations carried out for symmetric pairs $(\overline{G}, G)$ in the next section, we need a formula for the suspension of an invariant polynomial $\Phi \in I(\overline{G})^+$ in terms of the $\mathfrak{g}$-DG-algebra structure of $\Lambda \bar{\mathfrak{g}}^*$ defined by the splitting $\theta : \bar{\mathfrak{g}} \to \mathfrak{g}$. As this formula is of independent interest, we derive it under the sole assumption that the sequence (5.1) admits a G-invariant splitting θ.

Consider the diagram

(5.15)
$$\begin{tikzcd}
\Lambda \mathfrak{g}^* \arrow[r, "\Lambda\theta^*"] \arrow[d, shift right, "u"'] & \Lambda \bar{\mathfrak{g}}^* \arrow[d, shift right, "\bar{u}"'] \\
W(\mathfrak{g}) \arrow[u, shift right, "\pi"'] \arrow[ur, "\Delta(\theta)"] \arrow[r, shift left, "k(\theta)"] & W(\bar{\mathfrak{g}}) \arrow[u, shift right, "\bar{\pi}"'] \arrow[l, shift left, "W(i)"] \\
I(G) \arrow[u, hook] & I(\overline{G}) \arrow[l, "i^*"'] \arrow[u, hook, "j"']
\end{tikzcd}$$

in which u and $\bar{u}$ denote the canonical (universal) connections in $W(\mathfrak{g})$ and $W(\bar{\mathfrak{g}})$, and $k(\theta)$ the Weil homomorphism of the $\mathfrak{g}$-connection $\bar{u} \circ \Lambda\theta^* : \Lambda\mathfrak{g}^* \to W(\bar{\mathfrak{g}})$, i. e. the unique $\mathfrak{g}$-homomorphism $W(\mathfrak{g}) \to W(\bar{\mathfrak{g}})$ such that $k(\theta) \circ u = \bar{u} \circ \Lambda\theta^* : \Lambda\mathfrak{g}^* \to W(\bar{\mathfrak{g}})$ By the universal property of $W(\mathfrak{g})$, we have then

$$\bar{\pi} \circ k(\theta) \circ u = \bar{\pi}(\bar{u} \circ \Lambda\theta^*) = \Lambda\theta^* = \Delta(\theta) \circ u$$

and therefore

$$(5.16) \qquad \Delta(\theta) = \bar{\pi} \circ k(\theta).$$

In other words : $k(\theta)$ is a lift of $\Delta(\theta)$ to $W(\bar{\mathfrak{g}})$. This induces in particular on G-basic elements

$$(5.17) \qquad h(\theta) = \bar{\pi}_G \circ k(\theta)_G, \qquad \text{where} \quad h(\theta) = \Delta(\theta)_G.$$

The Weil homomorphism $k(\theta) : W(\mathfrak{g}) \to W(\bar{\mathfrak{g}})$ in diagram (5.15) has a useful property with respect to the canonical Koszul filtrations $F(\mathfrak{g})$ on $W(\mathfrak{g})$ and $F(\bar{\mathfrak{g}})$ on $W(\bar{\mathfrak{g}})$ defined by 3.1.

(5.18) LEMMA. — *The Weil-homomorphism* $k(\theta)$ *satisfies*

$$k(\theta)\, F^{2(p+q')}(\mathfrak{g})\, W(\mathfrak{g}) \subset F^{2p}(\bar{\mathfrak{g}})\, W(\bar{\mathfrak{g}}) \qquad \textit{for} \quad p \geqq 0,$$

where $q' = [q/2]$, $q = \dim \bar{\mathfrak{g}}/\mathfrak{g}$.

Proof. — By definition, $k(\theta)$ is determined by $k(\theta) \mid \Lambda \mathfrak{g}^* \otimes 1 = \Lambda \theta^*$ and

$$k(\theta)(1 \otimes \tilde{\alpha}) = \alpha \mathrm{K}(\theta) \otimes 1 + 1 \otimes \widetilde{\overset{*}{\theta} a} \in \mathrm{W}(\bar{\mathfrak{g}}) \quad \text{for} \quad \tilde{\alpha} \in \mathrm{S}^1(\mathfrak{g})^*, \tag{5.19}$$

where

$$\Delta(\theta)\tilde{\alpha} = \alpha \mathrm{K}(\theta) = d_\Lambda \alpha\theta + \frac{1}{2}\alpha[\theta, \theta] \in \Lambda^2 \bar{\mathfrak{g}}^*.$$

By Lemma 3.22 $\alpha \mathrm{K}(\theta) \in \Lambda^2 \mathfrak{m}^*$ and hence

$$\alpha_1 \mathrm{K}(\theta) \wedge \ldots \wedge \alpha_s \mathrm{K}(\theta) = 0 \quad \text{for} \quad s > q'. \tag{5.19'}$$

As $k(\theta)$ is a DG-algebra homomorphism it is sufficient to verify (5.18) for elements of the form $1 \otimes \tilde{\alpha}_1 \ldots \tilde{\alpha}_{p+q'} \in 1 \otimes \mathrm{S}^{p+q'}(\mathfrak{g}^*) \subset \mathrm{W}(\mathfrak{g})$. By (5.19), (5.19') we have

$$k(\theta)(1 \otimes \tilde{\alpha}_1 \ldots \tilde{\alpha}_{p+q'}) = \sum_{s=0}^{q'} \sum_{\sigma = (i_1, \ldots, i_{p+q'}) \in \mathrm{S}_{p+q'}} \alpha_{i_1} \mathrm{K}(\theta) \wedge \ldots \wedge \alpha_{i_s} \mathrm{K}(\theta) \otimes \widetilde{\overset{*}{\theta} \alpha_{i_{s+1}}} \ldots \widetilde{\overset{*}{\theta} \alpha_{i_{p+q'}}},$$
$$i_1 < \ldots < i_s, \qquad i_{s+1} < \ldots < i_{p+q'},$$

where $\mathrm{S}_{p+q'}$ denotes the symmetric group in $p+q'$ variables. But $p+q'-s \geqq p$ and hence the above expression is contained in $\mathrm{F}^{2p}(\bar{\mathfrak{g}})\,\mathrm{W}(\bar{\mathfrak{g}})$. ∎

Consider now the universal homotopy operator $\tilde{\lambda}^1 : \mathrm{W}(\bar{\mathfrak{g}}) \to \mathrm{W}(\bar{\mathfrak{g}}) \otimes \mathrm{W}(\bar{\mathfrak{g}})$ (*see* the appendix or [25'], 5.54). Since id $\otimes \mathrm{W}(i)$ is a $\mathfrak{g}$-DG-homomorphism and so is the canonical map

$$\alpha = (\mathrm{id}, k(\theta)) : \mathrm{W}(\bar{\mathfrak{g}}) \otimes \mathrm{W}(\mathfrak{g}) \to \mathrm{W}(\bar{\mathfrak{g}}),$$

it follows that the composition

$$\alpha \circ (\mathrm{id} \otimes \mathrm{W}(i)) \circ \tilde{\lambda}^1 : \mathrm{W}(\bar{\mathfrak{g}}) \to \mathrm{W}(\bar{\mathfrak{g}})$$

is compatible with the operators $i(x)$, $\Theta(x)$ for $x \in \mathfrak{g}$ and therefore induces a map

$$\lambda^1 : \mathrm{W}(\bar{\mathfrak{g}}, \mathrm{G}) \to \mathrm{W}(\bar{\mathfrak{g}}, \mathrm{G})$$

of degree -1. The homotopy formula

$$d\tilde{\lambda}^1 + \tilde{\lambda}^1 d = \varepsilon_0 - \varepsilon_1 : \mathrm{W}(\bar{\mathfrak{g}}) \to \mathrm{W}(\bar{\mathfrak{g}}) \otimes \mathrm{W}(\bar{\mathfrak{g}})$$

implies immediately the homotopy formula

$$d\lambda^1 + \lambda^1 d = k(\theta)_{\mathrm{G}} \circ \mathrm{W}(i)_{\mathrm{G}} - \mathrm{id}. \tag{5.20}$$

Restricted to $\mathrm{I}(\bar{\mathrm{G}}) \overset{j}{\hookrightarrow} \mathrm{W}(\bar{\mathfrak{g}}, \mathrm{G})$ this implies by $d \mid \mathrm{I}(\bar{\mathrm{G}}) = 0$ the formula

$$d\lambda^1 = k(\theta)_{\mathrm{G}} \circ i^* - j : \mathrm{I}(\bar{\mathrm{G}}) \to \mathrm{W}(\bar{\mathfrak{g}}, \mathrm{G}), \tag{5.21}$$

where $i^* : \mathrm{I}(\bar{\mathrm{G}}) \to \mathrm{I}(\mathrm{G}) = \mathrm{W}(\mathfrak{g}, \mathrm{G})$. Let now $\Phi \in \mathrm{I}(\bar{\mathrm{G}})^{2p}$, $p > 0$ and choose any transgressive cochain $\mathrm{T}(i^* \Phi) \in \mathrm{W}(\mathfrak{g})$ of the restriction $i^* \Phi \in \mathrm{I}(\mathrm{G})^{2p}$, so that $d\,\mathrm{T}(i^* \Phi) = i^* \Phi$. Then we have the following result.

(5.22) PROPOSITION. — *Let* $\Phi \in I(\bar{G})^{2p}$, $p > 0$. *The element*

$$\bar{T}\Phi = k(\theta)T(i^*\Phi) - \lambda^1(\Phi) \in W(\bar{\mathfrak{g}}) \tag{5.23}$$

is a transgressive cochain of Φ, *and the suspension of* Φ *is given by*

$$\bar{\sigma}(\Phi) = [\Delta(\theta)T(i^*\Phi) - (\lambda^1\Phi)^{2p-1,0}] \in H^{2p-1}(\bar{\mathfrak{g}}), \tag{5.24}$$

where $\bar{\pi}_G(\lambda^1\Phi) = (\lambda^1\Phi)^{2p-1,0} \in (\Lambda^{2p-1}\mathfrak{m}^*)^G$ *is the component of bidegree* $(2p-1, 0)$ *of* $\lambda^1\Phi \in W(\bar{\mathfrak{g}}, G) = (\Lambda^\cdot \mathfrak{m}^* \otimes S^\cdot \bar{\mathfrak{g}}^*)^G$.

Formula (5.23) expresses a transgressive cochain of $\Phi \in I(\bar{G})^+$ in terms of the restriction $i^*\Phi \in I(G)$ and the canonical G-basic cochain $\lambda^1\Phi \in W(\bar{\mathfrak{g}}, G)$.

Proof. — The element $\bar{T}\Phi = k(\theta)T(i^*\Phi) - \lambda^1\Phi$ is transgressive to $\Phi \in I(\bar{G})$ exactly if $d\bar{T}\Phi = \Phi$ But by (5.21):

$$\begin{aligned} d\bar{T}(\Phi) &= dk(\theta)T(i^*\Phi) - d\lambda^1(\Phi) = k(\theta)\,dT(i^*\Phi) + (\Phi - k(\theta)_G(i^*\Phi)) \\ &= k(\theta)_G(i^*\Phi) + (\Phi - k(\theta)_G(i^*\Phi)) = \Phi \end{aligned}$$

since $dT(i^*\Phi) = i^*\Phi$. Consequently we obtain by (5.16) for the suspension

$$\bar{\sigma}(\Phi) = \bar{\pi}\bar{T}(\Phi) = \Delta(\theta)T(i^*\Phi) - \bar{\pi}_G(\lambda^1\Phi).$$

But $\bar{\pi}_G$ clearly projects $W(\bar{\mathfrak{g}}, G) = (\Lambda\,\mathfrak{m}^* \otimes S\,\bar{\mathfrak{g}}^*)^G$ to $(\Lambda\,\mathfrak{m}^*)^G$ along the ideal generated by $(S\,\bar{\mathfrak{g}}^*)^+$ and (5.24) follows. ■

Proposition (5.22) has the following consequences.

(5.25) COROLLARY. — *Suppose that the differential* $d_\Lambda : (\Lambda^{\text{even}}\mathfrak{m}^*)^G \to (\Lambda^{\text{odd}}\mathfrak{m}^*)^G$ *is surjective. Then*

(i) $\bar{\sigma}(\Phi) = [\Delta(\theta)T(i^*\Phi)] \in H^{\text{odd}}(\bar{\mathfrak{g}})$ *for* $\Phi \in I(\bar{G})^+$,

(ii) $h(\theta) \circ i^* = 0$,

(iii) $k(\theta)_G \circ i^* : I(\bar{G})^+ \to \bar{F}^2 W(\bar{\mathfrak{g}}, G)$.

If G is closed in $\bar{G}$, the assumption means that all $\bar{G}$-invariant forms of odd degree on the homogeneous space $\bar{G}/G$ are coboundaries of invariant forms. This assumption implies that $H^{\text{odd}}(\bar{\mathfrak{g}}, G) = 0$, and hence that $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ [in case $(\bar{\mathfrak{g}}, \mathfrak{g})$ is a reductive pair] are of equal rank [14]. But the converse need not be true. The condition is satisfied e. g. for symmetric pairs of equal rank, where $(\Lambda^{\text{odd}}\mathfrak{m}^*)^G = 0$ [*see* (5.11) and the beginning of section 6].

Proof. — By (5.24) the suspension $\bar{\sigma}(\Phi)$ is represented by the cocycle

$$z = \Delta(\theta)T(i^*\Phi) - \bar{\pi}_G(\lambda^1\Phi) \in \Lambda\,\bar{\mathfrak{g}}^*, \qquad \text{where} \quad \bar{\pi}_G(\lambda^1\Phi) \in (\Lambda\,\bar{\mathfrak{g}}^*)_G = (\Lambda^{\text{odd}}\mathfrak{m}^*)^G.$$

By assumption $\bar{\pi}_G(\lambda^1\Phi) = d\omega$ for some $\omega \in (\Lambda\,\mathfrak{m}^*)^G$ and (i) follows. (ii) holds since $0 = dz = d\Delta(\theta)T(i^*\Phi) = h(\theta)i^*\Phi$. (iii) is a consequence of (ii) and formula (5.17). ■

(5.26) COROLLARY. — *Suppose that for* $\Phi \in I(\bar{G})^+$ *we have*

$$i^*\Phi = \sum_j i^*(\Phi_j).\Psi_j, \qquad \textit{where} \quad \Phi_j \in I(\bar{G})^+ \quad \textit{and} \quad \Psi_j \in I(G)^+.$$

Then $\bar{\sigma}(\Phi) \in \mathrm{H}(\bar{\mathfrak{g}})$ *is represented by the* G-*basic cocycle*

$$\sum_j \bar{\pi}_{\mathrm{G}}(\lambda^1 \Phi_j).h(\theta)\Psi_j - \bar{\pi}_{\mathrm{G}}(\lambda^1 \Phi) \in (\Lambda\bar{\mathfrak{g}}^*)_{\mathrm{G}} = (\Lambda \mathfrak{m}^*)^{\mathrm{G}} \tag{5.27}$$

In particular for $\Phi \in \ker\,(i^* : \mathrm{I}(\bar{\mathrm{G}}) \to \mathrm{I}(\mathrm{G}))$ *the transgressive cochain*

$$\bar{\mathrm{T}}\Phi = -\lambda^1 \Phi \in \mathrm{W}(\bar{\mathfrak{g}}, \mathrm{G})$$

is G-*basic and the suspension* $\bar{\sigma}(\Phi)$ *is represented by the* G-*basic cocycle*

$$-\bar{\pi}_{\mathrm{G}}(\lambda^1 \Phi) \in (\Lambda\bar{\mathfrak{g}}^*)_{\mathrm{G}} = (\Lambda \mathfrak{m}^*)^{\mathrm{G}}.$$

Proof. — For $\Phi \in \mathrm{I}(\bar{\mathrm{G}})^+$ as in the Corollary, the cochain $\sum_j \mathrm{T}(i^*\Phi_j).\Psi_j \in \mathrm{W}(\mathfrak{g})$ is transgressive for $i^*\Phi$. Hence we may choose $\mathrm{T}(i^*\Phi) = \sum_j \mathrm{T}(i^*\Phi_j).\Psi_j$. By (5.23)

$$\bar{\mathrm{T}}\Phi = \sum_j k(\theta)\mathrm{T}(i^*\Phi_j).k(\theta)_{\mathrm{G}}\Psi_j - \lambda^1\Phi \in \mathrm{W}(\bar{\mathfrak{g}})$$

is a transgressive cochain for Φ. But by (5.23) applied to $i^*\Phi_j$ we have

$$\begin{aligned}\bar{\mathrm{T}}\Phi &= \sum_j (\bar{\mathrm{T}}\Phi_j + \lambda^1\Phi_j).k(\theta)_{\mathrm{G}}\Psi_j - \lambda^1\Phi \\ &= \sum_j \lambda^1\Phi_j.k(\theta)_{\mathrm{G}}\Psi_j - \lambda^1\Phi + \sum_j \bar{\mathrm{T}}\Phi_j.k(\theta)_{\mathrm{G}}\Psi_j \\ &= \sum_j \lambda^1\Phi_j.k(\theta)_{\mathrm{G}}\Psi_j - \lambda^1\Phi - \sum_j d(\bar{\mathrm{T}}\Phi_j.k(\theta)\mathrm{T}\Psi_j) + \sum_j \Phi_j.k(\theta)\mathrm{T}\Psi_j.\end{aligned}$$

Hence

$$\sum_j \lambda^1\Phi_j.k(\theta)_{\mathrm{G}}\Psi_j - \lambda^1\Phi + \sum_j \Phi_j.k(\theta)\mathrm{T}\Psi_j \tag{5.28}$$

is also a transgressive cochain for Φ, which projects under $\bar{\pi}$ onto the G-basic cocycle in (5.27) since $\bar{\pi}_{\mathrm{G}}(\Phi_j) = 0$, $\Phi \in \mathrm{I}(\bar{\mathrm{G}})^+$.

The formulas in the case $i^*\Phi = 0$ are a direct consequence of (5.23) and (5.24). ■

In the next section Corollary 5.25 will be applied to the computation of $\Delta(\theta)_*$ for symmetric pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of equal rank. Formulas (5.23) (5.24) for the transgression and suspension are valid for arbitrary pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$. In order to compute $\Delta(\theta)_*$ in such cases, a subtler analysis of $k(\theta)$ is needed.

6. Symmetric pairs of equal rank

In this section we consider reductive pairs $(\bar{\mathrm{G}}, \mathrm{G})$ which are symmetric of equal rank, and compute the characteristic homomorphism

$$\begin{tikzcd} \mathrm{H}(\mathrm{W}(\mathfrak{g})_{q'}) \arrow[r, "\Delta_*(\mathrm{P})"] \arrow[dr, "\Delta(\theta)_*"'] & \mathrm{H}_{\mathrm{DR}}(\Gamma\backslash\bar{\mathrm{G}}) \\ & \mathrm{H}(\bar{\mathfrak{g}}) \arrow[u, hook, "\gamma_*"'] \end{tikzcd} \tag{6.1}$$

of the foliation on $\Gamma\backslash\bar{\mathrm{G}}$ induced by the left cosets of G in $\bar{\mathrm{G}}$.

The spectral sequence in the appendix [Theorem A.9, (ii)] has for $E = W(\bar{\mathfrak{g}})$ the form

$$''E_2^{2p,q} = H^q(\bar{\mathfrak{g}}, G) \otimes I^{2p}(\bar{G}) \Rightarrow H^{2p+q}(W(\bar{\mathfrak{g}}, G)) \cong I^{2p+q}(G). \tag{6.2}$$

As $(\bar{\mathfrak{g}}, \mathfrak{g})$ is an equal rank pair, the fibre $''E_2^{0,q}$ is totally non-homologeons to zero, i. e. the edge-homomorphism

$$H(W(\bar{\mathfrak{g}}, G)) \cong I(G) \xrightarrow{h_*} H(\bar{\mathfrak{g}}, G) \tag{6.3}$$

is surjective [14] (Vol. III). By the Leray-Hirsch Theorem, the restriction map

$$i^* : \ I(\bar{G}) \to I(G) \tag{6.4}$$

is injective and

$$\ker h_* = \mathrm{Id}(i^* I(\bar{G})^+) \subset I(G), \tag{6.5}$$

i. e. $\ker h_*$ is equal to the ideal generated by the image under i^* of $I(\bar{G})^+$.

Since the pair $(\bar{G}, G)$ is symmetric, the surjectivity of the cohomology map h_* in (6.3) implies by Lemma (5.11), (iv) the surjectivity of the cochain map

$$H(W(\bar{\mathfrak{g}}, G)) \cong I(G) \xrightarrow{h=h(\theta)} (\Lambda\, m^*)^G \cong H(\bar{\mathfrak{g}}, G). \tag{6.6}$$

and

$$\ker h(\theta) = \mathrm{Id}(i^* I(\bar{G})^+) \subset I(G). \tag{6.7}$$

As $I(G)$ is evenly graded, it follows that $(\Lambda^{\mathrm{odd}}\, \mathfrak{m}^*)^G = 0$.

Remark. — This is the only point in our computation of the characteristic homomorphism where the symmetry assumption enters. The following computations apply more generally for any reductive pair for which the differential

$$d_\Lambda : \ (\Lambda^{\mathrm{even}}\, \mathfrak{m}^*)^G \to (\Lambda^{\mathrm{odd}}\, \mathfrak{m}^*)^G \quad \text{is surjective.}$$

We can now apply Corollary (5.25) to compute the suspension $\bar{\sigma}(\Phi) \in H(\bar{\mathfrak{g}})$ of an invariant polynomial $\Phi \in I(\bar{G})^+$ via its restriction $i^*(\Phi)$ to $I(G)^+$. This in turn can be used to relate the cocycles in the image of $\Delta(\theta)_*$ to the primitive elements of $H^{\cdot}(\bar{\mathfrak{g}})$ and thereby to obtain an estimate on the dimension of im $\Delta(\theta)_*$ [*see* Theorem (6.28) and Corollary (6.29)].

For this purpose let c_j and $y_j = \sigma(c_j)$ ($j = 1, \ldots, r$; $r = \operatorname{rank} \mathfrak{g} = \operatorname{rank} \bar{\mathfrak{g}}$) be a basis for the indecomposable elements in $I(G)$ and the primitive elements $P_{\mathfrak{g}}$ in $H(\mathfrak{g})$. The corresponding elements for $\bar{G}$ will be denoted $\bar{c}_j$ and $\bar{y}_j = \bar{\sigma}(\bar{c}_j)$. Then

$$I(G) \cong \mathbf{R}[c_1, \ldots, c_r] \quad \text{and} \quad H(\mathfrak{g}) \cong \Lambda\, P_{\mathfrak{g}} = \Lambda(y_1, \ldots, y_r)$$

and similarly for $I(\bar{G})$ and $H(\bar{\mathfrak{g}})$ [29]. In terms of these generators the restriction

$$i^* : \ \mathbf{R}[\bar{c}_1, \ldots, \bar{c}_r] \to \mathbf{R}[c_1, \ldots, c_r]$$

can now be expressed uniquely as follows. Let

$$\sum_{k=1}^{r} a_{jk}\, c_k, \qquad a_{jk} \in \mathrm{I}^0(\mathrm{G}) \cong \mathbf{R}$$

be the indecomposable part of $i^* \bar{c}_j$ and $\sum_{k=1}^{r} c_k . \varphi_{jk}$, $\varphi_{jk} \in \mathbf{R}[c_k, \ldots, c_r]^+ \subset \mathrm{I}(\mathrm{G})^+$ the decomposable part of $i^* \bar{c}_j$, i. e.:

$$i^* \bar{c}_j = \sum_{k=1}^{r} c_k (a_{jk} + \varphi_{jk}). \tag{6.8}$$

The $(r \times r)$-matrices $a = [a_{jk}]$ and $\Phi = [\varphi_{jk}]$ contain all the information needed to compute $\Delta(\theta)$ as well as $\mathrm{T}\, \bar{c}_j$ and $\bar{\sigma}(\bar{c}_j)$. First it follows from the definitions:

(6.9) if $a_{jk} \neq 0$, then $\deg \bar{c}_j - \deg c_k = \deg a_{jk} = 0$;

(6.10) if $\varphi_{jk} \neq 0$, then $\deg \bar{c}_j - \deg c_k = \deg \varphi_{jk} > 0$.

Therefore

$$a_{jk} . \varphi_{jk} = 0 \qquad \text{for} \quad j, k = 1, \ldots, r. \tag{6.11}$$

By the injectivity of i^* we also have for every $j = 1, \ldots, r$:

(6.12) $a_{jk} + \varphi_{jk} \neq 0$ for at least one $k = 1, \ldots, r$.

(6.13) LEMMA. — *If* $\prod_{j=1}^{r} \varphi_{j, k_j} \neq 0$ *then*

$$\deg\left(\prod_{j=1}^{r} \varphi_{j, k_j}\right) = \sum_{j=1}^{r} \deg \varphi_{j, k_j} = 2q',$$

where $2q' = \dim \mathfrak{m} = \dim(\bar{\mathfrak{g}}/\mathfrak{g})$ *and* $\{k_1, \ldots, k_r\}$ *is a permutation of* $\{1, \ldots, r\}$.

Proof. — By (6.10) it follows that

$$\begin{aligned}\deg\left(\prod_{j=1}^{r} \varphi_{j, k_j}\right) &= \sum_{j=1}^{r} \deg(\varphi_{j, k_j}) \\ &= \sum_{j=1}^{r} (\deg \bar{c}_j - \deg c_{k_j}) = \sum_{j=1}^{r} \deg \bar{c}_j - \sum_{k=1}^{r} \deg c_k \\ &= (\dim \bar{\mathfrak{g}} + r) - (\dim \mathfrak{g} + r) = \dim(\bar{\mathfrak{g}}/\mathfrak{g}) =: 2q'.\end{aligned}$$

Here we used the fact that for any reductive Lie algebra $\mathfrak{g}$ the product $y_1 \wedge \ldots \wedge y_r$ of primitive generators is a top-dimensional non-trivial cohomology class in $\mathrm{H}(\mathfrak{g})$ and therefore

$$\sum_{k=1}^{r} \deg c_k = \sum_{k=1}^{r} (\deg \sigma(c_k) + 1) = \deg(y_1 \wedge \ldots \wedge y_r) + r = \dim \mathfrak{g} + r. \quad \blacksquare$$

In order to relate the cocycles in

$$\operatorname{im} \Delta(\theta)_* \subset \mathrm{H}^\cdot(\bar{\mathfrak{g}}) \cong \wedge^\cdot(\bar{y}_1, \ldots, \bar{y}_r)$$

with the primitive classes $\bar{y}_j = \sigma(\bar{c}_j)$, $j = 1, \ldots, r$, we realize the formula for $\sigma(\bar{c}_j)$ given in Corollary (5.25) in the A-complex of the g-DG-algebra $\Lambda\,\bar{\mathfrak{g}}^*$ (*see* appendix and [25'], chapter 5). With this formalism diagram (5.15) translates into the diagram

(6.14)
$$\begin{tikzcd}
\Lambda(y_1,\ldots,y_r) \arrow[rr] & & \Lambda(y_1,\ldots,y_r)\otimes(\Lambda\,\mathfrak{m}^*)^{G} \\
\Lambda(y_1,\ldots,y_r)\otimes\mathbf{R}[c_1,\ldots,c_r] \arrow[u, "\mathrm{id}\otimes\varepsilon"] \arrow[urr, "k", "\mathrm{id}\otimes h"'] \arrow[rr, "\mathrm{id}\otimes k(\theta)_G"] & & \Lambda(y_1,\ldots,y_r)\otimes W(\bar{\mathfrak{g}},G) \arrow[u, "\mathrm{id}\otimes\pi_G"'] \\
\mathbf{R}[c_1,\ldots,c_r] \arrow[u] & & \mathbf{R}[\bar{c}_1,\ldots,\bar{c}_r], \arrow[ll, "i^*"'] \arrow[u]
\end{tikzcd}$$

where h induces an isomorphism $h : \mathrm{I}(G)/\mathrm{Id}(i^*\,\mathrm{I}(\bar{G})^+) \cong (\Lambda\,\mathfrak{m}^*)^G$ by (6.5), (6.6) and

$$\pi_G : \quad W(\bar{\mathfrak{g}}, G) \cong (\Lambda\,\mathfrak{m}^* \otimes S\,\bar{\mathfrak{g}}^*)^G \to (\Lambda\,\mathfrak{m}^*)^G$$

is the projection along $(S\,\bar{\mathfrak{g}}^*)^+$.

First we observe that a transgressive chain $T\,i^*\,\bar{c}_j$ can be obtained in the following way. The cochains

(6.15)
$$z'_j = \sum_{k=1}^{r} y_k \otimes (a_{jk} + \varphi_{jk}) \in \Lambda(y_1, \ldots, y_r) \otimes \mathbf{R}[c_1, \ldots, c_r]$$

clearly satisfy

$$d_A\, z'_j = \sum_{k=1}^{r} 1 \otimes c_k (a_{jk} + \varphi_{jk}) = 1 \otimes i^*(\bar{c}_j),$$

and hence $\varphi(z'_j) \in W(\mathfrak{g})$ is a transgressive cochain $T\,i^*\,\bar{c}_j$ for $i^*\,\bar{c}_j$. But then by the naturality of φ we obtain

$$\Delta(\theta)\,T(i^*\bar{c}_j) = \Delta(\theta)\,\varphi(z'_j) = \varphi(\mathrm{id}\otimes h)(z'_j) = \varphi(z_j),$$

where

(6.16)
$$z_j = \sum_{k=1}^{r} y_k \otimes (a_{jk} + h\,\varphi_{jk}) \in \Lambda(y_1, \ldots, y_r) \otimes (\Lambda\,\mathfrak{m}^*)^G.$$

Hence we have proved

(6.17) PROPOSITION. — *The cochains* $z_j \in \Lambda(y_1, \ldots, y_r) \otimes (\Lambda\,\mathfrak{m}^*)^G$ *are cocycles representing the suspensions* $\bar{y}_j$ *of* $\bar{c}_j$:

$$\varphi_*([z_j]) = \bar{y}_j, \qquad j = 1, \ldots, r.$$

This enables us to write down for a pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ explicit formulas for $\bar{y}_j$ in terms of y_k, c_k whenever the matrices a and Φ are known.

(6.18) LEMMA:

$$z'_1\cdot\ldots\cdot z'_r = y_1\wedge\ldots\wedge y_r\otimes\det(a+\Phi)\in\Lambda^m(y_1,\ldots,y_r)\otimes\mathbf{R}[c_1,\ldots,c_r]^{2q'},$$

where $m=\dim\mathfrak{g}$, $\det(a+\Phi)\in\mathbf{R}[c_1,\ldots,c_r]^{2q'}$. *Hence* $z'_1\cdot\ldots\cdot z'_r$ *is a cocycle in* $\Lambda(y_1,\ldots,y_r)\otimes\mathbf{R}[c_1,\ldots,c_r]_{q'}$.

Proof:

$$z'_1\cdot\ldots\cdot z_r=\prod_{j=1}^{r}\left(\sum_k y_k\otimes(a_{jk}+\varphi_{jk})\right)=\sum_{k_1,\ldots,k_r=1}^{r} y_{k_1}\wedge\ldots\wedge y_{k_r}\otimes\prod_{j=1}^{r}(a_{j,k_j}+\varphi_{j,k_j})$$

$$=\sum_{\sigma\in S_r}\varepsilon_\sigma y_1\wedge\ldots\wedge y_r\otimes\prod_{j=1}^{r}(a_{j,k_{\sigma_j}}+\varphi_{j,k_{\sigma_j}})=y_1\wedge\ldots\wedge_{y_r}\otimes\det(a+\Phi).$$

The fact that deg det $(a+\Phi)=2q'$ follows from $\deg(z'_1\cdot\ldots\cdot z'_r)=\dim\bar{\mathfrak{g}}$ and $\deg(y_1\wedge\ldots\wedge y_r)=\dim\mathfrak{g}$. ■

From Proposition (6.17) and Lemma (6.18) it follows that the cocycle $z'_1\cdot\ldots\cdot z'_r$ maps under $\Delta(\theta)\sim\mathrm{id}\otimes h$ into the cocycle $z_1\cdot\ldots\cdot z_r$ representing the non-trivial cohomology class

$$\bar{y}_1\wedge\ldots\wedge\bar{y}_r\in\mathrm{H}^m(\bar{\mathfrak{g}})\cong\Lambda(\bar{y}_1,\ldots,\bar{y}_r).$$

(6.19) THEOREM. — *The invariant polynomial* $\det(a+\Phi)\in\mathbf{R}[c_1,\ldots,c_r]^{2q'}$ *determines a complement to the ideal* $\mathrm{Id}^{2q'}(i^*\bar{c}_1,\ldots,i^*\bar{c}_r)$, i. e. $\det(a+h\,\Phi)\in(\Lambda^{2q'}\mathfrak{m}^*)^{\mathrm{G}}\cong\mathbf{R}$ *determines a non-trivial characteristic number.*

Such non-trivial characteristic numbers always exist in the equal rank case [Corollary (5.10′)]. Theorem (6.19) gives an explicit procedure for the construction of such a number in terms of the matrix $a+\Phi$.

As in the proof of Lemma (6.18) one obtains more generally for the cochain $z'_{i_1}\cdot\ldots\cdot z'_{i_s}$, $i_1<\ldots<i_s$ the formula

(6.20) $$z'_{i_1}\cdot\ldots\cdot z'_{i_s}=\sum_{1\leqq k_1<\ldots<k_s\leqq r} y_{k_1}\wedge\ldots\wedge y_{k_s}\otimes\det(a+\Phi)_{i_\alpha,k_\beta)\alpha,\beta=1,\ldots,s}.$$

It is desirable to have a formula expressing the monomials

$$y_{k_1}\wedge\ldots\wedge y_{k_s}\otimes\Phi\in\mathrm{A}(\mathrm{W}(\mathfrak{g})_{q'})$$

in terms of the elements $z'_{i_1}\cdot\ldots\cdot z'_{i_s}$. This can be done under the following simplifying assumptions:

(6.21) $$\deg c_i<\deg c_k\quad\text{for}\quad 1\leqq i<k\leqq r;$$

(6.22) there exists $\bar{c}_j$ such that

$$i^*\bar{c}_j=\sum_{k=1}^{r}c_k\cdot\varphi_{jk}(c_k,\ldots,c_r)$$

with $\deg\varphi_{jk}=2q'$ for the non-zero φ_{jk}, $k=1,\ldots,r$.

Condition (6.22) implies that $i^* \bar{c}_j$ is decomposable ($a_{jk} = 0$ for $k = 1, \ldots, r$) and that $\deg \bar{c}_j > 2q'$. (6.21) together with the injectivity of i^* implies further that there exists exactly one k, $1 \leqq k \leqq r$ such that $\varphi_{jk} \neq 0$:

$$(6.23) \qquad i^* \bar{c}_j = c_k . \varphi_{jk}, \qquad \deg \varphi_{jk} = 2q'.$$

Similarly it follows from (6.11), (6.19) and (6.21) that (up to a sign)

$$(6.24) \qquad \det(a+\Phi) = \prod_{\substack{i=1\\ i\neq j}}^{r} a_{i,k_i} . \varphi_{jk} \neq 0,$$

where $(k_i)_{i \neq j}$ is the unique order-preserving mapping

$$(1, \ldots, \hat{j}, \ldots, r) \xrightarrow{\approx} (1, \ldots, \hat{k}, \ldots, r).$$

Hence for $i \neq j$

$$(6.25) \qquad i^* \bar{c}_i = a_{i,k_i} c_{k_i} + \sum_{k=1}^{r} c_k \varphi_{ik}(c_k, \ldots, c_r),$$

with $a_{i,k_i} \neq 0$.

(6.26) LEMMA. — *The monomials*

$$z'_{i_1} \cdot \ldots \cdot z'_{i_s} \cdot z'_j \in \Lambda(y_1, \ldots, y_r) \otimes \mathbf{R}[c_1, \ldots, c_r]_{q'}$$

for $1 \leqq i_1 < \ldots < i_s \leqq r$, $0 \leqq s \leqq r-1$ *and* $j \neq i_\alpha$ *are cocycles given by*

$$(6.27) \qquad \begin{aligned} z'_{i_1} \cdot \ldots \cdot z'_{i_s} z'_j &= \sigma(i^* \bar{c}_{i_1}) \cdot \ldots \cdot \sigma(i^* \bar{c}_{i_s}) . z'_j \\ &= a_{(i)} y_{k_{i_1}} \wedge \ldots \wedge y_{k_{i_s}} \wedge y_k \otimes \varphi_{jk}, \end{aligned}$$

where $a_{(i)} = \pm \prod_{\alpha=1}^{s} a_{i_\alpha, k_{i_\alpha}} \neq 0$. *Furthermore the class* $\bar{c}_j$ *satisfying condition* (6.22) *is unique.*

This follows at once from (6.23), (6.25) and the fact that $\deg \varphi_{jk} = 2q'$. Uniqueness of $\bar{c}_j$ follows from (6.27), since $\sigma(i^* \bar{c}_j) = 0$ by (6.22). ■

Using the structure of

$$\mathrm{H}(\mathrm{W}(\mathfrak{g})_{q'}) \cong \mathrm{H}(\Lambda(y_1, \ldots, y_r) \otimes \mathbf{R}[c_1, \ldots, c_r]_{q'})$$

given in Theorem (A.22) we can now conclude that $\operatorname{im} \Delta(\theta)_*$ must contain a large number of linearly independent classes.

(6.28) THEOREM. — *Let* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *be a reductive pair of equal rank satisfying condition* (6.22). *Then*

$$\mathrm{Id}(\bar{y}_j) \subset \operatorname{im} \Delta(\theta)_* \subseteq \mathrm{H}(\bar{\mathfrak{g}}) = \Lambda(\bar{y}_1, \ldots, \bar{y}_r).$$

(6.29) COROLLARY. — *The cohomology class* $\bar{y}_{i_1} \wedge \ldots \wedge \bar{y}_{i_s} \wedge \bar{y}_j \in \mathrm{H}(\bar{\mathfrak{g}}) \cong \Lambda(\bar{y}_1, \ldots, \bar{y}_r)$

is the image under $\Delta(\theta)_*$ *of the secondary characteristic class*

$$\sum_{k=1}^{r} \varphi_* [\sigma(i^* \bar{c}_{i_1}) \wedge \ldots \wedge \sigma(i^* \bar{c}_{i_s}) \wedge y_k \otimes \varphi_{jk}] \in \mathrm{H}(\mathrm{W}(\mathfrak{g})_{q'}).$$

In particular, the generator $\bar{y}_1 \wedge \ldots \wedge \bar{y}_r \in \mathrm{H}^m(\bar{\mathfrak{g}})$ *is the image under* $\Delta(\theta)_*$ *of the cocycle*

$$z'_1 \cdot \ldots \cdot z'_r = y_1 \wedge \ldots \wedge y_r \otimes \det(a+\Phi)$$

and hence $\det(a+\Phi) \notin \mathrm{Id}^{2q'}(i^* \bar{c}_1, \ldots, i^* \bar{c}_r)$.

Thus the conclusion of Theorem (6.19) also holds true under the assumptions of Theorem (6.28).

(6.30) Corollary. — $\dim \operatorname{im} \Delta(\theta)_* \geqq 2^{r-1}$, *whereas* $\dim \mathrm{H}(\bar{\mathfrak{g}}) = 2^r$.

Proof of Theorem (6.28). — For a symmetric pair of equal rank satisfying in addition condition (6.21) this follows from (6.16), proposition (6.17) and Lemma (6.26). In the general case one observes that by (5.24) and (6.16) the class $\bar{y}_i$ is represented by the cocycle

(6.31) $$z_i - 1 \otimes \bar{\pi}_{\mathrm{G}} \lambda^1(\bar{c}_i) \in \Lambda(y_1, \ldots, y_r) \otimes (\Lambda \mathfrak{m}^*)^{\mathrm{G}}.$$

But since $z_j = \sum_k y_k \otimes h\varphi_{jk}$, with $\deg \varphi_{jk} = 2q'$ for $\varphi_{jk} \neq 0$, it follows as before for $1 \leqq i_1 < \ldots < i_s \leqq r$, $j \neq i_\alpha$, that

$$\prod_{\alpha=1}^{s} (z_{i_\alpha} - 1 \otimes \bar{\pi}_{\mathrm{G}} \lambda^1(\bar{c}_{i_\alpha}))(z_j - 1 \otimes \bar{\pi}_{\mathrm{G}} \lambda^1 \bar{c}_j) = \sum_{k=1}^{r} \left(\prod_{\alpha=1}^{s} \sigma(i^* \bar{c}_{i_\alpha}) \right) \wedge y_k \otimes h\varphi_{jk},$$

where $\sigma(i^* \bar{c}_i) = \sum_l a_{il} \cdot y_l$. Hence the cocycle

$$\sum_{k=1}^{r} \left(\prod_{\alpha=1}^{s} \sigma(i^* \bar{c}_{i_\alpha}) \right) \wedge y_k \otimes \varphi_{jk} \in \Lambda(y_1, \ldots, y_r) \otimes \mathbf{R}[c_1, \ldots, c_r]_{q'}$$

maps under $\Delta(\theta) \sim \mathrm{id} \otimes h$ into a cocycle representing $\bar{y}_{i_1} \wedge \ldots \wedge \bar{y}_{i_s} \wedge \bar{y}_j$, $s \geqq 0$. ∎

Theorem (6.28) gives a lower bound for $\Delta(\theta)_*$. Before turning to a determination of $\operatorname{im} \Delta(\theta)_*$ we discuss the implication of conditions (6.21), (6.22) on the structure of $\mathrm{I}(\mathrm{G})/\mathrm{Id}(i^* \mathrm{I}(\mathrm{G})^+) \cong \mathrm{H}(\bar{\mathfrak{g}}, \mathrm{G})$.

Let $1 \leqq t \leqq r$ be the smallest integer such that $\deg c_i > 2q'$ for $i > t$. By using (6.25) to eliminate the generators c_i, $i \neq k$ modulo $\mathrm{Id}(i^* \mathrm{I}(\bar{\mathrm{G}})^+) = \mathrm{Id}(i^* \bar{c}_1, \ldots, i^* \bar{c}_r)$, one obtains by induction on i that

(6.32) $$c_i \equiv 0 \quad \text{for} \quad 1 \leqq i < k, \quad t < i \leqq r;$$

and

(6.33) $$c_i \equiv \varkappa_i c_k^{\lambda_i} \quad \text{for } k < i \leqq t, \qquad \text{with } \lambda_i > 1.$$

This implies that

$$\varphi_{jk} \equiv \varkappa \, c_k^{\lambda}. \tag{6.34}$$

By Theorem (6.19) and (6.24) it follows that $\varphi_{jk} \not\equiv 0$, hence $\varkappa \neq 0$ and $c_k^{\lambda} \not\equiv 0$, $\lambda . \deg c_k = 2q'$, while $c_k^{\lambda+1} \equiv 0$ since $\mathrm{H}(\bar{\mathfrak{g}}, \mathrm{G})^q = 0$ for $q > 2q'$. Thus

(6.35) PROPOSITION. — *For a reductive pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of equal rank satisfying* (6.21) *and* (6.22), *we have*

$$\mathrm{H}(\bar{\mathfrak{g}}, \mathrm{G}) \cong \mathrm{I}(\mathrm{G})/\mathrm{Id}(i^* \mathrm{I}(\bar{\mathrm{G}})^+) \cong \mathbf{R}[\tilde{c}_k]/(\tilde{c}_k^{\lambda+i}),$$

i. e. $\mathrm{H}(\bar{\mathfrak{g}}, \mathrm{G})$ *is a truncated polynomial algebra in one generator* $\tilde{c}_k = h_*(c_k)$, *the residue class of* c_k *modulo* $\mathrm{Id}(i^* \bar{c}_1, \ldots, i^* \bar{c}_r)$.

If $(\bar{\mathrm{G}}, \mathrm{G})$ is symmetric, then in particular $(\Lambda \, \mathfrak{m}^*)^{\mathrm{G}} \cong \mathbf{R}[\tilde{c}_k^{\lambda}]/(\tilde{c}_k^{\lambda+1})$. By a theorem of Bott the compact symmetric spaces of rank 1 have this cohomology structure. These are the spheres $\mathrm{SO}(r+1)/\mathrm{SO}(r)$, the complex and quaternion projective spaces

$$\mathrm{SU}(r+1)/\mathrm{U}(r) \quad \text{and} \quad \mathrm{Sp}(r+1)/\mathrm{Sp}(r) \times \mathrm{Sp}(1),$$

and the Cayley projective plane. The non-compact dual spaces have of course the same cohomology of invariant forms. We compute at the end of this section $\Delta(\theta)_*$ explicitly for the pairs

$$(\bar{\mathrm{G}}, \mathrm{G}) = (\mathrm{SU}(r+1), \mathrm{U}(r)) \quad \text{with} \quad \bar{\mathrm{G}}/\mathrm{G} = \mathbf{P}^r\mathbf{C}$$

and

$$(\mathrm{G}, \mathrm{G}) = (\mathrm{SO}(2r+1), \mathrm{SO}(2r)) \quad \text{with} \quad \bar{\mathrm{G}}/\mathrm{G} = \mathrm{S}^{2r}.$$

The other cases are treated similarly.

The degree of the generator c_k is intimately related to the existence of non-trivial secondary invariants which are rigid. It is known that the classes in the image of the canonical homomorphisms

$$\mathrm{H}(\mathrm{W}(\mathfrak{g})_{q'+l}) \xrightarrow{\rho_l} \mathrm{H}(\mathrm{W}(\mathfrak{g})_{q'}), \qquad l \geqq 1$$

are rigid, i. e. $\Delta(\theta)_* \circ \rho_{l_*}$ is invariant under deformation of the data defining $\Delta(\theta)_*$ (*see* Heitsch [16], and [25], section 8.7). We first have the following general result, valid for all symmetric pairs of equal rank.

(6.36) LEMMA. — *Suppose that* $\mathrm{H}(\bar{\mathfrak{g}}, \mathrm{G}) \cong (\Lambda \, \mathfrak{m}^*)^{\mathrm{G}}$ *is generated as an algebra by elements of degree* $\leqq 2l$. *Then the composition*

$$\mathrm{H}^+(\mathrm{W}(\mathfrak{g})_{q'+l}) \xrightarrow{\rho_{l*}} \mathrm{H}^+(\mathrm{W}(\mathfrak{g})_{q'}) \xrightarrow{\Delta(\theta^*)} \mathrm{H}^+(\bar{\mathfrak{g}})$$

is zero.

Proof. — A monomial cocycle

$$z_{(i,j)} = y_{(i)} \otimes c_{(j)} \in \Lambda(y_1, \ldots, y_r) \otimes \mathbf{R}[c_1, \ldots, c_r]_{q'}$$

[*see* Theorem (A.22)] is in the image of $(\rho_l)_*$ if and only if $dz_{(i,j)} \in F^{2(q'+l+1)}$ in the untruncated algebra, i. e.

$$\deg c_{i_1} \geqq 2(q'+1-(p-l)), \qquad \text{where} \quad 2p = \deg c_{(j)}.$$

By assumption we can write $c_{(j)} \equiv \sum_j \varphi'_j . \varphi''_j$ modulo $\ker h = \mathrm{Id}\,(i^* \mathrm{I}(\overline{\mathrm{G}})^+)$ with $\deg \varphi''_j \leqq 2\,l$. Therefore $\deg \varphi'_j \geqq 2(p-l) \geqq 0$. It follows that the elements $y_{(i)} \otimes \varphi'_j$ are cocycles modulo the filtration ideal $F^{2(q'+1)}$, and that in $A(\Lambda\,\bar{\mathfrak{g}}^*)$:

$$\Delta(\theta)_*[z_{(i,j)}] = \sum_j [y_{(i)} \otimes h(\varphi'_j)].[1 \otimes h(\varphi''_j)] = 0,$$

since $1 \otimes h(\varphi)''_j$ is a coboundary in $\Lambda(y_1, \ldots, y_r) \otimes (\Lambda\,\mathfrak{m}^*)^{\mathrm{G}}$.

Applied to the situation in Proposition (6.35) we obtain for $\deg c_k = 2\,l$:

(6.37) THEOREM. — (i) *If $l = 1$, then all the rigid classes in $\mathrm{H}^+(\mathrm{W}(\mathfrak{g})_{q'})$ are mapped to zero under $\Delta(\theta)_*$.*

(ii) *If $l > 1$, then the rigid class $[z'_j] = [y_k \otimes \varphi_{jk}]$ is mapped into $\bar{y}_j$ under $\Delta(\theta)_*$ and hence defines a non-trivial rigid secondary class.*

From (6.35), (6.36) and Theorem (A.22) in the appendix we see now that im $\Delta(\theta)_*$ is spanned by the images of the monomial classes

$$[z_{(i)}] \in \mathrm{H}^{\cdot}(\Lambda(y_1, \ldots, y_r) \otimes \mathbf{R}[c_1, \ldots, c_r]_{q'})$$

of the form

$$(6.38) \qquad z_{(i)} = y_{i_1} \wedge \ldots \wedge y_{i_s} \otimes c_k^p, \qquad 0 < s \leqq r,$$

where $0 \leqq p \leqq \lambda$ is the least integer satisfying

$$(6.39) \qquad \deg c_{i_1} \geqq 2(q'+1-p.l)$$

and where $i_1 \leqq k$ for $p > 0$. These cocycles $z_{(i)}$ are of three distinct types, namely

(*a*) $i_1 < k$, $k \neq i_\alpha$ for $\alpha = 1, \ldots, s$, with $p = \lambda$;

(*b*) $i_1 \leqq k$, $k = i_\alpha$ for some $\alpha = 1, \ldots, s$, with $p = \lambda$;

(*c*) $i_1 > t$, with $p = 0$.

In case (*a*) we may write

$$\Delta(\theta)\,z_{(i)} = (y_{i_1} \wedge \ldots \wedge y_{i_s} \otimes \tilde{c}_k^{\lambda-1}) \otimes (1 \otimes \tilde{c}_k),$$

where the first factor is a cocycle by (6.32) and (6.33), whereas $1 \otimes \tilde{c}_k = d(y_k \otimes 1)$ and hence $\Delta(\theta)\,z_{(i)} \sim 0$ in

$$\Lambda(y_1, \ldots, y_r) \otimes (\Lambda\,\mathfrak{m}^*)^{\mathrm{G}} \cong \Lambda(y_1, \ldots, y_r) \otimes \mathbf{R}[\tilde{c}_k]/(\tilde{c}_k^{\lambda+1}).$$

In case (b) $z_{(i)}$ can be written in the form

$$y_{k_{i_1}} \wedge \ldots \wedge y_{k_{i_{s-1}}} \wedge y_k \otimes c_k^\lambda,$$

where the k_i are as in (6.24). This cocycle maps under $\Delta(\theta)$ into a non-zero multiple of $z_{i_1} \cdot \ldots \cdot z_{i_{s-1}} \cdot z_j$ by Lemma (6.26) and (6.34), and thus $\Delta(\theta)_* [z_{(i)}]$ coincides up to a non-zero factor with the class $\bar{y}_{i_1} \wedge \ldots \wedge \bar{y}_{i_{s-1}} \wedge \bar{y}_j \in \mathrm{H}(\bar{\mathfrak{g}})$.

In the case (c) finally we have cocycles of the type

$$z_{(i)} = y_{i_1} \wedge \ldots \wedge y_{i_s} \otimes 1 \qquad \text{with} \quad i_1 > t,$$

i. e. $\deg c_{i_1} > 2q'$.

These cycles form a subalgebra

$$\Lambda^\cdot(y_{t+1}, \ldots, y_r) \subset \mathrm{H}^\cdot(\Lambda^\cdot(y_1, \ldots, y_r) \otimes \mathbf{R}[c_1, \ldots, c_r]_{q'}) = \mathrm{H}(\mathrm{W}(\mathfrak{g})_{q'})$$

which is mapped isomorphically under $\Delta(\theta)_*$, defining therefore a subalgebra

$$\Lambda^\cdot(y_{t+1}, \ldots, y_r) \subset \mathrm{H}^\cdot(\Lambda(y_1, \ldots, y_r) \otimes \mathbf{R}[\tilde{c}_k]/(\tilde{c}_k^{\lambda+1})) \cong \Lambda(\bar{y}_1, \ldots, \bar{y}_r).$$

We can now summarize these results.

(6.40) Theorem. — *Let* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *be a symmetric pair of equal rank* r *and satisfying* (6.21) *and* (6.22). *Then*

$$\operatorname{im} \Delta(\theta)_* = \mathrm{Id}(\bar{y}_j) + \Lambda^\cdot(y_{t+1}, \ldots, y_r) \subseteq \mathrm{H}^\cdot(\mathfrak{g}),$$

where $\bar{y}_j = \bar{\sigma}(\bar{c}_j)$ *is the primitive class of* $\bar{\mathfrak{g}}$ *defined by the distinguished generator* c_j *in* (6.22) *and* $y_i = \sigma(c_i)$, $i = t+1, \ldots, r$ *are the primitive classes of* $\mathfrak{g}$ *defined by the generators* c_i *satisfying* $\deg c_i > 2q' = \dim(\bar{\mathfrak{g}}/\mathfrak{g})$. *More precisely, the cocycles.*

$$z_{(i)} = y_{k_{i_1}} \wedge \ldots \wedge y_{k_{i_s}} \wedge y_k \otimes c_k^\lambda, \qquad 1 \leqq i_1 < \ldots < i_s \leqq r, \quad i_\alpha \neq j,$$

and $z_{(i)} = y_{i_1} \wedge \ldots \wedge y_{i_s} \otimes 1$, $t < i_1 < \ldots < i_s \leqq r$, *are mapped onto a basis of* $\operatorname{im}\Delta(\theta)_*$.

This result is applied to the following examples.

(6.41) *Example.* — $(\bar{\mathrm{G}}, \mathrm{G}) = (\mathrm{SU}(r+1), \mathrm{U}(r))$. Here

$$\mathrm{U}(r) \cong \mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(1)) \subset \mathrm{SU}(r+1),$$

i. e. $\mathrm{U}(r)$ is realized in $\mathrm{SU}(r+1)$ by the matrices $\left[\begin{array}{c|c} \mathrm{A} & 0 \\ \hline 0 & \lambda^{-1} \end{array}\right]$ with $\mathrm{A} \in \mathrm{U}(r)$, $\lambda = \det \mathrm{A}$. This is a symmetric pair of equal rank r with $\mathbf{P}^r\mathbf{C} \cong \mathrm{SU}(r+1)/\mathrm{U}(r)$. The restriction map i^* is given by

(6.42)
$$\begin{array}{ccc} \mathrm{I}(\mathrm{SU}(r+1)) & \xrightarrow{i^*} & \mathrm{I}(\mathrm{U}(r)) \\ \wr\| & & \wr\| \\ \mathbf{R}[\bar{c}_2, \ldots, \bar{c}_{r+1}] & \to & \mathbf{R}[c_1, \ldots, c_r] \end{array}$$

where the c_i (resp. $\bar{c}_{i+1}$) are the Chern polynonomials given by

$$c_t(\mathrm{A}) = \sum_{i=0}^{r} c_i(\mathrm{A})\, t^i = \det\left(\mathrm{Id} - \frac{t}{2\pi i}\mathrm{A}\right), \qquad \mathrm{A} \in \mathfrak{u}(r).$$

As $i^* \bar{c}_{i+1} = c_{i+1} - c_1 c_i$ for $i = 1, \ldots, r-1$, and $i^* \bar{c}_{r+1} = -c_1 c_r$, the matrix $a + \Phi$ is given by

$$(6.43) \qquad a + \Phi = \begin{bmatrix} -c_1 & 1 & & 0 \\ \vdots & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ -c_r & 0 & & 0 \end{bmatrix}$$

with determinant $(-1)^r \cdot c_r$.

It is clear that (6.21) is satisfied as $\deg c_i = 2i$ and $i^* \bar{c}_{r+1} = -c_1 c_r$, $\deg c_r = 2r = 2q'$ satisfies (6.22). Thus we have $\lambda = j = r$ and $k = 1$. Furthermore we have

$$(\Lambda\, \mathfrak{m}^*)^{\mathrm{U}(r)} \cong \mathbf{R}[\alpha]/(\alpha^{r+1}),$$

where $\alpha = h(c_1)$ is the residue class of c_1 mod $(i^* \bar{c}_2, \ldots, i^* \bar{c}_{r+1})$ and $h(c_i) = \alpha^i$. It follows now from our preceding results that the primitive elements

$$\bar{y}_{i+1} = \bar{\sigma}(\bar{c}_{i+1}) \in \mathrm{H}(\mathfrak{su}(r+1)) \cong \Lambda(\bar{y}_2, \ldots, \bar{y}_r, \bar{y}_{r+1}), \qquad i = 1, \ldots, r$$

are realized by the cocycles $z_{i+1} = y_{i+1} \otimes 1 - y_1 \otimes \alpha^i$, $i < r$ and $z_{r+1} = -y_1 \otimes \alpha^r$ in the A-complex $\Lambda(y_1, \ldots, y_r) \otimes \mathbf{R}[\alpha]/(\alpha^{r+1})$. From Proposition (6.17), (6.27) and Theorem (6.40) it follows now that $\operatorname{im} \Delta_*(\theta) = \mathrm{Id}(\bar{y}_{r+1})$ is exactly of dimension 2^{r-1} in $\mathrm{H}(\mathfrak{su}(r+1))$. More precisely, the cohomology classes defined by

$$z_{(i)} = y_1 \wedge y_{i_1} \wedge \ldots \wedge y_{i_s} \otimes c_1^r,$$

$2 \leqq i_1 < \ldots < i_s \leqq r$, $0 \leqq s \leqq r-1$ in $\mathrm{H}(\mathrm{W}(\mathfrak{u}(r))_r)$ map under $\Delta(\theta)_*$ onto the basis elements $(-1)^{s+1} \bar{y}_{i_1} \wedge \ldots \wedge \bar{y}_{i_s} \wedge \bar{y}_{r+1}$ of $\mathrm{Id}(\bar{y}_{r+1}) \subset \mathrm{H}(\mathfrak{su}(r+1))$. The geometric interpretation of $\Delta(\theta)_*$ is as follows.

The canonical homomorphism

$$h : \mathrm{I}(\mathrm{U}(r)) \cong \mathbf{R}[c_1, \ldots, c_r] \to \mathrm{H}(\mathbf{P}^r\mathbf{C}, \mathbf{R}) \cong \mathrm{H}(\mathfrak{su}(r+1), \mathrm{U}(r)) \cong \mathbf{R}[\alpha]/(\alpha^{r+1})$$

given by $h(c_i) = \alpha^i$ is the characteristic homomorphism of the $\mathrm{U}(r)$-bundle

$$\mathrm{P}' = \mathrm{SU}(r+1) \xrightarrow{\pi'} \mathrm{SU}(r+1)/\mathrm{U}(r) \cong \mathbf{P}^r\mathbf{C}.$$

Moreover the $\mathrm{U}(1)$-bundle

$$\det_* \mathrm{P}' = \mathrm{SU}(r+1)/\mathrm{SU}(r) \cong \mathrm{S}^{2r+1} \to \mathrm{SU}(r+1)/\mathrm{U}(r) \cong \mathbf{P}^r\mathbf{C}$$

has the U (1)-action given by

$$(z_0, \ldots, z_r) \circ \lambda = (z_0 \lambda^{-1}, \ldots, z_r \lambda^{-1}),$$
$$(z_0, \ldots, z_r) \in S^{2r+1}, \qquad \lambda \in U(1)$$

and thus $\det_* P'$ coincides with the principal U (1)-bundle associated with the canonical line bundle $H \to \mathbf{P}^r \mathbf{C}$. Thus

$$\alpha = h(c_1) = c_1(P') = c_1(\det_* P') = c_1(H)$$

and α identifies with the canonical generator of $H^2(\mathbf{P}^r\mathbf{C}, \mathbf{Z})$. The isotropy representation $\rho : U(r) \to U(\mathfrak{m})$ is given by $\rho = \mathrm{id} \otimes \det$ on $\mathfrak{m} = \mathfrak{su}(r+1)/\mathfrak{u}(r) \cong \mathbf{C}^r$. If we use the notation $E(n) = E \underset{\mathbf{C}}{\otimes} H^n$ for any $\mathbf{C}$-vectorbundle on $\mathbf{P}^r\mathbf{C}$, $n \in \mathbf{Z}$, it follows from the above formulas that

$$\text{(6.45)} \qquad T_{\mathbf{P}^r} \cong P' \underset{U(r)}{\times} \mathfrak{m}_\rho \cong (P' \underset{U(r)}{\times} \mathbf{C}^r) \underset{\mathbf{C}}{\otimes} H,$$

and hence

$$\text{(6.46)} \qquad P' \underset{U(r)}{\times} \mathbf{C}^r \cong T_{\mathbf{P}^r}(-1),$$

i. e. P' is the unitary frame bundle of $T_{\mathbf{P}^r}(-1) \cong T_{\mathbf{P}^r} \otimes H^*$. In fact $T_{\mathbf{P}^r}(-1)$ occurs in an exact sequence

$$0 \to H^* \to \varepsilon_{r+1} \to T_{\mathbf{P}^r}(-1) \to 0$$

and thus its characteristic classes are given by

$$c_t(T_{\mathbf{P}^r}(-1) = c_t(-[H^*]) = c_t([H^*])^{-1} = (1-\alpha t)^{-1} = \sum_{m=0}^{r} \alpha^m t^m.$$

We want to compute the generalized characteristic homomorphism of the (complex) normal bundle $Q_{U(r)}$ of the left-coset foliation of SU $(r+1)$ by U (r). This (trivial) vectorbundle is associated to the foliated bundle

$$P = SU(r+1) \times U(r) \to SU(r+1)$$

via the representation ρ by (2.15) and is equivalently given by

$$Q_{U(r)} \cong \pi'^* T_{\mathbf{P}^r} \cong \pi'^* P' \underset{U(r)}{\times} \mathfrak{m}_\rho.$$

By Theorem 3.7′ in the basic case we need to compute the induced homomorphim $\rho^* : I(U(r)) \to I(U(r))$. A direct calculation shows

$$\rho^* c_t = \sum_{m=0}^{r} \left(\sum_{j=0}^{m} \binom{r-j}{m-j} c_1^{m-j} c_j \right) t^m$$

and hence

$$\text{(6.47)} \qquad \rho^* c_m = \sum_{j=0}^{m} \binom{r-j}{m-j} c_1^{m-j} c_j, \qquad m = 1, \ldots, r.$$

Thus

$$\rho^* c_1 = (r+1) c_1 \qquad \text{and} \qquad \rho^* c_m \equiv c_m (\operatorname{mod} \mathrm{I}^{+2}),$$

and it follows that the map induced by ρ on the suspensions $y_j = \sigma(c_j)$ is given by

$$\rho^* y_1 = \rho^*(\sigma c_1) = \sigma(\rho^* c_1) = (r+1) y_1$$

and similarly

$$\rho^* y_j = y_j, \qquad j > 1.$$

Note that (6.47) gives us in particular the characteristic classes of $\mathrm{T}_{\mathrm{P}^r} \cong \mathrm{P}' \underset{\mathrm{U}(r)}{\times} \mathfrak{m}_\rho$:

$$(6.48) \quad c_t(\mathrm{T}_{\mathrm{P}^r}) = h(\rho^* c_t) = \sum_{m=0}^{r} \left(\sum_{j=0}^{m} \binom{r-j}{m-j} \alpha^m \right) t^m = \sum_{m=0}^{r} \binom{r+1}{m} \alpha^m t^m = (1+\alpha t)^{r+1}.$$

Summarizing our calculations, we obtain from Corollary (6.29) and Theorem (6.40) the following result for the normal bundle $\mathrm{Q}_{\mathrm{U}(r)}$.

(6.49) THEOREM. — *Let $\mathrm{Q}_{\mathrm{U}(r)}$ be the foliated complex normal bundle of the foliation of* SU $(r+1)$ *defined by the right action of* U (r) *with quotient space* $\mathbf{P}^r\,\mathbf{C}$. *Then the image of the generalized characteristic homomorphism*

$$\Delta_*^+(\mathrm{Q}_{\mathrm{U}(r)}) : \quad \mathrm{H}^+(\mathrm{W}(\mathfrak{u}(r))_r) \to \mathrm{H}^+(\mathrm{SU}(r+1)) \cong \Lambda^+(\bar{y}_2, \ldots, \bar{y}_{r+1})$$

is spanned by the linearly independent classes

$$\Delta_*(z_{(i)}) = \varkappa \bar{y}_{i_1} \wedge \ldots \wedge \bar{y}_{i_s} \wedge \bar{y}_{r+1},$$

where $z_{(i)} = y_1 \wedge y_{i_1} \wedge \ldots \wedge y_{i_s} \otimes c_1^r \in \mathrm{W}(\mathfrak{u}(r))_r$ *for* $2 \leqq i_1 < \ldots < i_s \leqq r$, $0 \leqq s \leqq r-1$ *and* $\varkappa = (-1)^{s+1} \cdot (r+1)^{r+1}$. *In particular*

$$\operatorname{im} \Delta_*^+(\mathrm{Q}_{\mathrm{U}(r)}) = \mathrm{Id}(\bar{y}_{r+1}) \subset \mathrm{H}^+(\mathrm{SU}(r+1)).$$

(6.50) *Example.* $(\bar{\mathrm{G}}, \mathrm{G}) = (\mathrm{SO}(2r+1), \mathrm{SO}(2r))$, with quotient

$$\mathrm{SO}(2r+1)/\mathrm{SO}(2r) \approx \mathrm{S}^{2r}.$$

In this case the restriction map i^* is given by

$$\begin{array}{ccc} \mathrm{I}(\mathrm{SO}(2r+1)) & \xrightarrow{i^*} & \mathrm{I}(\mathrm{SO}(2r)) \\ \wr\Vert & & \wr\Vert \\ \mathbf{R}[\bar{p}_1, \ldots, \bar{p}_r] & \to & \mathbf{R}[p_1, \ldots, p_{r-1}, e_r] \end{array}$$

where the $\bar{p}_i$, p_i are the Pontrjagin polynomials and e_r is the Pfaffian. As $i^*(\bar{p}_j) = p_j$ for $j = 1, \ldots, r-1$ and $i^*(\bar{p}_r) = e_r^2$, the matrix $a + \Phi$ is given by the diagonal matrix

$$(6.51) \qquad a + \Phi = \begin{bmatrix} 1 & & & \\ & \ddots & 0 & \\ & 0 & 1 & \\ & & & e_r \end{bmatrix}$$

with determinant e_r.

The class $\bar{p}_r$ satisfies (6.22) as $i^* \bar{p}_r = e_r^2$, $\deg e_r = 2r = 2q'$. (6.21) is satisfied for r odd. For r even there are two classes of degree $2r$ in $\mathrm{I}(\mathrm{SO}(2r))$, namely $p_{r/2}$ and e_r, but this does not affect our calculations in this case. Thus we have $j = k = r$ and $\lambda = 1$. Furthermore $(\Lambda \mathfrak{m}^*)^{\mathrm{SO}(2r)} \cong \mathbf{R}[\tilde{e}_r]/(e_r^2)$, where $h(e_r) = \tilde{e}_r$ is the residue class of e_r mod $(p_1, \ldots, p_r)$ [twice a generator of $\mathrm{H}^{2r}(\mathrm{S}^{2r}, \mathbf{Z})$]. From our preceding calculation it follows that the primitive generators $\bar{y}_i$ of $\mathrm{H}(\mathrm{SO}(2r+1))$ are realized by the cocycles $\bar{y}_i \sim y_i \otimes 1$, $i = 1, \ldots, r-1$, and $\bar{y}_r \sim x \otimes \tilde{e}_r$, $x = \sigma(e)$. Observing that the isotropy representation $\rho : \mathrm{SO}(2r) \to \mathrm{SO}(\mathfrak{m})$ is the identity, we obtain from Corollary (6.29) and Theorem (6.40) the following result for the normal bundle

$$\mathrm{Q}_{\mathrm{SO}(2r)} \cong \pi'^* \mathrm{T}_{\mathrm{S}^{2r}},$$

$\pi' : \mathrm{P}' = \mathrm{SO}(2r+1) \to \mathrm{SO}(2r+1)/\mathrm{SO}(2r) \approx \mathrm{S}^{2r}$.

(6.52) THEOREM. — *Let* $\mathrm{Q}_{\mathrm{SO}(2r)}$ *be the normal bundle of the foliation of* $\mathrm{SO}(2r+1)$ *by the left cosets of* $\mathrm{SO}(2r)$ *with quotient* S^{2r}. *The image of the generalized characteristic homomorphism*

$$\Delta_*(\mathrm{Q}_{\mathrm{SO}(2r)}) : \ \mathrm{H}(\mathrm{W}(\mathfrak{so}(2r))_r) \to \mathrm{H}(\mathrm{SO}(2r+1) \cong \Lambda(\bar{y}_1, \ldots, \bar{y}_r)$$

is spanned by the linearly independent classes

$$\Delta_*(z_{(i)}) = \bar{y}_{i_1} \wedge \ldots \wedge \bar{y}_{i_s} \wedge \bar{y}_r, \tag{6.53}$$

where $z_{(i)} = y_{i_1} \wedge \ldots \wedge y_{i_s} \wedge x \otimes e_r \in \mathrm{W}(\mathfrak{so}(2r))_r$ *for* $1 \leqq i_1 < \ldots < i_s \leqq r-1$, $0 \leqq s \leqq r-1$, *and*

$$\Delta_*(y_{k_1} \wedge \ldots \wedge y_{k_\alpha} \otimes 1) = \bar{y}_{k_1} \wedge \ldots \wedge \bar{y}_{k_\alpha}, \tag{6.54}$$

where $[r/2]+1 \leqq k_1 < \ldots < k_\alpha \leqq r-1$. *In particular*

$$\operatorname{im} \Delta_*(\mathrm{Q}_{\mathrm{SO}(2r)}) = \mathrm{Id}(\bar{y}_r) \otimes \Lambda(\bar{y}_{[r/2]+1}, \ldots, \bar{y}_{r-1}) \subset \mathrm{H}(\mathrm{SO}(2r+1))$$

and thus

$$\dim(\operatorname{im} \Delta_*) = \begin{cases} 2^{r-1} + 2^{[r/2]-1}, & r \text{ even}, \\ 2^{r-1} + 2^{[r/2]}, & r \text{ odd}. \end{cases}$$

Note that for $r > 1$ the non-trivial class $\Delta_*(x \otimes e_r) = \bar{y}_r$ is rigid by Theorem (6.37). This applies to deformations through basic adapted connections on the normal bundles involved.

7. Interrelation between the previously discussed invariants

In this section we finally show how for certain pairs $(\overline{\mathrm{G}}, \mathrm{G})$ the invariants previously discussed are interrelated.

We begin with a connected semi-simple Lie group $\overline{\mathrm{G}}$ with no compact factor, $\mathrm{K}_{\overline{\mathrm{G}}} \subset \overline{\mathrm{G}}$ a maximal compact subgroup, and $\Gamma \subset \overline{\mathrm{G}}$ a discrete, uniform and torsion free subgroup.

Let G be a subgroup of $\bar{G}$. Its maximal compact subgroup is $K_G = K_{\bar{G}} \cap G$ and we assume that the canonical map $K_{\bar{G}}/K_G \xrightarrow{\cong} \bar{G}/G$ is a diffeomorphism.

The total space of the foliated G-bundle

$$P = \Gamma \backslash \bar{G} \times_{K_G} G \to M = \Gamma \backslash \bar{G}/K_G \tag{7.1}$$

maps canonically into the total space of the flat $\bar{G}$-bundle

$$\bar{P} = \Gamma \backslash \bar{G} \times_{K_{\bar{G}}} \bar{G} \cong \bar{G}/K_{\bar{G}} \times_\Gamma \bar{G} \to X = \Gamma \backslash \bar{G}/K_{\bar{G}} \tag{7.2}$$

by the map $\varphi : P \to \bar{P}$ induced by the inclusion $\mathrm{id} \times \mathrm{incl} : \bar{G} \times G \to \bar{G} \times \bar{G}$. The map φ is clearly G-equivariant and hence induces a map $M = P/G \to \bar{P}/G$ of orbit spaces.

(7.3) PROPOSITION. — *Under the assumption* $K_{\bar{G}}/K_G \xrightarrow{\cong} \bar{G}/G$, *the canonical map* $\varphi : P \to \bar{P}$ *induces an isomorphism of* G*-bundles*

$$\begin{array}{ccc} P & \xrightarrow[\cong]{\varphi} & \bar{P} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\cong} & \bar{P}/G \end{array} \tag{7.4}$$

Proof. — Since both sides are principal G-bundles and φ is a G-map, it suffices to show that the induced map of the base spaces

$$\varphi/G : \quad \Gamma \backslash \bar{G} \times_{K_G} G/G \equiv \Gamma \backslash \bar{G}/K_G \to \Gamma \backslash \bar{G} \times_{K_{\bar{G}}} \bar{G}/G \cong \bar{G}/K_{\bar{G}} \times_\Gamma \bar{G}/G$$

is a diffeomorphism. But we have canonically

$$\Gamma \backslash \bar{G}/K_G \cong \Gamma \backslash \bar{G} \times_{K_{\bar{G}}} K_{\bar{G}}/K_G.$$

The assumption $K_{\bar{G}}/K_G \xrightarrow{\cong} \bar{G}/G$ implies now the desired result. ■

It follows that the foliation of the bundle $P \to M$ (7.1) defined by the diagonal action of G, coincides with the foliation on the bundle $\bar{P} \to \bar{P}/G$ induced from the flat bundle structure on $\bar{P} \to X$. Here the foliation on the space $\bar{P}/G$ is the quotient of the foliation on the space $\bar{P}$ given by the flat connection in $\bar{P}$.

For the natural bundle Q_G of the foliation on the base space M the first description shows that

$$Q_G = P \times_G \bar{\mathfrak{g}}/\mathfrak{g}, \tag{7.5}$$

where $\rho : G \to GL(\bar{\mathfrak{g}}/\mathfrak{g})$ is the adjoint representation of G in $\bar{\mathfrak{g}}/\mathfrak{g}$. The second description shows that

$$Q_G = T(\pi), \tag{7.6}$$

where $T(\hat{\pi})$ is the tangent bundle along the fiber of the canonical projection $\hat{\pi} : \bar{P}/G \to \bar{P}/\bar{G} = X$ induced by $\pi : P \to X$.

With our previous constructions we have then the following result.

(7.7) THEOREM. — *Let $\bar{G}$ be a connected semi-simple Lie group with finite center and no compact factor, $K_{\bar{G}} \subset \bar{G}$ a maximal compact subgroup; and $\Gamma \subset \bar{G}$ a discrete, uniform and torsion free subgroup. Let $G \subset \bar{G}$ be a subgroup such that $K_{\bar{G}}/K_G \xrightarrow{\cong} \bar{G}/G$, where $K_G = K_{\bar{G}} \cap G$.*

Let q be the codimension of the canonical G-foliation on $\Gamma\backslash\bar{G}/K_G$, with normal bundle Q_G, $q = \dim \bar{\mathfrak{g}}/\mathfrak{g}$. Let $\theta : \bar{\mathfrak{g}} \to \mathfrak{g}$ be a K_G-equivariant splitting of the exact sequence

(7.8) $$0 \to \mathfrak{g} \xrightarrow{i} \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\mathfrak{g} = \mathfrak{m} \to 0.$$

Then the generalized characteristic homomorphism $\Delta(Q_G)$ on the cochain level factorizes as in the following commutative diagram

(7.9) $$\begin{tikzcd}
W(\mathfrak{gl}(\mathfrak{m}), SO(\mathfrak{m}))_q \arrow[d, "W(\rho)"] \arrow[ddr, "\Delta(Q_G)"] & \\
W(\mathfrak{g}, K_G)_q \arrow[d, "\Delta(\theta)_{K_G}"] \arrow[dr, "\Delta(P)"] & \\
(\Lambda \bar{\mathfrak{g}}^*)_{K_G} \arrow[r, "\gamma"] \arrow[d, "j^*"] & \Omega(\Gamma\backslash\bar{G}/K_G) \arrow[d, "\hat{\pi}_*"] \\
(\Lambda \bar{\mathfrak{g}}^*)_{K_{\bar{G}}} \arrow[r, "\Delta(\bar{P}) = \bar{\gamma}"] & \Omega(\Gamma\backslash\bar{G}/K_{\bar{G}})
\end{tikzcd}$$

In this diagram, $\Delta(P)$ is the characteristic homomorphism of the foliated G-bundle P in (7.1) with its canonical K_G-reduction. $\Omega(\Gamma\backslash\bar{G}/K_G)$ denotes the de Rham complex of $\Gamma\backslash\bar{G}/K_G$. γ denotes the canonical inclusion of the $\bar{G}$-invariant forms on $\bar{G}/K_G$ into the de Rham complex of $\Gamma\backslash\bar{G}/K_G$. The two top triangles are then commutative by Theorem 3.7′.

$\Delta(\bar{P})$ is the characteristic homomorphism of the flat $\bar{G}$-bundle $\bar{\bar{P}}$ in (7.2) with its canonical $K_{\bar{G}}$-reduction, namely the canonical inclusion $\bar{\gamma}$ of the $\bar{G}$-invariant forms on $G/K_{\bar{G}}$ into the de Rham comp ex $\Omega(\Gamma\backslash\bar{G}/K_{\bar{G}})$ of $\Gamma\backslash\bar{G}/K_{\bar{G}}$. It induces an injective cohomology map $\Delta(\bar{\bar{P}})_*$ by Theorem (4.7).

The map $j_* = i(\xi_1 \wedge \ldots \wedge \xi_q)$ denotes the interior product with the unique q-vector $\xi_1 \wedge \ldots \wedge \xi_q \in \Lambda^q \mathfrak{k}_{\bar{G}}/\mathfrak{k}_G$ normalized by $i(\xi_1 \wedge \ldots \wedge \xi_q)\,\eta = 1$ for an invariant unit volume η on $K_{\bar{G}}/K_G$. If $j^* : (\Lambda \mathfrak{g}^*)_{K_{\bar{G}}} \to (\Lambda \bar{\mathfrak{g}}^*)_{K_G}$ denotes the canonical inclusion, then the derivation property of the interior product $i(\xi)$ leads to the following formula useful for computations:

$$j_*(j^*\alpha.\beta) = \alpha.j_*(\beta) \tag{7.10}$$

for $\alpha \in (\Lambda\, \bar{\mathfrak{g}}^*)_{K_{\bar{G}}}$, $\beta \in (\Lambda\, \bar{\mathfrak{g}}^*)_{K_G}$.

The map π_* denotes integration over the fiber $K_{\bar{G}}/K_G \cong \bar{G}/G$ of the canonical map

$$\hat{\pi}: \quad P/G = \Gamma\backslash\bar{G}/K_G \to \bar{P}/\bar{G} = \Gamma\backslash\bar{G}/K_{\bar{G}}.$$

The bottom rectangle in (7.9) is commutative (up to sign) (*see* [14], Vol. II, p. 243).

The point of (7.9) is that the left hand vertical map is given completely in Lie algebra terms by the adjoint representation ρ in $\mathfrak{m}$, the split θ, and the canonical map j_*. This applies e. g. to the following situations.

Let U/K be an irreducible hermitian space of the compact type. The complexification G_c of U is a simple complex group acting by holomorphic diffeomorphisms transitively on U/K. The isotropy group L is a complex subgroup of G_c and $U/K \cong G_c/L$. U is maximal compact in G_c and K maximal compact in L [17], (ch. VIII). The foliated L-bundle $P = \Gamma\backslash G_c \times_K L$ is mapped into the flat G_c-bundle $\bar{P} = \Gamma\backslash G_c \times_U G_c$ by an L-equivariant map, which induces over $\Gamma\backslash G_c/K$ an isomorphism of (real) L-bundles. The (complex) foliation of G_c by the cosets of L induces on $\Gamma\backslash G_c/K$ a (real) foliation with complex normal bundle Q_L, associated to P by the adjoint representation of L in $\mathfrak{g}_c/\mathfrak{l}$. The bundle Q_L is the tangent bundle along the complex fiber U/K of the projection $\Gamma\backslash G_c/K \to \Gamma\backslash G_c/U$. The evaluation of $\Delta(Q_L)_*$ involves then a K-equivariant splitting $\theta: \mathfrak{g}_c \to \mathfrak{l}$ of the exact sequence $0 \to \mathfrak{l} \to \mathfrak{g}_c \to \mathfrak{g}_c/\mathfrak{l} \to 0$. Note that no such L-invariant splitting exists by [21] (Prop. 9.14). The computation of $\Delta(Q_L)_*$ by the methods here discussed leads to non-trivial secondary invariants.

Further applications of Theorem (7.7) can be found in [25′], Theorem 7.93 and Theorem 7.95. Added in proof : These results are proved in the authors' paper. "On the linear independence of certain cohomology classes of BΓ".

APPENDIX

Computation of the universal generalized characteristic classes

In this appendix we discuss the universal algebras $H(W(\mathfrak{g}, \mathfrak{h})_k)$ of generalized characteristic classes. We describe more generally an algorithm for the computation of the cohomology $H(E_{\mathfrak{h}})$ of the $\mathfrak{h}$-basic elements $E_{\mathfrak{h}}$ in a $\mathfrak{g}$-DG-algebra E with connection. There are two spectral sequences converging to $E_{\mathfrak{h}}$. S. Halperin has independently obtained some of these results under slightly less restrictive hypotheses on E [14] (Vol. III). For $E = W(\mathfrak{g})_k$ this leads to the computation of $H(W(\mathfrak{g}, \mathfrak{h})_k)$ announced in [23].

We need the notion of a $\mathfrak{g}$-DG-algebra $A^{\cdot}$ with respect to a Lie algebra $\mathfrak{g}$ (all algebras are over the same ground field of characteristic zero). Such an algebra is an associative DG-algebra which is equipped with derivations $\Theta(x)$ of degree 0, $i(x)$ of degree-1 for $x \in \mathfrak{g}$, satisfying the identies $i(x)^2 = 0$, $i[x, y] = [\Theta(x), i(y)]$ and $\Theta(x) = di(x) + i(x)d$. The bracket of derivations is here used with the degree convention

$$[D, D'] = DD' + (-1)^{dd'+1} D' D$$

for derivations D, D′ of degress d, d'. For a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ the $\mathfrak{h}$-invariant resp. $\mathfrak{h}$-basic elements [killed by $\Theta(x)$ resp. $\Theta(x)$, $i(x)$ for $x \in \mathfrak{h}$] are denoted by $A^{\mathfrak{h}}$ resp. $A_{\mathfrak{h}}$.

Commutative $\mathfrak{g}$-DG-algebras were introduced in [8]. W$(\mathfrak{g})$ is canonically a $\mathfrak{g}$-DG-algebra. A (formal) connection in a (commutative) $\mathfrak{g}$-DG-algebra E is characterized by a $\mathfrak{g}$-DG-homomorphism $W(\mathfrak{g}) \to E$. Restricted to $\mathfrak{g}^* = \Lambda^1(\mathfrak{g}^*)$ this map gives a connection form $\omega : \mathfrak{g}^* \to E^1$, restricted to $\mathfrak{g}^* = S^1(\mathfrak{g}^*)$ its curvature form $K(\omega) : \mathfrak{g}^* \to E^2$, which in turn determine the map $k(\omega) : W(\mathfrak{g}) \to E$, the Weil homomorphism of the formal connection ω. For a principal G-bundle $P \to M$ the de Rham complex is a $\mathfrak{g}$-DG-algebra (more precisely a G-DG-algebra). To a connection in P corresponds by dualization a formal connection $\omega : \mathfrak{g}^* \to \Gamma(P, \Omega_P^1)$ with curvature $K(\omega) : \mathfrak{g}^* \to \Gamma(P, \Omega_P^2)$ and Weil homomorphism $k(\omega) : W(\mathfrak{g}) \to \Gamma(P, \Omega_P^{\cdot})$. To simplify matters we consider in this appendix everything from the Lie algebra point of view (ignoring group actions).

We describe a method for the computation of the cohomology $H(E_{\mathfrak{h}})$ for a commutative $\mathfrak{g}$-DG-algebra E with connection k. We assume that $(\mathfrak{g}, \mathfrak{h})$ is a reductive pair of Lie algebras and that E satisfies the condition:

(A.1) E^q is a direct sum of finite-dimensional simple $\mathfrak{g}$-modules for $q \geqq 0$.

First we define the graded algebra

$$\text{(A.2)} \qquad A^{\cdot}(E, \mathfrak{h}) = \Lambda^{\cdot} P_{\mathfrak{g}}^{\cdot} \otimes E_{\mathfrak{g}}^{\cdot} \otimes I^{\cdot}(\mathfrak{h}),$$

where $P_{\mathfrak{g}}^{\cdot} \subset \Lambda^{\cdot}(\mathfrak{g}^*)^{\mathfrak{g}}$ denotes the graded subspace of primitive elements of $\mathfrak{g}$. Let $\tau_{\mathfrak{g}} : P_{\mathfrak{g}} \to I(\mathfrak{g})^+$ be a transgression for $\mathfrak{g}$. A differential d_A is defined on A as a derivation of degree 1, which is zero on $I(\mathfrak{h})$, equal to the restriction of d_E on $E_{\mathfrak{g}}$, and on $\Lambda P_{\mathfrak{g}}$ uniquely characterized by

$$d_A(x) = 1 \otimes h(c) \otimes 1 - 1 \otimes 1 \otimes i^*(c) \qquad \text{for} \quad x \in P_{\mathfrak{g}}.$$

Here $h : I(\mathfrak{g}) \to E_{\mathfrak{g}}$ denotes the restriction of the (Weil-homomorphism of the) connection $k : W(\mathfrak{g}) \to E$ to $\mathfrak{g}$-basic elements, and this map is applied to the element $c = \tau_{\mathfrak{g}}(x) \in I(G)^+$ to which x transgresses. $i^* : I(\mathfrak{g}) \to I(\mathfrak{h})$ denotes the canonical restriction. $d_A^2 = 0$ is trivially verified. The DG-algebra (A, d_A) is functorial with respect to connection preserving homomorphisms of $\mathfrak{g}$-DG-algebras and with respect to inclusions $\mathfrak{h}' \subset \mathfrak{h}$.

Next we define a homomorphism of DG-algebras

$$\text{(A.3)} \qquad \varphi(E, \mathfrak{h}) : \quad A(E, \mathfrak{h}) \to (E \otimes W(\mathfrak{h}))_{\mathfrak{h}}$$

which is natural in E, $\mathfrak{h}$. On $\mathrm{I}(\mathfrak{h})$ this map is induced by the canonical map $\mathrm{W}(\mathfrak{h}) \to \mathrm{E} \otimes \mathrm{W}(\mathfrak{h})$. On $\mathrm{E}_\mathfrak{g}$ the map is induced by the canonical map $\mathrm{E} \to \mathrm{E} \otimes \mathrm{W}(\mathfrak{h})$. On $\Lambda\, \mathrm{P}_\mathfrak{g}$ the map is the canonical extension of

$$-\lambda^1(\mathrm{E}, \mathfrak{h}) \circ \tau_\mathfrak{g}, \qquad \text{where} \quad \lambda^1(\mathrm{E}, \mathfrak{h}) = (k \otimes \mathrm{W}(i)) \circ \tilde{\lambda}^1.$$

Here $\mathrm{W}(i)$ is induced by $i : \mathfrak{h} \subset \mathfrak{g}$ and $\tilde{\lambda}^1 = \lambda^1(\mathrm{W}(\mathfrak{g}), \mathfrak{g})$ is the universal homotopy operator

$$\tilde{\lambda}^1 : \quad \mathrm{W}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g}) \otimes \mathrm{W}(\mathfrak{g}) ([26], \text{(Theorem 1.12)}, [25'], (5.54))$$

which restricted to $\mathrm{I}(\mathfrak{g}) = \mathrm{W}(\mathfrak{g}, \mathfrak{g})$ satisfies by [26] (1.15) or [25'], (5.69)

$$d\tilde{\lambda}^1(\Phi) = 1 \otimes \Phi - \Phi \otimes 1 \qquad \text{for} \quad \Phi \in \mathrm{I}(\mathfrak{g})^+.$$

Note for later use that for $\mathfrak{h} = 0$ the map $\mathrm{W}(i) = \varepsilon : \mathrm{W}(\mathfrak{g}) \to \mathrm{F}$ is the augmentation to the groundfield F. For the map $\lambda^1(\mathrm{W}(\mathfrak{g}), 0) = (\mathrm{id} \otimes \varepsilon) \circ \tilde{\lambda}^1$ we obtain therefore

$$d\lambda^1(\mathrm{W}(\mathfrak{g}), 0)(\Phi) = (\mathrm{id} \otimes \varepsilon) \circ d\tilde{\lambda}^1(\Phi) = -\Phi \qquad \text{for} \quad \Phi \in \mathrm{I}(\mathfrak{g})^+.$$

Hence

(A.4) $$\mathrm{T} = -\lambda^1(\mathrm{W}(\mathfrak{g}), 0) \; : \; \mathrm{I}(\mathfrak{g})^+ \to \mathrm{W}(\mathfrak{g})^\mathfrak{g}$$

is a universal transgression operator. Consequently

(A.5) $$\sigma = -\lambda^1(\Lambda\, \mathfrak{g}^*, 0) = -\pi \circ \lambda^1(\mathrm{W}(\mathfrak{g}), 0) = \pi \circ \mathrm{T}$$

is the suspension $\sigma : \mathrm{I}(\mathfrak{g})^+ \to (\Lambda\, \mathfrak{g}^*)^\mathfrak{g}$, where

$$\pi : \quad \mathrm{W}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g})/\mathrm{F}^2\,\mathrm{W}(\mathfrak{g}) = \Lambda\, \mathfrak{g}^*$$

is the canonical projection.

The combination of the natural transformation (A.3) with the canonical homology equivalence $\alpha : (\mathrm{E} \otimes \mathrm{W}(\mathfrak{h}))_\mathfrak{h} \to \mathrm{E}_\mathfrak{h}$ induced by the identity on E and the $\mathfrak{h}$-connection $k \circ k(\theta)$ in E [8] leads to the following result.

(A.6) THEOREM. — *The homomorphism* (A.3) *induces an isomorphism*

$$\mathrm{H}(\mathrm{A}(\mathrm{E}, \mathfrak{h})) \xrightarrow[\cong]{\varphi(\mathrm{E}, \mathfrak{h})_*} \mathrm{H}((\mathrm{E} \otimes \mathrm{W}(\mathfrak{h}))_\mathfrak{h}) \cong \mathrm{H}(\mathrm{E}_\mathfrak{h}).$$

For $\mathfrak{h} = 0$ this is a result of Chevalley [8], [30]. This Theorem is proved by introducing filtrations on $\mathrm{A}(\mathrm{E}, \mathfrak{h})$ and $(\mathrm{E} \otimes \mathrm{W}(\mathfrak{h}))_\mathfrak{h}$ which are preserved by $\varphi(\mathrm{E}, \mathfrak{h})$, and establishing that φ induces an isomorphism of the initial terms of the associated spectral sequences. The following two multiplicative filtrations are used.

First the canonical filtration on $\mathrm{W}(\mathfrak{h})$ induces a filtration on $\mathrm{I}(\mathfrak{h})$ and hence on $\mathrm{A}(\mathrm{E}, \mathfrak{h})$ *via* $\mathrm{I}(\mathfrak{h})$, and further on $\mathrm{E} \otimes \mathrm{W}(\mathfrak{h})$ and hence on $(\mathrm{E} \otimes \mathrm{W}(\mathfrak{h}))_\mathfrak{h}$ *via* $\mathrm{W}(\mathfrak{h})$. These even filtrations will be denoted $\mathrm{F}^{2p}(\mathfrak{h}) = {'\mathrm{F}^{2p}}$ and are called $\mathfrak{h}$-filtrations of the respective DG-algebras.

Next consider the canonical filtration on E given by

$$\text{(A.7)} \qquad F^p E^{\cdot} = \bigoplus_{j \geqq p} ((E^{i(\mathfrak{g})})^j . E)^{\cdot},$$

where $E^{i(\mathfrak{g})}$ denotes the elements in E killed by $i(x)$, $x \in \mathfrak{g}$. This induces on $E_{\mathfrak{g}}$ the filtration $F^p E_{\mathfrak{g}}^{\cdot} = \bigoplus_{j \geqq p} E_{\mathfrak{g}}^j$, and hence a (decreasing) filtration on $A(E, \mathfrak{h})$, and further on $E \otimes W(\mathfrak{h})$ and hence on $(E \otimes W(\mathfrak{h}))_{\mathfrak{h}}$ *via* E. These filtrations will be denoted by $F^p(\mathfrak{g}) = {}''F^p$ and are called $\mathfrak{g}$-filtrations of the respective DG-algebras.

The natural homomorphism φ is filtration preserving for both the $\mathfrak{h}$-and the $\mathfrak{g}$-filtrations.

(A.8) THEOREM. — (i) φ *induces for the even spectral sequences associated to the* $\mathfrak{h}$-*filtrations an isomorphism on the* ${}'E_{2r}$-*level for* $r \geqq 1$.

(ii) *The composition of* φ *with the canonical map* $(E \otimes W(\mathfrak{h}))_{\mathfrak{h}} \to E_{\mathfrak{h}}$ *is filtration preserving with respect to the* $\mathfrak{g}$-*filtrations on* $A(E, \mathfrak{h})$ *and* $E_{\mathfrak{h}}$. *It induces for the associated spectral sequences an isomorphism on the* ${}''E_r$-*level for* $r \geqq 1$.

Each of these facts proves Theorem (A.6). Together with the computation of the initial terms of the spectral sequences associated to the $\mathfrak{h}$-and $\mathfrak{g}$-filtrations on $A(E, \mathfrak{h})$ we obtain the following result.

(A.9) THEOREM. — (i) *There is a multiplicative even spectral sequence*

$$'E_2^{2p, q} = H^q(E) \otimes I^{2p}(\mathfrak{h}) \quad \Rightarrow \quad H^{2p+q}(E_{\mathfrak{h}}).$$

(ii) *There is a multiplicative spectral sequence*

$$''E_2^{p, q} = H^q(\mathfrak{g}, \mathfrak{h}) \otimes H^p(E_{\mathfrak{g}}) \quad \Rightarrow \quad H^{p+q}(E_{\mathfrak{h}}).$$

A geometric analogon of the previous results concerns the G-DG-algebra $\Gamma(P, \Omega_P^{\cdot})$ for an ordinary principal G-bundle $P \to M$ with compact group G. For a closed subgroup $H \subset G$ the natural homomorphisms

$$A(\Gamma(P, \Omega_P^{\cdot}), H) \to (\Gamma(P, \Omega_P^{\cdot}) \otimes W(\mathfrak{h}))_H \to \Gamma(P, \Omega_P^{\cdot})_H$$

induce isomorphisms in homology. The spectral sequences discussed above have as geometric analoga the Serre spectral sequences of the fibrations

$$P \to T_H \times_H P \xrightarrow{\pi'} B_H, \qquad G/H \to P/H \xrightarrow{\pi''} M,$$

where $T_H \to B_H$ denotes a universal H-bundle.

These theorems apply to $E = W(\mathfrak{g})_k$, $k \geqq 0$. It is convenient to set $W(\mathfrak{g})_\infty = W(\mathfrak{g})$. Then we obtain the following result [23] (*see* also [25′], (5.85)).

(A.10) THEOREM. — *For* $0 \leqq k \leqq \infty$ *the cohomology* $H(W(\mathfrak{g}, \mathfrak{h})_k)$ *can be computed as the cohomology of the* DG-*algebra*

$$\text{(A.11)} \qquad A(W(\mathfrak{g})_k, \mathfrak{h}) = \Lambda^{\cdot} P_{\mathfrak{g}}^{\cdot} \otimes I^{\cdot}(\mathfrak{g})_k \otimes I^{\cdot}(\mathfrak{h}).$$

There are multiplicative spectral sequences

$$\text{(A.12)} \qquad {'E}_2^{2p,q} = H^q(W(\mathfrak{g})_k) \otimes I^{2p}(\mathfrak{h}) \Rightarrow H^{2p+q}(W(\mathfrak{g}, \mathfrak{h})_k),$$

$$\text{(A.13)} \qquad {''E}_2^{2p,q} = H^q(\mathfrak{g}, \mathfrak{h}) \otimes I^{2p}(\mathfrak{g})_k \Rightarrow H^{2p+q}(W(\mathfrak{g}, \mathfrak{h})_k).$$

In the second spectral sequence ${''E}_2^{2p+1,q} = 0$.

For $k = \infty$ we have ${'E}_2^{2p,q} = 0$ for $q > 0$ and an edge isomorphism

$$I(\mathfrak{h}) \xrightarrow{\cong} H(W(\mathfrak{g}, \mathfrak{h})).$$

Therefore

$$\text{(A.14)} \qquad {''E}_2^{2p,q} = H^q(\mathfrak{g}, \mathfrak{h}) \otimes I^{2p}(\mathfrak{g}) \to I^{2p+q}(\mathfrak{h}),$$

where the edge homomorphism $I^{2p}(\mathfrak{g}) \to I^{2p}(\mathfrak{h})$ is the restriction map. Since $I(\mathfrak{g}) = S(\mathfrak{g}^*)^{\mathfrak{g}}$, the initial term equals $H^q(\mathfrak{g}, \mathfrak{h}; S^p(\mathfrak{g}^*))$.

For the case of a connected group G and maximal compact subgroup K this gives e. g. a spectral sequence

$$\text{(A.15)} \qquad H^q(\mathfrak{g}, K;\ S^p(\mathfrak{g}^*)) \Rightarrow I(K).$$

The initial term can by the Van Est Theorem [11] be replaced by the continuous cohomology $H_c(G, S^p(\mathfrak{g}^*))$, whereas the end term is by the universal Chern-Weil homomorphism isomorphic to $H(B_K)$. Under these replacements (A.15) coincides with the spectral sequence

$$H_c^q(G, S^p(\mathfrak{g}^*)) \Rightarrow H(B_K)$$

considered in [6] and [33].

The computation of $H(A(E, \mathfrak{h}))$ can be further simplified. For this purpose we need the Samelson space $\hat{P} \subset P_{\mathfrak{g}}$ of the reductive pair $(\mathfrak{g}, \mathfrak{h})$ ([8], [14]). We use the condition

$$\text{(C)} \qquad \dim \hat{P} = \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{h}$$

and the stronger condition

(CS) There exists a transgression τ for $\mathfrak{g}$ such that

$$\ker(I(\mathfrak{g}) \to I(\mathfrak{h})) = \operatorname{ideal}(\tau \hat{P}) \subset I(\mathfrak{g}).$$

This condition is satisfied for symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ or $\mathfrak{h} \subset \mathfrak{g}$ of equal rank and implies (C) [23] (*see* also [25′], chapter 5).

We define now for a commutative $\mathfrak{g}$-DG-algebra E with connection and satisfying (A.1) a graded algebra

$$\text{(A.16)} \qquad \hat{A}^{\cdot}(E) = \Lambda^{\cdot} \hat{P}^{\cdot} \otimes E_{\mathfrak{g}}$$

with differential characterized by $d_{\hat{A}}(x) = 1 \otimes h(c)$ for $x \in \hat{P}$, $c = \tau x \in I(\mathfrak{g})$ and equal to the restriction of d_E on E_g. Under the condition (CS) for the pair $(\mathfrak{g}, \mathfrak{h})$ the canonical map

$$\text{(A.17)} \qquad \hat{A}(E) \overset{i}{\hookrightarrow} A(E, \mathfrak{h})$$

given by $x \otimes c \to x \otimes c \otimes 1$ is compatible with the differentials. Consider further the canonical homomorphism $j : I(\mathfrak{h}) \to A(E, \mathfrak{h})$ and the induced cohomology map

$$\text{(A.18)} \qquad \beta = (i_*, j_*) : \quad H(\hat{A}(E)) \otimes I(\mathfrak{h}) \to H(A(E, \mathfrak{h})).$$

Note that $\hat{A}(E)$ and hence $H(\hat{A}(E))$ is an $I(\mathfrak{g})$-module via the characteristic homomorphism. We have then the following result [23] (also [25′], (5.107)).

(A.19) THEOREM. — *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a reductive pair satisfying* (CS). *The homomorphism* (A.18) *factorizes through an isomorphism*

$$\hat{\beta} : \quad H(\hat{A}(E)) \otimes_{I(\mathfrak{g})} I(\mathfrak{h}) \xrightarrow{\cong} H(A(E, \mathfrak{h})).$$

This result applies to $E = W(\mathfrak{g})_k$. Let $\hat{A}_k = \hat{A}(W(g)_k)$. Then we get the following consequence [23] (also [25′], (5.108)).

(A.20) THEOREM. — *For* $0 \leqq k \leqq \infty$ *there are isomorphisms*

$$H(\hat{A}_k) \otimes_{I(\mathfrak{g})} I(\mathfrak{h}) \xrightarrow{\cong} H(A(W(\mathfrak{g})_k, \mathfrak{h}) \cong H(W(\mathfrak{g}, \mathfrak{h})_k).$$

To describe a basis of $\hat{H}(A_k)$ over the groundfield F of characteristic zero, recall that $I(\mathfrak{g}) \cong F[c_1, \ldots, c_r]$, $r = \operatorname{rank} \mathfrak{g}$ (ordered such that $\deg c_i \leqq \deg c_{i+1}$) and P_g has a basis of elements transgressing to $c_1, \ldots, c_r$ respectively. Let

$$r' = \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{h} = \dim \hat{P}$$

and $y_1, \ldots, y_{r'}$ a basis of $\hat{P}$ such that y_i transgresses to c_{α_i} $(\alpha_1 \leqq \ldots \leqq \alpha_{r'})$. For $\mathfrak{h} = 0$ we have in particular $\hat{P} = P_g$, $r' = r$ and $\alpha_i = i$ for all i. With these notations

$$\hat{A}_k = \Lambda(y_1, \ldots, y_{r'}) \otimes F[c_1, \ldots, c_r]_k,$$

where $dy_i = c_{\alpha_i}$. We use the following conventions:

$$\text{(A.21)} \quad \begin{cases} y_{(i)} = y_{i_1} \wedge \ldots \wedge y_{i_s} & \text{for } (i) = (i_1, \ldots, i_s), \quad 1 \leqq i_1 < \ldots < i_s \leqq r' \quad (s > 0); \\ y_{(i)} = 1 & \text{for } (i) = \varnothing \quad (s = 0); \\ c_{(j)} = c_1^{j_1} \ldots c_r^{j_r} & \text{for } (j) = (j_1, \ldots, j_r), \quad 0 \leqq j_i; \\ 2p = \deg c_{(j)} = \sum_{i=1}^{r} j_i \deg c_i. \end{cases}$$

Then we have the following result (this is the result of [23] with a slight change of notation adapted to the present purposes, *see* also [25′], (5.110)).

(A.22) THEOREM. — *An* F-*basis of* $H(\hat{A}_k)$ *is given by the classes of the monomial cocycles* $z_{(i,j)} = y_{(i)} \otimes c_{(j)}$ *satisfying the conditions:*

(*a*) $0 \leqq 2p \leqq 2k$, $0 \leqq s \leqq r'$;

(*b*) $\deg c_{\alpha_{i_1}} = \deg y_{i_1} + 1 \geqq 2(k+1-p)$ *if* $(i) \neq \varnothing$;

(*c*) $j_{\alpha_l} = 0$ *for* $l < i_1$ *if* $(i) \neq \mathrm{O}$ *and* $j_{\alpha_l} = 0$ *for all* l *if* $(i) = \varnothing$.

Remarks. — (i) The monomials $z_{(\varnothing,j)}$ $(s=0)$ form a basis of the primary classes (induced from $I(\mathfrak{g}) \to W(\mathfrak{g}, \mathfrak{h})_k$).

(ii) The monomials $z_{(i,j)}$ for $s > 0$ form a basis for the secondary classes. (iii) The classes for $s > 0$ and $p = 0$ are the classes $y_{(i)} \otimes 1$ with $\deg y_{i_1} + 1 \geqq 2(k+1)$.

Note that the degrees of the secondary classes $[z_{(i,j)}]$ in $H(\hat{A}_k)$ satisfy the inequality

$$(A.23) \qquad 2k+1 \leqq \deg z_{(i,j)} \leqq 2k+m, \qquad m = \dim \mathfrak{g}.$$

In fact $s > 0$ guarantees the occurance of at least the element y_{i_1} and hence

$$\deg z_{(i,j)} + 1 \geqq 2p + \deg y_{i_1} + 1 \geqq 2(k+1).$$

The other inequality follows from the fact that $\deg c_{(j)} \leqq 2k$ and $\deg y_{(i)} \leqq m$ (which equals the sum of the degrees of all primitive generators).

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