Comments on a paper of Brown and Guivarc’h : “Espaces de Poisson des groupes de Lie”

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COMMENTS ON A PAPER OF BROWN AND GUIVARC'H

BY CALVIN C. MOORE (1) AND JONATHAN ROSENBERG (2)

In a recent paper [2], Brown and Guivarc'h announce a proof of the following conjecture from [1]: Let G be a connected Lie group with radical R such that G/R has finite center; then G is type T in the sense of [1] if and only if the eigenvalues of \( \text{ad} (X) \) restricted to the Lie algebra \( \mathfrak{z}(R) \) of R are purely imaginary for every X in the Lie algebra \( \mathfrak{z}(G) \) of G. The proof given, however, has a gap in it, and in particular the crucial Proposition 4 is clearly false as stated. The difficulty occurs in the next to last sentence of the proof of this Proposition for it is surely possible for G \( \cap \) V to leave invariant a compact set in \( \mathfrak{g}p(V) \), for instance a one point set consisting of an affine subspace containing G \( \cap \) V. We shall show how the difficulty can be repaired by modifying both Propositions 4 and 5; in the end, the modified version is a bit more direct than the original version. We also show that the condition in the theorem that G/R have finite center is necessary; in fact, we show that the universal covering group of SL \( _2(R) \) fails to have property T.

Specifically, Proposition 4 should be modified to read as follows:

**Proposition 4**. — Let G be a connected Lie group contained in the affine group of a vector space V. If G \( \supseteq \) V, and if G is type T, then G is type R.

**Proof.** — The given proof applies directly except that the affine Grassmann manifold \( \mathfrak{g}r(V) \) (use some letter other than p) must be chosen so that 0 < r < dim V which is possible by the proof of Proposition 3. The next to last sentence of the proof must be changed; the point is that if a compact subset C of \( \mathfrak{g}r(V) \) is invariant under a subspace \( V' \) of V, then C must consist of affine subspaces parallel to \( V' \). In particular, if \( V' = V \), we have an impossibility since r < dim V. This completes the proof.

Now Proposition 5 has to be strengthened as follows:

**Proposition 5**. — Let G be a connected Lie group with radical R (which is non-compact) and nil-radical N. Then there exists a homomorphism h of G onto a group h (G) such that

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the kernel of \( h \) operates unipotently on \( \mathcal{L}(R) \) and such that either: (i) \( h(G) \subseteq \text{GA}(V) \) for some vector space \( V \) and \( h(N) \supseteq V \), or else, (ii) \( h(G) \) is a solvable group.

**Proof.** — Exactly as in the paper, one reduces to the case when \( N \) is a vector group. Let \( \mathcal{P} \) be a Levi factor for \( \mathcal{L}(G) \) so that \( \mathcal{L}(G) = \mathcal{L}(R) + \mathcal{P} \). Now let \( \mathcal{L}(N_0) \) be the subspace of \( \mathcal{L}(N) \) where \( \mathcal{P} \) acts trivially and let \( N_0 \) be the corresponding vector subgroup of \( N \). If \( N = N_0 \), then as \( \mathcal{P} \) acts trivially on \( \mathcal{L}(R)/\mathcal{L}(N) \) and as \( \mathcal{P} \) is semisimple, \( \mathcal{P} \) acts trivially on \( \mathcal{L}(R) \) so that \( \mathcal{L}(G) \) is the Lie algebra direct sum of \( \mathcal{L}(R) \) and \( \mathcal{P} \). The commutator subalgebra of \( \mathcal{L}(G) \) is \([\mathcal{L}(R),\mathcal{L}(R)] + \mathcal{P} = \mathcal{L}(N) + \mathcal{P} \) which acts nilpotently on the radical \( \mathcal{L}(R) \). Hence, the commutator subgroup \([G,G] \) of \( G \) acts unipotently on \( \mathcal{L}(R) \) and hence so does its closure \( G_i \). In this case, we choose \( h \) to be the projection of \( G \) onto \( G/G_i \) and (ii) holds.

Now if \( N_0 \neq N \), we note that \( N_0 \) is a normal subgroup of \( G \) since \( N \) is abelian and since \( \mathcal{L}(R/N) \) is central in \( \mathcal{L}(G/N) \). Since \( \mathcal{P} \) is semisimple and acts trivially on \( \mathcal{L}(R)/\mathcal{L}(N) \), we may find a subspace \( \mathcal{A} \) of \( \mathcal{L}(R) \) complementary to \( \mathcal{L}(N) \) which is centralized by \( \mathcal{P} \). Since \([\mathcal{A},\mathcal{A}] \) is also centralized by \( \mathcal{P} \), it is contained in \( \mathcal{L}(N_0) \). Dividing out by \( N_0 \), let \( G' = G/N_0 \), \( R' = R/N_0 \), \( N' = N/N_0 \), and let \( \mathcal{A}' \simeq \mathcal{A} \), \( \mathcal{P}' \simeq \mathcal{P} \) be the images of \( \mathcal{A} \) and \( \mathcal{P} \) in \( \mathcal{L}(G') \). (Note that \( R' \) is the radical of \( G' \), but that \( N' \) may be smaller than the nil-radical of \( G' \).) Then \( \mathcal{A}' \) is an abelian subalgebra and \( \mathcal{P}' + \mathcal{A}' \) is a complement to \( \mathcal{L}(N') \) so that \( \mathcal{L}(G') \) is the semi-direct product of \( \mathcal{L}(N') \) and \( \mathcal{P}' + \mathcal{A}' \). Now let \( H \) be the connected subgroup of \( G \) with Lie algebra \( \mathcal{P}' + \mathcal{A}' \), and let \( H \) be its closure. Then \( \overline{H} \cap N' \) consists of elements \( n \) such that \( \text{Ad}(n) \) is trivial on \( \mathcal{L}(N') \) and on \( \mathcal{L}(G')/\mathcal{L}(N') \) which stabilize \( \mathcal{P}' + \mathcal{A}' \). That implies that \( \text{Ad}(n) \) is the identity, or in other words, that \( n \) is in the center of \( G' \). However, by the construction of \( N_0 \), and semi-simplicity of \( \mathcal{P}' \), this implies that \( n = e \). Thus, \( \overline{H} \cap N' = \{ e \} \) so \( H = \overline{H} \) is closed and \( G' \) is the semi-direct product of \( N' \) and \( H \).

We choose our vector space \( V \) to be \( N' \); for \( g \in G \), let \( g' \) be its image in \( G' \) and write \( g' = \tau(g) \rho(g) \) with \( \tau(g) \in V \), and \( \rho(g) \in H \). Now we let \( h(g)v = g'vg'^{-1} + \tau(g) \) for \( v \in V \); then \( h \) is a homomorphism of \( G \) into \( \text{GA}(V) \). Moreover \( h(N) = V \) and the kernel of \( h \) consists of elements \( g \in G \) whose projection in \( G' \) lies in \( H \) and which act trivially on \( V \). Thus the kernel surely acts unipotently on \( \mathcal{L}(R) \) and (i) holds. Proposition 5' is proved.

The proof of the main theorem now proceeds as in [2] if \( h \) satisfies (i) and is trivial if \( h \) satisfies (ii).

We turn now to the second point about necessity of the condition that \( G/R \) have finite center. Let \( G \) be semisimple with center \( Z \). By Proposition V.1 of [1] \( G \) will fail to have property \( T \) if and only if there is an open semigroup \( S \) in \( G \) such that \( S S^{-1} \cap Z \) has infinite index in \( Z \). Now let \( G \) be universal covering group of \( G_0 = \text{SL}_2(R) \), so that \( Z = \mathbb{Z} \), the integers, and let \( S_0 \) be the open semigroup of \( \text{SL}_2(R) \) consisting of matrices with all entries strictly positive. It is known that \( S_0 S_0^{-1} \) meets the center of \( G_0 \) in only one point. Now on page 46 of [3], there is constructed a very explicit cross section \( : G_0 \to G \) for the group extension so that the corresponding cocycle \( b \) from \( G_0 \times G_0 \) into \( Z \) defined
by \( s(g)s(h) = b(g, h)s(gh) \) is explicitly computable. The cross section \( s \) is continuous and hence a homeomorphism on a dense open set \( D \), specifically, the dense double coset of the triangular subgroup of \( G_0 \). It is clear that \( S_0 \subset D \), and a direct calculation using the formulas on page 46 of [3] shows that the cocycle is trivial on \( S_0 \times S_0 \) and that \( s(g^{-1}) = s(g)^{-1} \) for \( g \in S_0 \). It follows that \( s \) is a homomorphism on \( S_0 \) and that \( S = s(S_0) \) is an open semigroup in \( G \), and that \( SS^{-1} \cap Z = \{ e \} \). Thus \( G \) fails to have property T.

REFERENCES


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Calvin C. Moore
and
Jonathan Rosenberg,
Department of Mathematics,
University of California,
Berkeley, California 94720,
U.S.A.