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COHOMOLOGY OF THE INFINITESMAL SITE

BY ARTHUR OGUS

Introduction

In his famous letter to Tate, Grothendieck attempted to develop a p-adic cohomology for varieties in characteristic $p > 0$ by considering a site (the "infinitesmal site") whose objects were the nilpotent thickenings of open subsets of the given variety. By the end of the letter he had realized that this cohomology was "too rigid" and that better results would be obtained if one considered nilpotent thickenings endowed with divided powers (the "crystalline site"). Since then Berthelot has quite fully developed the crystalline theory, which Mazur has used to prove Katz's conjectures [7].

There remains the question of the meaning of the cohomology of the infinitesmal site. We shall prove that, for proper schemes X over an algebraically closed field k, the infinitesmal cohomology of $X \mid W_n(k)$ is nothing else than the étale cohomology with coefficients in $W_n(k)$, a much older, equally "unsatisfactory", p-adic cohomology.

In view of the relationship between the infinitesmal site and F-descent, this result is not too surprising. Care is required, however, because this result is false for nonproper schemes. Our approach to infinitesmal cohomology is through the ring of differential operators, a technique used in characteristic zero by differential geometers and by Katz, Hartshorne, and others. (A somewhat different approach, using the Čech-Alexander complex, is sketched at the end of paragraph 4.)

In paragraph 1 we discuss the behaviour of differential operators on formal schemes. Most of this section can be skipped by readers interested only in smooth lifted schemes, except for (1.10), which compares cohomology of affine (formal) schemes $\mathscr{X}$ with cohomology computed over $\Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$, and (1.14), a useful "invariance under base change" result. In paragraph 2 we set up a spectral sequence which, in characteristics 0 and p, degenerates, but in entirely different ways. Then in paragraph 3 we use the spectral sequence to compute $\mathrm{Ext}^i_{\mathscr{D}}(\mathcal{O}_{\mathscr{X}}\ \mathcal{O}_{\mathscr{X}})$, in various cases, including the comparison with étale cohomology. The fourth section contains the proof that this Ext group "is" the cohomology of the infinitesmal topos, a result we were surprised not to find in the literature. Finally in paragraph 5 we prove a curious result about the hypercohomology of the de Rham complex of the formal neighborhood of a subvariety of projective space, – a result which is, in a way, a ridiculous analogue of the strong Lefschetz theorem.

I would like to say here that our results can be viewed in the following way : If X is a k-scheme, where char : $k = p > 0$, there is a morphism of ringed topoi,

$$(X \mid W_n(k))_{\mathrm{cris}} \to (X \mid W_n(k))_{\mathrm{inf}},$$

and hence on the limit of cohomology : $H^i_{\mathrm{inf}}(X \mid W, \mathcal{O}_{X|W}) \to H^i_{\mathrm{cris}}(X \mid W, \mathcal{O}_{X|W})$. Then our results can be interpreted as saying that if $X \mid k$ is proper, this map is an isomorphism onto the "unit root part". One is tempted to ask if there are intermediate sites, whose cohomologies correspond to the pieces of other slopes. The feasibility of this approach to understanding the slope filtration remains, however, in doubt. Forthcoming work of S. Bloch using K-theory seems much more promising.

Many thanks go to the referee and to P. Berthelot for numerous useful suggestions.

Differential Operators on Formal Schemes

All "algebras" will be commutative with identity element, and all "algebra homomorphisms" will preserve the identity element, unless otherwise specified. If A is an R-algebra and M and N are A-modules, $\mathrm{Diff}^n_{A/R}(M, N)$ will denote the set of differential operators $M \to N$ of order $\leqq n$ ([3], IV, 16.8). Recall that if P is another A-module, composition of maps induces a composition

$$\mathrm{Diff}^n_{A/R}(M, N) \times \mathrm{Diff}^m_{A/R}(N, P) \to \mathrm{Diff}^{m+n}_{A/R}(M, P).$$

This is better written

$$\mathrm{Diff}^m_{A/R}(N, P) \times \mathrm{Diff}^n_{A/R}(M, N) \to \mathrm{Diff}^{m+n}_{A/R}(M, P)$$

by $(g, f) \to g \circ f$. If

$$\mathrm{Diff}_{A/R}(M, N) = \bigcup_0^\infty \mathrm{Diff}^n_{A/R}(M, N), \qquad \text{etc.,}$$

then $\mathrm{Diff}_{A/R}(M, N)$ is a left $\mathrm{Diff}_{A/R}(N, N)$-module and a right $\mathrm{Diff}_{A/R}(M, M)$-module, where $\mathrm{Diff}_{A/R}(N, N) \subseteq \mathrm{End}_R(N, N)$ is a subring. In particular $\mathrm{Diff}_{A/R}(M, N)$ has two A-module structures, one via $A \to \mathrm{Diff}^0_{A/R}(N, N) \to \mathrm{Diff}_{A/R}(N, N)$ and one via $A \to \mathrm{Diff}^0_{A/R}(M, M) \to \mathrm{Diff}_{A/R}(M, M)$. We shall write the first of these on the left and the second on the right. Thus if $\partial : M \to N$ is an operator, $a\partial : M \to N$ means $M \to N \xrightarrow{a} N$ and ∂a means $M \xrightarrow{a} M \xrightarrow{\partial} N$.

Since we want to work on formal schemes we need to study the behaviour of Diff under completion. It turns out to be quite simple:

(1.1) PROPOSITION. — *Suppose* A *is a noetherian* R-*algebra,* $I \subseteq A$ *is an ideal, and* $\hat{A}$ *is its* I-*adic completion. Then:*

(1.1.1) *For each n, there are natural isomorphisms:*

$$\hat{P}^n_{A/R} \to \hat{P}^n_{\hat{A}/R} \to \varprojlim_k P^n_{A_k/R}$$

where P^n *is the ring of principal parts of order* $\leqq n$ [3] *and* $\hat{P}^n$ *means completion with respect to the ideal* IP^n.

(1.1.2) *If* A/R *is of finite type and if* M *and* N *are finite type* A*-modules, there are natural isomorphisms:*

$$\hat{A} \otimes_A \mathrm{Diff}^n_{A/R}(M, N) \cong \mathrm{Diff}^n_{\hat{A}/R}(\hat{M}, \hat{N})$$

and

$$\hat{A} \otimes_A \mathrm{Diff}_{A/R}(M, N) \cong \mathrm{Diff}_{\hat{A}/R}(\hat{M}, \hat{N}).$$

Proof. — For each k, there is a natural map $P^n_{A/R}/I^{k+1} P^n_{A/R} \to P^n_{A_k/R}$ which I claim induces an isomorphism in the inverse limit. To construct the inverse, observe that a differential operator $M \to N$ of order $\leqq n$ maps $I^{n+k+1} M$ into $I^{k+1} N$, and hence induces a map $M_{n+k} \to N_k$ for all k. Apply this to the universal operator of order $\leqq n$: $A \to P^n_{A/R}$ to get a differential operator $A_{n+k} \to P^n_{A/R}/I^k P^n_{A/R}$ hence $P^n_{A_{n+k}/R} \to P^n_{A/R}/I^k P^n_{A/R}$. It is easily verified that this induces the inverse isomorphism in the limit. This shows that $\hat{P}^n_{A/R} \to \varprojlim P^n_{A_k/R}$ is an isomorphism, and if we apply this result with $\hat{A}$ in place of A, we get (1.1.1).

For the next statement recall that if A/R is of finite type, $P^n_{A/R}$ is a finite type A-module. Then

$$\begin{aligned}
\mathrm{Diff}^n_{\hat{A}/\hat{R}}(\hat{M}, \hat{N}) &\cong \mathrm{Hom}_{\hat{A}}[P^n_{\hat{A}/R} \otimes_{\hat{A}} \hat{M}, \hat{N}] \\
&\cong \mathrm{Hom}_{\hat{A}}[(\hat{A} \otimes_A P^n_{A/R}) \otimes_{\hat{A}} \hat{M}, \hat{N}] \\
&\cong \mathrm{Hom}_{\hat{A}}[\hat{A} \otimes_A P^n_{A/R} \otimes_A M, \hat{N}] \\
&\cong \mathrm{Hom}_A[P^n_{A/R} \otimes_A M, \hat{N}] \cong \hat{A} \otimes \mathrm{Diff}^n_{A/R}(M, N).
\end{aligned}$$

Taking direct limits, we finish the proof. □

Next we have to study the behaviour of Diff with respect to localization and base change. Everything we say is quite well-known except for the presence of the completions, and using (1.1) one can reduce to the familiar case. Therefore we omit most of the proofs.

(1.2) COROLLARY. — *Suppose* B *is an étale* A*-algebra and* $\hat{B}$ *is the* I*-adic completion of* B, *where* A/R *is of finite type and* $I \subseteq A$ *is an ideal. Then there is a natural isomorphism:*

$$\hat{B} \otimes_{\hat{A}} \mathrm{Diff}_{\hat{A}/R}(\hat{M}, \hat{N}) \to \mathrm{Diff}_{\hat{B}/R}(\hat{B} \otimes_{\hat{A}} \hat{M}, \hat{B} \otimes_{\hat{A}} \hat{N})$$

for any two finitely generated A*-modules* M *and* N. □

If A/R is smooth (hence of finite type), it is well known that Diff (A/R) = D (A/R) is projective as a left A-module. It follows that $D(\hat{A}/R) = \hat{A} \otimes_A \mathrm{Diff}(A/R)$ is projective as a left $\hat{A}$-module. We use this fact extensively in the sequel.

(1.3) Proposition. — *Suppose* R' *is an* R*-algebra, with* R *and* R' *noetherian,* B/R *is smooth, and* $I \subseteq B$ *is an ideal. Let* $\hat{B}$ *be the* I*-adic completion of* B *and* $\hat{B}'$ *be the* I*-adic completion of* $B' = R' \otimes_R B$ (*or equivalently of* $R' \otimes_R \hat{B}$).

(1.3.1) *If* $D = \operatorname{Diff}(\hat{B}/R)$ *and* $D' = \operatorname{Diff}(\hat{B}'/R)$, *there is a natural isomorphism:*

$$\hat{B}' \otimes_{\hat{B}} D \cong D'.$$

(1.3.2) *If* M *is a left* D*-module,*

$$M' \underset{\text{def}}{=} B' \otimes_B M \cong D' \otimes_D M,$$

and for any D'*-module* E',

$$\operatorname{Hom}_{D'}(M', E') \cong \operatorname{Hom}_D(M, E').$$

(1.3.3) *If either* B' *or* M *is flat over* B, *we have natural isomorphisms:* $\operatorname{Ext}^i_{D'}(M', E') \cong \operatorname{Ext}^i_D(M, E')$ *for all* i.

Proof. — We omit the proofs of the first two statements. If $F^{\cdot}$ is a resolution of N by free D-modules, $\hat{B}' \otimes_{\hat{B}} F^{\cdot} \cong D' \otimes_D F^{\cdot}$ is a complex of free D'-modules, and either flatness hypothesis implies that it is a resolution of M'. Applying $\operatorname{Hom}_{D'}(\ , E')$ and (1.3.2), the result follows. □

We are now ready to define the functors we will study. Notice that A is, tautologically, a left $D = D(A/R)$-module. The functor $\operatorname{Hom}_D(A, \ )$ can be identified with the functor $E \to \{ x \in E : \partial x = 0 \text{ whenever } \partial 1 = 0 \}$. The derived functors of this functor are of course $\operatorname{Ext}_D(A, \ )$, but they deserve a special name:

(1.4) Definition. — If E is a D-module, $H^i_D(E) = \operatorname{Ext}^i_D(A, E)$.

(1.5) Corollary. — *With the notations of* (1.3) *and* (1.4), *there are natural isomorphisms:*

$$H^i_D(E') \cong H^i_{D'}(E'). \quad \square$$

In order to calculate $H^*_D(E)$ we shall use the following standard construction of injectives in the category of left D-modules. If M is any A-module, let $L(M) = \operatorname{Hom}_A[D, M]$, the set of A-linear homomorphisms $\varphi : D \to M$, where D is regarded as a left A-module. Then $L(M)$ is a D-module by the formula $(\partial\varphi)(\delta) = \varphi(\delta\partial)$ — one must check carefully that $\partial\varphi$ is still A-linear and that $\partial'(\partial\varphi) = (\partial'\partial)(\varphi)$. Beware that if $L(M)$ is regarded as an A-module through its D-module structure $(a\varphi)(\delta) \neq a(\varphi(\delta))$. There is a canonical A-linear map $\pi : L(M) \to M : \varphi \to \varphi(1)$, and the functor L is right adjoint to the forgetful functor which takes D-modules to A-modules: Given an A-linear map $f : E \to M$, the unique D-linear map $L(f) : E \to L^{\cdot}(M)$ such that $\pi \circ L(f) = f$ is given by $L(f)(x)(\partial) = f(\partial x)$.

(1.6) Proposition. — *Suppose* A/R *is smooth,* R *is noetherian, and* $\hat{A}$ *is the* I*-adic completion of* A. *Then the functor* L *is exact and takes injectives to injectives. If* E *is a* *left* $D(\hat{A}/R)$*-module and* M *is an* $\hat{A}$*-module, there are natural isomorphisms:*

$$\mathrm{Ext}^i_{\hat{A}}(E, M) \cong \mathrm{Ext}^i_{D}(E, L(M)).$$

In particular $H^i_D(L(M)) = 0$ *if* $i > 0$.

Proof. — Since the forgetful functor is exact and L is its right adjoint, L takes injectives to injectives. The additional hypotheses imply that D is projective as a left A-module, i. e., that L is exact. The rest follows. □

We now want to sheafify the above constructions. We shall find it convenient to work in the étale topology as well as the Zariski topology. If $f : \mathscr{X} \to S$ is any morphism of ringed spaces, let $\mathscr{D}_0 = \mathscr{D}_0(\mathscr{X}/S)$ be the presheaf which to any open U assigns the ring $\mathrm{Diff}_R(A, A)$, where $R = \Gamma(U, f^{-1}(\mathscr{O}_S))$ and $A = \Gamma(U, \mathscr{O}_{\mathscr{X}})$. Let $\mathscr{D} = \mathscr{D}(\mathscr{X}/S)$ be the associated sheaf.

Before we describe the sheaves $\mathscr{D}$, a word about the étale topology on a formal scheme, as defined in [4]. If A is I-adically complete, and if B_0 is étale over $A_0 = A/I$, then for each n there is a unique étale A_n-algebra B_n lifting A_0, and $B = \varprojlim B_n$ is said to be étale over A, although it is not of finite type (or the I-adic completion of a finite type A-algebra, in general). We shall call such an étale map "special" iff there exists a finite type and étale A-algebra C with $\hat{C} = B$. It is easy to see that any étale B/A is, locally in the Zariski topology on B, special.

(1.8) PROPOSITION. — *Suppose* A *is a finite type noetherian* R*-algebra, and* $\hat{A}$ *is its* I*-adic completion.* $\mathscr{X} = \mathrm{Spf}\,\hat{A}$. *Then:*

(1.8.1) *If* $V \to \mathscr{X}$ *is étale and affine, with* $V = \mathrm{Spf}\,B$, *there are natural isomorphisms:*

$$B \otimes_{\hat{A}} D(\hat{A}/R) \to D(B/R) \to \Gamma(V, \mathscr{D}).$$

(1.8.2) *If* $V \to \mathscr{X}$ *is a quasi-compact affine étale covering, the Čech complex* $\mathscr{C}(V, \mathscr{D})$ *is a resolution of* $\Gamma(\mathscr{X}, \mathscr{D}) \cong D(\hat{A}/R)$.

(1.8.3) $H^i(\mathscr{X}, \mathscr{D}) = H^i(\mathscr{X}_{\acute{e}t}, \mathscr{D}) = 0$ *for* $i > 0$.

Proof. — If V is special, say $V = \mathrm{Spf}(\hat{C})$ with $C/\hat{A}$ finite type and étale, by (1.2) $D(\hat{C}/R) = \hat{C} \otimes_{\hat{A}} D(\hat{A}/R)$. If $V_.$ is a covering by special étales, the sequence $\hat{A} \to \mathscr{C}^{\cdot}(V, \mathscr{O}_V)$ is exact and consists of flat $\hat{A}$-modules, so that tensoring with $D(\hat{A}/R)$ we obtain an exact sequence $D(\hat{A}/R) \to \mathscr{C}^{\cdot}(V, D)$. Since this is true with $\hat{A}$ replaced by a basis of open sets, it follows that $\Gamma(\mathscr{X}, \mathscr{D}) \cong D(\hat{A}/R)$, and that the analogue is true for a special étale V in place of $\mathscr{X}$. This shows that the natural map $B \otimes_{\hat{A}} D(\hat{A}/R) \to \Gamma(V, \mathscr{D})$ is an isomorphism if V is special, and the general case follows easily. One quickly deduces (1.8.2) and (1.8.3) by standard arguments. It remains only to prove that $D(B/R) \to \Gamma(V, \mathscr{D})$ is an isomorphism for any étale V, and injectivity will suffice. If $\partial \in D(B/R)$ goes to zero, then there is a special étale cover V_i such that $\partial|_{V_i} = 0$ for all i, hence in particular if $\beta \in B$, $\partial(\beta)|_{V_i} = 0$ for all i, so $\partial(\beta) = 0$. □

Recall that a sheaf $\mathscr{F}$ of $\mathscr{O}_{\mathscr{X}}$-modules on $\mathscr{X} = \operatorname{Spf} A$ is "quasi-coherent" if for each affine open $V \to \mathscr{X}$ with $V = \operatorname{Spf} B$, $\mathscr{F}(V) = B \otimes_A \mathscr{F}(\mathscr{X})$. In particular, $\mathscr{D}$ is quasi-coherent. If the $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{F}$ comes from a sheaf $\mathscr{F}$ of $\mathscr{D}$-modules, then quasi-coherence as an $\mathscr{O}_{\mathscr{X}}$-module implies, and hence is equivalent to, quasi-coherence as a $\mathscr{D}$-module, because

$$\mathscr{F}(V) = B \otimes_A \mathscr{F}(\mathscr{X}) \cong B \otimes_A \mathscr{D}(\mathscr{X}) \otimes_{\mathscr{D}(\mathscr{X})} \mathscr{F}(\mathscr{X}) \cong \mathscr{D}(V) \otimes_{\mathscr{D}(\mathscr{X})} \mathscr{F}(\mathscr{X}).$$

Note that this is so both in the étale and Zariski topologies.

(1.9) Proposition. — *Suppose $\mathscr{X} = \operatorname{Spf} A$ and $\mathscr{E}$ and $\mathscr{F}$ are $\mathscr{O}_{\mathscr{X}}$-modules, with $\mathscr{F}$ quasi-coherent. Then if* A *is the completion of a finite type* R*-algebra, and if we work in either the étale or the Zariski topology:*

(1.9.1) *There is a spectral sequence:*

$$E_2^{pq} = \operatorname{Ext}_A^p(\mathscr{F}(\mathscr{X}), H^q(\mathscr{X}, \mathscr{E})) \Rightarrow \operatorname{Ext}^i_{\mathscr{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{E}).$$

(1.9.2) *If $\mathscr{E}$ and $\mathscr{F}$ are also sheaves of $\mathscr{D}$-modules, there is a spectral sequence:*

$$E_2^{pq} = \operatorname{Ext}^p_{\mathscr{D}(\mathscr{X})}(\mathscr{F}(\mathscr{X}), H^q(\mathscr{X}, \mathscr{E})) \Rightarrow \operatorname{Ext}^i_{\mathscr{D}}(\mathscr{F}, \mathscr{E}).$$

Proof. — There are natural maps

$$\operatorname{Hom}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{E}) \to \operatorname{Hom}_A(\mathscr{F}(\mathscr{X}), \mathscr{E}(\mathscr{X}))$$

and

$$\operatorname{Hom}_{\mathscr{D}}(\mathscr{F}, \mathscr{E}) \to \operatorname{Hom}_{\mathscr{D}(\mathscr{X})}(\mathscr{F}(\mathscr{X}), \mathscr{E}(\mathscr{X})),$$

which are easily seen to be isomorphisms, from the quasi-coherence of $\mathscr{F}$. Since the functor $H^0(\mathscr{X}, \ )$ takes injectives to injectives, we can apply the spectral sequence of a composite functor to get the spectral sequences. □

(1.10) Corollary. — *With the hypotheses above, assume that* $H^q(\mathscr{X}, \mathscr{E}) = 0$ *if* $q > 0$. *Then there are nataral isomorphisms:*

$$\operatorname{Ext}^i_A(\mathscr{F}(\mathscr{X}), \mathscr{E}(\mathscr{X})) \cong \operatorname{Ext}^i_{\mathscr{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{E})$$

and

$$\operatorname{Ext}^i_{\mathscr{D}(\mathscr{X})}(\mathscr{F}(\mathscr{X}), \mathscr{E}(\mathscr{X})) \cong \operatorname{Ext}^i_{\mathscr{D}}(\mathscr{F}, \mathscr{E}). \quad \square$$

We shall need to construct explicit injective resolutions using the sheaf $\underline{\operatorname{Hom}}_{\mathscr{O}_{\mathscr{X}}}[\mathscr{D}, \mathscr{E}] = \mathscr{L}(\mathscr{E})$; note that this sheaf is not quasi-coherent because $\mathscr{D}$ is not coherent.

(1.11) Proposition. — *Suppose $\mathscr{X} = \operatorname{Spf} \hat{A}$, where* A/R *is of finite type.*

(1.11.1) *The functor $\mathscr{L}$ takes injectives to injectives and is right adjoint to the forgetful functor from $\mathscr{D}$-modules to $\mathscr{O}_{\mathscr{X}}$-modules.* $\Gamma(\mathscr{X}, \mathscr{L}(\mathscr{E})) \cong L(\Gamma(\mathscr{X}, \mathscr{E}))$ for any *$\mathscr{O}_{\mathscr{X}}$-module $\mathscr{E}$. If* A/R *is smooth then:*

(1.11.2) *If* $\mathscr{E}$ *is a sheaf of* $\mathcal{O}_{\mathscr{X}}$*-modules such that* $\mathrm{H}^q(\mathscr{X}, \mathscr{E}) = 0$ *for* $q > 0$, *the same is true of* $\mathscr{L}(\mathscr{E})$.

(1.11.3) *If* $\mathscr{E}$ *is a sheaf of* $\mathcal{O}_{\mathscr{X}}$*-modules such that* $\mathrm{H}^q(\mathscr{X}, \mathscr{E}) = 0$ *for* $q > 0$, *there are natural isomorphisms:* $\mathrm{Ext}^i_{\mathscr{D}}(\mathscr{F}, \mathscr{L}(\mathscr{E})) \to \mathrm{Ext}^i_{\mathcal{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{E})$ *for any quasi-coherent* $\mathscr{D}$*-module* $\mathscr{F}$.

Proof. — The first statement is standard except for the last part, which follows from (1.9) and the quasi-coherence of $\mathscr{D}$. For the next statement, use the previous corollary and the fact that $\mathscr{D}(\mathscr{X})$ is a projective $\hat{\mathrm{A}}$-module to see that $\mathrm{Ext}^i_{\mathcal{O}_{\mathscr{X}}}(\mathscr{D}, \mathscr{E}) = 0$ if $i > 0$, and hence that $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{\mathscr{X}}}(\mathscr{D}, \mathscr{E}) = 0$. Then the spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(\mathscr{X}, \underline{\mathrm{Ext}}^q_{\mathcal{O}_{\mathscr{X}}}(\mathscr{D}, \mathscr{E})) \Rightarrow \mathrm{Ext}^i_{\mathcal{O}_{\mathscr{X}}}(\mathscr{D}, \mathscr{E})$$

degenerates and converges to zero, so that $\mathrm{H}^p(\mathscr{X}, \mathscr{L}(\mathscr{E})) = 0$ if $p > 0$. The last statement now follows from the corollary and (1.6). □

Now we can deduce some results about nonaffine formal schemes. For simplicity we consider only formal schemes arising globally from smooth quasi-compact, quasi-separated ordinary schemes, although we could easily consider formal schemes obtained by gluing these.

Thus we consider a smooth morphism $\mathrm{X} \to \mathrm{S}$, $\mathrm{S} = \mathrm{Spec}\,\mathrm{R}$, with R noetherian, and let $\mathscr{X}$ be the formal completion along a closed subset $\mathrm{Y} \subsetneq \mathrm{X}$. If $\mathscr{F}$ and $\mathscr{E}$ are sheaves of $\mathscr{D}$-modules we shall want to consider $\mathrm{Ext}^i_{\mathscr{D}}(\mathscr{F}, \mathscr{E})$ and in particular $\mathrm{H}^i_{\mathscr{D}}(\mathscr{X}, \mathscr{E}) = \mathrm{Ext}^i_{\mathscr{D}}(\mathcal{O}_{\mathscr{X}}, \mathscr{E})$.

(1.13) Proposition. — *Suppose* $\mathscr{E}$ *and* $\mathscr{F}$ *are sheaves of* $\mathscr{D}$*-modules on* $\mathscr{X}$, *such that* $\mathscr{F}$ *is quasi-coherent and* $\mathrm{H}^q(\mathrm{U}, \mathscr{E}) = \mathrm{H}^q(\mathrm{U}_{\text{ét}}, \mathscr{E}) = 0$ *for* $q > 0$ *and every affine open* $\mathrm{U} \subseteq \mathscr{X}$. *Then the natural maps:*

$$\mathrm{Ext}^i_{\mathscr{D}}(\mathscr{X}_{\mathrm{zar}}, \mathscr{F}, \mathscr{E}) \to \mathrm{Ext}^i_{\mathscr{D}}(\mathscr{X}_{\mathrm{et}}, \mathscr{F}, \mathscr{E})$$

are isomorphisms for all i.

Proof. — We have already seen in (1.10) that this is true if $\mathscr{X}$ is affine; the general case follows by considering the Čech spectral sequence of an affine open hypercovering. □

(1.14) Proposition. — *Suppose* R′ *is a noetherian* R*-algebra and* $f: \mathscr{X}' \to \mathscr{X}$, *with* $\mathscr{X}'$ *the formal completion of* $\mathrm{X}' = \mathrm{X} \times_{\mathrm{R}} \mathrm{R}'$ *along* $\mathrm{Y}' = \mathrm{Y} \times_{\mathrm{R}} \mathrm{R}'$. *If* $\mathscr{E}'$ *is a* $\mathscr{D}' = \mathscr{D}(\mathscr{X}'/\mathrm{R}')$*-module such that* $\mathrm{H}^q(\mathrm{U}', \mathscr{E}') = 0$ *for open affines* U′, *and if* $\mathscr{F}$ *is a quasi-coherent* $\mathscr{D}$*-module and is flat as an* $\mathcal{O}_{\mathscr{X}}$*-module, the natural maps:*

$$\mathrm{Ext}^i_{\mathscr{D}'}(f^*\mathscr{F}, \mathscr{E}') \to \mathrm{Ext}^i_{\mathscr{D}}(\mathscr{F}, f_*\mathscr{E}')$$

are isomorphisms for all i.

Proof. — Note that $f^*\mathscr{F}$ is quasi-coherent on $\mathscr{X}$ and that $\mathrm{H}^q(\mathrm{V}, f_*\mathscr{E}') = 0$ for $q > 0$ for $\mathrm{V} \subsetneq \mathscr{X}$ affine, because f is affine. Since the above result follows from (1.3) and (1.10) if $\mathscr{X}$ is affine, we deduce the general case from the Čech spectral sequence as above. □

(1.15) Corollary. — *If* $\mathscr{E}'$ *is a* $\mathscr{D}'$*-module with* $\mathrm{H}^q(\mathrm{V}', \mathscr{E}') = 0$ *for any affine* V′, *there are natural isomorphisms:*

$$\mathrm{H}^i_{\mathscr{D}'}(\mathscr{X}', \mathscr{E}') \to \mathrm{H}^i_{\mathscr{D}}(\mathscr{X}, f_*\mathscr{E}') \quad \textit{for all } i. \quad \square$$

2. A Spectral Sequence

For the time being let us work in the affine situation with $D^n = \mathrm{Diff}^n(A/R)$, $D = \mathrm{Diff}(A/R)$. If N and M are two left D-modules, let

$$H_n(N, M) = \mathrm{Hom}_{D^n}(N, M)0 = \{f : N \to M : f(\partial x) = \partial f(x) \text{ for all } x \in N \text{ and } \partial \in D^n\}.$$

Then

$$\mathrm{Hom}_A(N, M) = H_0(N, M) \supseteq H_1(N, M) \supseteq \dots$$

and

$$\bigcap_{n=0}^{\infty} H_n(N, M) = \mathrm{Hom}_D(N, M).$$

Thus if $H_{.}(N)$ is the functor which takes D-modules to the inverse system of R-modules $\{H_n(N, M)\}$, we have expressed $\mathrm{Hom}_{.}(N, \ )$ as the composite $\varprojlim \circ H(N, \ )$. Note that we could also have used $H_k(N, \ ) = \mathrm{Hom}_{D^{n_k}}(N, \ )$, if (n_k) is any increasing sequence of integers.

We shall show that there is a spectral sequence for the composite $\varprojlim \circ H_{.}(N, \ )$, and this spectral sequence will be our main tool in the calculation of infinitesmal cohomology. We must show that any M has a resolution by injective objects I such that $H_{.}(N, I)$ is acyclic for $\varprojlim$. Note that we will *never* have a Mittag-Leffler condition, so a more delicate argument is needed.

(2.1) LEMMA. – *Suppose* Q *is an* A-*module and* N *is a* D-*module. Then* $R^i \varinjlim H_{.}(N, L(Q)) = 0$ *for* $i > 0$.

Proof. – Since the index set is the set N of positive integers, we need only show that $R^1 \varprojlim H_{.}(N, L(Q)) = 0$, i. e. that each H_i is complete in the topology defined by $H_i \supseteq H_{i+1} \supseteq \dots$ [8]. If $x \in N$, and $f \in H_0$, let $f_x \in L(Q) = \mathrm{Hom}_A[D, Q]$ be the corresponding element. Then $f \in H^k$ iff $\partial f_x = f_{\partial x}$ whenever $\partial \in D^k$, i. e. iff

$$f_x(\delta\, \partial) = f_{\partial x}(\delta) \text{ for all } \delta \in D, \text{ all } \partial \in D^k, \text{ and all } x \in N.$$

Suppose (f^j) is a Cauchy sequence in H_i; by passing to a sub-sequence, if necessary, we may assume that $f^j - f^k \in H_k$ for $j \geqq k$. This condition says that

$$f_x^j(\delta\partial) - f_{\partial x}^j(\delta) = f_x^k(\delta\partial) - f_{\partial x}^k(\delta) \qquad \text{for} \quad j \geqq k \geqq \mathrm{ord}(\partial),$$

and in particular we see that $f_x^j(\partial) - f_{\partial x}^j(1)$ is independent of j for $j \geqq \mathrm{ord}(\partial)$. Denote this value by $f_x(\partial)$; I claim that f is a limit of (f^j) in H_i.

First we must verify that $f_x \in L(Q)$, i. e. that it is A-linear. If $a \in A$ and $\partial \in D$, then for large j,

$$f_x(a\,\partial) = f_x^j(a\,\partial) - f_{a\partial x}^j(1) = f_x^j(a\,\partial) - f_{\partial x}^j(a).$$

Now since f_x^j and $f_{\partial x}^j$ are A-linear, this is

$$= a\,(f_x^j(\partial)) - a\,(f_{\partial x}^j(1)) = a\,f_x(\partial).$$

Next we check that $f \in \mathrm{H}_i$. If $\partial \in \mathrm{D}^i$ and $\delta \in \mathrm{D}$, and if $j \geqq \max(i, \operatorname{ord}(\delta\partial))$, we have:

$$f_x(\delta\,\partial) = f_x^j(\delta\,\partial) - f_{\delta\partial x}^j(1) = f_{\partial x}^j(\delta) - f_{\delta\partial x}^j(1) = f_{\partial x}(\delta).$$

Finally we must check that the sequence f^k converges to f, i. e. that $f - f^k \in \mathrm{H}_k$. If $\partial \in \mathrm{D}^k$ and if $\delta \in \mathrm{D}$, then for large j:

$$\begin{aligned} f_x(\delta\,\partial) - f_x^k(\delta\,\partial) &= f_x^j(\delta\,\partial) - f_{\delta\partial x}^j(1) - f_x^k(\delta\,\partial), \qquad \text{(and since } f^j - f^k \in \mathrm{H}^k\text{)} \\ &= f_{\partial x}^j(\delta) - f_{\delta\partial x}^j(1) - f_{\partial x}^k(\delta) \\ &= f_{\partial x}(\delta) - f_{\partial x}^k(\delta). \quad \square \end{aligned}$$

(2.2) PROPOSITION. — *Suppose* A/R *is of finite type and* $\hat{\mathrm{A}}$ *is the* I-*adic completion of* A. *Then if* N *and* M *are* D-*modules, there are exact sequences:*

$$0 \to \mathrm{R}^1 \varprojlim \operatorname{Ext}_{\mathrm{D}}^{i-1,\cdot}(\mathrm{N}, \mathrm{M}) \to \operatorname{Ext}_{\mathrm{D}}^i(\mathrm{N}, \mathrm{M}) \to \varprojlim \operatorname{Ext}_{\mathrm{D}}^{i,\cdot}(\mathrm{N}, \mathrm{M}) \to 0,$$

where $\operatorname{Ext}^{i,\cdot}(\mathrm{N},\ )$ *means the inverse system of right derived functors of* $\operatorname{Hom}^{\cdot}(\mathrm{N},\ ) = \mathrm{H}.(\mathrm{N},\ )$.

Proof. — This will just be the spectral sequence of a composite functor, once we know that any D-module M is contained in an injective which is acyclic for $\varprojlim$. This is easy: Find an injective A-module Q containing M; then there is a unique D-linear map $i : \mathrm{M} \to \mathrm{L}(\mathrm{Q})$ such that $\pi \circ i = \text{incl}$, so i is injective. $\square$

Next we shall sheafify the above construction. For technical reasons, we must consider only the functors $\mathrm{H}_{\mathscr{D}}^i(\mathscr{X}, \mathscr{E})$. Define a functor $\Phi_{\bullet}$ from the category of $\mathscr{D}$-modules on the formal scheme $\mathscr{X}/\mathrm{S}$ to the category of inverse systems of sheaves of abelian groups on $\mathscr{X}$ by $\Phi_n(\mathscr{E}) = \underline{\operatorname{Hom}}_{\mathscr{D}^n}(\mathscr{O}_{\mathscr{X}}, \mathscr{E})$. Then if Γ is the functor which takes an inverse system of sheaves $\mathrm{S}_{\bullet}$ to $\varprojlim \Gamma(\mathscr{X}, \mathrm{S}_{\bullet}) \cong \Gamma(\mathscr{X}, \varprojlim \mathrm{S}_{\bullet})$, we see that

$$\Gamma \circ \Phi_{\bullet} = \Gamma(\mathscr{X}, \varprojlim \Phi_{\bullet}(\mathscr{E})) = \Gamma(\mathscr{X}, \underline{\operatorname{Hom}}_{\mathscr{D}}(\mathscr{O}_{\mathscr{X}}, \mathscr{E})) = \operatorname{Hom}_{\mathscr{D}}(\mathscr{O}_{\mathscr{X}}, \mathscr{E}) = \Gamma_{\mathscr{D}}(\mathscr{X}, \mathscr{E}).$$

(2.3) PROPOSITION. — *There is a spectral sequence*

$$\mathrm{E}_2^{pq} = \mathrm{R}^p\Gamma^{\cdot}\mathrm{R}^q\Phi_{\bullet}(\mathscr{E}) \quad \Rightarrow \quad \mathrm{H}_{\mathscr{D}}^i(\mathscr{X}, \mathscr{E}),$$

for any sheaf of $\mathscr{D}$-*modules* $\mathscr{E}$ *on* $\mathscr{X}$.

Proof. — If we can show that any $\mathscr{E}$ can be embedded in an injective $\mathscr{D}$-module $\mathscr{F}$ such that $\Phi_{\bullet}(\mathscr{F})$ is acyclic for Γ, this will just be the spectral sequence of a composite functor. Thus it will suffice to prove that if Q is an injective $\mathscr{O}_{\mathscr{X}}$-module, $\Phi_{\bullet}\mathscr{L}(\mathrm{Q})$ is Γ-acyclic.

First let us reinterpret the functor Φ. We have a natural map: $\mathscr{D} \to \mathscr{O}_{\mathscr{X}} \to 0$ given by $\alpha(\partial) = \partial(1)$, which is a homorphism of left $\mathscr{D}$-modules, and hence we have an exact sequence of left $\mathscr{D}$-modules:

$$0 \to \mathscr{M} \to \mathscr{D} \to \mathscr{O}_{\mathscr{X}} \to 0.$$

Note that $\mathcal{M}$ is quasi-coherent, and if $\mathcal{E}$ is any $\mathcal{D}$-module, we get an exact sequence:

$$0 \to \underline{\mathrm{Hom}}_{\mathcal{D}}(\mathcal{O}_{\mathcal{X}}, \mathcal{E}) \to \mathcal{E} \to \underline{\mathrm{Hom}}_{\mathcal{D}}(\mathcal{O}_{\mathcal{M}}, \mathcal{E}).$$

Thus, we can think of $\underline{\mathrm{Hom}}_{\mathcal{D}}(\mathcal{O}_{\mathcal{X}}, \mathcal{E})$ as the sheaf of all $\mathcal{X} \in \mathcal{E}$ such that $\partial x = 0$ for all sections ∂ of $\mathcal{M}$. Similarly, $\overline{\mathrm{Hom}}_{\mathcal{D}^n}(\mathcal{O}_{\mathcal{X}}, \mathcal{E})$ is the sheaf of all x such that $\partial x = 0$ for all sections ∂ of $\mathcal{M} \cap \mathcal{D}^n$, or equivalently, the sheaf of all x such that $\partial x = 0$ for all ∂ in the left ideal $\mathcal{M}_n$ generated by $\mathcal{M} \cap \mathcal{D}^n$. Thus, $\Phi_n(\mathcal{E}) \cong \underline{\mathrm{Hom}}_{\mathcal{D}}(\mathcal{D}/\mathcal{M}_n, \mathcal{E})$. Since $\mathcal{M}$ and $\mathcal{D}^n$ are quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules, so are $\mathcal{M} \cap \mathcal{D}^n$, and $\mathcal{M}_n$, and $\mathcal{D}/\mathcal{M}_n$.

We are going to use the spectral sequence for the composite $\mathbf{\Gamma} = \Gamma \circ \varinjlim$, as explained in [8]. Note that $\mathrm{R}^i \varprojlim$ is *not*, in general, the sheaf associated to the presheaf $\mathrm{U} \to \mathrm{R}^i \varprojlim(\mathrm{U}, \ )$. Suppose Q is an injective $\mathcal{O}_{\mathcal{X}}$-module and $\mathrm{Q}_{.} = \Phi_{.}(\mathcal{L}(\mathrm{Q}))$. I claim that $\mathrm{R}^i \varprojlim \mathrm{Q}_{.} = 0$ for $i > 0$. We must check two conditions.

(*a*) For a basis of opens U, $\mathrm{H}^i(\mathrm{U}, \mathrm{Q}_n) = 0$ if $i > 0$. Indeed,

$$\mathrm{Q}_n \cong \underline{\mathrm{Hom}}_{\mathcal{D}}[\mathcal{D}/\mathcal{M}_n, \mathcal{L}(\mathrm{Q})] \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}}}[\mathcal{D}/\mathcal{M}_n, \mathrm{Q}]$$

which is flasque since Q is injective, so this is true for any open set U.

(*b*) For a basis of opens U, $\mathrm{R}^1 \varprojlim \mathrm{Q}_{.}(\mathrm{U}) = 0$. If $\mathrm{U} = \mathrm{Spf}\,\mathrm{A}$ is affine and $\mathrm{D} = \mathcal{D}(\mathrm{U})$, etc., we have from the quasi-coherence of $\mathcal{D}/\mathcal{M}_n$ that:

$$\mathrm{Q}_n(\mathrm{U}) \cong \mathrm{Hom}_{\mathrm{D}}(\mathrm{D}/\mathrm{M}_n, \mathrm{L}(\mathrm{Q}(\mathrm{U}))).$$

Thus $\mathrm{R}^1 \varprojlim \mathrm{Q}_{.}(\mathrm{U}) = 0$ by (2.1).

Thus the spectral sequence for the composite $\mathbf{\Gamma} = \Gamma \circ \varprojlim$ degenerates, and we have $\mathrm{R}^i \mathbf{\Gamma} = \mathrm{H}^i(\mathcal{X}, \varprojlim \mathrm{Q}_{.})$. But $\varprojlim \mathrm{Q}_{.} \cong \underline{\mathrm{Hom}}_{\mathcal{D}}(\mathcal{O}_{\mathcal{X}}, \mathcal{L}(\mathrm{Q})) \cong \mathrm{Q}$ and since Q is injective, these vanish. □

In characteristic 0, $\mathcal{D}^1$ generates $\mathcal{D}$, so that $\Phi_1 \cong \varprojlim \Phi_{.}$ and the spectral sequence degenerates stupidly. In characteristic $p > 0$ however, we shall see that if $\nu_n = p^n - 1$ for $n \geqq 0$ and $\Phi_n \ \mathrm{Hom}_{\mathcal{D}^{\nu_n}}(\mathcal{O}_{\mathcal{X}}, \ )$, the functor $\Phi_{.}$ is exact. Thus the spectral sequence will again degenerate, this time in a more interesting way. We obtain:

(2.4) THEOREM. — *Suppose* $p\,\mathcal{O}_{\mathrm{S}} = 0$, X/S *is smooth, and* $\mathcal{X}$ *is the formal completion of* X *along* $\mathrm{Y} \subset \mathrm{X}$. *Then the functor* $\Phi_{.}$ (*indexed as above*) *is exact, and for any sheaf* $\mathcal{E}$ *of* $\mathcal{D}$ *modules there are exact sequences:*

$$0 \to \mathrm{R}^1 \varprojlim \mathrm{H}^{i-1}(\mathcal{X}, \Phi_{.}(\mathcal{E})) \to \mathrm{H}^i_{\mathcal{D}}(\mathcal{X}, \mathcal{E}) \to \varprojlim \mathrm{H}^i(\mathcal{X}, \Phi_{.}\mathcal{E}) \to 0.$$

Proof. — We must show that each Φ_n is exact. This is a local question, so we can introduce coordinates on X, $t_1 \ldots t_d$. Then there exist well-known projection operators

$\pi^n \in \mathscr{D}$ which universally project $\mathscr{E}$ onto $\Phi_{v_n}(\mathscr{E})$. If we work with one coordinate $t = t_k$ at a time, we take

$$\pi_k^n = \sum_{i=0}^{p^n-1} (-1)^i t^i \nabla_i, \qquad \text{where} \quad \nabla_i = \frac{1}{i!}(\partial/\partial t)^i.$$

It is well known that $\nabla_1 \pi_k^n = 0$; we leave it as an exercise in the binomial theorem to verify that $\nabla_i \pi_k^n = 0$ if $i \leqq p^n - 1$. Then $\pi^n = \prod_{k=1}^{d} \pi_k^n$ is the desired projection. The exactness of Φ_n follows immediately. To get the exact sequence above, use the spectral sequence of the composite $\mathbf{\Gamma} = \varprojlim \circ \Gamma$ and the previous proposition. $\square$

We shall call the first term in the sequence above the "unstable submodule" of $H^i_{\mathscr{D}}(\mathscr{X}, \mathscr{E})$, and shall call the last term the "stable quotient" of $H^i_{\mathscr{D}}(\mathscr{X}, \mathscr{E})$. Needless to say, the unstable submodule is unpleasant; if it does not vanish one should, no doubt, work with a pro-object instead of a module. In the next section we shall investigate the exact sequence with some care.

3. The Comparison Theorem

We now turn our attention to the case $\mathscr{E} = \mathscr{O}_{\mathscr{X}}$. Then one expects that $H^i_{\mathscr{D}}(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is determined by the action of Frobenius on $H^i(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$. This turns out to be the case, at least with certain hypotheses on $\mathscr{X}$. We shall essentially consider two cases; first $\mathscr{X} = X$, a smooth proper scheme over R, and next any proper formal scheme $\mathscr{X}$ over a field k. The first case corresponds to a family of smooth varieties; we shall see that the monodrony around supersingular points creates unpleasant behavior in $H^i_{\mathscr{D}}(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$.

We begin with some general nonsense about p-linear endomorphisms. Let R be a regular local ring of characteristic $p > 0$, let $\sigma : R \to R$ be the p-th power map, which we assume to be finite (for instance, if R is "geometric"). A σ-linear endomorphism of an R-module M is an additive map $\psi : M \to M$ such that $\psi(ax) = a^\sigma \psi(x)$ if $a \in R, x \in M$. We deduce an R-linear map $\varphi_1 : M_1 = R \otimes_\sigma M \to M : a \otimes x \to a\psi(x)$, and in fact an inverse system $(M_., \varphi_.)$ of R-linear maps, where $M_i = R \otimes_{\sigma^i} M$, and $M_{i+k} \to M_i$ is induced by ψ^k.

Let $N_i = \operatorname{Im} \varphi_i : M_i \to M$, the R-module spanned by $\operatorname{Im} \psi^i$, and let $M^s = \bigcap_{i=1}^{\infty} N^i$. Obviously φ maps M^s into M^s; we shall call (M^s, ψ^s) the "stable submodule" of (M, ψ), and the quotient of M by M^s "(M^n, ψ^n)", the "unstable quotient" of (M, ψ).

(3.1) THEOREM. — *Suppose* M *is a finite type and torsion free* R-*module. Then:*

(3.1.1) *The map* $\dot{\pi}_0 : \varprojlim M_. \to M$ *is injective, with image* M^s.

(3.1.2) *There are natural isomorphisms:*

$$M^s = \varprojlim N_. \xrightarrow{\cong} \varprojlim M_.$$

and

$$R^1 \varprojlim N_. \xrightarrow{\cong} R^1 \varprojlim M.$$

(3.1.3) *The submodule* (M^s, ψ^s) *of* (M, ψ) *is stable, i. e. the map* $R \otimes_\sigma M^s \to M^s$ *is an isomorphism.*

(3.1.4) *The quotient* (M^n, ψ^n) *of* (M, ψ) *is unstable, i. e.* $\varprojlim M^n_. = 0$.

Moreover there is a natural isomorphism:

$$R^1 \varprojlim M_. \xrightarrow{\cong} R^1 \varprojlim M^n.$$

Proof. — If R is a field, this is all well-known. In fact we have:

(3.2) LEMMA. — *If* R *is a field, then:*

(1) *There exists an integer* r *such that* $N_{i+r} = N_r$ *for all* $i \geqq 0$.

(2) *The maps* $\varphi^n_{i+k,i} : M^n_{i+k} \to M^n_i$ *are zero for* $k > r$.

(3) *The maps* $\varphi^s_{i+k,i} : M^s_{i+k} \to M^s_i$ *are isomorphisms for all* i, k.

Sketch. — The first statement is clear, because M is finite dimensional, and immediately implies (2). Since the modules M^s_i are all of the same dimension, surjectivity suffices to prove (3), and this is easy.

(3.3) LEMMA. — *Let* $K_i = \text{Ker}(\varphi_i : M_i \to M)$. *Then there is an* r *such that the map* $K_{i+r} \to K_i$ *is zero for all* i.

Proof. — First assume R is a field, and let r be as in the first lemma. Then if $x \in M_{i+r}$, $\varphi_{i+r,i}(x)$ lies in M^s_i, and since the map $M^s_i \to M$ is injective, $\varphi_{i+r,i}(x) = 0$ if $x \in K_{i+r}$, as claimed. Now in general, since M is torsion free, so are each M_i and each K_i, so the result follows by tensoring with the quotient field of R.

Now we can prove the theorem. We have an exact sequence of inverse systems:

$$0 \to K_. \to M_. \to N_. \to 0.$$

By Lemma (3.3), the inverse system is essentially zero, so that

$$\varprojlim K_. = R^1 \varprojlim K_. = 0.$$

This proves (3.1.2) and (3.1.1), because $\varprojlim N_. = \bigcap N_. = M^s$.

Next observe that R is a flat, finite type R-module via σ, and hence is free and of finite type, so that the natural map $t : R \otimes_\sigma \varprojlim M_i \to \varprojlim R \otimes_\sigma M_i$ is an isomorphism. There is an isomorphism (equality) $R \otimes_\sigma M_i \to M_{i+1}$ for each i, hence an isomorphism

$\delta: \varprojlim R \otimes_\sigma M_i \to \varprojlim M_{i+1}$, and finally there is a shifting isomorphism $T: \varprojlim M_{i+1} \to \varprojlim M_i$ sending $(x_1, x_2, \ldots)$ to $(\varphi(x_1), x_1, x_2, \ldots)$. Composing these, we get an isomorphism $\varphi: R \otimes \varprojlim M_{\bullet} \to \varprojlim M_{\bullet}$. Since $\pi_0 \circ \varphi(a \otimes x_{\bullet}) = a\,\psi\,\pi_0(x_{\bullet})$, we get a diagram:

$$\begin{tikzcd}
\varprojlim R \otimes_\sigma M_i \arrow[d, "\delta"', "\cong"] & R \otimes_\sigma \varprojlim M_{\bullet} \arrow[l, "T\cong"'] \arrow[r, "\mathrm{id}\otimes\pi_0"] \arrow[d, "\varphi"] & R \otimes M^s \arrow[d, "\varphi_1^s"] \\
\varprojlim M_{i+1} \arrow[r, "T\cong"] & \varprojlim M_{\bullet} \arrow[r, "\pi_0", "\cong"'] & M^s
\end{tikzcd}$$

It follows that φ_1^s is an isomorphism, proving (3.1.3). Finally, looking at the exact sequence of inverse systems:

$$0 \to M_{\bullet}^s \to M_{\bullet} \to M_{\bullet}^n \to 0$$

and observing that the maps in $M_{\bullet}^s$ are isomorphisms and that $\varprojlim M_{\bullet}^s \to \varprojlim M_{\bullet}$ is an isomorphism, we see that $\varprojlim M_{\bullet}^n = 0$ and that $R^1 \varprojlim M_{\bullet} \xrightarrow{\cong} R^1 \varprojlim M^n$. $\square$

(3.4) *Remark.* — $R^1 \varprojlim N_i \cong \prod_i \hat{N}_i/N_i$, where $\hat{N}_i$ is the $N_{\bullet}$-adic completion of N_i [8] — a very unpleasant object indeed. For poetic reasons we shall call this the "complete unstable quotient of M".

(3.5) *Remark.* — If φ is *linear*, it is an easy exercise to prove the above result, without the assumption of torsion freeness. However in our case the assumption is necessary because if R is a local ring and M is the residue field with its Frobenius endomorphism, $M_i \cong R/\mathfrak{m}^{(p^i)}$ and $\varprojlim M_i = \hat{R}$.

(3.6) THEOREM. — *Suppose* X *is smooth and proper over the spectrum* S *of a regular local ring* R *of characteristic* $p > 0$. *Then the inverse system* $H^i(X, \Phi_{\bullet}(\mathcal{O}_X))$ *is isomorphic to the inverse system deduced by linearizing the* σ*-linear endomorphism* F_X^* *of* $H^i(X, \mathcal{O}_X)$. *If in addition* σ *is finite, we can conclude:*

(3.6.1) *The stable quotient of* $H^i_{\mathcal{D}}(X/S, \mathcal{O}_X)$ *is isomorphic to the stable submodule of* $(H^i(X, \mathcal{O}_X), F^*)$, *if* $H^i(X, \mathcal{O}_X)$ *is torsion free.*

(3.6.2) *The unstable submodule of* $H^i_{\mathcal{D}}(X/S, \mathcal{O}_X)$ *is isomorphic to the complete unstable quotient of* $(H^{i-1}(X, \mathcal{O}_X), F_X^*)$, *if* $H^{i-1}(X, \mathcal{O}_X)$ *is also torsion free.*

Proof. — Let $X^n = \operatorname{Spec} R \times_{\sigma^n} X$, with $f^n: X \to X^n$ the R-morphism induced by F_X^n. It is well-known that f^n is a homeomorphism and that there is a natural isomorphism:

$$f_*^n \operatorname{Hom}_{\mathcal{D}^{V_n}}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_{X^n}.$$

Thus $\Phi_n(\mathcal{O}_X) \cong \mathcal{O}_{X^n}$, so we have

$$H^i(X, \Phi_n \mathcal{O}_X) \cong H^i(X^n, \mathcal{O}_{X^n}).$$

Since σ^n is flat, this is canonically isomorphic to $R \otimes_{\sigma^n} H^i(X, \mathcal{O}_X)$. $\square$

(3.7) *Example*. — Let R be a noetherian ring of characteristic $p > 0$, X an elliptic curve over $S = \underline{\mathrm{Spec}}\, R$; then $H^0(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{O}_X)$ are each one dimensional. There are three cases to consider: If all the fibers are ordinary, i. e., if the Hasse invariant h is a unit in R, then each map $\varphi_i : R \otimes_{\sigma^i} H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is an isomorphism. Thus $H^1(X, \mathcal{O}_X)$ is stable, so $H^1_{\mathscr{D}}(X/S, \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X)$ and $H^2_{\mathscr{D}}(X/S, \mathcal{O}_X) = 0$. [Of course, $H^0(X, \mathcal{O}_X)$ is always stable.] If the Hasse invariant h is not a unit in R, the filtration $N_{\bullet}$ of $H^1(X, \mathcal{O}_X)$ is equivalent to the (h)-adic filtration, so that

$$H^1_{\mathscr{D}}(X/S, \mathcal{O}_X) = \varprojlim N_{\bullet} = 0,$$

and

$$H^2_{\mathscr{D}}(X/S, \mathcal{O}_X) = R^1 \varprojlim N_{\bullet} \cong (\hat{R}/R)^N,$$

where $\hat{}$ means h-adic completion. Notice that this vanishes if R is complete h-adically or if $h = 0$, but in general it is very unpleasant. If we consider instead the pro-object defined by $N_{\bullet}$ one gets something slightly more reasonable. Namely, this pro-object is the functor $M \to \varinjlim \mathrm{Hom}[N_{\bullet}, M] \cong M_h$, the localization of M by h. This functor is not representable by an R-module, but if $S = \underline{\mathrm{Spec}}\, R$ and $U = \mathrm{Spec}\, R_h$, with $j : U \to S$ the inclusion, $M_h \cong \mathrm{Hom}_U[\mathcal{O}_U, i^* \tilde{M}] \cong \mathrm{Hom}_X(i_! \mathcal{O}_U, \tilde{M})$, where $i_! \mathcal{O}_U$ is the extension of $\mathcal{O}_U$ by zero. Thus, the pro-object "$\varprojlim N_{\bullet}$ is represented by the non-quasi-coherent sheaf $i_! \mathcal{O}_U$.

Let us now consider the case in which the base $S = \mathrm{Spec}\, V_n$ where V is a discrete valuation ring of mixed characteristic p, with uniformizing parameter π and $V_n = V/\pi^n V$. We shall assume that the residue field k is separably closed and that $[k : k^p]$ is finite.

(3.8) THEOREM. — *Let* X/S *be smooth and proper, let* $Y \to X$ *be a closed subset, and let* $\mathscr{X}$ *be the formal completion of* X *along* Y. *Then there is a natural isomorphism:* $H^i(Y_{\text{ét}}, f^{-1}(\mathcal{O}_S)) \to H^i_{\mathscr{D}}(\mathscr{X}/S, \mathcal{O}_{\mathscr{X}})$ *for all i, where* $\mathscr{D} = \mathscr{D}(\mathscr{X}/S)$.

Proof. — It is easy to see, and well-known, that the natural map

$$f^{-1}(\mathcal{O}_S) \to \underline{\mathrm{Hom}}_{\mathscr{D}}(\mathcal{O}_{\mathscr{X}}, \mathcal{O}_{\mathscr{X}})$$

in an isomorphism, both on $\mathscr{X}_{\text{zar}}$ and on $\mathscr{X}_{\text{ét}}$. Henceforth we shall work uniformly on the latter. The edge homomorphism of the spectral sequence

$$E_2^{pq} = H^p(\mathscr{X}, \underline{\mathrm{Ext}}^q_{\mathscr{D}}(\mathcal{O}_{\mathscr{X}} \mathcal{O}_{\mathscr{X}})) \quad \Rightarrow \quad \mathrm{Ext}^i_{\mathscr{D}} \mathscr{D}(\mathcal{O}_{\mathscr{X}}, \mathcal{O}_{\mathscr{X}})$$

is therefore a map $H^i(\mathscr{X}_{\text{ét}}, f^{-1}(\mathcal{O}_S)) \to H^i_{\mathscr{D}}(\mathscr{X}/S, \mathcal{O}_{\mathscr{X}})$ and since $\mathscr{X}_{\text{ét}}$ and $Y_{\text{ét}}$ are the same site, we get our map. The rest of the proof can be interpreted as showing that

$$H^q(\mathscr{X}_{\text{ét}}, \underline{\text{Ext}}^1_{\mathscr{D}}(\mathcal{O}_{\mathscr{X}}, \mathcal{O}_{\mathscr{X}})) = 0$$

for all q, but we do not do this directly.

First let us reduce to the case of a field. We have an exact sequence: $0 \to K \to V_n \to V_{n-1} \to 0$, where K is a one-dimensional k-vector space, and since $\mathscr{X}$ is flat over S, an exact sequence $0 \to K \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathscr{X}} \to \mathcal{O}_{\mathscr{X}} \to \mathcal{O}_{\mathscr{X}_{n-1}} \to 0$; note that these are all $\mathscr{D}$-modules. We can write this as $0 \to i_* \mathcal{O}_{\mathscr{X}_1} \to \mathcal{O}_{\mathscr{X}} \to j_* \mathcal{O}_{\mathscr{X}_{n-1}} \to 0$, where $\mathscr{X}_k$ is $\mathscr{X} \times_S S_k$, $S_k = \text{Sp}\, V_k$, i and j are the inclusions. Now if $\mathscr{D}_k = \mathscr{D}(\mathscr{X}_k/S_k)$, we have

$$\text{Hom}_{\mathscr{D}}(\mathcal{O}_{\mathscr{X}}, i_*\mathcal{O}_{\mathscr{X}_k}) \cong \text{Hom}_{\mathscr{D}_k}(\mathcal{O}_{\mathscr{X}_k}, \mathcal{O}_{\mathscr{X}_k}) \cong f^{-1}(\mathcal{O}_{S_k}),$$

and hence we obtain by functoriality of the edge homorphism, a long exact sequence:

$$\begin{array}{ccccccc}
\dots \to H^i_{\mathscr{D}}(\mathscr{X}/S, i_*\mathcal{O}_{\mathscr{X}_1}) & \longrightarrow & H^i_{\mathscr{D}}(\mathscr{X}/S_n, \mathcal{O}_{\mathscr{X}}) & \longrightarrow & H^i_{\mathscr{D}}(\mathscr{X}, j_*\mathcal{O}_{\mathscr{X}_{n-1}}) & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\dots \to H^i(Y_{\text{ét}}, f^{-1}(\mathcal{O}_{S_1})) & \to & H^i(Y_{\text{ét}}, f^{-1}(\mathcal{O}_{S_n})) & \to & H^i(Y_{\text{ét}}, f^{-1}(\mathcal{O}_{S_{n-1}})) & \to & \dots
\end{array}$$

Thus we will be able to prove our claim by induction on n, provided we can show that the maps $H^i(Y_{\text{ét}}, f^{-1}(\mathcal{O}_{S_1})) \to H^i_{\mathscr{D}}(\mathscr{X}/S, i_* \mathcal{O}_{\mathscr{X}_1})$ are isomorphisms. But we have a natural isomorphism, by the base changing result (1.15): $H^i_{\mathscr{D}}(\mathscr{X}/S, i_* \mathcal{O}_{\mathscr{X}_1}) \cong H^i_{\mathscr{D}_1}(\mathscr{X}_1/S_1, \mathcal{O}_{\mathscr{X}_1})$, so we have reduced to the case $S = k$.

Since now $p\, \mathcal{O}_S = 0$, we can apply the spectral sequence of paragraph 2. We cannot use (3.1) directly to analyze $(H^i(\mathscr{X}, \mathcal{O}_{\mathscr{X}}), F^*_{\mathscr{X}})$, however, because it will not be finite dimensional over k. But since each $Y_\nu = \text{Sp}_{\mathscr{X}}\, \mathcal{O}_{\mathscr{X}}/\mathscr{I}_Y^{\nu+1}$ is proper over k, $H^i(Y, \mathcal{O}_{Y_\nu})$ is finite dimensional and $H^i(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \cong \varprojlim H^i(Y, \mathcal{O}_{Y_\nu})$. Moreover F acts on each Y_ν, and we have:

(3.9) LEMMA. — *For each* ν, *the map* $(H^i(Y, \mathcal{O}_{Y_\nu})^s, F^*_\nu) \to (H^i(Y, \mathcal{O}_Y)^s, F^*)$ *is an isomorphism, and the "complete unstable quotient" of* $(H^i(Y, \mathcal{O}_{Y_\nu})\, F^*_\nu)$ *vanishes.*

Proof. — The second statement is clear, because for any inverse system of finite dimensional k-vector spaces, $R^1 \varprojlim = 0$. For the first statement, consider the exact sequence:

$$0 \to I_\nu \to \mathcal{O}_{Y_\nu} \to \mathcal{O}_Y \to 0$$

which induces an exact sequence of modules with σ-linear endomorphisms:

$$\dots \to H^i(Y, I_\nu) \to H^i(Y, \mathcal{O}_{Y_\nu}) \to H^i(Y, \mathcal{O}_Y) \to H^{i+1}(Y, I_\nu) \to \dots$$

Since the above is an exact sequence of finite dimensional k-vector spaces, we get an exact sequence of the corresponding stable submodules, so it is enough to show that $H^i(Y, I_\nu)^s = 0$ for all i. Since $(F^i)^* = 0$ for $\nu \gg 0$ on $H^i(Y, I_\nu)$, this is clear. □

Now for each non-negative integer i, let $M_{\nu i} = k \otimes_{\sigma^i} H^i(Y, \mathcal{O}_{Y_\nu})$. Thus we have a doubly indexed inverse system of finite dimensional k-vector spaces. We get two spectral sequences for $\varprojlim_{N\times N}$ (which may be viewed as the Leray spectral sequences for the projections $N\times N \rightrightarrows N$). Both are degenerate because $R^1 \varprojlim_{N} V^{\cdot} = 0$ for an inverse system of finite dimensional vector spaces. Thus, the first of these tells us that

$$\varprojlim_{N\times N} M_{\nu i} \cong \varprojlim_{\nu}\varprojlim_{i} M_{\nu i} \cong \varprojlim_{\nu} (M_\nu)^s \cong (M_0)^s \cong H^i(Y, \mathcal{O}_Y)^s,$$

and

$$R^1 \varprojlim_{N\times N} M_{\nu i} = R^1 \varprojlim_{\nu} \varprojlim_{i} M_{\nu i} = R^1 \varprojlim_{\nu} (M_\nu)^s = 0.$$

The second tells us that

$$\begin{aligned}\varprojlim_{N\times N} M_{\nu i} &\cong \varprojlim_{i}\varprojlim_{\nu} M_{\nu i}\\ &\cong \varprojlim_{i}\varprojlim_{\nu} (k \otimes_{\sigma^i} H^i(Y, \mathcal{O}_{Y_\nu})) \cong \varprojlim_{i} (k \otimes_{\sigma^i} \varprojlim_{\nu} H^i(Y, \mathcal{O}_\nu)),\end{aligned}$$

since k is finite over k via σ^i. This is just

$$\varprojlim_{i} (k \otimes_{\sigma^i} H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})) = H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})^s.$$

Also,

$$R^1 \varprojlim_{N\times N} M_{\nu i} = R^1 \varprojlim_{i} (k \otimes_{\sigma^i} H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})).$$

We deduce from our first calculation that the R^1 vanishes and that the natural map $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})^s \to H^i(Y, \mathcal{O}_Y)^s$ is an isomorphism. Thus from (2.4) and (3.6) we obtain canonical isomorphisms (valid whenever $[k : k^p]$ is finite):

(3.9.1) $$H^i_{\mathcal{D}}(\mathcal{X}/S, \mathcal{O}_{\mathcal{X}}) \to H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})^s \to H^i(Y, \mathcal{O}_Y)^s.$$

It is well know that over a separably closed field k, the natural map $H^i(Y_{\acute{e}t}, k) \to H^i(Y, \mathcal{O}_Y)^s$ is an isomorphism ([SGA 7, Exp. XXII, § 2]). Modulo an idenitification between canonical maps, this concludes the proof. □

4. Cohomology of the Infinitesmal Site

In this section we shall construct a canonical resolution of a D-module E, which, it will appear, is related to the Čech-Alexander resolution used by Grothendieck [2]. We need this resolution to prove that the functors $H^i_{\mathcal{D}}$ calculate cohomology in the infinitesmal topos. What is surprising about these results is that they have not yet appeared in the literature (1).

(1) I recently learned that (4.4), at least, will appear in a paper by M. Sweedler.

The construction we use is almost completely formal: Let A be a commutative ring with identity, D a non-commutative A-algebra, with $\theta : A \to D$ the given homomorphism. Suppose that $\theta(1)$ is a two sided identity for D. If $\partial \in D$ and $a \in A$, write $a\,\partial$ for $\theta(a)\,\partial$ and ∂a for $\partial\theta(a)$, so that D becomes a (unitary) A-module in two different ways. We again make use of the functor $L = \mathrm{Hom}_A[D,\ \ ]$, adjoint to the forgetful functor F from the category of left D-modules to the category of A-modules. Recall that here D is regarded as an A-module from the left.

We define inductively functors C^k from ((A-modules)) to ((left-D modules)) by $C^0 = L$ and $C^k = L \circ F \circ C^{k-1}$. A little care shows that $C^k(E)$ can be identified with the set of all functions $\varphi : D \times D \times \ldots \times D \to E$ such that:

$$(4.0.1) \qquad \varphi(\partial_0, \ldots, \partial_i, a\,\partial_{i+1}, \ldots, \partial_k) = \varphi(\partial_0, \ldots, \partial_i a, \partial_{i+1}, \ldots, \partial_k)$$

and

$$\varphi(a\,\partial_0, \partial_1, \ldots, \partial_k) = a\,\varphi(\partial_0, \ldots, \partial_k),$$

for any $a \in A$. The D-module structure is through the extreme right: if $\delta \in D$ and $\varphi \in C^k(E)$, $(\delta\varphi)(\partial_0, \ldots, \partial_k) = \varphi(\partial_0, \ldots \partial_k\,\delta)$.

(4.1) LEMMA. – *Suppose* F (D) *and* F (N) *are projective* A-*modules. Then for any* A-*module* E, $C^k(E)$ *is acyclic for* $\mathrm{Hom}_D(N,\ \ )$.

Proof. – As in paragraph 1 we observe that L is exact because D is projective, and also that L takes injectives to injectives because its adjoint F is exact. Thus

$$\mathrm{Ext}^i_D(N, L(Q)) \cong \mathrm{Ext}^i_A(F(N), Q),$$

for any N, Q, and hence

$$\mathrm{Ext}^i_D(N, L(Q)) = 0 \quad \text{for} \quad i > 0$$

for any A-module Q, in particular, for $FC^{k-1}(E)$. □

(4.2) PROPOSITION. – *Suppose* E *is a left* D-*module. Then* $C^{\cdot}(E) = \bigoplus_{k=0}^{\infty} C^k(E)$ *has a natural structure of a complex of* D-*modules, and there is a natural* D-*linear quasi-isomorphism* $E \to C^{\cdot}(E)$.

Proof. – Define $d : C^k(E) \to C^{k+1}(E)$ by:

$$(4.2.1) \qquad (d\varphi)(\partial_0, \ldots, \partial_{k+1}) = \partial_0\,\varphi(\partial_1, \ldots, \partial_{k+1}) + \sum_{i=1}^{k+1} (-1)^i \varphi(\partial_0, \ldots, \partial_{i-1}\partial_i, \ldots, \partial_{k+1}).$$

We let the reader verify that d is well-defined, D-linear, and satisfies $d^2 = 0$. There is a natural D-linear augmentation $\varepsilon : E \to C^0(E)$ given by $(\varepsilon(x))(\partial) = \partial x$; note that if $\pi : C^0(E) \to E$ is the adjunction map, $\pi \circ \varepsilon = \mathrm{id}$ (but π is not D-linear). In fact there is a homotopy (which is not D-linear) $R : C^{\cdot}(E) \to C^{\cdot -1}(E)$ given by

$$(R^k\varphi)(\partial_0, \ldots, \partial_{k-1}) = \varphi(\partial_0, \ldots, \partial_{k-1}, 1)$$

with $R^0 : C^0(E) \to E$ equal to π. Checking:

$$(dR\varphi)(\partial_0, \ldots, \partial_k) = \partial_0(R\varphi)(\partial_1, \ldots, \partial_k) + \sum_{i=1}^{k-1} R\varphi(\partial_0, \ldots, \partial_{i-1}\partial_i, \ldots, \partial_k)$$
$$= \partial_0\varphi(\partial_1, \ldots, \partial_k, 1) + \sum_{i=1}^{k-1} \varphi(\partial_0, \ldots, \partial_{i-1}\partial_i, \ldots, \partial_k, 1)$$

while

$$(R\,d\varphi)(\partial_0, \ldots, \partial_k) = (d\varphi)(\partial_0, \ldots, \partial_k, 1)$$
$$= \partial_0\varphi(\partial_1, \ldots, \partial_k, 1) + \sum_{i=1}^{k} (-1)^i \varphi(\partial_0, \ldots, \partial_{i-1}\partial_i, \ldots, \partial_k, 1)$$
$$+(-1)^{k+1}\varphi(\partial_0, \ldots, \partial_k 1).$$

Thus $(-1)^{k+1} R\,d + (-1)^k dR = id$ so R is a chain homotopy. It follows of course that ε is a quasi-isomorphism. □

Combining the two previous results we immediately obtain:

(4.3) COROLLARY. – *With the hypotheses of* (4.1) *and* (4.2) *above, the complex* $\mathrm{Hom}_D(N, C^\cdot(E))$ *represents the derived functor* $R\,\mathrm{Hom}_D(N, E)$. *In particular its i-th cohomology group is* $\mathrm{Ext}^i_D(N, E)$. □

The corollary applies if A is itself a left D-module, and in this case we shall describe the complex $R\,\mathrm{Hom}_D(A, E)$ more explicitly. We have a surjective D-linear map $\alpha : D \to A$ given by $\alpha(\partial) = \partial(1)$, and in terms of this we can define a new complex $(L^\cdot(E), d)$, for any left D-module E, by $L^0(E) = E$, $L^k(E) = C^{k-1}(E)$ for $k > 0$, and with $d : L^k(E) \to L^{k+1}(E)$ given by:

$$d^0(x)(\partial) = \partial x - \alpha(\partial)x$$
$$d^k(\lambda)(\partial_0, \ldots, \partial_k) = \partial_0\lambda(\partial_1, \ldots, \partial_k) + \sum_{i=1}^{k} (-1)^i(\partial_0, \ldots, \partial_{i-1}\partial_i, \ldots, \partial_k)$$
$$+(-1)^{k+1}\lambda(\partial_0, \ldots, \partial_{k-1})\alpha(\partial_k).$$

(4.4) THEOREM. – *If* $F(D)$ *is a projective* A-*module, the complex* $(L^\cdot(E), d)$ *represents* $R\,\mathrm{Hom}_D(A, E)$.

Proof. – By the previous result, $R\,\mathrm{Hom}_D(A, E) \cong \mathrm{Hom}_A(A, C^\cdot(E))$. Since $C^k(E) \cong L\,F\,C^{k-1}(E)$, $\mathrm{Hom}_D(A, C^k(E)) \cong \mathrm{Hom}_A(A, C^{k-1}(E)) = L^k(E)$. It remains to check that the boundary map on $L^\cdot(E)$ induced from the boundary on $C^\cdot(E)$ is as described. To do this, use α to obtain an injection $i: Q \to L(Q)$, namely

$$Q \cong \mathrm{Hom}_A(A, Q) \xrightarrow{\cong} \mathrm{Hom}_D(A, L(Q)) \xrightarrow{\alpha^*} \mathrm{Hom}_D(D, L(Q)) \cong L(Q).$$

Explicitly, $i(g)(\partial) = \alpha(\partial)g$. We also have the adjunction map $\pi: L(Q) \to Q$ given by $\pi(\lambda) = \lambda(1)$; of course $\pi \circ i = \mathrm{id}_Q$. We thus have for each k a natural inclusion $i^k : L^k(E) \to C^k(E)$ with a section $\pi^k : C^k(E) \to L^k(E)$, and if $d_L : L^k(E) \to L^{k+1}(E)$ is induced from $d_C : C^k(E) \to C^{k+1}(E)$, it follows that $d_L = \pi^{k+1} \circ d_C \circ i^k$. Using the explicit formulas

$$(i^k\lambda)(\partial_0, \ldots, \partial_k) = \alpha(\partial_k)\lambda(\partial_0, \ldots, \partial_{k+1})$$

and

$$(\pi^k \varphi)(\partial_0, \ldots, \partial_{k-1}) = \varphi(\partial_0, \ldots, \partial_{k-1}, 1),$$

one easily computes d_L. □

(4.5) THEOREM. — *Suppose $\mathscr{E}$ is a coherent $\mathscr{D}(\mathscr{X}/\mathrm{S})$-module on the "smooth" formal* S-*scheme, where S is a noetherian scheme. Then there are natural isomorphisms:*

$$\mathrm{H}^i_{\mathscr{D}}(\mathscr{X}/\mathrm{S}, \mathscr{E}) \cong \mathbf{H}^i(\mathscr{X}, \mathrm{L}^{\cdot}(\mathscr{E})).$$

Proof. — Consider the complex $\mathscr{C}^{\cdot}(\mathscr{E})$. Since $\mathscr{E} \to \mathscr{C}^{\cdot}(\mathscr{E})$ *is* a quasi-isomorphism of complexes of D-modules, we have an isomorphism $\mathrm{R}\,\Gamma_{\mathscr{D}}(\mathscr{X}, \mathscr{E}) \to \mathrm{R}\,\Gamma_{\mathscr{D}}(\mathscr{X}, \mathscr{C}^{\cdot}(\mathscr{E}))$. Since $\mathrm{R}\,\Gamma_{\mathscr{D}} = \mathrm{R}\,\Gamma \circ \mathrm{R}\,\underline{\mathrm{Hom}}_{\mathscr{D}}(\mathcal{O},\ \ )$ we get an isomorphism :

$$\mathrm{R}\,\Gamma_{\mathscr{D}}(\mathscr{X}, \mathscr{E}) \to \mathrm{R}\,\Gamma \circ \mathrm{R}\,\underline{\mathrm{Hom}}_{\mathscr{D}}(\mathcal{O}_{\mathscr{X}}, \mathscr{C}^{\cdot}(\mathscr{E})).$$

By the previous result and (1.10), $\mathrm{R}\,\underline{\mathrm{Hom}}_{\mathscr{D}}(\mathcal{O}_{\mathscr{X}}, \mathscr{C}^{\cdot}(\mathscr{E}))$ is represented by

$$\mathrm{L}^{\cdot}(\mathscr{E}), \text{ and so } \mathrm{R}\,\Gamma_{\mathscr{D}}(\mathscr{X}, \mathscr{E}) \cong \mathrm{R}\,\Gamma(\mathscr{X}, \mathrm{L}^{\cdot}(\mathscr{E})).$$

This prove the theorem. □

Now suppose Y is an S-scheme embedable in a smooth S-scheme X, and suppose E is a coherent crystal on $(\mathrm{Y}/\mathrm{S})_{\mathrm{inf}}$ (without divided powers). Thus for every nilpotent immersion $\mathrm{U} \subsetneq \mathrm{T}$ where U is open in T, we have a coherent sheaf E_{T} on T, and for any $f : \mathrm{T}_1 \to \mathrm{T}_2$ inducing an open immersion $\mathrm{U}_1 \subsetneq \mathrm{U}_2$, we have an *isomorphism* $\mathrm{E}_{\mathrm{T}_1} \cong f^* \mathrm{E}_{\mathrm{T}_2}$; moreover these isomorphisms are compatible with composition. Then if $\mathscr{X}$ is the formal completion of X along Y, we get a coherent $\mathrm{D}(\mathscr{X}/\mathrm{S})$-module, as indicated by Grothendieck [2] and Berthelot [1]: Namely, for each X_ν, we have $\mathrm{E}_{\mathrm{X}_\nu}$, compatibly, so by [EGA], $\varprojlim \mathrm{E}_{\mathrm{X}_\nu} = \mathscr{E}$ is a coherent sheaf on $\mathscr{X}$. Moreover for each n, we have a nilpotent immersion $\mathrm{Y} \subsetneq \mathrm{P}^n(\mathrm{X}_\nu/\mathrm{S})$, and hence a *canonical* isomorphism $\pi_2^* \mathrm{E}_{\mathrm{X}_\nu} \cong \pi_1^* \mathrm{E}_{\mathrm{X}_\nu}$ for each ν, where the π_i are the projections $\mathrm{P}^n_{\mathrm{X}_\nu/\mathrm{S}} \to \mathrm{X}_\nu$. Passing to the limit we get [thanks to (1.1)] an isomorphism $\varepsilon_i : \mathrm{P}^n_{\mathscr{X}/\mathrm{S}} \otimes \mathscr{E} \to \mathscr{E} \otimes \mathrm{P}^n_{\mathscr{X}/\mathrm{S}}$, satisfying the usual cocyle condition. Then given any differential operator ∂ of order $\leqq n$, i. e. a map $\mathrm{P}^n_{\mathscr{X}/\mathrm{S}} \to \mathcal{O}_{\mathscr{X}}$, we get a map $\mathscr{E} \otimes \hat{\mathrm{P}}^n_{\mathscr{X}/\mathrm{S}} \to \mathscr{E}$, hence composing with the universal differential operator $\delta : \mathscr{E} \to \mathscr{E} \otimes \mathrm{P}^n_{\mathscr{X}/\mathrm{S}}$, an endomorphism of E. The cocyle conditions tell us that this map $\mathrm{Diff}(\mathrm{X}/\mathrm{S}) \to \mathrm{End}_{\mathcal{O}_s}(\mathrm{E})$ is a ring homomorphism. □

(4.6) THEOREM. — *With the above notations and hypotheses, we have natural isomorphisms:*

$$\mathrm{H}^i_{\mathrm{inf}}(\mathrm{Y}/\mathrm{S}, \mathrm{E}) \cong \mathrm{H}^i_{\mathscr{D}}(\mathscr{X}/\mathrm{S}, \mathscr{E}).$$

Proof. — According to Grothendieck's calculation, $\mathrm{H}^i_{\mathrm{inf}}(\mathrm{Y}/\mathrm{S}, \mathrm{E})$ may be calculated as the Zariski hypercohomology of the Čech-Alexander complex $\mathscr{C}^{\cdot}(\mathrm{X}/\mathrm{S}, \mathscr{E})$ [2]. So by theorem (4.5) above, it suffices to notice that this complex is just the complex $\mathrm{L}^{\cdot}(\mathscr{E})$ — a matter of inspection: Observe for instance that the first two terms of $\mathrm{L}^{\cdot}(\mathscr{E})$ are $\mathscr{E} \to \mathrm{Hom}_{\mathcal{O}_{\mathscr{X}}}(\mathscr{D}, \mathscr{E}) \cong \mathscr{E} \otimes \mathrm{P}_{\mathscr{X}/\mathrm{S}}$. Then I claim that, with this identification, the boundary d of L is given by $d_{\mathrm{L}}(\mathrm{X}) = \varepsilon(\delta(x)) - x \otimes 1$. It suffices to check this after "evaluating" with any differential operator ∂. But $(\varepsilon\,\delta(x), \partial) - (x \otimes 1, \partial) = \partial x - x\,\partial(1)$, which is indeed $d_{\mathrm{L}}(x)$. □

(4.7) COROLLARY. — *Suppose k is algebraically closed and Y/k is proper and embeddable in a smooth $W_n(k)$-scheme. Then there are natural isomorphisms:*

$$H^i(Y_{\acute{e}t}, Z/p^n Z) \otimes W_n(k) \to H^i(Y_{\acute{e}t}, W_n(k)) \to H^i_{\inf}(Y/S, \mathcal{O}_{Y/S}). \quad \square$$

The referee has pointed out that although the relationship between Z_p-étale cohomology and the unit root part of crystalline cohomology is folklore, no precise theorems seem to exist in the literature. Fortunately we can provide a quick proof, without killing torsion.

Actually it seems most natural to compare crystalline and infinitesmal cohomology directly. Let us begin by briefly describing the morphism $v_{X/S} : (X/S)_{cris} \to (X/S)_{inf}$, alluded to in the introduction. Here we are working over a noetherian base scheme S on which p is nilpotent and which is endowed with a P.D. ideal (I, γ) which extends to X. For details of these notions, we must refer to Berthelot's thesis [1]; the reader can keep in mind the special case $S = \operatorname{Spec} W_n(k)$, with k a perfect field and with the canonical divided power structure on the ideal $\pi = p\, W_n$.

The functor $v^*_{X/S} : (X/S)_{inf} \to (X/S)_{cris}$ is easy to describe: If F is a sheaf on Inf (X/S) and $(U, T, \delta) \in \operatorname{Cris}(X/S)$, then $(U, T) \in \operatorname{Inf}(X/S)$, and we can set $v^*_{X/S}(F)_{(U,T,\delta)} = F_{(U,T)}$. Clearly this construction commutes with inverse limits. If G is a sheaf on Cris (X/S) and $(U, T) \in \operatorname{Inf}(X/S)$, let $D = (U, D_U(T), [\ \ ])$ be the P.D. envelope of U in T, and let $v_{X/S^*}(G)_{(U,T)}$ be the sheaf G_D. The reader can easily check the necessary compatibilities and the fact that $v^*_{X/S}$ is left adjoint to v_{X/S^*}. Clearly $v_{X/S}$ defines a morphism of ringed topoi: $(X/S, \mathcal{O}_{X/S})_{cris} \to (X/S, \mathcal{O}_{X/S})_{inf}$. To describe the image of the induced map in cohomology, we need the following unequal characteristic version of (3.1):

(4.8) LEMMA. — *Let R be a local ring, $\sigma : R \to R$ a finite flat endomorphism such thas $R \otimes_\sigma k \cong k$. Using the notations of* (3.1) *for σ-linear endomorphisms, we have, if M it an Artininan R-module:*

(4.8.1) *The map* $\varprojlim M_{\bullet} \to M$ *is injective, with image* M^s.

(4.8.2) *There are natural isomorphisms:*

$$M^s \cong \varprojlim N_{\bullet} \cong \varprojlim M_{\bullet}$$

and

$$R^1 \varprojlim M_{\bullet} = R^1 \varprojlim N_{\bullet} = 0.$$

(4.8.3) *The submodule* (M^s, ψ^s) *is stable, and the quotient* (M^n, ψ^n) *unstable.*

(4.8.4) *The functors* $M \mapsto M^s$ *and* $M \mapsto M^n$ *are exact.*

Proof. — In fact, since $\otimes_\sigma R$ preserves lenghts, the analogue of (3.2) is even true, and the first statement follows easily. Mittag-Leffler conditions imply the vanishing of the $R^1 \varprojlim$. The exactness claim follows from this, or from the snake lemma and the fact that any homorphism from a stable module to an unstable one and which preserves ψ vanishes. $\square$

(4.9) THEOREM. — *Let k be a perfect field, and suppose X/k is smooth and proper and embeddable in a smooth scheme over $W_n(k)$. Then the map:*

$$v^*_{X/S}:\ H^i_{\inf}(X/W_n(k), \mathcal{O}_{X/W_n(k)}) \to H^i_{\mathrm{cris}}(X/W_n(k), \mathcal{O}_{X/W_n(k)})$$

is injective, with image the stable submodule of $H^i_{\mathrm{cris}}(X/W_n(k))$.

Proof. — We already know from (3.9.1) that the infinitesmal cohomology is stable. Using the exactness of extraction of stable submodules and the inductive technique of (3.8), we reduce to the case $n = 1$, i. e. $W_n(k) = k$. We have maps of ringed topoi:

$$(X_{\mathrm{zar}}, \mathcal{O}_X) \xrightarrow{i_{X/S}} (X_{\mathrm{cris}}, \mathcal{O}_{X/S}) \xrightarrow{v_{X/S}} (X_{\inf}, \mathcal{O}_{X/S}).$$

[For an explanation of $i_{X/S}$, see ([1], III, 3.3).] Since these maps are compatible with Frobenius, we obtain maps:

$$H^i_{\inf}(X/S, \mathcal{O}_{X/S})^S \xrightarrow{v^{*S}_{X/S}} H^i_{\mathrm{cris}}(X/S, \mathcal{O}_{X/S})^s \xrightarrow{i^{*S}_{X/S}} H^i_{\mathrm{zar}}(X, \mathcal{O}_X)^S.$$

According to (3.6) and (4.6) (and an unchecked compatibility), the composite is an isomorphism. Thus it suffices to show that $i^{*S}_{X/S}$ is injective. By the Poincaré lemma for crystals, we can identify $H^i_{\mathrm{cris}}(X/S, \mathcal{O}_{X/S})$ with de Rham cohomology $\mathbf{H}^i(X, \Omega^{\cdot}_{X/S})$ and the map $i^*_{X/S}$ with the edge homorphism $\mathbf{H}^i(X, \Omega^{\cdot}_{X/S}) \to H^i(X, \mathcal{O}_X)$. Thus, Ker $(i^*_{X/S})$ becomes $F^1_{\mathrm{Hodge}}\,\mathbf{H}^i(X, \Omega^{\cdot}_{X/S}) \subseteq \mathrm{Ker}(F^*_X)$, and since F^*_X is injective on $\mathbf{H}^i(X, \Omega^{\cdot}_{X/S})^S$, $i^{*S}_{X/S}$ is injective. □

(4.10) *Remark.* — The same formula holds after passing to the limit. Indeed, if (M_n, ψ_n) is an inverse system of Artinian W-modules with σ-linear endomorphisms and $(M, \psi) = \varprojlim (M_n, \psi_n)$, then since σ is finite $M_i = R \otimes_\sigma M \cong \varprojlim R \otimes_\sigma M_n$, so $M^S \cong \varprojlim M^S_n$. On the torsion free part of M, which we assume to be finitely generated, M^S can clearly be described as the unit root part. Thus we obtain the following result, which includes the torsion as well:

(4.11) COROLLARY. — *If k is algebraically closed, we have natural isomorphisms:*

$$\begin{tikzcd} H^i_{\inf}(X/W) \arrow[rr] & & H^i_{\mathrm{cris}}(X/W)^S \\ & H^i_{\text{ét}}(X, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} W \arrow[ul] \arrow[ur] & \end{tikzcd}$$

(4.12) *Remark.* — The referee suggests that I point out how the exactness of the functor Φ. in (2.4) and the exact sequence (2.4) can be deduced from F-descent, using the Čech-Alexander complex. If A is an S-algebra, where $pS = 0$, let $A^{(p^n)} = S \otimes_{F^n_S} A$, where F_S is the absolute Frobenius endomorphism of S, and let A have the structure of $A^{(p^n)}$-algebra by the relative Frobenius $F_{A/S}$. There is a natural surjection $A \otimes_S A \to A \otimes_{A^{(p^n)}} A$, and one easily sees that its kernel is generated by the set of elements $(a \otimes 1 - 1 \otimes a)^{p^n}$ with $a \in A$. In other words, there is a natural isomorphism between $A \otimes_{A^{(p^n)}} A$ and $(A \otimes_S A)/I^{(p^n)}$, where I is the ideal of the diagonal and $I^{(p^n)}$ is the ideal generated by the

p^n-*th* powers of its elements. The same is true if we take products with more factors, and we deduce an isomorphism of complexes:

$$\begin{array}{ccccc} A \rightrightarrows & (A \otimes_S A)/I^{(p^n)} & \substack{\to\\\to\\\to} & (A \otimes_S A \otimes_S A)/I^{(p^n)} & \substack{\to\\\to\\\to\\\to} \dots \\ \wr\| & \wr\| & & \wr\| & \\ A \rightrightarrows & A \otimes_{A^{(p^n)}} A & \substack{\to\\\to\\\to} & A \otimes_{A^{(p^n)}} A \otimes_{A^{(p^n)}} A & \substack{\to\\\to\\\to\\\to} \end{array}$$

Now if A/S is smooth, a calculation with local coordinates shows that the map $A \to A^{(p^n)}$ is faithfully flat, so that the complexes above are both resolutions of $A^{(p^n)}$. One now can see the exactness of $\Phi_\bullet$ in the affine setting. However, if we use Čech-Alexander resolutions to calculate the cohomology of an arbitrary crystal on Inf (X/S), we can derive the exact sequence (2.4) directly in this context from the isomorphism of complexes above. Details of the necessary sheafifications, completions, and Mittag-Leffler arguments are left to the mythical interested reader.

5. **Formal de Rham Cohomology**

In this section we discuss the hypercohomology of the formal completion of projective space along a closed subset. The proofs are logically independent of the rest of this paper, but are related spiritually. We let $\mathbf{P}$ be n-dimensional projective space over an algebraically closed field k of positive characteristic.

(5.1) THEOREM. — *Suppose* $Y \subseteq \mathbf{P}$ *is Cohen-Macaulay of pure dimension d, and* $\mathscr{P}$ *is the formal completion of* $\mathbf{P}$ *along* Y. *Then:*

(5.1.1) *The map* $L : \mathbf{H}^{i-1}(\mathscr{P}, \Omega^\bullet_{\mathscr{P}/k}) \to \mathbf{H}^{i+1}(\mathscr{P}, \Omega^\bullet_{\mathscr{P}/k})$ *induced by cup-product with* $c_1(\mathscr{O}_{\mathscr{P}}(1))$ *is injective for* $i \leqq d$.

(5.1.2) *There are natural maps:*

$$\oplus_{i+j=k} \mathbf{H}^i_{\mathrm{DR}}(\mathbf{P}, \Omega^\bullet_{\mathbf{P}/k}) \otimes_{\mathbf{Z}/p\mathbf{Z}} H^j(Y_{\text{ét}}, \mathbf{Z}/p\,\mathbf{Z}) \to \mathbf{H}^k(\mathscr{P}, \Omega^\bullet_{\mathscr{P}/k})$$

which are isomorphisms if $k < d$ *and injective if* $k = d$.

(5.1.3) *The spectral sequence* $E_1^{pq} = H^q(\mathscr{P}, \Omega^\bullet_{\mathscr{P}/k}) \Rightarrow \mathbf{H}^i(\mathscr{P}, \Omega^\bullet_{\mathscr{P}/k})$ *is partially degenerate, namely* $d_r^{p,q} = 0$ *if* $r \geqq 1$ *and* $p+q < d$.

Proof. — This theorem was motivated by a misbegotten attempt to prove the strong Lefschetz theorem in characteristic 0 using the Barth vanishing theorem (*cf.* [9]). In characteristic p we can use the Cartier operation to "turn the vanishing around", as needed, but we succeed in obtaining the above theorem instead of strong Lefschetz.

The proof makes use of the " primitive de Rham complex " $P^\bullet_{\mathbf{P}/k}$ explained in [9]. Let $G = k[X_0, \dots, X_n]$ be the homogeneous coordinate ring of $\mathbf{P}$, let $G^{(p)} = k[T_0, \dots, T_n]$, and define $\varphi : G^{(p)} \to G$ by $T_i \to X_i^p \dots$ the "relative Frobenius map F". For each i, we have the "inverse Cartier isomorphism," if we identify T_i with $1 \otimes X_i$. $\mathscr{C}^{-1} : \Omega^{i(p)}_{G/k} \to H^i(\Omega^\bullet_{G/k})$ [6]. Now recall that $\Omega^\bullet_G$ is a graded complex, where $\deg dX_i = 1$, and $\mathscr{C}^{-1}(dT_i) = X_i^{p-1}\, dX_i$, $\mathscr{C}^{-1}(\alpha\omega) = \varphi(\alpha)\, \mathscr{C}^{-1}(\omega)$ if $\alpha \in G^{(p)}$, $\omega \in \Omega^i_{G^{(p)}/k}$.

Thus it follows that $\deg \mathscr{C}^{-1}(\omega) = p \deg(\omega)$. Moreover $\mathscr{C}^{-1}$ is compatible with localization, and so for any $f \in G^{(p)}$, we have an isomorphism of graded $G^{(p)}$-modules: $(\Omega^i_{G^{(p)}/k})_f \to H^i[(\Omega^{\cdot}_G)_{\varphi(f)}]$. Taking the degree zero parts of both sides we obtain an isomorphism:

$$P^i_{\mathbf{P}^{(p)}/k}(D^+(f)) \to F_* H^i(P^{\cdot}(D^+(\varphi(f)))).$$

Since this map is compatible with further localization we obtain an isomorphism of sheaves: $P^i_{\mathbf{P}^{(p)}/k} \to F_* \mathscr{H}^i(P^{\cdot}_{\mathbf{P}/k})$.

Now if $Y \subseteq \mathbf{P}$ is a closed subscheme defined by an ideal I, we obtain by base change a closed subscheme $Y^{(p)} \subseteq \mathbf{P}^{(p)}$ defined by $\mathscr{I}^{(p)}$, of course $F: \mathbf{P} \to \mathbf{P}^{(p)}$ and $Y \to Y^{(p)}$ are homeomorphisms. Since F^* maps $I^{(p)}$ into I and $\mathscr{C}^{-1}$ is $\mathscr{O}_{\mathbf{P}^{(p)}/k}$-linear we see that we get induced an isomorphism:

$$\mathscr{C}^{-1}: \quad P^i_{\mathscr{P}^{(p)}/k} \to F_* \mathscr{H}^i(P^{\cdot}_{\mathscr{P}/k}) \quad \text{for all } i, \tag{5.2}$$

where $\mathscr{P}^{(p)}$ is the completion of $\mathbf{P}^{(p)}$ along $Y^{(p)}$.

The isomorphism above and the following results are the key ingredients to our proof.

(5.3) THEOREM. — (Harstorne) *With the assumptions of* (5.1), *there are natural maps, for any locally free sheaf* F *on* **P**:

$$\oplus_{i+j=k} H^i(\mathbf{P}, F) \otimes_{\mathbf{Z}/p\mathbf{Z}} H^j(Y_{\text{ét}}, \mathbf{Z}/p\,\mathbf{Z}) \to H^k(\hat{\mathbf{P}}, \hat{F}).$$

These maps are isomorphisms for $k < d$ and injective for $k = d$.

Proof. — Hartshorne proves the above result with $F = \mathscr{O}_{\mathbf{P}^n}(l)$ for any l in [5]; we have replaced $H^i(Y, \mathscr{O}_Y)^S$ by $H^i(Y_{\text{ét}}, \mathbf{Z}/p\ \mathbf{Z})$. The general case follows by subtle dévissage (*cf.* [9]). □

(5.4) LEMMA. — *Let* $F^{\cdot}_{\text{con}}\,\mathbf{H}$ *denote the filtration which goes with the spectral sequence of hypercohomology*

$$E_2^{pq} = H^p(X, \mathscr{H}^q) \quad \Rightarrow \quad \mathbf{H}^{\cdot}(X).$$

Then

$$F^i_{\text{con}}\,\mathbf{H}^i(\mathscr{P}, P^{\cdot}_{\mathscr{P}k}) = \mathbf{H}^i(\mathscr{P}, P^{\cdot}_{\mathscr{P}/k}) \qquad \textit{for} \quad i \leqq d,$$

and

$$F^d_{\text{con}}\,\mathbf{H}^i(\mathscr{P}, P^{\cdot}_{\mathscr{P}/k}) = \mathbf{H}^i(\mathscr{P}, P^{\cdot}_{\mathscr{P}/k}) \qquad \textit{for} \quad i \geqq d.$$

Proof. — We have

$$E_2^{pq} = H^p(\mathscr{P}, \mathscr{H}^q(P^{\cdot}) \cong H^p(\mathscr{P}^{(p)}, F_*\mathscr{H}^q(P^{\cdot})) \cong H^p(\mathscr{P}^{(p)}, P^q_{\mathscr{P}^{(p)}/k}),$$

from isomorphism (5.2). Since the Frobenius σ of k is an isomorphism, one can easily deduce that this is $k \otimes_\sigma H^p(\mathscr{P}, P^q_{\mathscr{P}/k})$. But $P^q_{\mathscr{P}/k} = \Lambda^q P^1_{\mathscr{P}/k}$ and $P^1_{\mathbf{P}/k} \cong \oplus_0^n \mathscr{O}_{\mathscr{X}}(-1)$; thus $P^q_{\mathscr{P}/k}$ is a direct sum of line bundles of negative degrees. It follows from (5.3) that $H^p(\mathscr{P}, P^q_{\mathscr{P}/k}) = 0$ if $p < d$ and $q > 0$, so the same holds for E_2^{pq}. The lemma follows. □

Next we use the exact sequence of complexes on $\mathscr{P}$ [9, 1.6.1]

$$0 \to \Omega^{\cdot}_{\mathscr{P}/k} \to P^{\cdot}_{\mathscr{P}/k} \to \Omega_{\mathscr{P}/k}[-1] \to 0.$$

Recall that the coboundary map of the associated long exact sequence

$$\mathbf{H}^i(\mathscr{P}, \Omega^{\cdot}_{\mathscr{P}/k}[-1]) \cong \mathrm{H}^{i-1}(\mathscr{P}, \Omega^{\cdot}_{\mathscr{P}/k}) \to \mathrm{H}^{i+1}(\mathscr{P}, \Omega^{\cdot}_{\mathscr{P}/k}) \quad \text{is L.}$$

Trivially, $\mathrm{E}_2^{pq} = \mathrm{H}^p(\mathscr{P}, \mathscr{H}^q(\Omega^{\cdot}_{\mathscr{P}/k}[-1])) = 0$ if $q < 1$, so $\mathrm{F}^i_{\mathrm{con}}\, \mathbf{H}^i(\mathscr{P}, \Omega^{\cdot}_{\mathscr{P}/k}[-1]) = 0$. Since the map $\mathbf{H}^i(\mathscr{P}, \mathrm{P}^{\cdot}_{\mathscr{P}/k}) \to \mathbf{H}^i(\mathscr{P}, \Omega^{\cdot}_{\mathscr{P}/k}[-1])$ preserves the filtration $\mathrm{F}^{\cdot}_{\mathrm{con}}$, it follows immediately from (5.4) that it vanishes for $i \leqq d$. The long exact sequence of hypercohomology therefore proves (5.1.1).

To prove (5.1.2) observe that $\Omega^{\cdot}_{\mathscr{P}/k}$ is a complex of $\mathbf{Z}/p\mathbf{Z}$-modules and $\mathbf{Z}/p\mathbf{Z}$-linear maps on $\mathscr{P}_{\text{ét}}$, so that we have a cup-product:

$$\mathbf{H}^i(\mathscr{P}_{\text{ét}}, \Omega^{\cdot}_{\mathscr{P}/k}) \otimes \mathbf{H}^j(\mathscr{P}_{\text{ét}}, \mathbf{Z}/p\,\mathbf{Z}) \to \mathbf{H}^{i+j}(\mathscr{P}_{\text{ét}}, \Omega^{\cdot}_{\mathscr{P}/k}).$$

Using the isomorphisms:

$$\mathrm{H}^k(\mathscr{P}, \Omega^{\cdot}_{\mathscr{P}/k}) \cong \mathrm{H}^k(\mathscr{P}_{\text{ét}}, \Omega^{\cdot}_{\mathscr{P}/k}) \qquad \text{and} \qquad \mathrm{H}^j(\mathrm{Y}_{\text{ét}}, \mathbf{Z}/p\,\mathbf{Z}) \cong \mathrm{H}^j(\mathscr{P}_{\text{ét}}, \mathbf{Z}/p\,\mathbf{Z})$$

and the map $\mathbf{H}^i(\mathbf{P}, \Omega^{\cdot}_{\mathbf{P}/k}) \to \mathrm{H}^i(\mathscr{P}_{\text{ét}}, \Omega^{\cdot}_{\mathbf{P}/k})$, we obtain the maps in (5.1.2). These maps are compatible with the "first" spectral sequence of hypercohomology and the maps of Hartshorne's (5.3). We deduce from (5.3) isomorphisms

$$\oplus_j \mathrm{H}^{q-j}(\mathbf{P}, \Omega^p_{\mathbf{P}/k}) \otimes \mathrm{H}^j(\mathrm{Y}_{\text{ét}}, \mathbf{Z}/p\,\mathbf{Z}) \to \mathrm{H}^{\cdot}(\mathscr{P}, \Omega^p_{\mathscr{P}/k}) \qquad \text{for} \quad q < d,$$

compatible with the differentials of the spectral sequence. Statement (5.1.2) follows immediately, and so does (5.1.3), because the differentials on the left all vanish. □

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