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WEYL GROUP OF A CUSPIDAL PARABOLIC

BY A. W. KNAPP (*)

1. Introduction

A subalgebra $\mathfrak{s}$ of a real semisimple Lie algebra $\mathfrak{g}$ is *parabolic* if $\mathfrak{s}^{\mathbf{C}}$ contains a maximal solvable subalgebra of $\mathfrak{g}^{\mathbf{C}}$. With respect to any Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a parabolic subalgebra $\mathfrak{s}$ has a *Langlands decomposition* $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ defined as follows: Let θ be the Cartan involution ($+1$ on $\mathfrak{k}$ and -1 on $\mathfrak{p}$), let $B_\theta(x, y) = -B(x, \theta y)$, where B is the Killing form, and require that the Langlands decomposition be orthogonal with respect to B_θ. The algebra $\mathfrak{n}$ is defined by the identity $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{s} \cap \theta\,\mathfrak{s}$, and $\mathfrak{a}$ is taken as the intersection of $\mathfrak{p}$ and the center of $\mathfrak{m} \oplus \mathfrak{a}$. Then $\mathfrak{m}$ is reductive, $\mathfrak{a}$ is abelian and commutes with $\mathfrak{m}$, $\mathfrak{n}$ is nilpotent, and $\mathfrak{m} \oplus \mathfrak{a}$ normalizes $\mathfrak{n}$.

Following Harish-Chandra ([4], [5]), we say that the parabolic subalgebra $\mathfrak{s}$ is *cuspidal* if the rank of $\mathfrak{k} \cap \mathfrak{m}$ equals the rank of $\mathfrak{m}$. In this case the sum of $\mathfrak{a}$ and a maximal abelian subalgebra of $\mathfrak{k} \cap \mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{g}$; conversely every Cartan subalgebra of $\mathfrak{g}$, up to conjugacy, can be obtained by this construction from a cuspidal parabolic subalgebra. The cuspidal parabolics are distinguished among the general parabolics by the important role they play in the Plancherel formula for semisimple groups [4].

The finitely many distinct cuspidal parabolics with a given $\mathfrak{m} \oplus \mathfrak{a}$ are not all conjugate, in general. This phenomenon is a reflection of the fact that a suitably defined "Weyl group" is not necessarily transitive on the Weyl chambers, as is seen already in $\mathfrak{sl}(3, \mathbf{R})$. The point of this paper is to expose the structure of this "Weyl group" completely, showing, among other things, that it is a Weyl group in the traditional sense.

The Weyl group $W(\mathfrak{a})$ in question is defined in terms of any analytic group G having Lie algebra $\mathfrak{g}$. Let K be the analytic subgroup corresponding to $\mathfrak{k}$. Then $W(\mathfrak{a})$ is the quotient of the normalizer $N_K(\mathfrak{a})$ of $\mathfrak{a}$ in K by the centralizer $Z_K(\mathfrak{a})$ of $\mathfrak{a}$ in K.

The *roots* of $\mathfrak{a}$ are defined in the standard way: For each linear functional β on $\mathfrak{a}$, we let

$$\mathfrak{g}_\beta = \{X \in \mathfrak{g} \mid [H, X] = \beta(H)X \text{ for all } H \in \mathfrak{a}\}.$$

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The non-zero functionals β for which $\mathfrak{g}_\beta$ is not 0 are the roots. Then the negative of a root is a root, $\mathfrak{n}$ is the direct sum of the root spaces for the positive roots, $\theta\,\mathfrak{n}$ is the direct sum for the negative roots, and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \theta\,\mathfrak{n}$. The set of roots need not form a root system, as one can see in simple examples (e. g., [13], p. 71). Exceptf or the definition of *useful root*, which is deferred to Paragraph 4, we can now state our main theorem.

MAIN THEOREM. — *The useful roots of* $\mathfrak{a}$ *form a (possibly non-reduced) root system* Δ_0 *in a subspace of* $\mathfrak{a}$. *A reflection* p_β *of a root of* $\mathfrak{a}$ *is in* $W(\mathfrak{a})$ *if and only if* $t\,\beta$ *is useful for some* $t > 0$; *if and only if* β *itself is useful in case* $\mathfrak{g}$ *has no split* G_2 *factors. Moreover,* $W(\mathfrak{a})$ *coincides with the Weyl group of* Δ_0.

This result will be used in joint work, based on [9], with E. M. Stein. It is a key algebraic step in establishing a necessary and sufficient condition for the irreducibility of the representations occurring in the Plancherel formula of G.

The whole paper is built around the proof of the Main Theorem. In Paragraph 2 we derive and recall some results about the more familiar situation in which $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. In Paragraph 3, we introduce a notion of conjugation that connects this special case with the general case and allows us to define useful roots in Paragraph 4. The main ideas of the proof are contained in two reduction lemmas in Paragraph 5, and from there the proof splinters into several cases. The generic cases are handled in Paragraph 6, and the exceptional cases are handled in Paragraph 7.

2. Essential and inessential roots of $\mathfrak{a}_p$

Let $\mathfrak{a}_p$ be any maximal abelian subspace of $\mathfrak{p}$, and let $\mathfrak{m}_p = Z_{\mathfrak{k}}(\mathfrak{a}_p)$. Forming roots with respect to $\mathfrak{a}_p$, introducing an ordering, and letting $\mathfrak{n}_p$ be the sum of the root spaces for the positive roots, we obtain a minimal parabolic subalgebra $\mathfrak{m}_p \oplus \mathfrak{a}_p \oplus \mathfrak{n}_p$. It is well known that this subalgebra is cuspidal and that the roots of $\mathfrak{a}_p$ do form a root system, possibly non-reduced (twice a root may be a root).

The roots of $\mathfrak{a}_p$ have been studied extensively and lead, for example, to the classification of real semisimple Lie algebras. *See* [1] and [12], for example. In this section we shall obtain some limitations on the interactions among such roots. These limitations, contained in Propositions 5 and 7 below, seem to have a general usefulness in allowing one to pin-point the central problem quickly in various questions about roots. They are the starting point not only for this paper but also for the proofs of the results of [7], whose details will be given on a subsequent occasion.

For this section only, it will be useful to deal with a Cartan subalgebra of $\mathfrak{g}$ obtained from $\mathfrak{a}_p$. Let $\overline{X}$ be the conjugate of X in $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{g}$. Fix a maximal abelian subspace $\mathfrak{h}_0$ of $\mathfrak{m}_p$; then $\mathfrak{h} = \mathfrak{a}_p \oplus \mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}$. Form roots relative to $\mathfrak{h}$ (i. e., roots of $\mathfrak{g}^{\mathbf{C}}$ relative to $\mathfrak{h}^{\mathbf{C}}$); roots are real on $\mathfrak{a}_p$ and imaginary on $\mathfrak{h}_0$ and so belong to $\mathfrak{a}'_p + i\,\mathfrak{h}'_0$. The restrictions to $\mathfrak{a}_p$ of the roots of $\mathfrak{h}$ are (0 and) the roots of $\mathfrak{a}_p$. Frequently we shall decompose a root of $\mathfrak{h}$ as $\alpha = \alpha_R + \alpha_I$, where α_R is 0 or the root of $\mathfrak{a}_p$ (the projection on $\mathfrak{a}'_p$) and α_I is the projection on $i\,\mathfrak{h}'_0$. The inner product on $\mathfrak{a}' + i\,\mathfrak{h}'_0$ is denoted $\langle \cdot, \cdot \rangle$, and $\mathfrak{a}'$ and $i\mathfrak{h}'_0$ are orthogonal. A compatible ordering on the roots of $\mathfrak{h}$ is chosen so that $\mathfrak{a}_p$ comes before $i\mathfrak{h}_0$; then the restriction to $\mathfrak{a}_p$ of a positive α is $\geqq 0$.

We recall the following three ways of constructing roots of $\mathfrak{h}$ from other roots:

1. CONJUGATION. — Define $\bar{\alpha}(\mathrm{H}) = \overline{\alpha(\overline{\mathrm{H}})}$. If α is a root of $\mathfrak{h}$, then $\bar{\alpha}$ is a root and $\mathrm{X}_{\bar{\alpha}}$ can be taken as $\overline{\mathrm{X}}_\alpha$. If $\alpha = \alpha_\mathrm{R}+\alpha_\mathrm{I}$, then $\bar{\alpha} = \alpha_\mathrm{R}-\alpha_\mathrm{I}$.

2. CARTAN INVOLUTION. — Extend θ to be complex linear on $\mathfrak{g}^\mathbf{C}$. If α is a root of $\mathfrak{h}$, so is $\theta\alpha$; and $\mathrm{X}_{\theta\alpha}$ can be taken as $\mathrm{X}_{-\bar{\alpha}}$.

3. ROOT STRING. — If α and γ are roots of $\mathfrak{h}$, the α string containing γ is

$$\gamma - p\,\alpha, \ldots, \gamma+q\,\alpha \qquad \text{with} \quad p-q = 2\langle \gamma, \alpha\rangle/\langle \alpha, \alpha\rangle.$$

If $\langle \gamma, \alpha\rangle < 0$, then $\gamma+\alpha$ is a root. If $\gamma-\alpha$ is not a root, then $\gamma+\alpha$ is a root if and only if γ is not orthogonal to α.

LEMMA 1 (cf. [1], p. 4). — *Let $\alpha_\mathrm{R}\pm\alpha_\mathrm{I}$ be roots of $\mathfrak{h}$ with $\alpha_\mathrm{R} \neq 0$ and $\alpha_\mathrm{I} \neq 0$. Then $2\,\alpha_\mathrm{I}$ is not a root of $\mathfrak{h}$.*

Proof. — Let $\alpha = \alpha_\mathrm{R}+\alpha_\mathrm{I}$. If $2\,\alpha_\mathrm{I}$ is a root, then

$$2\,\alpha_\mathrm{I} = (\alpha_\mathrm{R}+\alpha_\mathrm{I})-(\alpha_\mathrm{R}-\alpha_\mathrm{I}) = \alpha+\theta\alpha$$

says $[\mathrm{X}_\alpha, \theta\,\mathrm{X}_\alpha] \neq 0$ and is a vector of $\mathfrak{m}_p^\mathbf{C}$. On the other hand,

$$\theta[\mathrm{X}, \theta\,\mathrm{X}] = [\theta\,\mathrm{X}, \mathrm{X}] = -[\mathrm{X}, \theta\,\mathrm{X}],$$

so that $[\mathrm{X}_\alpha, \theta\,\mathrm{X}_\alpha]$ is in $\mathfrak{p}^\mathbf{C} \cap \mathfrak{m}^\mathbf{C} = 0$.

LEMMA 2 (cf. [1], p. 9-10). — *If $\alpha_\mathrm{R}\pm\alpha_\mathrm{I}$ are roots of $\mathfrak{h}$ with $\alpha_\mathrm{R} \neq 0$, $\alpha_\mathrm{I} \neq 0$, and $2\,\alpha_\mathrm{R}$ not a root of $\mathfrak{a}_p$, then $\langle \alpha_\mathrm{R}+\alpha_\mathrm{I}, \alpha_\mathrm{R}-\alpha_\mathrm{I}\rangle = 0$ and so $|\alpha_\mathrm{R}|^2 = |\alpha_\mathrm{I}|^2$.*

Proof. — Combine root construction (3) with Lemma 1.

LEMMA 3. — *If $\alpha_\mathrm{R}\pm\alpha_\mathrm{I}$ are roots of $\mathfrak{h}$ with $\alpha_\mathrm{R} \neq 0$ and if $2\,\alpha_\mathrm{R}$ is a root of $\mathfrak{a}_p$, then $2\,\alpha_\mathrm{R}$ is a root of $\mathfrak{h}$ when extended by 0 on $\mathfrak{h}_0$.*

Proof. — This follows from Propositions 2.2 and 2.4 of [1], or it can be proved directly from Lemma 2.

We say that a root α_0 of $\mathfrak{a}_p$ is *essential* if neither α_0 nor $2\,\alpha_0$ is a root of $\mathfrak{h}$. Otherwise α_0 is *inessential*. *See* p. 266 of [7] for the etymology of these terms. From root construction 1 and Lemma 3, α_0 is essential if and only if the root space $\mathfrak{g}_{\alpha_0}$ is even-dimensional and $2\,\alpha_0$ is not a root of $\mathfrak{a}_p$. [From the classification, one then sees that α_0 is essential if and only if the real-rank-one subalgebra $\mathfrak{g}^{(\alpha_0)}$ generated by $\mathfrak{g}_{\alpha_0}$ and $\mathfrak{g}_{-\alpha_0}$ is isomorphic with $\mathfrak{so}(2\,k+1, 1)$.] If w is in $\mathrm{W}(\mathfrak{a}_p)$, then w exhibits $\mathfrak{g}^{(\alpha_0)}$ as conjugate to $\mathfrak{g}^{(w\alpha_0)}$, and it follows that $w\,\alpha_0$ and α_0 are both essential or both inessential.

The "cuspidal" hypothesis will enter our considerations through the following lemma (applied to $\mathfrak{m}$ instead of $\mathfrak{g}$), which is a sharp form of a result that is widely known ([11], Prop. 11). For convenience we shall state it for $\mathfrak{g}$ reductive, and we adopt the convention that the center of $\mathfrak{g}$ is incorporated into $\mathfrak{k}$ in the Cartan decomposition.

LEMMA 4. — *Let* $\mathfrak{g}$ *be reductive, and let* G *be an analytic group with Lie algebra* $\mathfrak{g}$. *If the rank of* $\mathfrak{k}$ *equals the rank of* $\mathfrak{g}$, *then there exists an element* w *in* K *such that* $\operatorname{Ad}(w)$ *is* $+1$ *on* $\mathfrak{k}$ *and* -1 *on* $\mathfrak{p}$. *This element* w *exhibits* -1 *as an element of the Weyl group* $W(\mathfrak{a}_p)$, *and in fact* -1 *is the product of commuting reflections relative to inessential roots of* $\mathfrak{a}_p$.

Proof. — The assumption is that there is a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$. Roots of $\mathfrak{b}$ are compact or non-compact according as the root space is in $\mathfrak{k}^{\mathbf{C}}$ or $\mathfrak{p}^{\mathbf{C}}$. If $i\pi H$ is in $\mathfrak{b}$ and X_α is in $\mathfrak{g}_\alpha$, then

$$\operatorname{Ad}(\exp i\pi H) X_\alpha = e^{\operatorname{ad} i\pi H} X_\alpha = e^{i\pi\alpha(H)} X_\alpha.$$

Let $\alpha_1, \ldots, \alpha_l$ be the simple roots of $\mathfrak{b}$, and define H_0 by the condition

$$\alpha_j(H_0) = \begin{cases} 1 & \text{if } \alpha_j \text{ non-compact,} \\ 0 & \text{if } \alpha_j \text{ compact,} \end{cases}$$

taking the component of H_0 in the center of $\mathfrak{g}$ as 0. Then

$$\alpha(H_0) \equiv \begin{cases} 1 \bmod 2 & \text{if } \alpha \text{ non-compact,} \\ 0 \bmod 2 & \text{if } \alpha \text{ compact,} \end{cases}$$

and $w = \exp i\pi H$ has the property

$$\operatorname{Ad}(w) X_\alpha = \begin{cases} -X_\alpha & \text{if } \alpha \text{ non-compact,} \\ X_\alpha & \text{if } \alpha \text{ compact.} \end{cases}$$

Moreover $\operatorname{Ad}(w)$ is the identity on $\mathfrak{b}$. Hence $\operatorname{Ad}(w)$ is $+1$ on $\mathfrak{k}$ and -1 on $\mathfrak{p}$. Since w is in K and $\mathfrak{a}_p$ is contained in $\mathfrak{p}$, w exhibits -1 as in $W(\mathfrak{a}_p)$. Also $\mathfrak{m}_p$ is contained in $\mathfrak{k}$, and therefore w exhibits the existence of a member of $W(\mathfrak{h})$ that is -1 on $\mathfrak{a}_p$ and $+1$ on $\mathfrak{h}_0$. Applying Lemma 63 of [8], we see that this member of $W(\mathfrak{h})$ is the commuting product of reflections relative to roots of $\mathfrak{h}$ that vanish on $\mathfrak{h}_0$. The restrictions of these roots to $\mathfrak{a}_p$ provide the required inessential roots of $\mathfrak{a}_p$ to complete the proof.

Returning to the case that $\mathfrak{g}$ is semi-simple, again let G be a connected group with Lie algebra $\mathfrak{g}$. Define $M_p = Z_K(\mathfrak{a}_p)$. The first proposition has been known by case-by-case inspection for some time, but a direct proof has never been given.

PROPOSITION 5. — *If* $\mathfrak{g}$ *is simple and* $\dim \mathfrak{a}_p = 1$, *then* M_p *is connected unless* $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbf{R})$.

Proof. — Let A_p, N_p, θN_p be the analytic subgroups of G with Lie algebras $\mathfrak{a}_p$, $\mathfrak{n}_p$, $\theta\mathfrak{n}_p$. The map that takes v in θN_p into the coset $\varkappa(v) M$ of K/M, where $\varkappa(v)$ is the K component of the Iwasawa decomposition $G = KA_pN_p$, is known to be a homeomorphism onto an open dense set. Since $\dim \mathfrak{a}_p = 1$, the Bruhat decomposition shows that the image of θN_p is all but one point of K/M_p. Therefore K/M_p is the one-point compactification of a Euclidean space and is a sphere. Since K is connected, M_p cannot be disconnected unless $\pi_1(K/M_p) \neq \{1\}$, i. e., unless the sphere K/M_p is one-dimensional. In this case $\mathfrak{g}$ is isomorphic with $\mathfrak{sl}(2, \mathbf{R})$.

Now suppose that the group G has a faithful matrix representation, so that its complexification $G^{\mathbf{C}}$ is well defined. To each root α_0 of $\mathfrak{a}_p$ we associate the corresponding member H_{α_0} of $\mathfrak{a}_p$ by means of the inner product on $\mathfrak{a}_p$. Let γ_{α_0} be the element of $G^{\mathbf{C}}$ defined by

$$\gamma_{\alpha_0} = \exp(2\pi i \langle \alpha_0, \alpha_0 \rangle^{-1} H_{\alpha_0}),$$

and let $\mathfrak{g}^{(\alpha_0)}$ be the real-rank-one simple Lie algebra generated by the root spaces $\mathfrak{g}_{\alpha_0}$ and $\mathfrak{g}_{-\alpha_0}$. The next lemma gives some properties of the elements γ_{α_0} that limit the possibilities for the algebras $\mathfrak{g}^{(\alpha_0)}$. *See* also [3] (p. 121) and [8] (p. 549).

LEMMA 6. — *(a) Each γ_{α_0} is in the center of M_p and satisfies $\gamma_{\alpha_0}^2 = 1$. Also γ_{α_0} is in M_0, the identity component of M_p, unless $\mathfrak{g}^{(\alpha_0)} \cong \mathfrak{sl}(2, \mathbf{R})$.*

(b) $M_p = M_0 F$, where F is the finite abelian group generated by the γ_{α_0}. Also M_p is connected unless $\mathfrak{g}_{\alpha_0} \cong \mathfrak{sl}(2, \mathbf{R})$ for some α_0 such that $\alpha_0/2$ is not a root of $\mathfrak{a}_p$.

(c) Let α_0 and α_1 be roots of $\mathfrak{a}_p$, and let p_{α_1} be the root reflection for α_1 in $W(\mathfrak{a}_p)$. If w is any representative of p_{α_1} in K, then $w\gamma_{\alpha_0} w^{-1} = \gamma_{\alpha_0}\gamma_{\alpha_1}^l$, where $l = 2\langle \alpha_0, \alpha_1 \rangle / \langle \alpha_0, \alpha_0 \rangle$.

(d) If α_0 is inessential and $2\alpha_0$ is not a root of $\mathfrak{a}_p$ and $G^{\mathbf{C}}$ is simply-connected, then $\gamma_{\alpha_0} \neq 1$.

Proof. — The first halves of (*a*) and (*b*) are well known. [*See* [10] (p. 93) for (*b*).] The second half of (*a*) follows from Proposition 5, and the second half of (*b*) follows from the second half of (*a*) since then $F \subseteq M_0$. For (*c*) we have

$$w\gamma_{\alpha_0} w^{-1} = \gamma_{p_{\alpha_1}\alpha_0},$$

and (*c*) follows by direct calculation from the definition since $\gamma_{\alpha_1}^2 = 1$. In (*d*) the assumption is that α_0 is a root when extended to be 0 on $\mathfrak{h}_0$. Changing the ordering, we may assume that α_0 is a simple root of $\mathfrak{h}$ in the new ordering, say $\alpha_0 = \alpha_i$. Let Λ_i be the basic dominant weight with $2\langle \Lambda_i, \alpha_j \rangle / \langle \alpha_j, \alpha_j \rangle = \delta_{ij}$. Since $G^{\mathbf{C}}$ is simply-connected, Λ_i is integral. Form the associated finite-dimensional representation of $G^{\mathbf{C}}$. Its value on

$$\gamma_{\alpha_i} = \exp(2\pi i \langle \alpha_i, \alpha_i \rangle^{-1} H_{\alpha_i})$$

is

$$\exp\left(\frac{2\pi i \langle \Lambda_i, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\right) = \exp \pi i = -1.$$

So $\gamma_{\alpha_i} \neq 1$.

In the next proposition, the Dynkin diagram means the Dynkin diagram of the simple roots of $\mathfrak{a}_p$. Parts of the proposition are due to Araki [1] (p. 10-11).

PROPOSITION 7. — *Let α_1 and α_2 be distinct simple roots of $\mathfrak{a}_p$.*

(a) If α_1 and α_2 are connected by a single line in the Dynkin diagram, then $\mathfrak{g}^{(\alpha_1)}$ and $\mathfrak{g}^{(\alpha_2)}$ are isomorphic. Moreover, either both α_1 and α_2 are essential or else $\mathfrak{g}^{(\alpha_1)} \cong \mathfrak{g}^{(\alpha_2)} \cong \mathfrak{sl}(2, \mathbf{R})$.

(b) If $\langle \alpha_1, \alpha_2 \rangle \neq 0$ and $2\alpha_2$ is a root of $\mathfrak{a}_p$, then α_1 is essential.

(c) If α_1 is essential and α_2 is inessential and if $2\alpha_2$ is not a root of $\mathfrak{a}_p$, then the integer $2\langle \alpha_1, \alpha_2 \rangle / \langle \alpha_1, \alpha_1 \rangle$ is even.

(*d*) *If* $\mathfrak{g}$ *is simple and if* $\mathfrak{g}_{\alpha_0} \cong \mathfrak{sl}(2, \mathbf{R})$ *for every simple root of* $\mathfrak{a}_p$, *then* $\mathfrak{g}$ *is split over* $\mathbf{R}$, *i. e.,* $\mathfrak{a}_p$ *is a Cartan subalgebra.*

(*e*) (Wallach) *If every root of* $\mathfrak{a}_p$ *is essential, then* $\mathfrak{g}$ *has just one conjugacy class of Cartan subalgebras. Consequently if* $\mathfrak{a}_p \neq 0$, *then* rank $\mathfrak{k} \neq$ rank $\mathfrak{g}$.

(*f*) *If* $\mathfrak{g}$ *has* G_2 *as Dynkin diagram, then* $\mathfrak{g}$ *is split over* $\mathbf{R}$ *or else every root of* $\mathfrak{a}_p$ *is essential.*

Proof. – In (*a*), $p_{\alpha_1} p_{\alpha_2}(\alpha_1) = \alpha_2$. If w is a representative of $p_{\alpha_1} p_{\alpha_2}$ in K, then $\mathfrak{g}^{(\alpha_2)} = \mathrm{Ad}(w)\, \mathfrak{g}^{(\alpha_1)}$ exhibits the isomorphism. For the second statement we may therefore assume α_1 and α_2 are both inessential. Since they are connected by a single line, neither $2\alpha_1$ nor $2\alpha_2$ is a root of $\mathfrak{a}_p$. By Lemma 6 *d*, we may assume that $\gamma_{\alpha_1} \neq 1$ and $\gamma_{\alpha_2} \neq 1$. Now α_1, being inessential, is a root of $\mathfrak{h}$, and we let X_{α_1} and $X_{-\alpha_1}$ lie in the root spaces relative to $\mathfrak{h}$ of α_1 and $-\alpha_1$. Then $\{H_{\alpha_1}, X_{\alpha_1}, X_{-\alpha_1}\}$ spans a subalgebra of $\mathfrak{g}^{\mathbf{C}}$ isomorphic to $\mathfrak{sl}(2, \mathbf{C})$. Since $\bar{\alpha}_1 = \alpha_1$, H_{α_1} is in $\mathfrak{a}_p$ and $\mathbf{C} X_{\alpha_1}$ and $\mathbf{C} X_{-\alpha_1}$ are closed under conjugation. Thus the intersection of this subalgebra with $\mathfrak{g}$ is isomorphic with $\mathfrak{sl}(2, \mathbf{R})$. Find the corresponding homomorphism of SL (2, $\mathbf{R}$) into G, and let s be the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then s is an element of K such that $\mathrm{Ad}(s) = -1$ on $\mathbf{R} H_\alpha$ and $\mathrm{Ad}(s) = +1$ on the orthogonal complement in $\mathfrak{h}$. By Lemma 6 *c*, $s \gamma_{\alpha_2} s^{-1} = \gamma_{\alpha_2} \gamma_{\alpha_1}$. Choose an irreducible representation σ of the compact group M_p that is $\neq 1$ on γ_{α_1}. Then

$$s^{-1} \sigma(\gamma_{\alpha_2}) = \sigma(s \gamma_{\alpha_2} s^{-1}) = \sigma(\gamma_{\alpha_2} \gamma_{\alpha_1}) = -\sigma(\gamma_{\alpha_2}).$$

On $\mathfrak{h}_0$, however,

$$s^{-1} \sigma(H) = \sigma(\mathrm{Ad}(s) H) = \sigma(H).$$

Thus γ_{α_2} is not in $\exp \mathfrak{h}_0$, which contains the center of M_0. Then γ_{α_2} cannot be in M_0, and Lemma 6 *a* shows that $\mathfrak{g}^{(\alpha_2)} \cong \mathfrak{sl}(2, \mathbf{R})$. Since $\mathfrak{g}^{(\alpha_1)} \cong \mathfrak{g}^{(\alpha_2)}$, $\mathfrak{g}^{(\alpha_1)} \cong \mathfrak{sl}(2, \mathbf{R})$.

For (*b*) we run through the above argument again. We may assume α_1 is inessential. Since $2\alpha_2$ is a root of $\mathfrak{a}_p$, $2\alpha_1$ cannot be. Thus Lemma 6 *d* shows we may assume $\gamma_{\alpha_1} \neq 1$, and the argument applies. The conclusion is that $\mathfrak{g}^{(\alpha_2)} \cong \mathfrak{sl}(2, \mathbf{R})$, contradicting the fact that $2\alpha_2$ is a root of $\mathfrak{a}_p$. Hence α_1 must be essential.

For (*c*), α_2 is a root of $\mathfrak{h}$ and so is some $\alpha_1 + \alpha_I$. By Lemma 2, $|\alpha_1 + \alpha_I|^2 = 2|\alpha_1|^2$. Therefore

$$\frac{1}{2}\left(\frac{2\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_1, \alpha_1 \rangle}\right) = \frac{2\langle \alpha_1 + \alpha_I, \alpha_2 \rangle}{\langle \alpha_1 + \alpha_I, \alpha_1 + \alpha_I \rangle}$$

is an integer.

For (*d*), the assumption $\mathfrak{g}^{(\alpha_0)} \cong \mathfrak{sl}(2, \mathbf{R})$ for every simple root α_0 of $\mathfrak{a}_p$ says that every simple root of $\mathfrak{h}$ with a non-zero restriction to $\mathfrak{a}_p$ vanishes on $i\,\mathfrak{h}_0$. Thus every root of $\mathfrak{h}$ vanishes on all of $\mathfrak{a}_p$ or all of $i\,\mathfrak{h}_0$. Since $\mathfrak{g}$ is simple and the roots span $\mathfrak{a}_p + i\,\mathfrak{h}_0$, we conclude that $\mathfrak{h}_0 = 0$ and $\mathfrak{a}_p$ is a Cartan subalgebra.

For (*e*), let $\mathfrak{h}_1$ be a Cartan subalgebra not conjugate to $\mathfrak{h}$. We may assume that $\mathfrak{h}_1$ is θ-stable and that $\mathfrak{a}_1 = \mathfrak{p} \cap \mathfrak{h}$ is contained in $\mathfrak{a}_p$ with positive codimension. Let $\mathfrak{b}_1$ be the orthogonal complement (with respect to B_θ) of $\mathfrak{a}_1$ in $\mathfrak{h}_1$, and let $\mathfrak{a}_0$ be the orthogonal complement of $\mathfrak{a}_1$ in $\mathfrak{a}_p$. Denoting by α a typical root of $\mathfrak{a}_p$, we see that the centralizer of $\mathfrak{a}_1$ is

$$\mathfrak{a}_p + \mathfrak{m}_p + \sum_{\alpha \mid \mathfrak{a}_1 \equiv 0} \mathfrak{g}_\alpha.$$

Then $\mathfrak{b}_1$ and $\mathfrak{a}_0 + \mathfrak{h}_0$ are Cartan subalgebras of the reductive subalgebra

$$\mathfrak{a}_0 + \mathfrak{m}_p + \sum_{\alpha \mid \mathfrak{a}_1 \equiv 0} \mathfrak{g}_\alpha.$$

We shall apply Lemma 4 to this subalgebra. We can do so since $\mathfrak{b}_1$ is contained in the $\mathfrak{k}$-part of the algebra and $\mathfrak{a}_0$ is a maximal abelian subspace of the $\mathfrak{p}$-part of the algebra. The lemma shows that there exists an inessential root of $\mathfrak{a}_0$ for the subalgebra. That is, some α whose restriction to $\mathfrak{a}_1$ is $\equiv 0$ is also 0 on $\mathfrak{h}_0$. Then α is an inessential root of $\mathfrak{a}_p$.

Finally for (f), let α_1 and α_2 be the simple roots of $\mathfrak{a}_p$. Results (c) and (d) show that the only possibility not covered by (f) is that both α_1 and α_2 are inessential and $\mathfrak{g}^{(\alpha_1)}$ or $\mathfrak{g}^{(\alpha_2)}$ is different from $\mathfrak{sl}(2, \mathbf{R})$. Once again we run through the argument of (a) and we find that this possibility is ruled out.

3. Properties of conjugation

Let $\mathfrak{g}$ be semi-simple, and let $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Langlands decomposition of a cuspidal parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{a}_M$ be a maximal abelian subspace of $\mathfrak{m} \cap \mathfrak{p}$, and let $\mathfrak{a}_p = \mathfrak{a} \oplus \mathfrak{a}_M$. Then $\mathfrak{a}_p$ is a maximal abelian subspace of $\mathfrak{p}$, and the theory of paragraph 2 applies. The roots of $\mathfrak{a}$ are the non-zero restrictions to $\mathfrak{a}$ of the roots of $\mathfrak{a}_p$, and the roots of $(\mathfrak{m}, \mathfrak{a}_M)$ are the roots of $(\mathfrak{g}, \mathfrak{a}_p)$ that vanish on $\mathfrak{a}$. Introduce a compatible ordering on the roots of $\mathfrak{a}_p$ so that $\mathfrak{a}$ comes before $\mathfrak{a}_M$. Then the restriction to $\mathfrak{a}$ of a positive α is $\geqq 0$, and a simple root of $(\mathfrak{g}, \mathfrak{a}_p)$ that vanishes on $\mathfrak{a}$ is a simple root of $(\mathfrak{m}, \mathfrak{a}_M)$, and conversely.

We define a conjugation on linear functionals on $\mathfrak{a}_p$, denoted by bar, as $+1$ on the $\mathfrak{a}$ part and -1 on the $\mathfrak{a}_M$ part. This will be the only notion of conjugation used in the rest of the paper; the different one used in Paragraph 2 will not reappear. If α is a linear functional on $\mathfrak{a}_p$, let $\alpha = \alpha_R + \alpha_I$ be its decomposition relative to $\mathfrak{a}_p = \mathfrak{a} \oplus \mathfrak{a}_M$.

The next lemma is an unpublished result of Harish-Chandra. A result with an analogous proof appears on page 117 of [13].

LEMMA 8. — *Every element of* $N_K(\mathfrak{a})$ *decomposes as the product* zn, *where* n *is in* $N_K(\mathfrak{a}_p)$ *and* z *is in* $Z_K(\mathfrak{a})$. *Consequently every element of* $W(\mathfrak{a})$ *can be extended to an element of* $W(\mathfrak{a}_p)$. *The extension in* $W(\mathfrak{a}_p)$ *normalizes* $\mathfrak{a}$ *and* $\mathfrak{a}_M$ *and can be chosen so as to preserve the positive roots of* $(\mathfrak{m}, \mathfrak{a}_M)$; *in this case the extension is unique. Conversely an element of* $W(\mathfrak{a}_p)$ *is an extension of an element of* $W(\mathfrak{a})$ *if and only if it normalizes* $\mathfrak{a}$ *and* $\mathfrak{a}_M$.

Proof. — Let x be in $N_K(\mathfrak{a})$ and consider $\mathrm{Ad}(x)\,\mathfrak{a}_M$. Since x is in K and $\mathfrak{a}_M$ is in $\mathfrak{p}$, $\mathrm{Ad}(x)\,\mathfrak{a}_M$ is in $\mathfrak{p}$. On the other hand, let X be in $\mathfrak{m} \subseteq Z_{\mathfrak{g}}(\mathfrak{a})$. Then for $H \in \mathfrak{a}$ we have

$$[\mathrm{Ad}(x)X, H] = \mathrm{Ad}(x)[X, \mathrm{Ad}(x^{-1})H] \subseteq \mathrm{Ad}(x)[X, \mathfrak{a}] = 0,$$

and $\mathrm{Ad}(x)X$ is in $Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a}$. Since $\mathrm{Ad}(x)$ acts orthogonally and normalizes $\mathfrak{a}$, $\mathrm{Ad}(x)X$ is in $\mathfrak{m}$. Thus $\mathrm{Ad}(x)\,\mathfrak{a}_M$ is in $\mathfrak{m} \cap \mathfrak{p}$ and is a maximal abelian subspace of $\mathfrak{m} \cap \mathfrak{p}$. Hence we can find z in $K \cap M_e$ (where M_e is the analytic subgroup corresponding to $\mathfrak{m}$) such that $\mathrm{Ad}(z^{-1})\,\mathrm{Ad}(x)\,\mathfrak{a}_M = \mathfrak{a}_M$. If $x = zn$, then z is in $M_e \subseteq Z(\mathfrak{a})$ and $\mathrm{Ad}(n)\,\mathfrak{a}_M = \mathfrak{a}_M$. Since x and z normalize $\mathfrak{a}$, so does n. Thus $x = zn$ is the required decomposition. By composing with an element of $W(\mathfrak{m}, \mathfrak{a}_M)$, we can assume that the positive roots of $(\mathfrak{m}, \mathfrak{a}_M)$ are preserved by the composition, and then the extension is certainly unique. The converse is obvious.

Now we use the "cuspidal" hypothesis. The $\mathfrak{a}_M$ Dynkin diagram of $\mathfrak{m}$ is a subset of the $\mathfrak{a}_p$ Dynkin diagram of $\mathfrak{g}$.

LEMMA 9. — *The parabolic subalgebra being cuspidal, the following statements are true:*

(*a*) *The roots of* $\mathfrak{a}_p$ *are closed under conjugation.*

(*b*) *The* $\mathfrak{a}_M$ *Dynkin diagram of* $\mathfrak{m}$ *has no simple component of type* A_n *for some* $n > 1$ *or of type* D_{2n+1} *for some* $n > 1$ *or of type* E_6.

Proof. — For (*a*) let M_e be the analytic subgroup corresponding to $\mathfrak{m}$. Since $\mathrm{rank}(\mathfrak{k} \cap \mathfrak{m}) = \mathrm{rank}\,\mathfrak{m}$, we can apply Lemma 4 to $\mathfrak{m}$. Then there exists an element w in $K \cap M$ so that $\mathrm{Ad}(w)$ is -1 on $\mathfrak{a}_M$. Since M centralizes $\mathfrak{a}$, $\mathrm{Ad}(w)$ is $+1$ on $\mathfrak{a}$. Therefore conjugation is implemented by $\mathrm{Ad}(w)$, and the roots of $\mathfrak{a}_p$ are preserved. For (*b*) A_n, D_{2n+1}, and E_6 are all single-line diagrams. Proposition 7, parts (*a*, *d*, *e*), says the corresponding simple component of $\mathfrak{m}$ is split over **R**. In view of Lemma 4, the lemma follows from the observation that -1 is not in the Weyl group for a complex simple Lie algebra of types A_n $(n > 1)$, D_{2n+1} $(n > 1)$, or E_6. (Lemma 63 of [8] is handy in the verification for E_6.)

4. Useful roots of $\mathfrak{a}$

In the notation of Paragraph 3, let α be a root of $\mathfrak{a}_p$, so that $\bar{\alpha}$ is again a root of $\mathfrak{a}_p$. Clearly $|\alpha| = |\bar{\alpha}|$, and thus there are the following possibilities:

(i) $2\langle \alpha, \bar{\alpha} \rangle / |\alpha|^2 = -2$, in which case $\alpha = -\bar{\alpha}$ and hence α is a root of $(\mathfrak{m}, \mathfrak{a}_M)$.

(ii) $2\langle \alpha, \bar{\alpha} \rangle / |\alpha|^2 = -1$, in which case $\alpha+\bar{\alpha}$ is a root of $\mathfrak{a}_p$ that is its own conjugate (and so vanishes on $\mathfrak{a}_M$).

(iii) $2\langle \alpha, \bar{\alpha} \rangle / |\alpha|^2 = 0$, in which case α is orthogonal to $\bar{\alpha}$.

(iv) $2\langle \alpha, \bar{\alpha} \rangle / |\alpha|^2 = 2$, in which case $\alpha = \bar{\alpha}$ and α vanishes on $\mathfrak{a}_M$.

(v) $2\langle \alpha, \bar{\alpha} \rangle / |\alpha|^2 = 1$, in which case $\alpha-\bar{\alpha} = \delta_\alpha$ is a root of $(\mathfrak{m}, \mathfrak{a}_M)$.

In cases (i)-(iv) we say that α is *useful*. A root of $\mathfrak{a}$ is *useful* if it is the restriction to $\mathfrak{a}$ of some useful root of $\mathfrak{a}_p$.

The prototype for case (v) is $\mathfrak{g} = \mathfrak{sl}(3, \mathbf{R})$ with $\mathfrak{a}$ one-dimensional and $\mathfrak{m} = \mathfrak{sl}(2, \mathbf{R})$. The six $\mathfrak{a}_p$ roots are $\pm\alpha$, $\pm\bar{\alpha}$ and $\pm\delta_\alpha$. The Weyl group $W(\mathfrak{a}_p)$ is the obvious 6-element group, and Lemma 8 shows that $W(\mathfrak{a})$ is trivial. Hence although the $\mathfrak{a}$-roots form a one-dimensional root system $\{\beta, -\beta\}$, $W(\mathfrak{a})$ falls short of being the Weyl group of a one-dimensional root system.

PROPOSITION 10. — (*a*) *If* α *is a root of* $\mathfrak{a}_p$ *that is not useful, then* δ_α *defines a simple ideal of* $\mathfrak{m}$ *of type* A_1 (*relative to its* $\mathfrak{a}_M$ *roots*), *and one of* $\pm\delta_\alpha$ *is a simple root of* $\mathfrak{a}_p$.

(*b*) *If* $\mathfrak{g}$ *has no ideal of type* G_2 (*relative to its* $\mathfrak{a}_p$ *roots*) *and if* $\alpha = \alpha_R + \alpha_I$ *is a useful root of* $\mathfrak{a}_p$, *then every root of* $\mathfrak{a}_p$ *whose restriction to* $\mathfrak{a}$ *is* α_R *is useful.*

(*c*) *If* $\mathfrak{g}$ *has no ideal of type* G_2 (*relative to its* $\mathfrak{a}_p$ *roots*) *and if* $\alpha = \alpha_R + \alpha_I$ *is a root of* $\mathfrak{a}_p$ *not useful, then the only multiples of* α_R *that are roots of* $\mathfrak{a}$ *are* $\pm\alpha_R$.

(*d*) *If* $\mathfrak{g}$ *is simple and* $\dim \mathfrak{a} = 1$, *then either every root of* $\mathfrak{a}_p$ *is useful or* $\mathfrak{g}$ *is of type* A_2 *or* G_2 (*relative to its* $\mathfrak{a}_p$ *roots*).

Proof. — Suppose that $\alpha = \alpha_R + \delta_\alpha/2$ is not useful. We begin by proving (*b*) and

$$(a') \qquad |\alpha|^2 = |\delta_\alpha|^2 = \frac{4|\alpha_R|^2}{3}.$$

For the proof of (*a'*), we have

$$\frac{2\langle \alpha, \bar{\alpha}\rangle}{|\alpha|^2} = 1$$

or

$$2\left(|\alpha_R|^2 - \frac{|\delta_\alpha|^2}{4}\right) = |\alpha_R|^2 + \frac{|\delta_\alpha|^2}{4},$$

whence

$$|\alpha_R|^2 = \frac{3|\delta_\alpha|^2}{4}$$

or

$$|\alpha|^2 = |\delta_\alpha|^2 = \frac{4|\alpha_R|^2}{3}$$

as asserted. For (*b*) let $\beta = \alpha_R + \gamma$ be a root of $\mathfrak{a}_p$. Then

$$|\beta|^2 = |\alpha_R|^2 + |\gamma|^2 = \frac{3|\alpha|^2}{4} + |\gamma|^2$$

by (*a'*). The assumption excluding G_2 implies that $|\beta|^2 = c|\alpha|^2$ with $c = 1/4$, $1/2$, $1, 2$, or 4. But the condition $|\gamma|^2 \geqq 0$ rules out $c = 1/4$ and $c = 1/2$. Then $|\gamma|^2 = |\alpha|^2/4$, $5|\alpha|^2/4$, or $13|\alpha|^2/4$ in the three cases $c = 1$, 2, or 4. To see that β is not useful, we compute

$$\frac{2\langle \beta, \bar{\beta}\rangle}{|\beta|^2} = \frac{2(|\alpha_R|^2 - |\gamma|^2)}{|\alpha_R|^2 + |\gamma|^2} = \frac{2((3/4)|\alpha|^2 - |\gamma|^2)}{(3/4)|\alpha|^2 + |\gamma|^2}$$

in the three cases and find that it is 1, $-1/2$, or $-5/4$. The last two are not integers, and only the first remains. Thus β is not useful, and (*b*) is proved.

For (*a*), we may clearly assume that $\mathfrak{g}$ has no factor of type G_2. If γ is a root $(\mathfrak{m}, \mathfrak{a}_M)$ not $\pm\delta_\alpha$, we are to show that γ is orthogonal to δ_α. If γ is not orthogonal, we may assume there is a root $\beta \neq \alpha$ or $\bar{\alpha}$, say $\beta = \alpha+\gamma$ without loss of generality, with $\mathfrak{a}$-component α_R. Then (*b*) shows that β is not useful and (*a'*) shows that $|\beta| = |\alpha|$. Then

$$\frac{2\langle\alpha_R, \beta_R\rangle}{|\alpha|^2} = \frac{(1/2)\langle\alpha+\bar{\alpha}, \beta+\bar{\beta}\rangle}{|\alpha|^2} = \frac{\langle\alpha, \beta+\bar{\beta}\rangle}{|\alpha|^2} = \frac{1}{2}\left(\frac{2\langle\alpha, \beta\rangle}{|\alpha|^2} + \frac{2\langle\alpha, \bar{\beta}\rangle}{|\alpha|^2}\right).$$

Here α is not $\pm\beta$ or $\pm\bar{\beta}$ and so each term in the parentheses on the right is 0, $+1$, or -1. Therefore

$$\frac{2\langle\alpha_R, \beta_R\rangle}{|\alpha|^2} = 1, \quad \frac{1}{2}, \quad 0, \quad -\frac{1}{2}, \quad \text{or} \quad -1.$$

However, (*a'*) shows that

$$\frac{2\langle\alpha_R, \beta_R\rangle}{|\alpha|^2} = \frac{2|\alpha_R|^2}{|\alpha|^2} = 2\frac{3}{4} = \frac{3}{2},$$

which is not in the list. Hence we have a contradiction, and we conclude β is $\bar{\alpha}$ and γ is $-\delta_\alpha$.

In (*c*), let $\alpha = \alpha_R+\gamma$ be not useful and let β be a root of $\mathfrak{a}_p$ whose $\mathfrak{a}$-component is a non-zero multiple of α_R. We may assume that β is of type (iii), (iv), or (v) in the list at the start of this section, since existence of roots of type (ii) implies existence of roots of type (iv).

If β is of type (iii), then β is orthogonal to $\bar{\beta}$. We may assume that $\beta = c\,\alpha_R+\gamma'$ with $c > 0$. Then (*a'*) implies

$$|\beta|^2 = 2c^2|\alpha_R|^2 = \frac{3}{2}c^2|\alpha|^2.$$

Now $|\beta|^2 = c'|\alpha|^2$ with $c' = 1/4$, $1/2$, 1, 2, or 4, and thus

$$c^2 = \frac{1}{6}, \quad \frac{1}{3}, \quad \frac{2}{3}, \quad \frac{4}{3}, \quad \text{or} \quad \frac{8}{3}.$$

But

$$\frac{2\langle\alpha+\bar{\alpha}, \beta\rangle}{|\alpha|^2} = \frac{2\langle 2\alpha_R, c\,\alpha_R+\gamma'\rangle}{4|\alpha_R|^2/3} = 3c$$

must be an integer, and we have a contradiction. Thus no such β exists.

If β is of type (iv), we may assume that $\beta = c\,\alpha_R$ with $c > 0$. Then

$$\frac{2\langle\beta, \alpha\rangle}{|\alpha|^2} = \frac{2\langle c\,\alpha_R, \alpha_R+\gamma\rangle}{4|\alpha_R|^2/3} = \frac{3c}{2}$$

and

$$\frac{2\langle \beta, \alpha \rangle}{|\beta|^2} = \frac{2c\langle \alpha_{\mathrm{R}}, \alpha_{\mathrm{R}}+\gamma \rangle}{c^2|\alpha_{\mathrm{R}}|^2} = \frac{2}{c},$$

and their product is 3. Since $\mathfrak{g}$ is not of type G_2, no such β exists.

Finally if β is of type (v), then we may assume β is not useful and $\beta = c\,\alpha_{\mathrm{R}}+\gamma'$ with $c > 0$. Then $|\beta|^2 = c^2|\alpha|^2$ by (a'), and $c^2 = 1/4,\ 1/2,\ 1,\ 2$, or 4. Also

$$\frac{2\langle \alpha+\bar{\alpha}, \beta \rangle}{|\alpha|^2} = \frac{2\langle 2\alpha_{\mathrm{R}}, c\,\alpha_{\mathrm{R}}+\gamma' \rangle}{4|\alpha_{\mathrm{R}}|^2/3} = 3c$$

is an integer, and $c = 1$ or 2. But

$$\frac{2\langle \alpha+\bar{\alpha}, \beta \rangle}{|\beta|^2} = \frac{2\langle 2\alpha_{\mathrm{R}}, c\,\alpha_{\mathrm{R}}+\gamma' \rangle}{4c^2|\alpha_{\mathrm{R}}|^2/3} = \frac{3}{c}$$

is an integer and thus $c \neq 2$. Hence $c = 1$ and result (c) is proved.

For (d) suppose $\mathfrak{g}$ is simple and not of type G_2. If $\mathfrak{g}$ has a not useful root of $\mathfrak{a}_p$, say $\alpha = \alpha_{\mathrm{R}}+\delta_\alpha/2$, then ($c$) and the one-dimensionality of $\mathfrak{a}$ imply that $\pm\alpha_{\mathrm{R}}$ are the only roots of $\mathfrak{a}$. By (b), there are no useful roots of $\mathfrak{a}_p$ other than the roots of $\mathfrak{a}_{\mathrm{M}}$. Let $\beta = \alpha_{\mathrm{R}}+\delta_\beta/2$ be a not useful root of $\mathfrak{a}_p$ different from α and $\bar{\alpha}$. The proof of (c) when β is of type (v) shows that $|\beta| = |\alpha|$ and $2\langle \alpha+\bar{\alpha}, \beta \rangle/|\alpha|^2 = 3$. Since $\mathfrak{g}$ is not of type G_2, the only possibilities are

$$\frac{2\langle \alpha, \beta \rangle}{|\alpha|^2} = 2 \qquad \text{and} \qquad \frac{2\langle \bar{\alpha}, \beta \rangle}{|\bar{\alpha}|^2} = 1$$

or vice-versa. In the first case, $|\alpha| = |\beta|$ implies $\beta = \alpha$, and in the second case, we obtain $\beta = \bar{\alpha}$. We conclude that the only roots of $\mathfrak{a}_p$ with non-zero $\mathfrak{a}$-component are $\pm\alpha$ and $\pm\bar{\alpha}$. Result (a) and the simplicity of $\mathfrak{g}$ imply that $\pm\delta_\alpha$ are the only other roots of $\mathfrak{a}_p$. Hence $\mathfrak{g}$ is of type A_2.

Remarks. — We mentioned that $\mathfrak{sl}(3, \mathbf{R})$ provides the prototype of a root that is not useful. If $\dim \mathfrak{a} = 1$, Propositions 10 d and 7 show how close $\mathfrak{sl}(3, \mathbf{R})$ is to providing the only example. It is instructive to examine split G_2 to see how (b), (c), and (d) in Proposition 10 fail for this Lie algebra.

The sense in which useful roots of $\mathfrak{a}_p$ are useful is that they provide elements of the Weyl group $W(\mathfrak{a})$.

LEMMA 11. — *Let $\alpha = \alpha_{\mathrm{R}}+\gamma$ be a useful root of $\mathfrak{a}_p$ with $\alpha_{\mathrm{R}} \neq 0$. Then there exists a member of the Weyl group $W(\mathfrak{a}_p)$ that leaves $\mathfrak{a}$ and $\mathfrak{a}_{\mathrm{M}}$ stable and equals the reflection $p_{\alpha_{\mathrm{R}}}$ on $\mathfrak{a}$.*

Proof. — The element is, in the various cases: (ii) $p_{\alpha+\bar{\alpha}}$, (iii) $p_\alpha p_{\bar{\alpha}}$, and (iv) p_α.

Remark. — If s is an element of $W(\mathfrak{a}_p)$ that leaves $\mathfrak{a}$ stable and if α is a useful root of $\mathfrak{a}_p$, then $s\,\alpha$ will be useful also, since $\overline{s\,\alpha} = s\,\bar{\alpha}$.

Proposition 12. — *The useful roots of* $\mathfrak{a}$ *form a root system in a subspace of* $\mathfrak{a}$.

Proof. — Lemma 11 and the remark show that the set of useful roots of $\mathfrak{a}$ is left stable by its own reflections. In view of [2] (p. 142), the proof will be complete if we show that whenever $\alpha = \alpha_R + \alpha_I$ and $\beta = \beta_R + \beta_I$ are useful, then $2 \langle \alpha_R, \beta_R \rangle / | \alpha_R |^2$ is an integer. We proceed according to the type of α, assuming as we may that $\alpha_R \neq 0$.

If α is of type (ii), then $\alpha + \bar{\alpha} = 2\alpha_R$ is a root of $\mathfrak{a}_p$ and

$$\frac{2 \langle 2\alpha_R, \beta \rangle}{| 2\alpha_R |^2} = \frac{1}{2} \frac{2 \langle \alpha_R, \beta_R \rangle}{| \alpha_R |^2}$$

is an integer; hence $2 \langle \alpha_R, \beta_R \rangle / | \alpha_R |^2$ is an integer.

If α is of type (iii), then

$$| \alpha_R |^2 = \frac{| \alpha |^2}{2}$$

and

$$\begin{aligned} \frac{2 \langle \alpha_R, \beta_R \rangle}{| \alpha_R |^2} &= \frac{2 \langle \alpha, \beta_R \rangle}{| \alpha |^2} + \frac{2 \langle \bar{\alpha}, \beta_R \rangle}{| \alpha |^2} \\ &= \frac{\langle \alpha, \beta \rangle}{| a |^2} + \frac{\langle \alpha, \bar{\beta} \rangle}{| \alpha |^2} + \frac{\langle \bar{\alpha}, \beta \rangle}{| \alpha |^2} + \frac{\langle \bar{\alpha}, \bar{\beta} \rangle}{| \alpha |^2} \\ &= \frac{2 \langle \alpha, \beta \rangle}{| \alpha |^2} + \frac{2 \langle \alpha, \bar{\beta} \rangle}{| \alpha |^2}, \end{aligned}$$

which is an integer.

If α is of type (iv), then $\alpha_R = \alpha$ and

$$\frac{2 \langle \alpha_R, \beta_R \rangle}{| \alpha_R |^2} = \frac{2 \langle \alpha, \beta \rangle}{| \alpha |^2},$$

which is an integer.

Corollary. — *Let* β *be a root of* $\mathfrak{a}$ *such that* $t\beta$ *is not a root of* $\mathfrak{a}$ *for* $0 < t < 1$. *Then the only possibilities for the set of positive* t *such that* $t\beta$ *is a root of* $\mathfrak{a}$ *are* $\{ 1 \}$, $\{ 1, 2 \}$, *and* $\{ 1, 2, 3 \}$.

Proof. — We may assume $\mathfrak{g}$ is simple. Suppose $\mathfrak{g}$ is not of type G_2. Propositions 10 *c* and 12 then show the only possibilities are $\{ 1 \}$ and $\{ 1, 2 \}$. In $\mathfrak{g}$ of type G_2, the only new possibilities that can occur are when $\dim \mathfrak{a} = 1$, and there are two cases according as a short or long root of $\mathfrak{a}_p$ is taken as a root of $\mathfrak{m}$. Easy computation gives the sets $\{ 1, 2 \}$ and $\{ 1, 2, 3 \}$ in the two cases, respectively.

5. Reduction lemmas

We retain the notation of Paragraph 3 and come to a consideration of the Main Theorem, stated in Paragraph 1. Proposition 12 shows that the useful roots of $\mathfrak{a}$ form a root system Δ_0. Let W_0 be the Weyl group of Δ_0. Lemma 11 shows that $W_0 \subseteq W(\mathfrak{a})$. The point of the Main Theorem is that equality holds in this inclusion.

Once the equality is established, the rest follows easily. In fact, we may assume that $\mathfrak{g}$ is simple. The only reflections in a Weyl group (involutions with one-dimensional eigenspace for -1) are those relative to roots of the system, by Lemma 63 of [8]. Hence if β is a root of $\mathfrak{a}$ and p_β is in W ($\mathfrak{a}$), then $t\,\beta$ is useful for some $t \neq 0$. If $\mathfrak{g}$ is not split G_2, Proposition 7 *f* and the cuspidal hypothesis show that either $\mathfrak{g}$ is not of any type G_2 or $\mathfrak{m} = 0$. In the first case, Proposition 10 *c* shows we may take $t = 1$. In the second case, our cuspidal parabolic subalgebra is minimal and every root is useful; thus we may take $t = 1$.

Thus the Main Theorem is proved as soon as it is shown that equality holds in the inclusion $W_0 \subseteq W(\mathfrak{a})$. In demonstrating this equality, we shall make extensive use of the following lemma due to Chevalley [6] (p. 249).

LEMMA 13 (Chevalley). — *If an element p of a Weyl group leaves a set* E *pointwise fixed, then p is the product of root reflections each leaving* E *pointwise fixed.*

With the aid of Lemma 13, the reverse inclusion $W(\mathfrak{a}) \subseteq W_0$ will be handled by the following considerations : Each element of W ($\mathfrak{a}$) has a representative in W ($\mathfrak{a}_p$) by Lemma 8, and two such representatives differ by an element of the Weyl group of ($\mathfrak{m}$, $\mathfrak{a}_M$) by Lemma 13. Therefore each element of W ($\mathfrak{a}^+$) uniquely determines an outer automorphism of the root system of ($\mathfrak{m}$, $\mathfrak{a}_M$). We study W ($\mathfrak{a}$) by studying these outer automorphisms. For purposes of computation, we may assume that the restriction to $\mathfrak{a}_M$ of the member of W ($\mathfrak{a}$) leaves stable the positive roots of $\mathfrak{m}$. The proof will splinter into a number of cases, all of which will be handled by one or the other of the two reduction lemmas below.

LEMMA 14. — *If each outer automorphism of* ($\mathfrak{m}$, $\mathfrak{a}_M$) *achievable by means of* W ($\mathfrak{a}$) *is achievable by means of* W_0, *then* $W(\mathfrak{a}) = W_0$.

Proof. — Let p be the representative in W ($\mathfrak{a}_p$) of an element of W ($\mathfrak{a}$), so chosen that p preserves the positive roots of $\mathfrak{m}$. The assumption is that there exists s in W ($\mathfrak{a}_p$) so that $s\,|_{\mathfrak{a}}$ is in W_0 and $s\,|_{\mathfrak{a}_M} = p\,|_{\mathfrak{a}_M}$. Then $s^{-1}\,p$ is in W ($\mathfrak{a}_p$) and leaves $\mathfrak{a}_M$ pointwise fixed. By Lemma 13, $s^{-1}p$ is the product of reflections in roots of $\mathfrak{a}_p$, each leaving $\mathfrak{a}_M$ pointwise fixed. All such roots are roots of type (iv), hence useful. Thus $s^{-1}\,p = r$ with $r\,|_{\mathfrak{a}}$ in W_0, and $p\,|_{\mathfrak{a}} = sr\,|_{\mathfrak{a}}$ is in W_0.

LEMMA 15. — *If every root of* $\mathfrak{a}_p$ *is useful, in particular if* $\mathfrak{m}$ *has no ideals of type* A_1, *then* $W(\mathfrak{a}) = W_0$.

Proof. — Let w in W ($\mathfrak{a}_p$) leave $\mathfrak{a}$ and $\mathfrak{a}_M$ stable. To see that $w\,|_{\mathfrak{a}}$ is in W_0, choose a member s_0 of W_0 so that $s_0^{-1}\,(w\,|_{\mathfrak{a}})$ leaves the set of positive roots of $\mathfrak{a}$ stable; this choice is possible because the usefulness of every root makes W_0 transitive on the set of Weyl chambers of $\mathfrak{a}$. Let s be a member of W ($\mathfrak{a}_p$) with $s\,|_{\mathfrak{a}} = s_0$. Next choose t in the Weyl group of $\mathfrak{m}$ so that $t^{-1}\,s^{-1}\,w$ leaves stable the positive roots of $\mathfrak{m}$. The compatibility of the orderings imply that $t^{-1}\,s^{-1}\,w$ leaves stable the set of positive roots of $\mathfrak{a}_p$. Thus $t^{-1}\,s^{-1}\,w = 1$. Hence $w = st$ and $w\,|_{\mathfrak{a}} = s\,|_{\mathfrak{a}} = s_0$ is in W_0.

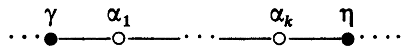
6. Proof of Main Theorem

In proving that $W(\mathfrak{a}) = W_0$, we shall think in terms of using Lemma 14 as often as convenient. To expedite matters we shall use Dynkin diagrams of the roots of $\mathfrak{a}_p$. Clearly we may assume $\mathfrak{g}$ is simple, so that the diagram is connected. In the diagram, dots, of course, represent simple roots. Shaded dots are the simple roots of $\mathfrak{m}$, and white dots are the other simple roots. The expression "..." in a diagram means "any permissible Dynkin diagram, possibly empty."

First there are some general considerations concerning the automorphisms of the root system of $\mathfrak{m}$. By Lemma 9 *b*, $\mathfrak{m}$ has no ideals of type A_n with $n > 1$ or D_n with n odd and $\geqq 3$ or E_6, and the diagram of $\mathfrak{g}$, being connected, cannot have more than one subsystem with a multiple line or triple point. Hence the only possible sources of automorphisms of the roots of $\mathfrak{m}$ are: (1) permutations of the factors A_1 among the shaded dots; (2) automorphisms of a single factor D_n with n even and $\geqq 4$; (3) combinations of (1) and (2).

We shall see that phenomena (1) and (2) can always be isolated from each other. Note that in any permutation (1) the length of each shaded root equals the length of its image under the permutation.

Our procedure will be to consider various configurations that might occur within the Dynkin diagram and then to prove the theorem by showing that the list of configurations is exhaustive. For each configuration we give a Roman-numeral label, the diagram, the constraints, the conclusions about the configuration, and the proof of the the conclusions.

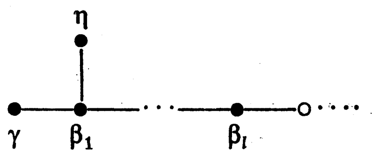
(I) 

We assume that $k \geqq 1$, $\{\alpha_1, \ldots, \alpha_k\}$ is connected by single lines, γ and η are isolated in $\mathfrak{m}$, none of $\alpha_1, \ldots, \alpha_k$ is connected to any roots of $\mathfrak{m}$ other than γ and η; we allow one of $\alpha_1, \ldots, \alpha_k$ to be a triple point. Then there exists an element of W_0 whose outer automorphism of $\mathfrak{m}$ transposes γ and η and leaves the other simple roots of $\mathfrak{m}$ fixed.

Proof. – By [2] (p. 160), $\alpha = \gamma+\alpha_1+\ldots+\alpha_k$ is a root of $\mathfrak{a}_p$. Also $\bar{\alpha}$ is $\alpha_1+\ldots+\alpha_k+\eta$ because $\alpha-\bar{\alpha} = \gamma-\eta$ is spanned by γ and η and $\alpha+\bar{\alpha} = \gamma+2(\alpha_1+\ldots+\alpha_k)+\eta$ is orthogonal to γ and η and all the other simple roots of $\mathfrak{m}$. Computation shows that α and $\bar{\alpha}$ are orthogonal. We claim that $p_\alpha p_{\bar{\alpha}}$, which is then in W_0, is the required element. In fact,

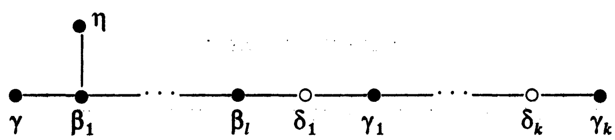
$$p_\alpha p_{\bar{\alpha}}(\gamma) = p_\alpha(\bar{\alpha}+\gamma) = \bar{\alpha}+(\gamma-\alpha) = \eta$$

and similarly $p_\alpha p_{\bar{\alpha}}(\eta) = \gamma$. Since α and $\bar{\alpha}$ are orthogonal to the other simple roots of $\mathfrak{m}$, $p_\alpha p_{\bar{\alpha}}$ leaves them fixed. Hence it has the required properties ([1]).

(II) 

([1]) This argument becomes more transparent if one thinks of the case that $\mathfrak{g}$ is $\mathfrak{sl}(n, \mathbf{R})$ and uses the known form of the roots.

We assume that $l \geqq 0$ and that $\beta_1, \ldots, \beta_l$ are all shaded dots connected by single lines. We conclude that if γ and η are moved at all by the outer automorphism of $\mathfrak{m}$ associated to an element of $W(\mathfrak{a})$, then they are transposed, and there exists an element transposing them if and only if the diagram is not



with $k \geqq 1$; when the element exists, the transposition itself (leaving the other simple roots of $\mathfrak{m}$ fixed) is achievable by W_0.

Proof. – The diagram in question is of type D_n, and we may take $\gamma = e_{n-1} - e_n$, $\eta = e_{n-1} + e_n$ in the usual notation for roots of D_n. The Weyl group of D_n contains only permutations and certain sign changes; so if γ and η are mapped to simple roots, $\{\gamma, n\}$ is mapped to itself. Next, assume the diagram is not the exceptional one. Choose j as large as possible so that $e_j - e_{j+1}$ is a white dot and also $e_{j-1} - e_j$ is a white dot or does not exist. Take $\alpha = e_j - e_n$. Easy computation shows that $\bar{\alpha} = e_j + e_n$, and clearly α is orthogonal to $\bar{\alpha}$. Then $p_\alpha p_{\bar{\alpha}}$, which is in W_0, interchanges γ and η and leaves the other simple roots of $\mathfrak{m}$ fixed. Hence $p_\alpha p_{\bar{\alpha}}$ has the required properties.

Conversely assume the diagram is the exceptional one. We are to show that the transposition of γ and η cannot occur. Composing the situation with (I), we see that it is enough to show that there is no element of $W(\mathfrak{a}_p)$ that fixes

$$e_1 - e_2, \qquad e_3 - e_4, \qquad \ldots, \qquad e_{2k-3} - e_{2k-2},$$

fixes

$$e_{2k-1} - e_{2k}, \qquad e_{2k} - e_{2k+1}, \qquad \ldots, \qquad e_{n-2} - e_{n-1},$$

and interchanges $e_{n-1} - e_n$ and $e_{n-1} + e_n$. By Lemma 13, such an element must be a product of reflections that fix the two fixed sets, i. e.,

$$p_{e_1+e_2}, \quad p_{e_3+e_4}, \quad \ldots, \quad p_{e_{2k-3}+e_{2k-2}}.$$

Thus such an element cannot perform the interchange. Hence (II) is proved.

Proof of Main Theorem when $\mathfrak{g}$ *is not of type* E_6, E_7, *or* E_8. – If the diagram of $\mathfrak{g}$ has no triple point, the only possible outer automorphisms of $\mathfrak{m}$ are permutations of the factors A_1. The lengths must be preserved in any permutation, and (I) says all permutations preserving length are achieved by W_0. Lemma 14 then disposes of these cases.

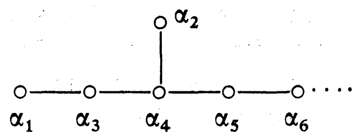
Suppose $\mathfrak{g}$ is of type D_n, $n \geqq 4$. We describe the outer automorphisms achievable by $W(\mathfrak{a})$. All such are achievable by W_0, by (I) or (II). Namely if $\mathfrak{m}$ contains some D_k with $k \geqq 3$, the automorphism group in question is all permutations of factors A_1 [handled by (I)], together with a transposition of the D_k if we are not in the exceptional case [handled by (II)]. (Even if the D_k is D_4, no other automorphism of D_4 is allowed.) If $\mathfrak{m}$ contains no D_k, the group is either the full permutation group on the factors A_1

or, in case (II) when $l = 0$, the permutation group on the factors other than γ and η, possibly direct sum with the transposition of γ and η.

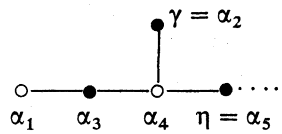
From the classification of Dynkin diagrams, we are done unless the diagram of $\mathfrak{g}$ is of type E_6, E_7, or E_8.

7. Exceptional cases

To handle the cases that the Dynkin diagram of roots of $\mathfrak{a}_p$ for $\mathfrak{g}$ is of type E_6, E_7, E_8, we shall first list some additional configurations and the conclusions for each, then show that they are exhaustive, and finally prove the conclusions for each configuration. Following Bourbaki [2] (p. 260-268), we label the Dynkin diagram of $\mathfrak{g}$ as

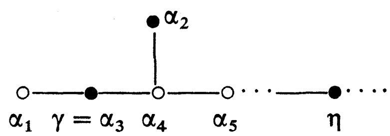


(III A)



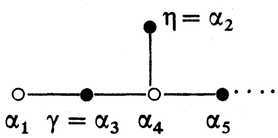
Then the transposition $\gamma \leftrightarrow \eta$ can always be achieved by W_0.

(III B)

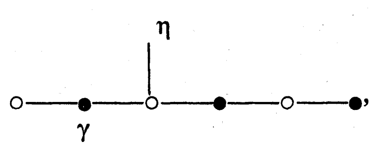


Then the transposition $\gamma \leftrightarrow \eta$ can always be achieved by W_0.

(III C)

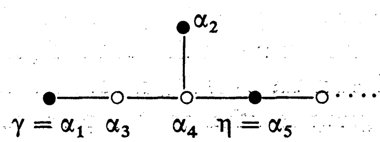


Then the transposition $\gamma \leftrightarrow \eta$ can be achieved by W_0 except in E_7 with diagram

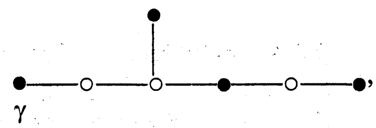


in which case no automorphism defined by $W(\mathfrak{a})$ moves γ.

(III D)

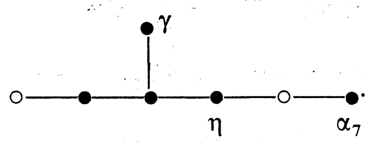


Then the transposition $\gamma \leftrightarrow \eta$ can be achieved in W_0 except in E_7 with diagram



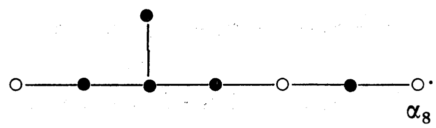
in which case no automorphism defined by $W(\mathfrak{a})$ moves γ.

(IV A)



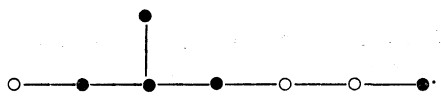
Then the transposition $\gamma \leftrightarrow \eta$ can be achieved by W_0, but no other automorphism of the roots of $\mathfrak{m}$ can be achieved by $W(\mathfrak{a})$.

(IV B)



Then the full 6-element automorphism group of D_4 can be achieved by W_0.

(IV C)



Then the full 6-element automorphism group of D_4 can be achieved by W_0.

Proof of Main Theorem in E_6, E_7, E_8. — Lemma 9 shows that all the components in the diagram for $\mathfrak{m}$ are of type A_1, D_4, or D_6. Lemma 15 shows that we may assume there is at least one factor A_1, and this condition then excludes D_6. If the factor D_4 occurs, Lemma 15 shows we may assume there is a factor A_1 as well, and then the only possible configurations are (IV A), (IV B), (IV C). Each conclusion for these configurations is that all the automorphisms achievable by $W(\mathfrak{a})$ are achievable by W_0. Thus the theorem in this case follows from Lemma 14.

We are left with the case that $\mathfrak{m}$ has only factors of type A_1. By (I) we may assume that α_2 is shaded, hence that α_4 is white. Suppose α_1 is white. If also α_3 is white, (I) finishes the argument; thus assume α_3 is shaded. If α_5 is white, then $\alpha_2 \leftrightarrow \alpha_3$ can be achieved by W_0, by (I); then (III B) and possibly (I) again show that all permutations are in the group of automorphisms achievable by M_0, and we can apply Lemma 14. If α_5 is shaded, $\alpha_2 \leftrightarrow \alpha_5$ can be achieved by W_0, by (III A); then (III C) and possibly (I) show that the permutations achievable by $W(\mathfrak{a})$ are achievable by W_0.

Finally suppose α_1 is shaded. Then α_3 is white, and α_2 may be transposed with α_5 or larger α by means of W_0, by (I). If α_5 is white, then $\alpha_1 \leftrightarrow \alpha_2$ can be achieved by W_0, by (I), and so the full permutation group can be achieved by W_0. If α_5 instead is shaded, matters are settled by (III D). Thus the theorem follows in all these cases from Lemma 14.

In the proofs of the assertions about each configuration, we shall omit a number of routine computations.

Proof for (III A) *and for* (IV A) *existence.* — Let $\alpha = \alpha_1+\alpha_2+\alpha_3+\alpha_4$. Then $\bar{\alpha} = \alpha_1+\alpha_3+\alpha_4+\alpha_5$, α and $\bar{\alpha}$ are orthogonal, and $p_\alpha p_{\bar{\alpha}}$ is the element of W_0 yielding the transposition $\gamma \leftrightarrow \eta$ and leaving the other simple roots of $\mathfrak{m}$ fixed.

Proof for (III B). — Whatever η is, we use $p_\alpha p_{\bar{\alpha}}$ for a suitable α such that α is orthogonal to $\bar{\alpha}$ as the required element of W_0. If $\eta = \alpha_6$, take

$$\alpha = \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6$$

in the notation at the beginning of this section, so that

$$\bar{\alpha} = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5.$$

If $\eta = \alpha_7$, take

$$\alpha = \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7$$

and

$$\bar{\alpha} = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6,$$

while if $\eta = \alpha_8$, take

$$\alpha = \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$$

and

$$\bar{\alpha} = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7.$$

Proof for (III C) *existence.* — Assuming we are not in the exceptional case, let k be the least index > 5 such that α_k is white and α_{k+1} either is white or does not exist. We imitate abstractly the argument for (II) when $l = 0$. Let

$$\alpha = \alpha_3+\alpha_4+\alpha_5+\ldots+\alpha_k,$$

so that

$$\bar{\alpha} = \alpha_2+\alpha_4+\alpha_5+\ldots+\alpha_k.$$

Then α is orthogonal to $\bar{\alpha}$, and $p_\alpha p_{\bar{\alpha}}$ is the required element of W_0.

Proof for (III D) *existence.* — Assuming we are not in the exceptional case, let k be the least index > 5 such that α_k is white and α_{k+1} either is white or does not exist. Let

$$\alpha = \alpha_k+\ldots+\alpha_5+2\alpha_4+\alpha_3+\alpha_2+\alpha_1,$$

so that

$$\bar{\alpha} = \alpha_k+\ldots+2\alpha_5+2\alpha_4+\alpha_3+\alpha_2.$$

Then α is orthogonal to α, and $p_\alpha p_{\bar{\alpha}}$ is the required element of W_0.

Proof for (IV B). — The transposition $\alpha_2 \leftrightarrow \alpha_5$ is handled as in (IV A), and the transposition $\alpha_2 \leftrightarrow \alpha_3$ is handled as in (III C) with $k = 8$. Hence the full 6-element group is achieved by W_0.

Proof for (IV C). — The transposition $\alpha_2 \leftrightarrow \alpha_5$ is handled as in (IV A), and the transposition $\alpha_2 \leftrightarrow \alpha_3$ is handled as in (III C) with $k = 6$. Hence the full 6-element group is achieved by W_0.

Proof for non-existence in (III C) *and* (IV A). — We assume we are in the exceptional case of E_7 with α_2, α_3, α_5, α_7, and possibly α_4 shaded. We are to show that α_3 remains fixed under each automorphism achieved by $W(\mathfrak{a})$. From the existence in (IV A) or (III A), according as α_4 is shaded or not, together with (I), it is enough to show that the transposition $\alpha_2 \leftrightarrow \alpha_3$ cannot be achieved by a member of $W(\mathfrak{a}_p)$ that leaves the other simple roots of $\mathfrak{m}$ fixed, particularly α_5 and α_7. From [2] (p. 264), such a Weyl group element must map

$$\varepsilon_1+\varepsilon_2 \leftrightarrow \varepsilon_2-\varepsilon_1, \qquad \varepsilon_4-\varepsilon_3 \leftrightarrow \varepsilon_4-\varepsilon_3, \qquad \varepsilon_6-\varepsilon_5 \leftrightarrow \varepsilon_6-\varepsilon_5$$

and hence also $2\varepsilon_2 \leftrightarrow 2\varepsilon_2$. By Lemma 13, the element is a product of reflections in roots orthogonal to ε_2, $\varepsilon_4-\varepsilon_3$, and $\varepsilon_6-\varepsilon_5$. The ε_2 says that the complicated roots do not enter. Hence the only eligible reflections are with respect to $\varepsilon_4+\varepsilon_3$, $\varepsilon_6+\varepsilon_5$, and $\varepsilon_8-\varepsilon_7$. These all fix $\varepsilon_1+\varepsilon_2$, contradiction.

Proof of non-existence in (III D). — We assume we are in the exceptional case of E_7 with α_1, α_2, α_5, and α_7 shaded. By (I) the simple roots of $\mathfrak{m}$ other than α_1 are permuted transitively by means of W_0. To prove that α_1 is fixed by any automorphism achieved by $W(\mathfrak{a})$, it is enough to show that the transposition $\alpha_1 \leftrightarrow \alpha_2$, with α_5 and α_7 fixed, cannot be achieved by $W(\mathfrak{a})$. From [2] (p. 264), such a Weyl group element must map

$$\frac{1}{2}(\varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_5-\varepsilon_4-\varepsilon_3-\varepsilon_2+\varepsilon_1) \leftrightarrow \varepsilon_2+\varepsilon_1, \qquad \varepsilon_4-\varepsilon_3 \leftrightarrow \varepsilon_4-\varepsilon_3,$$
$$\varepsilon_6-\varepsilon_5 \leftrightarrow \varepsilon_6-\varepsilon_5.$$

By Lemma 13, the element is a product of reflections in roots orthogonal to

$$\varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_5-\varepsilon_4-\varepsilon_3+\varepsilon_2+3\varepsilon_1, \qquad \varepsilon_4-\varepsilon_3 \qquad \text{and} \qquad \varepsilon_6-\varepsilon_5.$$

The only positive such roots are

$$\frac{1}{2}(\varepsilon_8-\varepsilon_7 \pm (\varepsilon_6-\varepsilon_5) \pm (\varepsilon_4-\varepsilon_3) \pm (\varepsilon_2-\varepsilon_1)),$$

and reflection in each of these leaves α_2 fixed, contradiction.

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