ANNALES SCIENTIFIQUES DE L'É.N.S.

DAN BURNS
MICHAEL RAPOPORT

On the Torelli problem for kählerian $K-3$ surfaces

Annales scientifiques de l'É.N.S. 4e série, tome 8, nº 2 (1975), p. 235-273
<http://www.numdam.org/item?id=ASENS_1975_4_8_2_235_0>

© Gauthier-Villars (Éditions scientifiques et médicales Elsevier), 1975, tous droits réservés.

L'accès aux archives de la revue « Annales scientifiques de l'É.N.S. » (http://www.elsevier.com/locate/ansens) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. scient. Éc. Norm. Sup.,
4[e] série, t. 8, 1975, p. 235-274.

ON THE TORELLI PROBLEM FOR KÄHLERIAN K-3 SURFACES

BY DAN BURNS, JR (*) AND MICHAEL RAPOPORT

TABLE OF CONTENTS

Introduction

In this article we generalize to kählerian K-3 surfaces the recent beautiful solution of the Torelli problem for algebraic K-3 surfaces due to Piatetskii-Shapiro and Shafarevitch [0]. The result has been conjectured in [0].

Our version is:

THEOREM 1. — *Let* X *and* X′ *be two kählerian* K-3 *surfaces. Let*

$$\varphi^* : \ H^2(X, Z) \xrightarrow{\sim} H^2(X', Z)$$

be an isomorphism between the (lattices of) 2nd cohomology groups which

(i) *preserves the Hodge structures,*

(ii) *sends the cone* $V^+(X)$ *to* $V^+(X')$, *and*

(iii) *sends (the class of) an effective divisor of self-intersection* -2 *to an effective cycle.*

Then φ^* *is induced by a unique isomorphism* $\varphi : X' \xrightarrow{\sim} X$.

(*) Partially supported by N. S. F.

Here $V^+(X)$ is the connected component of

$$V(X) = \{x \in H^{1,1}(X) \cap H^1(X, \mathbf{R}) \mid x^2 > 0\}$$

containing a Kähler class of X (*cf.* § 2).

The main difficulty in extending the proof of the Torelli theorem in [0] to kählerian K-3 surfaces, is caused by the fact that the moduli space of K-3 surfaces of kählerian type is not a Hausdorff space.

The proof in [0] of the corresponding result for algebraic K-3 surfaces proceeds roughly as follows:

(*a*) One first proves the Special Torelli theorem. To formulate it, call an algebraic K-3 surface a *special Kummer surface* if it is the Kummer surface associated to an abelian surface containing an elliptic curve.

SPECIAL TORELLI THEOREM. — *Let* X *be a special Kummer surface and let* X′ *be an algebraic* K-3 *surface. Let* φ^* *be an isomorphism between* $H^2(X, \mathbf{Z})$ *and* $H^2(X', \mathbf{Z})$ *preserving Hodge structures and effective cycles. Then* φ^* *is induced by a unique isomorphism between* X *and* X′.

(*b*) One considers the period mapping from the moduli space M of *polarized* algebraic K-3 surfaces to the corresponding moduli space of polarized Hodge structures Ω:

$$\tau : M \to \Omega.$$

The Torelli theorem in [0] is equivalent to the assertion that the morphism τ is injective.

The local Torelli theorem ensures that τ is étale, and the Special Torelli theorem implies that τ is one-to-one on the subset of M corresponding to special Kummer surfaces.

Now one shows that this subset of M is dense. This implies that τ is an open embedding.

Our proof of theorem 1 proceeds quite analogously and the idea that this could be done is due to P. Deligne.

Here is the outline of the proof.

First we observe that we can modify the hypothesis in the Special Torelli theorem (this strengthened version will still be called by the same name):

We need only assume that X′ is kählerian and that φ^*, instead of preserving *all* effective cycles, only preserves the effective cycles of self-intersection -2, but also sends the cone $V^+(X)$ into the cone $V^+(X')$.

Next, let M be the moduli space of kählerian K-3 surfaces with trivialized cohomology. Let Ω be the moduli space of corresponding Hodge structures. By the local Torelli theorem (*cf.* e.g. [10]) the period mapping

$$\tau : M \to \Omega$$

is étale. Now we construct a moduli space $\tilde{\Omega}$ of Hodge structures of the previous type *equipped with additional data* such that we get a commutative diagram

$$\begin{tikzcd} & \tilde{\Omega} \arrow[d, "\pi"] \\ M \arrow[ur, "\tilde{\tau}"] \arrow[r, "\tau"'] & \Omega \end{tikzcd}$$

where the "forgetful morphism" π is étale. The space $\tilde{\Omega}$ is so constructed that

$$\tau^{-1}(\tilde{\tau}(s)) = \{s\}, \qquad s \in M,$$

if and only if theorem 1 holds for $X = X_s$ (and arbitrary X').

A point of $\tilde{\Omega}$ corresponds to

– a Hodge structure $H = H^{0,2} + H^{1,1} + H^{2,0}$ of type (1, 20, 1) on an even, unimodular lattice L of rank 22 and signature (3, 19);

– a choice V^+ of one of the two connected components of the set

$$V = \{ x \in H^{1,1} \cap L \otimes R \mid x^2 > 0 \};$$

– a partition $P = \Delta^+ \cup -\Delta^+$ of the set

$$\Delta = \{ \delta \in H^{1,1} \cap L \mid \delta^2 = -2 \}$$

such that, if

$$\delta_1, \ldots, \delta_k \in \Delta^+ \qquad \text{and} \qquad \delta = \sum_{i=1}^{k} n_i \delta_i \in \Delta,$$

each $n_i \geqq 0$, then $\delta \in \Delta^+$.

One shows, in the same way as in [0], that the subset of M corresponding to special Kummer surfaces is everywhere dense. The following result allows us to conclude from the fact that $\tilde{\tau}$ is injective on a dense subset of M that $\tilde{\tau}$ is injective and thus conclude the proof of theorem 1:

MAIN LEMMA. — *Let* S *be a (contractible) analytic manifold. Let* $p : X \to S$ *and* $p : X' \to S$ *be two families of kählerian* K-3 *surfaces. Let*

$$\varphi^* \ : \ R^2 p_*(Z) \xrightarrow{\sim} R^2 p'_*(Z)$$

be an isomorphism of the relative second cohomology lattices which respects the Hodge structures and which for every point $s \in S$ *sends effective cycles of self-intersection* -2 *on* X_s *into effective cycles on* X'_s. *If* φ^* *is induced by an isomorphism*

$$\varphi_t \ : \ X'_t \xrightarrow{\sim} X_t$$

for all points t in a dense subset $T \subset S$, *there exists a unique isomorphism*

$$\begin{tikzcd} \varphi : X' \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

inducing φ^* *and* φ_t $(t \in T)$.

Actually, this Main Lemma together with the results in paragraph 2 (esp. lemma 2.4) yields the following generalization to *families* of K-3 surfaces of theorem 1. (It is for this version of theorem 1 that the construction of the moduli varieties is essential.)

THEOREM 1′. — *Let* S *be a connected analytic space. Let* $p: X \to S$ *and* $p': X' \to S$ *be two families of kählerian* K-3 *surfaces. Let*

$$\varphi^* : R^2 p_*(\mathbf{Z}) \to R^2 p'_*(\mathbf{Z})$$

be an isomorphism of the relative second cohomology lattices which

(i) *respects Hodge structures,*

(ii) *sends* $V^+(X_s)$ *to* $V^+(X'_s)$ *for one* (*and hence, every*) $s \in S$, *and*

(iii) *for every* $s \in S$, *sends effective cycles of self-intersection* -2 *into effective cycles.*

Then φ^* *is induced by a unique isomorphism*

$$\begin{tikzcd} X' \arrow[rr, "\varphi"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

As mentioned above, the moduli space M is non-separated (i. e. non-Hausdorff). The Main lemma essentially asserts that the *morphism*

$$\tilde{\tau} : M \to \tilde{\Omega}$$

is separated.

The basic reason for the non-separatedness of M is the existence of different simultaneous resolutions of double points in a family (cf. [1], [3], [6]). The basic example (due to Atiyah [3]) is the following: Let $p: X \to D$ be a family of smooth quartic surfaces over the punctured unit disc which acquires an ordinary double point over the origin. After making the base change

$$D' \to D$$

by extracting the square root of the local parameter on D we get a threefold X′ with a unique singular point P′ in the special fibre X'_0:

$$\begin{tikzcd} P' \in X_0 \arrow[r, hook] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \in D' \arrow[r, equal, phantom] & D' \arrow[r] & D \end{tikzcd}$$

Atiyah shows that there exist modifications $\tilde{X}$ of X′ which replace the point P′ by a curve C. The 3-fold $\tilde{X}$ fibres in a smooth way over D′:

$$\begin{tikzcd} C \subset \tilde{X}_0 \subset \tilde{X} \arrow[dr] & \\ & X' \arrow[d] \\ 0 \arrow[uu, phantom] \arrow[r] & D' \end{tikzcd}$$

But this process can be done in two different ways, hence we obtain two different families of smooth surfaces, $\tilde{X}_1$ and $\tilde{X}_2$, which coincide outside $0 \in D'$. This means that $\tilde{X}_1$ and $\tilde{X}_2$ define two different morphisms from D' into the moduli space M (fix some trivialization of the relative cohomology of $\tilde{X}_1$ and $\tilde{X}_2$) which coincide over $D' - \{0\}$: this is only possible if M is non-separated.

We refer to paragraph 7 where we prove a theorem which shows that this example is indeed the main reason for the phenomenon of non-separatedness in the moduli of unpolarized non-ruled algebraic surfaces over **C**.

It need hardly be pointed out that the present article is an afterthought about [0].

We wish to thank M. Artin, P. Deligne, R. P. Langlands, and T. Zink for their help.

1. Uniqueness Assertion of the Main Theorem

PROPOSITION 1.1. — *Let* φ *be an automorphism of a* K-3 *surface* X. *If the induced automorphism* φ^* *on* $H^2(X, \mathbf{Z})$ *is trivial,* φ *is the identity.*

Proof. — The Kuranishi family of X exists; denote by $p : \mathscr{X} \to \mathscr{U}$ a representative. Then φ induces an automorphism φ of $p : \mathscr{X} \to \mathscr{U}$.

Let Ω be the space of Hodge-structures of type (1, 20, 1) on an even 22-dimensional unimodular lattice L of signature (3, 19) (cf. § 2 below). By the local Torelli theorem, the period map, determined by the choice of an isomorphism of lattices between $H^2(X_0, \mathbf{Z})$ and L,

$$\tau : \quad \mathscr{U} \to \Omega$$

is étale. The assumption that $\varphi^* = \mathrm{id}$ implies that

$$\tau \circ \varphi = \tau.$$

We may thus choose a representative $p : \mathscr{X} \to \mathscr{U}$ such that the induced morphism

$$\tau : \quad \mathscr{U} \to \Omega$$

is an open embedding; φ acts on $\mathscr{U}$ such that $\tau \circ \varphi = \tau$.

Now we make use of the fact that the proposition is true if X_0 is an *algebraic* K-3 surface (cf. [0]). The Hodge structures corresponding to algebraic surfaces form a set of points in Ω which is everywhere dense (Kodaira [9] and Tjurina [15], ch. IX, cf. also § 4 below). So φ induces the identity morphism on X_t for a dense set of points $t \in \mathscr{U}$. Since X is separated, this implies that φ is the identity morphism.

Q. E. D.

2. Construction of the Relevant Moduli Spaces

All unimodular even lattices of rank 22 and signature (3, 19) are isomorphic; we fix one of them and call it L.

Let

$$\Omega = SO(2) \times O(1, 19) \backslash O(3, 19) = SO(2) \times SO(1, 19) \backslash SO(3, 19)$$

Ω is a 20-dimensional smooth complex-analytic manifold which parametrizes the Hodge-structures H of type (1, 20, 1) on L such that the Hodge-filtration

$$F: \ H^{2,0} \subset H^{2,0}+H^{1,1} \subset L \otimes \mathbf{C}$$

verifies $(H^{0,2})^{\perp} = H^{0,2}+H^{1,1}$ and such that $\omega.\overline{\omega} > 0$ for $\omega \in H^{2,0}$. Such H. S. 's will be called *admissible.*

This the first moduli space we require. The next one is given by the following theorem. In its statement, we call a proper and smooth morphism $p: X \to S$ a K-3 *surface* (resp. a K-3 *surface of kählerian type*) *over* S if all its fibres are K-3 surfaces (resp. K-3 surfaces of kählerian type).

THEOREM 2.1. — *The functor which to an analytic space* S *associates the set of isomorphism classes of* K-3 *surfaces of kählerian type* $p: X \to S$ *over* S *together with a trivialization as quadratic lattice of the* (*relative*) *second cohomology group* $\alpha: R^2 p_* (\mathbf{Z}) \xrightarrow{\sim} L$, *is representable by a smooth* 20-*dimensional analytic space* M.

Proof. — Let X_0 be a K-3 surface, and let $p: X \to U$ be the Kuranishi family of X_0. By taking U sufficiently small, we may assume that X_s is kählerian for every $s \in U$ [12]. Fix a trivialization $\alpha: R^2 p_* \mathbf{Z} \xrightarrow{\sim} L$ of the relative second cohomology lattice, and let $\tau: U \to \Omega$ be the period mapping so determined. By the local Torelli theorem, τ is a local isomorphism. For every point $t \in \Omega$ sufficiently close to $\tau(0)$, the space Ω is isomorphic to the Kuranishi space of the K-3 surface, corresponding to the Hodge structure H_t on L. So, if U is small enough, for every $s \in U$, U is the Kuranishi space of X_s. By the uniqueness result of paragraph 1, for s and s' in U, X_s and $X_{s'}$ are not isomorphic (as varieties with trivialized cohomology). We now obtain M by glueing all the U's obtained as above identifying points corresponding to K-3 surfaces isomorphic as varieties with trivialized cohomology.

Q. E. D.

VARIANT 2.2. — *The functor which to an analytic space* S *associates the set of isomorphism classes of* K-3 *surfaces over* S *together with a trivialization* (*as a lattice*) *of the* (*relative*) *second cohomology group*

$$\alpha: \ R^2 p_*(\mathbf{Z}) \xrightarrow{\sim} L$$

is representable by a smooth 20-*dimensional analytic space* M.

Indeed, this is shown by the previous discussion.

Q. E. D.

(It is still unknown whether there are any non-kählerian K-3 surfaces.)

From now on we retain the notation $\tau: M \to \Omega$ for the period mapping. τ associates to a pair $(X, \alpha: H^2(X, \mathbf{Z}) \xrightarrow{\sim} L)$ the (admissable) Hodge structure on L induced by the Hodge structure on $H^2(X, \mathbf{Z})$ via α.

Before defining the third moduli space $\tilde{\Omega}$, we have to insert a few preliminary remarks. We refer to [5] (esp. exercises to Chapt. V, § 4, and [16]).

Let H be an admissible Hodge structure on L. We denote by $H^{1,1}_{\mathbf{R}}$ the elements in $H^{1,1}$ fixed under complex conjugation in $L \otimes \mathbf{C}$, and set $H^{1,1}_{\mathbf{Z}} = H^{1,1} \cap L$. Set

$$V = \{ x \in H^{1,1}_{\mathbf{R}} \mid x^2 > 0 \}.$$

Since the form on $H^{1,1}_{\mathbf{R}}$ has signature (1, 19), V is the disjoint union of two cones, V^+ and $-V^+$. Let

$$\Delta = \{ x \in H^{1,1}_{\mathbf{Z}} \mid x^2 = -2 \}.$$

For $\delta \in \Delta$, let s_δ be the reflection of the vector space $H^{1,1}_{\mathbf{R}}$:

$$s_\delta : \quad x \to x + (x.\delta)\,\delta.$$

These reflections generate a group W operating properly discontinuously on V^+. A fundamental domain for the action of W on V^+ is given by a convex polyhedron V^+_P bounded by the (possibly infinitely many) hyperplanes

$$H_\delta = \{ x \in H^{1,1}_{\mathbf{R}} \mid (\delta.x) = 0 \} \qquad (\delta \in \Delta).$$

Such a convex polyhedron V^+_P defines a system of generators of W — namely those s_δ, for $\delta \in \Delta$ such that V^+_P lies on the positive side of H_δ — and a partition

$$P : \quad \Delta = \Delta^+ \cup -\Delta^+$$

where

$$\Delta^+ = \{ \delta \in \Delta \mid (\delta.x) > 0, \ \forall x \in V^+_P \}.$$

This partition has the property that

(★) If $\delta_1, \ldots, \delta_n \in \Delta^+$, and $\delta = \sum_{i=1}^{n} r_i \delta_i \in \Delta$ ($r_i > 0$, integers), then $\delta \in \Delta^+$.

Conversely, any such partition P of Δ verifying (★) defines a fundamental domain V^+_P, where

$$V^+_P = \{ x \in V^+ \mid (x.\delta) > 0, \ \forall \delta \in \Delta^+ \};$$

indeed, V^+_P is contained in a fundamental domain, is bounded by hyperplanes, and is non-empty. (Of course, if $\Delta = \varnothing$, then $V^+_P = V^+$.)

Let H_s ($s \in S$) be a holomorphic family of admissible Hodge structures parametrized by an analytic space S, together with a continuously varying choice of a connected component V^+_s of V_s.

PROPOSITION 2.3. — *Let $x_0 \in V^+_{s_0}$. Then there exists an open neighborhood* K *of x_0 in* $L \otimes \mathbf{R}$ *and an neighborhood* U *of s_0 in* S *such that for all $s \in U$, the only hyperplanes* H_δ ($\delta \in \Delta_s$) *going through* K *are those for which* $\delta \in \Delta_{s_0}$.

Proof. — The orthogonal complement $x_0^\perp \cap H^{1,1}_{s_0 \mathbf{R}}$ is negative-definite (of dimension 19) We extend $-(\ ,\ )$ from $x_0^\perp \cap H^{1,1}_{s_0 \mathbf{R}}$ into a euclidean norm $\|\ \|_0$ on $L \otimes \mathbf{R}$.

For r an arbitrary positive real number we can find a neighborhood U of s_0 such that all $\delta \in \Delta_s$ ($s \in U$) which do not lie in Δ_{s_0} verify

$$\|\delta\|_0 > r.$$

Indeed, the set

$$\Delta(r) = \{\delta \in L \mid \delta^2 = -2; \|\delta\|_0 \leqq r\}$$

is finite. On the other hand, for a given $\delta \in \Delta(r)$, $\delta \notin \Delta_{s_0}$, the subset of S:

$$\{s \in S \mid \delta \in H_s^{1,1}\}$$

is a closed subset not containing s_0.

For $s \in S$ and $x \in L \otimes \mathbf{R}$, we define $L_2(x, s) \subset L \otimes \mathbf{R}$ by

$$L_2(x, s) = \{y \in H_{s_{\mathbf{R}}}^{1,1} \mid y.x = 0\}$$

and set $L_1(x, s) = L_2(x, s)^{\perp}$ = orthogonal complement of $L_2(x, s)$ in $L \otimes \mathbf{R}$.

Let $K \subset L \otimes \mathbf{R}$ be an open neighborhood of x_0 and let $U \subset S$ be an open neighborhood of s_0 such that

(i) for all $x \in K$ and for all $s \in U$, we have a decomposition

$$L \otimes \mathbf{R} = L_1(x, s) \oplus L_2(x, s)$$

of $L \otimes \mathbf{R}$ into a 3-dimensional positive-definite vectorspace $L_1(x, s)$ and a 19-dimensional negative-definite vectorspace $L_2(x, s)$;

(ii) the restriction of the bilinear form on $L \otimes \mathbf{R}$ to $L_2(x, s)$ is arbitrarily close to $-\| \;\|_0$ [i. e., $(1/c)(v.v) \leqq -\|v\|_0^2 \leqq c(v.v)$, for all $v \in L_2(x, s)$ with c arbitrarily close to 1].

To see that such K and U exist, notice first that, since $H_{s_0\mathbf{R}}^{1,1}$ has signature (1, 19) and $x_0^2 > 0$, we see that $L_2(x_0, s_0)$ is negative-definite of dimension 19; since $L \otimes \mathbf{R}$ has signature (3, 19), $L_1(x_0, s_0)$ is positive-definite of dimension 3; finally the condition (ii) for $L_2(x_0, s_0)$ is obviously verified by construction of $\| \;\|_0$. But $L_2(x, s)$ (together with the induced bilinear form) varies continuously with x and s (in a Grassmanian). Thus the existence of K and U is clear.

Now let $x \in K$ and assume that x lies on a hyperplane H_δ with $\delta \in \Delta_s$, $s \in U$. Then $\delta = \delta_2 \in L_2(x, s)$ and

$$\|\delta\|_0^2 = \|\delta_2\|_0^2 < -\frac{1}{c}(\delta_2, \delta_2) = \frac{2}{c}.$$

If now $\delta \notin \Delta_{s_0}$, then by the initial remarks we can make U so small that $\|\delta\|_0 > r$, $r \in \mathbf{R}_+$. By making r very large, we see that such δ cannot exist.

Q. E. D.

COROLLARY 2.4. — *Let $s_0 \in S$ and let x_0, $x'_0 \in V^+_{s_0}$. Then there exists an open neighborhood* U *of s_0 such that for all hyperplanes H_δ, $\delta \in \Delta_s$, $s \in U$ which separate x_0 and x'_0 (i. e. x_0 and x'_0 lie on different sides of H_δ) one has*

$$\delta \in \Delta_{s_0}.$$

Proof. — Join x_0 to x'_0 by a line segment contained in $V^+_{s_0}$. For every point x on this segment, we choose K_x and U_x according to the previous proposition. A finite number of the $K_x, K_{x_1}, \ldots, K_{x_n}$ will cover the line segment, and $U = \bigcap_{i=1}^{n} U_{x_i}$ is the required U.

We can now construct the third moduli space. Let $\tilde{\Omega}$ be the functor which to an analytic space S associates :

(1) A holomorphically varying Hodge structure $\underline{H}$ parametrized by S.

(2) A continuously varying choice of one (of the two) connected components.

$$V^+_s \subset V_s \subset H^{1,1}_{s_{\mathbf{R}}}.$$

(3) For every point $s \in S$, a partition

$$P_s : \quad \Delta_s = \Delta^+_s \cup -\Delta^+_s \quad \text{verifying } (\bigstar).$$

Data (3) are required to verify the following continuity condition :

For every point $s_0 \in S$ and every $c_0 \in V^+_{P_{s_0}}$ there exists an open neighborhood K of c_0 in $L \otimes \mathbf{R}$ and an open neighborhood U of s_0 in S such that for every $s \in U$ we have

$$\Delta^+_s = \{ \delta \in \Delta_s \mid (\delta . c) > 0 \text{ for all } c \in K \}.$$

THEOREM 2.5. — *The "forgetful morphism" of functors*

$$\pi : \quad \tilde{\Omega} \to \Omega$$

is relatively representable by an étale morphism of analytic spaces (the fibres of π are *not* necessarily finite).

In particular, $\tilde{\Omega}$ is representable by a smooth 20-*dimensional complex-analytic space.*

Before proving this theorem, we indicate how to obtain the commutative diagram (mentioned in the introduction)

$$\begin{tikzcd} & \tilde{\Omega} \arrow[d, "\pi"] \\ M \arrow[ur, "\tilde{\tau}"] \arrow[r, "\tau"'] & \Omega \end{tikzcd}$$

Namely, $\tilde{\tau}$ associates to $(p : X \to S, \alpha : R^2 p_* (\mathbf{Z}) \xrightarrow{\sim} L)$, a family of kählerian surfaces with trivialized cohomology :

(1) The family of Hodge structures on L obtained by pulling back via α the Hodge structures on $H^2 (X_s, \mathbf{Z})$ $(s \in S)$.

(2) The connected component V_s^+ of $V_s \subset H_s^{1,1}$ which contains a Kähler class.

(3) For every point $s \in S$, the partition P_s where Δ_s^+ = *effective* cycles of self-intersection -2.

We check the continuity condition imposed on data (3): Given $s_0 \subset S$ and a Kähler class c_0 on K, there exists a neighborhood $K \subset L \otimes \mathbf{R}$ of c_0 and a neighborhood U of s_0 such that if $c \in K \cap H_{s_\mathbf{R}}^{1,1}$ $(s \in U)$, then c is a Kähler class of X_s (Kodaira-Spencer). Furthermore, the effective cycles Δ_s^+ are given by

$$\Delta_s^+ = \{ \delta \in \Delta_s \mid (\delta . c) > 0 \}.$$

If now $x_0 \in V_{P_{s_0}}^+$, then, since $c_0 \in V_{P_{s_0}}^+$, by corollary 2.4 there exists an open neighborhood U of s_0 such that for all $s \in U$:

$$\{ \delta \in \Delta_s \mid (\delta . x_0) > 0 \} = \{ \delta \in \Delta_s \mid (\delta . c_0) > 0 \} = \Delta_s^+.$$

Proof of theorem 2.5. — We first show that the functor Ω' which to an analytic space S associates data (1) and (2) is representable. Indeed, we have

LEMMA 2.5. — $\Omega' \simeq SO(2) \times SO(1, 19)^0 \backslash SO(3, 19)$. *In particular, Ω' is a trivial 2-sheeted covering of Ω (i.e., Ω' has 2 connected components).*

Proof. — Ω' is clearly representable by a 2-sheeted etale covering of Ω.

The group SO (1, 19) has 2 connected components; an easy calculation shows that the elements not contained in the connected component of the identity interchange the connected components of $V_s \subset H_{s_\mathbf{R}}^{1,1}$, $s \in \Omega$. SO (3, 19) has 2 connected components such that

$$SO(3, 19)^0 \cap SO(1, 19) = SO(1, 19)^0.$$

Hence SO (3, 19) acts transitively on Ω' and the assertion follows.

Q. E. D.

For any point $s \in \Omega'$ and any element $c \in V_s^+$ we choose an open neighborhood K of c inside $L \otimes \mathbf{R}$ and an open neighborhood U of s in Ω' with the properties given by proposition 2.3. We glue U (arising in connection with s_0, c_0, K_0) and U' (arising in connection with s_1, c_1, K_1) along the sublocus consisting of points s where c_0 and c_1 are not separated by a hyperplane H_δ for $\delta \in \Delta_s$. It follows from corollary 2.3 that this sublocus is open both in U and U'. In particular the resulting space $\tilde{\Omega}$ is an analytic space which is etale over Ω. It is clear that this analytic space indeed represents the functor Ω. Let $\tilde{s} \in \tilde{\Omega}$. Then to $\tilde{s}$ we can associate $s \in \Omega$, U, c, K as above.

This defines:

(1) $H_{\tilde{s}} \underset{\overline{Df}}{=} H_s$ = an admissible Hodge structure on L.

(2) $V_{\tilde{s}}^+ \subset V_s$ is the connected component of V_s containing c.

(3) $\Delta_{\tilde{s}}^+ = \{ \delta \mid (\delta . c) > 0 \}$.

The continuity condition imposed on data (3) is clear by construction. Also, the set of points $\tilde{s}$ above s corresponds to the different choices of $V_{\tilde{s}}^{+} \subset C_s$ and of partitions $P_{\tilde{s}}$ of Δ_s into Δ_s^{+} and $-\Delta_s^{+}$ verifying (★).

Q. E. D.

We conclude this paragraph with the following lemma which shows the relevance of the moduli space $\tilde{\Omega}$ to our Torelli theorem:

LEMMA 2.7. — *Let* $s \in \tilde{\Omega}$. *If* $\tilde{\tau}^{-1}(s)$ *consists of exactly one point* $t \in M$ *the Torelli theorem is true for the* K-3 *surface* $X = X_t$.

Proof. — Let $X' = X_{t'}$ be a kählerian K-3 surface and let $\varphi^* : H^2(X, \mathbf{Z}) \xrightarrow{\sim} H^2(X', \mathbf{Z})$ be an isomorphism which verifies the hypotheses of theorem 1 of the introduction. Since X and X′ have trivialized cohomology groups, φ^* induces an isomorphism, still denoted φ^*, between admissible H.S'.s H_X and $H_{X'}$ on L.

We know that φ^* maps V_X^+ into $V_{X'}^+$, and induces a bijection between Δ_X^+ and $\Delta_{X'}^+$. In other words we have that $\tilde{\tau}(t) = \tilde{\tau}(t')$, implying by assumption that $t = t'$, i.e. the Torelli theorem for X.

Q. E. D.

3. The Special Torelli Theorem

The purpose of this paragraph is to show that, in the case of *algebraic* K-3 surfaces, our theorem 1 was already proved in [0] and to elucidate the hypotheses of that theorem.

LEMMA 3.1. — *Let* X *be an algebraic* K-3 *surface. Then any element* $c \in V_P^+(X) \cap H_{\mathbf{Z}}^{1,1}(X)$ *is the class of an ample divisor.*

Proof. — By the Riemann-Roch formula, c or $-c$ is the class of an effective divisor; since $c \in V_P^+(X)$, c must be effective. By the Nakai-Moisezon criterion for ampleness, it suffices to show that for a = the class of an irreducible effective divisor on X, we have $a.c > 0$.

If $a^2 < 0$, we have $a^2 = -2$, and $a.c > 0$, by the definition of $V_P^+(X)$.

If $a^2 \geqq 0$, then $a.c \geqq 0$. By the Hodge Index Theorem, we may take a rational basis $(d_1, \ldots, d_\alpha)$ of $H_{\mathbf{Z}}^{1,1}(X) \otimes \mathbf{Q}$ with $d_1 = c$, $d_i^2 < 0$ $(i = 2, \ldots, p)$, and $d_i . d_j = 0$ $(i \neq j)$. Write

$$a = \alpha_1 d_1 + \ldots + \alpha_p d_p \qquad (\alpha_i \in \mathbf{Q}).$$

If $a.c = 0$, then $\alpha_1 = 0$, and $a^2 < 0$. This contradiction proves the lemma.

COROLLARY 3.2. — *Let* X *and* X′ *be algebraic* K-3 *surfaces. Let*

$$\varphi^* : \quad H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$$

be an isomorphism of lattices preserving Hodge structures. The following three statements are equivalent :

(i) *The isomorphism* φ^* *preserves the classes of effective divisors.*

(ii) *The isomorphism* φ^* *takes the class of an ample divisor on* X *into the class of an ample divisor on* X'.

(iii) *The isomorphism* φ^* *maps* $V_P^+(X)$ *into* $V_P^+(X')$.

Proof. — (i) $\Rightarrow$ (iii) : the assumption (i) on φ^* implies that φ^* takes $V_P^+(X)$ into $V_P^+(X')$ or $-V_P^+(X')$. Applying φ^* to the class of an ample divisor on X, we see that $\varphi^*(V_P^+(X)) \subset V_P^+(X')$. The implication (iii) $\Rightarrow$ (ii) follows from the previous lemma; the direction (ii) $\Rightarrow$ (i) is equally easy, and is proved on [0] (§ 5).

This corollary, joined with the results of paragraph 5 in [0] imply the following result:

SPECIAL TORELLI THEOREM. — *Let* X *be a special Kummer surface and let* X' *be a* K-3 *surface. If there exists an isomorphism of lattices* $\varphi^* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ *which preserves Hodge structures and transforms* $V_P^+(X)$ *into* $V_P^+(X')$, *then* φ^* *is induced by a unique isomorphism between* X *and* X'.

Proof. — Since there exists an element $c \in H_{\mathbf{Z}}^{1,1}(X')$ with $c^2 > 0$, X' is algebraic. Hence, we can apply the previous corollary and the results of [0] (§ 5).

4. The Density Theorem

In this paragraph we show that the proof of the density theorem in [0] works in our context. (Another proof is obtained by taking the conjunction of the density theorem proved in [0] with the fact that the algebraic surfaces are dense in M (*cf.* [9], [15]). Since the argument which follows reproves this last fact we give it in full.)

We call a kählerian K-3 surface X *exceptional* if X has (the maximum possible number) 20 linearly independent algebraic cycles. Then, of course, X is algebraic. If, furthermore X is a Kummer surface, then it is special Kummer surface. For a K-3 surface we denote by L_X the orthogonal complement in $H^2(X, \mathbf{Z})$ of the Néron-Severi group NS (X). (The elements in L_X are the "transcendental cycles".)

PROPOSITION 4.1. — *Let* B *be a positive-definite lattice of rank* 2 *such that* $b^2 \equiv 0\ (4)$, *for all* $b \in B$. *Then there exists a unique exceptional* K-3 *surface* X *for which* $L_X \simeq B$. *This surface* X *is a special Kummer surface.*

For the proof *see* [0] (§ 6).

For a point $s \in \Omega$, we denote by L_s the orthogonal complement in L of $H_{s\mathbf{Z}}^{1,1} \subset L$. The following theorem is the required density theorem: together with the previous proposition (and the special Torelli theorem) it implies that there is a dense subset $S \subset \Omega$ such that for any $\tilde{s} \in \pi^{-1}(S) \subset \tilde{\Omega}$ the set $\tilde{\tau}^{-1}(\tilde{s}) \subset M$ consists of exactly one point [of course, $\pi^{-1}(S)$ will then be dense in $\tilde{\Omega}$ and $\tau^{-1}(S)$ will be dense in M].

THEOREM 4.2. — *The set* S *of all points* $s \in \Omega$ *for which*

(1) $\mathrm{rg}\, L_s = 2$ (*in particular* L_s *is definite*), *and*

(2) $b^2 \equiv 0\,(4)$ *for all* $b \in L_s$

is dense in Ω.

Proof. — By the previous proposition the set S is non-empty.

Let G be the group of linear transformations of the vector space $L \otimes \mathbf{R}$ which preserve the bilinear form on $L \otimes \mathbf{R}$ up to a positive factor. Inside G, let Γ be the group of linear transformations of $L \otimes \mathbf{Q}$ which, together with their inverses, can be written in a basis of L such that the denominators of the matrix entries are relatively prime to 2. Then, as is known from the theory of algebraic groups, $\Gamma \subset G$ is everywhere dense (1). G acts transitively on Ω.

To complete the proof of the theorem it remains to show that S is stable under Γ. Let $s \in S$, $\gamma \in \Gamma$, and take $b \in L_{\gamma(s)}$. There is an odd integer q such that

$$qb = \gamma(a)$$

for some $a \in L_s$. Further, there are odd integers m, n such that

$$m(qb\,.\,qb) = m(\gamma(a)\,.\,\gamma(a)) = n(a\,.\,a).$$

Since $a \in L_s$, and m, n, q are odd, we get

$$b^2 \equiv 0 \bmod 4.$$

Q. E. D.

5. **Proof of the Main Lemma**

We retain the notation introduced in the statement of this lemma (*cf.* the introduction). The uniqueness assertion was proved in paragraph 1: therefore the problem is local on S.

---

(1) This may be seen, for instance, as follows : We write $G = \mathbf{R}_+^* . SO(L \otimes \mathbf{R})$. Clearly, the positive real numbers with odd denominator and odd numerator are dense in the first factor. To treat the second factor, note that it is the set of real points of an algebraic group $\mathcal{G}$ défined over $\mathbf{Q}$ and that it suffices to show that $\mathcal{G}(\mathbf{Q})$ lies dense in $\mathcal{G}(\mathbf{R}) \times \mathcal{G}(\mathbf{Q}_2)$ w. r. t. the product of the real and 2-adic topologies : it will then follow that the inverse image in $\mathcal{G}(\mathbf{Q})$ of the open subgroup $\mathcal{G}(\mathbf{Z}_2) \subset \mathcal{G}(\mathbf{Q}_2)$ via the second projection has dense image in $\mathcal{G}(\mathbf{R})$, which is what we needed.

To see the required density, use the fact (*cf.* J. DIEUDONNÉ, *Sur les groupes classiques*, Hermann, Paris, 1963) that every element $g^{(\infty)}$, resp. $g^{(2)}$, in $\mathcal{G}(\mathbf{R})$, resp. $\mathcal{G}(\mathbf{Q}_2)$, may be written as a product of (an even number of) reflections about non-isotropic hyperplanes. We may arrange that $g^{(\infty)}$ and $g^{(2)}$ are product of the same number of reflections

$$g^{(\infty)} = s_1^{(\infty)} \ldots s_n^{(\infty)}, \qquad g^{(2)} = s_1^{(2)} \ldots s_n^{(2)},$$

where $s_i^{(\infty)}$, resp. $s_i^{(2)}$, is a reflection about a non-isotropic hyperplane $H_i^{(\infty)}$ of $L \otimes \mathbf{R}$, resp. $H_i^{(2)}$ of $L \otimes \mathbf{Q}_2$. Now simultaneously approximate by a rational non-isotropic hyperplane H_i the i-th real, resp. 2-adic, hyperplane $H_i^{(\infty)}$, resp. $H_i^{(2)}$: the product $g = s_1 \ldots s_n$ of the corresponding reflections is arbitrarily close to $g^{(\infty)}$ and $g^{(2)}$.

Let $0 \in S$, and assume that S is contractible. Assume also that $X \times_S X'$ has a hermitian metric which induces a Kähler metric on $X_s \times X'_s$ for each $s \in S$; this is justified by [12].

LEMMA 5.1. — *Let $t_1, t_2, \ldots$ be a sequence of points in $T \subset S$, converging to 0. Let $\Gamma_{t_i} \subset X_{t_i} \times X'_{t_i}$ denote the graph of the isomorphism φ_{t_i}. Then, passing to a subsequence if necessary, we may assume that the Γ_{t_i} converge to a purely two-dimensional limit cycle $\Gamma_0 \subset X_0 \times X'_0$.*

REMARK 5.2. — The Γ_{t_i}'s define closed, positive, integral currents in $X \times_S X'$, and the limit above can be taken in the distributional sense. The limit Γ_0 will be a current of the same type, i. e. $\Gamma_0 = \sum a_j Z_j$, with the a_j's positive integers, and the Z_j's are (the currents of integration over) irreducible analytic subvarieties of $X_0 \times X'_0$.

Proof. — We wish to appeal to a result of E. Bishop [4] (*cf.* also [8] since we want to consider Γ_0 as a current, i. e. with appropriate multiplicities, not just as a limit set). The metric on $X \times_S X'$ gives a continuously varying Kähler class

$$\omega_s \in H^2(X_s \times X'_s, \mathbf{R}) \simeq H^2(X \times_S X', \mathbf{R}).$$

By the quoted references, it suffices to show that the 4-volumes (computed in the metric on $X \times_S X'$) of the analytic cycles Γ_{t_i} are bounded.
But

$$\mathrm{vol}(\Gamma_{t_i}) = [\Gamma_{t_i}] \cup \frac{\omega_{t_i}^2}{2} \in H^8(X \times_S X', \mathbf{R}) \simeq \mathbf{R}$$

and this last expression equals

$$[\varphi^*] \cup \frac{\omega_{t_i}^2}{2} \in H^8(X \times_S X', \mathbf{R}) \simeq \mathbf{R},$$

where $[\varphi^*] \in H^4(X \times_S X', \mathbf{Z})$ is the cohomology class of the isomorphism φ^* on cohomology. The function

$$s \to [\varphi^*] \cup \frac{\omega_s^2}{2} \in \mathbf{R}$$

is a real-valued continuous function on S, hence vol (Γ_{t_i}) is bounded for all t_i.

Q. E. D.

REMARK 5.3. — The cohomology class $[\Gamma_0]$ of the limit cycle Γ_0 equals

$$[\varphi^*] \in H^4(X \times_S X', \mathbf{Z}) \simeq H^4(X_0 \times X'_0, \mathbf{Z}).$$

LEMMA 5.4. — *In the notation of the previous lemma, the limit cycle Γ_0 has the following form:*

$$\Gamma_0 = \Delta_0 + \sum a_{ij} C_i \times C'_j; \qquad a_{ij} \in \mathbf{Z}, \geqq 0,$$

where Δ_0 is the graph of an isomorphism between X_0 and X'_0 and C_i, resp. C'_j, are irreducible curves on X_0 resp. X'_0.

Proof. — Since Γ_0 is purely 2-dimensional, we may write

$$\Gamma_0 = Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + Z_5,$$

where

- each irreducible component of Z_0 projects *onto* X_0 as well as X'_0;
- each irreducible component of Z_1 projects *onto* a *curve* in X_0 and X'_0;
- each irreducible component of Z_2 projects *onto* X'_0 and to a *point* in X_0;
- each irreducible component of Z_3 projects *onto* X_0 and to a *point* in X'_0;
- each irreducible component of Z_4 projects *onto* X_0 and onto a *curve* in X'_0;
- each irreducible component of Z_5 projects *onto* X'_0 and onto a *curve* in X_0.

A purely 2-dimensional cycle Z on $X_0 \times X'_0$ determines a cohomology class $[Z] \in H^4(X_0 \times X'_0, \mathbf{Z})$. A class $z \in H^4(X_0 \times X'_0, \mathbf{Z})$, in its turn, defines a linear map

$$z_* : \quad H^*(X_0, \mathbf{Z}) \to H^*(X'_0, \mathbf{Z})$$

according to the following rule:

Let $x \in H^i(X_0, \mathbf{Z})$ be a cohomology class. Then $p^*(x) \in H^i(X_0 \times X'_0, \mathbf{Z})$, and cupping with the cohomology class $z \in H^4(X_0 \times X'_0, \mathbf{Z})$ gives

$$p^*(x) \cup z \in H^{4+i}(X_0 \times X'_0, \mathbf{Z}).$$

Now applying the Gysin morphism

$$p'^* : \quad H^{4+i}(X_0 \times X'_0, \mathbf{Z}) \to H^i(X'_0, \mathbf{Z})$$

gives the desired image $z_*(x)$.

$$\text{Let } x = \sum_0^4 x_i \in H^*(X_0, \mathbf{Z}) = \bigoplus_0^4 H^i(X_0, \mathbf{Z}).$$

Then:

(*a*) If $Z = P \times X'_0$, where $p \in X_0$ is a point, then

$$[Z]_*(x) = \deg_4(x_4).\alpha' \in H^4(X'_0, \mathbf{Z}).$$

[α', resp. α, is the positive generator of $H^4(X'_0. \mathbf{Z})$, resp. $H^4(X_0, \mathbf{Z})$.]

(*b*) If $Z = X_0 \times P'$, where $P' \in X'_0$ is a point, then $[Z]_*(x) = \deg_0(x).1' \in H^0(X'_0, \mathbf{Z})$.

(*c*) If $Z = C \times C'$, where C, resp. C', is a curve on X_0, resp. X'_0, then

$$[Z]_*(x) = ([C].x_2).[C'] \in H^2(X'_0, \mathbf{Z})$$

(here $([C].x_2)$ denotes the cup product on H^2).

(*d*) If $Z \subset X_0 \times X'_0$ is an irreducible purely 2-dimensional analytic cycle projecting onto X_0 and X'_0 by generically finite maps of degree *d*, resp. *d'*, then

$$[Z]_*(1) = d'.1' \in H^0(X'_0, \mathbf{Z}),$$

and

$$[Z]_*(\alpha) = d.\alpha' \in H^4(X'_0, \mathbf{Z}).$$

(*e*) If $Z \subset X_0 \times X'_0$ is an irreducible purely 2-dimensional analytic cycle projecting onto X_0 by a generically finite map of degree d and onto a curve $C' \subset X'_0$, then

$$[Z]_*(1) = 0 \in H^0(X'_0, \mathbf{Z}),$$
$$[Z]_*(x_2) \in \mathbf{Z}.[C'] \subset H^2(X'_0, \mathbf{Z}),$$
$$[Z]_*(\alpha) = d.\alpha' \subset H^4(X'_0, \mathbf{Z}).$$

(*f*) If $Z \subset X_0 \times X'_0$ is an irreducible purely 2-dimensional analytic cycle projecting onto a curve $C \subset X_0$ and by a generically finite map of degree d' to X'_0, then

$$[Z]_*(1) = d'.1 \in H^0(X'_0, \mathbf{Z}),$$
$$[Z]_*(x_2) = ([C].x_2).x'_2,$$

$x'_2 \in H^2(X'_0, \mathbf{Z})$ depending on x_2.

$$[Z]_*(\alpha) = 0 \in H^4(X'_0, \mathbf{Z}).$$

Referring to the decomposition of Γ_0, this shows that Z_1, Z_2, Z_3, Z_4, Z_5 annihilate $H^{2,0}(X_0) \subset H^2(X_0, \mathbf{C})$. Since this subspace is non-trivial, Z_0 is non-empty. Write

$$Z_0 = \sum_i u_i.Z_{0i},$$

where Z_{0i} are the irreducible components of Z_0 of degrees d_i, resp. d'_i, over X_0, resp. X'_0.

By (*d*) above,

$$[Z_0]_*(1) = (\sum u_i.d'_i).1' \in H^0(X'_0, \mathbf{Z});$$

by (*b*) and (*f*) we conclude that

- $Z_2 = \varnothing$ and $Z_5 = \varnothing$;
- $Z_0 = Z_{0,1}$, say
- $Z_{0,1}$ projects birationally to X'_0.

Similarly, $Z_{0,1}$ projects birationally to X_0 and $Z_3 = \varnothing$ and $Z_4 = \varnothing$ (consider the action of Γ_0 on H^4).

But X_0 and X'_0 are absolutely minimal models; this implies that a birational map whose graph is contained in $X_0 \times X'_0$ is an isomorphism.

Q. E. D.

We continue in the proof of the main lemma by showing that

$$[\Gamma_0] = [\Delta_0] \in H^4(X_0 \times X'_0, \mathbf{Z}).$$

Since $X_0 \times X'_0$ is kählerian this will imply that all coefficients a_{ij} vanish so that $\Gamma_0 = \Delta_0$. To this end, we distinguish three cases according to the transcendence degree of X_0 (or, what amounts to the same, of X'_0).

Case 1: tr.deg. $(X_0) = 2$. — In this case X_0 and X'_0 are algebraic surfaces. Let $\eta \in H^2(X_0, \mathbf{Z})$ be the class of an ample divisor L on X_0.

Using the hypotheses on $[\Gamma_0]_* : H^2(X_0, Z) \to H^2(X'_0, Z)$ we conclude by the results of paragraph 3 that $\eta' = [\Gamma_0]_*(\eta)$ is the class of an ample divisor L′ on X'_0. We have, for any $n \gg 0$,

(i) $[\Gamma_0]_*(L^n) = L'^n$;

(ii) $\dim H^0(X_0, L^n) = \dim H^0(X'_0, L'^n)$.

Here (i) follows from the fact that numerical, homological, and rational equivalence all coincide on a K-3 surface; and (ii) results from the Riemann-Roch theorem.

Once these two facts are granted, the proof of theorem 2 in [13] shows that $\Gamma_0 = \Delta_0$.

Before treating the remaining two cases, we prove a lemma. Identify X_0 and X'_0 *via* Δ_0. Then Γ_0 becomes a cycle

$$\Gamma_0 = \Delta_0 + \sum a_{ij} (C_i \times C_j) \subset X_0 \times X_0,$$

where Δ_0 is the diagonal on $X_0 \times X_0$ and C_i and C_j [which is an abuse of notation for $\Delta_0^{-1}(C'_j)$] are curves on X_0.

Form the collection of curves on X_0:

$$E = \bigcup_i C_i \cup \bigcup_j C_j.$$

Decompose E into connected components

$$E = E_1 \cup E_2 \cup \ldots \cup E_k.$$

Then, if we denote by $E_{\mathbf{Z}}$ (resp. $E_{i\mathbf{Z}}$) the subgroup of $H^2(X_0, \mathbf{Z})$ spanned by the irreducible components of E (resp. E_i) we obtain a (not necessarily direct sum) decomposition

$$E_{\mathbf{Z}} = E_{1\mathbf{Z}} + \ldots + E_{k\mathbf{Z}}.$$

LEMMA 5.5. — *With the above notations,* $a_{ij} \neq 0 \Rightarrow C_i$ *and* C_j *lie in the same connected component* E_k *of* E.

In particular

$$[\Gamma_0]_*(E_{k\mathbf{Z}}) = E_{k\mathbf{Z}}.$$

Proof. — The last statement follows from the first one. Indeed, let $C \in E_{k\mathbf{Z}}$. Then $C.C_i = 0$, if $C_i \notin E_k$.

Thus

$$\begin{aligned}[\Gamma_0]_*(C) &= C + \sum_{i,j} a_{ij}(C.C_i).C_j \\ &= C + \sum_{\substack{i,j \\ C_i \in E_k}} a_{ij}(C.C_i).C_j \in E_{k\mathbf{Z}},\end{aligned}$$

since we assumed the first contention of the lemma.

Hence $[\Gamma_0]_*(E_{i\mathbf{Z}}) \subset E_{i\mathbf{Z}}$. The same argument shows

$$[\Gamma_0]_*^{-1}(E_{i\mathbf{Z}}) = [\Gamma_0]_*(E_{i\mathbf{Z}}) \subset E_{i\mathbf{Z}}.$$

Hence we are left to show the first statement.

Since $\Gamma_0 = \lim \Gamma_{t_i}$, and each Γ_{t_i} is connected, it is easy to see that the support of Γ_0 is connected. (This is a trivial analogue of Zariski's connectedness theorem.)

Let $C_i \times C_j$ be contained in the support of Γ_0. The condition that a point on $C_i \times C_j$ can be joined to Δ by a path lying in Γ_0 is expressed as follows: There exists a sequence of pairs of indices $(i, j) = (i_0, j_0), (i_1, j_1), \ldots, (i_r, j_r)$ such that

– $C_{i_k} \times C_{j_k} \subset \operatorname{supp}(\Gamma_0)$;

– $(C_{i_k} \times C_{j_k}) \cap (C_{i_{k+1}} \times C_{j_{k+1}}) \neq \varnothing$ for $k = 0, \ldots, r-1$ and $(C_{i_r} \times C_{j_r}) \cap \Delta_0 \neq \varnothing$.

This implies that there exists a path in E from a point on C_i to a point on C_j.

We have thus obtained: $a_{i,j} \neq 0 \Rightarrow C_i$ and C_j lie in the same connected component of E.

Q. E. D.

REMARK 5.6. — Let F_i be a collection of curves containing E_i and disjoint from E_j $(j \neq i)$; and let $F_{i\mathbf{Z}}$ be the subgroup of $H^2(X_0, \mathbf{Z})$ generated by the irreducible components of F_i. The preceding proof shows that

$$[\Gamma_0]_*(F_{i\mathbf{Z}}) = F_{i\mathbf{Z}}.$$

Case 2: tr. deg. $(X_0) = 1$. – In this case X_0 possesses an elliptic fibering

$$f : X_0 \to B$$

inducing an isomorphism between the field of meromorphic functions on X_0 and the function field of the non-singular algebraic curve B. Every irreducible, effective divisor on X_0 lies in a fiber of f. The intersection product in the Néron-Severi group of X_0 is negative semi-definite. The images α_i of the irreducible components of a reducible fiber of f form an "extended Dynkin diagram" in the sense of [5] (*see* also § 6 below). The vertices of the diagram are in one-to-one correspondence with the α_i, and α_i is joined to α_j by $\alpha_i . \alpha_j$ edges (for $i \neq j$). For all these facts *see* [9].

In our case, a connected component E_i of E must lie in a fiber F_i of f. The isomorphism $[\Gamma_0]_*$ preserves the cohomology class of the fibering $\beta \in H^2(X_0, \mathbf{Z})$, and hence, by lemma 5.5 and remark 5.6, $[\Gamma_0]_*$ induces an automorphism of the subgroup $F_{i\mathbf{Z}} \subset H^2(X_0, \mathbf{Z})$ generated by the classes of irreducible components of F_i. Thus, $[\Gamma_0]_*$ induces an automorphism of this lattice which verifies all the hypotheses of lemma 2 of the next paragraph. Hence $[\Gamma_0]_*$ acts trivially on the lattice in $H^2(X_0, \mathbf{Z})$ generated by any of the fibers of f, i. e. on the entire Néron-Severi group. Since β is the generator of the radical of the intersection form on the Néron-Severi group, $[\Gamma_0] = [\Delta_0] + n(\beta \times \beta)$.

The following lemma implies that $n = 0$:

LEMMA 5.7. — *Let* X_0 *be a surface of kählerian type and let*

$$\beta \in H^2(X_0, \mathbf{Z})$$

be the cohomology class of an effective divisor with $\beta^2 = 0$.

Let $\Gamma_0 \subset X_0 \times X_0$ *be a cycle such that*

$$[\Gamma_0]_* : \ H^2(X_0, \mathbf{Z}) \to H^2(X_0, \mathbf{Z})$$

is an automorphism of the cohomology lattice. If

$$[\Gamma_0] = [\Delta_0] + n.(\beta \times \beta), \qquad n \geqq 0,$$

then $[\Gamma_0] = [\Delta_0]$.

Proof. — By the formulae for $[\Gamma_0]_*$ previously employed,

$$[\Gamma_0]_* \ (x) = x + n.(\beta.x).\beta, \qquad x \in H^2(X_0, \mathbf{Z}),$$

and

$$[\Gamma_0]_*^{-1}(y) = y + n.(\beta.y).\beta, \qquad y \in H^2(X_0, \mathbf{Z}).$$

This shows

$$x = [\Gamma_0]_*^{-1}([\Gamma_0]_*(x)) = [\Gamma_0]_*^{-1}(x + n(x.\beta)\beta)$$
$$= x + n.(x\beta).\beta + n(x.\beta).\beta.$$

This being valid for all $x \in H_2(X_0, \mathbf{Z})$, this shows that $2n\beta = 0$, i. e. $n = 0$.

Q. E. D.

Case 3: tr. deg. $(X_0) = 0$. — In this case, X_0 contains only finitely many irreducible curves [9]; they are non-singular rational curves of self-intersection -2 (this follows from the genus formula). The intersection form on the Néron-Severi group is negative-definite.

We infer that the irreducible effective divisors on X_0 form the direct sum of Dynkin diagrams with roots of equal lengths (cf. [2]). [Again α_1 is joined to α_j by $(\alpha_i.\alpha_j)$ edges, $i \neq j$.] Let $F_1, \ldots, F_k$ denote the connected components of the collection F of all curves on X_0.

Lemma 5.5 and remark 5.6 imply $[\Gamma_0]_*(F)_{i\mathbf{Z}} = F_{i\mathbf{Z}}$. Lemma 1 of the next paragraph applied to the map $[\Gamma_0]_* : F_{i\mathbf{Z}} \to F_{i\mathbf{Z}}$, shows that $[\Gamma_0]_*$ is the identity on $F_{i\mathbf{Z}}$. Since the intersection form is definite on the Néron-Severi group, $[\Gamma_0] = [\Delta_0]$, as desired.

Conclusion of the Proof of the Main Lemma. — We have shown that $\Gamma_0 = \Delta_0$, so that X_0 and X'_0 are isomorphic by an isomorphism which induces the given isomorphim φ^* on cohomology. By the local Torelli theorem, there is an open neighborhood U of 0 in S and an isomorphism

$$\varphi_U : \ X' \times_S U \to X \times_S U$$

inducing the given isomorphism φ^*, and whose fiber at 0 is Δ_0. The uniqueness assertion of paragraph 1 shows that the fiber of φ_U over $t \in T \cap U$ is the original $\varphi_t : X'_t \xrightarrow{\sim} X_t$.

As remarked in the beginning of this paragraph, the assertion is local on S around 0; hence the Main Lemma is proved.

REMARK 5.8. — We conclude this paragraph with an example showing that conditions (ii) and (iii) of Theorem 1 are independent of one another. We use the terminology of paragraph 7.

Let X be a K-3 surface which contains a collection E of nodal curves arising from the resolution of a rational double point. The Weyl group of the associated root system acts on $H^2(X, \mathbb{Z})$ and preserves the Hodge structures and the cone $V^+(X)$. In particular, the "opposite involution" w_0 in this Weyl group acts on $H^2(X, \mathbb{Z})$. But w_0 sends the class of an effective cycle of self-intersection -2 supported on E into an anti-effective cycle. Hence, if all effective cycles of self-intersection -2 in X are supported on E, then $-w_0$ is an automorphism of $H^2(X, \mathbb{Z})$ which satisfies conditions (i) and (iii) of Theorem 1, but not (ii).

6. The Lemmas on Dynkin Diagrams

The purpose of this paragraph is to supply the lemmas necessary to complete the proof of the main lemma for non-algebraic surfaces: lemma 1 deals with the case arising from surfaces whose field of meromorphic functions has transcendence degree 0, and lemma 2 deals with the case of transcendence degree 1.

It should be noted that lemma 1 is a special case of lemma 2. We present lemma 1 because its proof, suggested to us by R. P. Langlands, avoids case-by-case checking (the proof actually applies to all reduced, irreducible root systems and not just the ones with symmetric Cartan matrix). We were unable, however, to find a proof of the more general lemma 2 which doesn't use the classification of root systems. The general reference for this paragraph is [5].

In this paragraph we adopt the conventions of [5]; in particular, the Cartan matrices of this paragraph are the *negatives* of the intersection matrices used elsewhere in the paper.

Let R be a reduced, irreducible root system in a real vector space V. *We suppose that all roots have the same length* (i. e. that R is of type A, D or E). Let $B = \{ \alpha_1, \ldots, \alpha_n \}$ be a fundamental system of simple roots, with Cartan matrix $N = (n_{ij})$, which is then symmetric. The corresponding dual roots $\{ \check{\alpha}_1, \ldots, \check{\alpha}_n \} \subset \check{V}$ satisfy the conditions

$$\langle \alpha_i, \check{\alpha}_j \rangle = n_{ij},$$

and the system of fundamental weights is given by $\{ \omega_1, \ldots, \omega_n \} \subset V$ such that

$$\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}.$$

Thus $\alpha_i = \sum_j n_{ij} \omega_j$. *The fundamental chamber* $C \subset V$ is defined by

$$C = \{ x \in V \mid \langle x, \check{\alpha}_i \rangle > 0, \, i = 1, \ldots, n \},$$

so that we have

$$C = \{ x \in V \mid x = \sum r_i \omega_i, \, r_i > 0, \, i = 1, \ldots, n \}.$$

Put on V the positive-definite inner product given by

$$(\alpha_i, \alpha_j) = \langle \check{\alpha_i}, \alpha_j \rangle.$$

The dual cone C^* to C is given by

$$C^* = \{ x \in V \mid (x, x') > 0, \forall x' \in C \},$$

i. e.:

$$C^* = \left\{ x \in V \mid x = \sum_{i=1}^{n} r_i \alpha_i, \ r_i > 0 \right\}.$$

We can now state the first lemma.

LEMMA 6.1. — *Let φ be a non-singular orthogonal transformation of* V *preserving the lattice* M *generated by the α_i, and preserving the cone* C^*. *If φ can be written in terms of the basis (α_i) of* V *as a matrix of the form* I-AN, *with* A *a matrix of non-negative integers, then φ is the identity.*

Proof. — Since φ preserves a lattice and is orthogonal, it is of finite order. Therefore, $\varphi(C^*) \subset C^*$ and $\varphi(M) \subset M$ imply

$$\varphi(C^*) = C^* \quad \text{and} \quad \varphi(M) = M.$$

Since the α_i are extermals of C^* and primitive elements of M, φ permutes the α_i among themselves. Let

$$\mu = I - \varphi,$$

i. e. the transformation μ is given by the matrix A.N with respect to the basis (α_i) of V. From the expression of the ω_i's in terms of the α_i

$$\omega_i = N^{-1}(\alpha_i),$$

we see

$$(\omega_k, \mu(\omega_i)) = (\omega_k, A(\alpha_i)) = \sum_j a_{ij}(\omega_k, \alpha_j) \geqq 0,$$

for each i, k, since each $a_{ij} \geqq 0$. Hence

$$\mu(C) \subset \overline{C^*},$$

where $\overline{C^*}$ = closure of C^*. But φ preserves lengths. This implies

$$(x, x) = (\varphi(x), \varphi(x)) = (x, x) - (x, \mu(x)) - (\mu(x), \varphi(x)).$$

Let $x \in C$. Then $\varphi(x) \in C$ and the last two terms are non-positive. This implies that

$$(x, \mu(x)) \quad = 0$$
$$(\mu(x), \varphi(x)) = 0,$$

so that $(\mu(x), \mu(x)) = 0$, i. e. $\mu(x) = 0$.

Q. E. D.

To state the second lemma, we form the vector space $\tilde{V}$ generated by $\alpha_1, \ldots, \alpha_n$ (as above) and a new basis vector β. We extend the previous bilinear form on V to $\tilde{V}$ via

$$(\alpha_i, \alpha_j) = n_{ij},$$
$$(\alpha_i, \beta) = (\beta, \beta) = 0.$$

This form is semi-definite, with radical spanned by β. Let $\tilde{\alpha} = \sum_{i=1}^{n} n_i \alpha_i$ be the highest positive root with respect to $(\alpha_1, \ldots, \alpha_n)$, and set $\alpha_0 = \beta - \tilde{\alpha}$.

It is known that each n_i is > 0. The matrix $\tilde{N} = (n_{ij})$, $n_{ij} = (\alpha_i, \alpha_j)$, $i, j = 0, 1, \ldots, n$, (which describes the above inner product) is called the *extended Cartan matrix*. Let

$$\tilde{C}^* = \left\{ x \in \tilde{V} \mid x = \sum_{i=0}^{n} r_i \alpha_i, r_i > 0 \right\}.$$

LEMMA 6.2. — *Let φ be an invertible linear transformation of $\tilde{V}$ which preserves the bilinear form, the lattice generated by $\alpha_0, \alpha_1, \ldots, \alpha_n$ and the cone $\tilde{C}^*$. If φ may be written in matrix form*

$$I - A . \tilde{N}$$

with respect to the basis $(\alpha_0, \ldots, \alpha_n)$, where A has non-negative coefficients, then φ is the identity.

Proof. — We first prove that φ is of finite order. Since $\tilde{N}(\beta) = 0$, we get that $\varphi(\beta) = \beta$. Let $x \in V$. Then

$$\varphi(x) = \bar{\varphi}(x) + l(x) . \beta,$$

where $\bar{\varphi} : V \to V$ is the linear transformation induced by φ on the quotient $V = \tilde{V}/\mathbf{R}.\beta$ by φ and $l : V \to \mathbf{R}$ is a linear form on V. Since $\bar{\varphi}$ preserves the lattice generated by the α_i in V and is orthogonal, it is of finite order. So there exists an n such that

$$\varphi^n(x) = x + l'(x) . \beta$$

for some linear form l' on V which takes non-negative values on C*. Let $\varphi' = \varphi^n$ and let β_i be the element which is mapped by φ' to α_i. Then there exist integers k_i such that

$$\alpha_i = \beta_i + k_i \beta,$$

i. e.:

$$\beta_i = \alpha_i - k_i \beta.$$

Thus, $k_i = l'(\alpha_i) \geqq 0$.

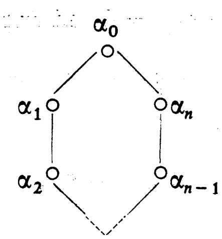
If k_i were non-zero, we would have that $\beta_i \in -\tilde{C}^*$ so that $\varphi'(\beta_i) \in -\tilde{C}^*$. But $\varphi'(\beta_i) = \alpha_i$, so all k_i vanish and $\varphi' = \mathrm{id}$. Hence φ is of finite order. Consequently φ preserves the cone $\tilde{C}^*$. Since the α_i are extremals of this cone and primitive they are permuted among themselves.

Now the α_i form an "extended Dynkin diagram" D, and we have to see that φ induces the trivial automorphism of $\tilde{D}$. [The vertices of $\tilde{D}$ are the α_i, and for $i \neq j$, α_i is joined to α_j by $-(\alpha_i . \alpha_j)$ edges.]

At this point we must rely on the classification of the possible $\tilde{D}$'s (types $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ of [5]), and check each type. Our result follows in each case from a consideration of congruences modulo the index of connectivity of the Dynkin diagram D associated to $\tilde{D}$ (gotten from $\tilde{D}$ by deleting α_0 and all edges abutting at α_0; equivalently, the Dynkin diagram associated to the root system $\alpha_1, \ldots, \alpha_n$). $\tilde{E}_8$ admits no non-trivial automorphism, which is fortunate since the index of connectivity of E_8 is 1!

Let (ε_{ij}) be the entries of the matrix $A . \tilde{N}$: since φ permutes the α_i's, each $\varepsilon_{i,j}$ can be only 0, or -1, and in any row, either all entries are 0, or two of them are non-zero and of opposite sign.

Case $\tilde{A}_n$ $(n \geqq 2)$:



If the i-th row of $A . \tilde{N}$ is non-trivial, we get

$$
\begin{aligned}
&-a_{i,n} \quad +2.a_{i,0} - a_{i,1} = \varepsilon_{i,0}, \\
&-a_{i,0} \quad +2.a_{i,1} - a_{i,2} = \varepsilon_{i,1}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
&-a_{i,n-1} + 2.a_{i,n} - a_{i,0} = \varepsilon_{i,n}.
\end{aligned}
$$

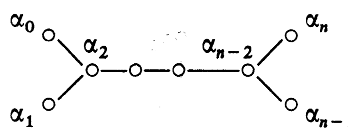
From this we get

$$\sum_{j=0}^{n} (j+1).\varepsilon_{i,j} = -(n+1).a_{i,n-1} + (n+1).a_{i,n}.$$

But the left hand side is a non-zero integer of absolute value $< n$; it cannot be divisible by $n+1$.

We leave it to the reader to treat the case $\tilde{A}_1$ (consider divisibility by 2).

Case $\tilde{D}_n$ $(n \geqq 4)$:



If φ induces a non-trivial automorphism of $\tilde{D}_n$ it has to move one of the vertices α_0, α_1, α_{n-1}, α_n. By symmetry we may suppose that $\varphi(\alpha_0) \neq \alpha_0$. Then we get

$$\begin{aligned}
+2.a_{0,0}-a_{0,2} &= \varepsilon_{0,0} = -1,\\
+2.a_{0,1}-a_{0,2} &= \varepsilon_{0,1},\\
-a_{0,0}-a_{0,1}+2.a_{0,2}-a_{0,3} &= \varepsilon_{0,2} = 0,\\
-a_{0,2}+2.a_{0,3}-a_{0,4} &= \varepsilon_{0,3} = 0,\\
&\ \vdots\\
-a_{0,n-4}+2.a_{0,n-3}-a_{0,n-2} &= \varepsilon_{0,n-3} = 0,\\
-a_{0,n-3}+2.a_{0,n-2}-a_{0,n} &= \varepsilon_{0,n-2} = 0,\\
-a_{0,n-2}+2.a_{0,n-1} &= \varepsilon_{0,n-1},\\
-a_{0,n-2}+2.a_{0,n} &= \varepsilon_{0,n}.
\end{aligned}$$

Now $\varphi(\alpha_0)$ has to be α_1: otherwise $\varepsilon_{0,1} = 0$; adding the first two equations we get

$$2.a_{0,0}+2.a_{0,1}-2a_{0,2} = -1.$$

This is impossible (divisilibity by 2).

If $\varphi(\alpha_0) = \alpha_1$, i. e. $\varepsilon_{0,1} = 1$, then $\varepsilon_{0,n-1} = \varepsilon_{0,n} = 0$. So $a_{0,n-2}$ is even. Add the last two equations to get

$$-2.a_{0,n-2}+2.a_{0,n-1}+2.a_{0,n} = 0,$$

i. e.:

$$a_{0,n-2} = a_{0,n-1}+a_{0,n}.$$

The equation for $\varepsilon_{0,n-2}$ now implies

$$a_{0,n-3} = a_{0,n-2}.$$

The equation for $\varepsilon_{0,n-3}$ implies

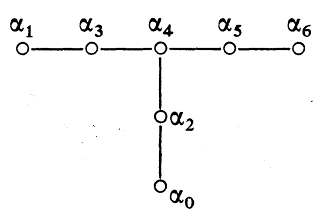
$$a_{0,n-4} = a_{0,n-3},$$

etc., until

$$a_{0,2} = a_{0,n-2}.$$

But the first equation shows that a_{02} is odd – a contradiction.

Case $\tilde{E}_6$:



If non-trivial, φ has to move one of α_0, α_1, α_6; by symmetry (and after renumbering) we may assume that $\varphi(\alpha_0) = \alpha_1$. We get

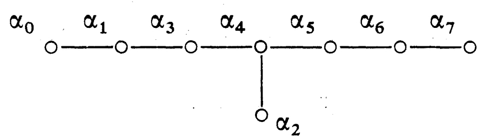
$$\begin{aligned} +2.a_{0,0}-a_{0,2} &= \varepsilon_{0,0} = -1, \\ 2.a_{0,1}-a_{0,3} &= \varepsilon_{0,1} = 1, \\ -a_{0,0}+2.a_{0,2}-a_{0,4} &= \varepsilon_{0,2} = 0. \end{aligned}$$

From these equations we get

$$\begin{aligned} -1 &= \varepsilon_{0,0}+2.\varepsilon_{0,2} = 3.a_{0,2}-2.a_{0,4}, \\ 0 &= \varepsilon_{0,6}+2.\varepsilon_{0,5} = 3.a_{0,5}-2.a_{0,4}. \end{aligned}$$

Hence $1 = 3.a_{0,5}-3.a_{0,2}$: a contradiction (divisibility by 3).

Case E_7:



If φ were non-trivial, it must interchange α_0 and α_7. This gives us the following system of equations :

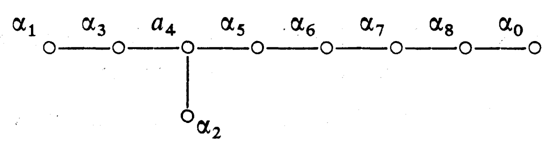
$$\begin{aligned} +2a_{0,0}-a_{0,1} &= \varepsilon_{0,0} = -1, \\ -a_{0,0}+2a_{0,1}-a_{0,3} &= \varepsilon_{0,1} = 0, \\ 2a_{0,2}-a_{0,4} &= \varepsilon_{0,2} = 0, \\ -a_{0,1}+2a_{0,3}-a_{0,4} &= \varepsilon_{0,3} = 0, \\ -a_{0,2}-a_{0,3}+2a_{0,4}-a_{0,5} &= \varepsilon_{0,4} = 0, \\ -a_{0,4}+2a_{0,5}-a_{0,6} &= \varepsilon_{0,5} = 0, \\ -a_{0,5}+2a_{0,6}-a_{0,7} &= \varepsilon_{0,6} = 0, \\ -a_{0,6}+2a_{0,7} \phantom{-a_{0,7}} &= \varepsilon_{0,7} = 1. \end{aligned}$$

Hence

$$\begin{aligned} \varepsilon_{0,0}+2\varepsilon_{0,1}+3\varepsilon_{0,3} &= -1 = 4a_{0,3}-3a_{0,4}, \\ \varepsilon_{0,7}+2\varepsilon_{0,6}+3\varepsilon_{0,5} &= 1 = 4a_{0,5}-3a_{0,4}. \end{aligned}$$

So $2 = 4a_{0,5} = 4a_{0,3}$: a contradiction (divisibility by 2).

Case $\tilde{E}_8$:



This diagram has no non-trivial automorphism.

Q. E. D.

7. Degeneration of Isomorphisms

In this paragraph we analyze non-separatedness of the moduli of unpolarized surfaces; recall that the moduli space M previously introduced is not a Hausdorff space (cf. the introduction).

We introduce the concept of an *elementary operation*. For the sake of definiteness we consider families of surfaces over the disc $D \subset \mathbf{C}$: Let

$$p : X \to D$$

be a smooth family of complex-analytic surfaces (i. e. p is smooth of relative dimension 2).

Let $C_0 \subset X_0$ be an irreducible, complete, non-singular rational curve of self-intersection -2; in what follows, we'll call such curves *nodal curves*. The curve C_0 can be blown down in the family X, i. e. there exists a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "p"'] & & \overline{X} \arrow[dl, "\bar{p}"] \\ & D & \end{tikzcd}$$

where

— $\bar{p}$ is a flat morphism;

— π is a proper morphism which induces the minimal desingularization of $\overline{X}_t$, for all $t \in D$;

— $\overline{X}_0$ is the surface obtained from X_0 by contracting C_0 to a point.

Furthermore, this diagram is unique, locally around $0 \in D$.

Indeed, the situation is local around C_0 inside X and is then unique. The above assertions follow thus by appealing to [2] and [6].

There are two possibilities: either $\overline{X}_t$ is singular for all $t \in D$ (in which case we say that C_0 extends to X), or $\overline{X}_t$ is non-singular for $t \neq 0$. By [6], if C_0 doesn't extend, we may resolve the family $\bar{p} : \overline{X} \to D$ in a different way: there exists a smooth morphism $p' : X' \to D$ whose fiber X'_0 is the minimal desingularization of $\overline{X}_0$. Hence X_0 and X'_0, as minimal desingularizations of $\overline{X}_0$, are canonically isomorphic even though the families X and X′ over D are distinct. The morphism p' is uniquely determined by these properties, locally around $0 \in D$.

The process which leads from X to X′ will be called *the elementary operation corresponding to the* (non-extending) *nodal curve* $C_0 \subset X_0$ (or just an *elementary operation*).

Note that, if $C'_0 \subset X'_0$ is the inverse image of the singular point on $\overline{X}_0$, then X is obtained from X′ by the elementary operation corresponding to $C'_0 \subset X'_0$. Hence the relation

"X′ is obtained from X by a finite number of elementary operations" defines an equivalence relation between families of smooth surfaces over D, always taken locally around $0 \in D$. Note that elementary operations can be defined in a similar way for a family of smooth complex-analytic surfaces over an arbitrary base space and for a family of smooth algebraic surfaces over an arbitrary base scheme (cf. [1]).

After these preliminary considerations we can state a conjecture:

CONJECTURE 7.1. — *Let k be an algebraically closed field and let* $S = \mathrm{Spec}(k[[t]])$; *let* 0, *resp.* η, *denote the special, resp. generic, point of* S.

Let

$$p : X \to S \quad \textit{and} \quad p' : X' \to S$$

be two proper and smooth morphisms of relative dimension 2 *whose fibres are absolutely minimal models [in particular, no fibre is (birationally equivalent to) a ruled surface]. If*

$$X'_\eta \simeq X_\eta,$$

then X′ *is obtained from* X *by a successive application of finitely many elementary operations.*

The statement of this conjecture may be modified: one may allow S to be a base scheme of higher dimension or one may formulate analogues for families of complex-analytic surfaces (cf., however, remark 7.9).

We can't prove this conjecture. However, if char $(k) = 0$, an affirmative answer to 7.1, is provided, via the Lefschetz principle, by theorem 2 below.

REMARK 7.2. — It is possible to formulate a conjecture similar to 7.1 without assuming that all fibers of p, resp. p', are absolutely minimal models, but retaining the assumption that none of the fibers is ruled. However, as simple examples show, the formulation 7.1 becomes wrong if X_0 is not assumed to be absolutely minimal.

The assumption that the fibers be absolutely minimal models is stable under deformations (the proof in [10] also covers the case of hyperelliptic surfaces).

REMARK 7.3. — A statement similar to conjecture 7.1 is wrong in the case of ruled surfaces. Indeed, it may happen (cf., e. g., [14], ch. 1) that the trivial family of rational ruled surfaces F_n over D^* jumps over the origin to F_m $(0 \leqq n < m; n \equiv m \pmod 2)$. Since only F_2 contains nodal curves, such a family cannot be obtained from the trivial family $F_n \times D$ by elementary operations.

In this example the two families under consideration don't have isomorphic fibres over the origin. There exist, however, families of ruled surfaces over D which are fiber by fiber isomorphic but which are not isomorphic. This is related to the fact that the see-saw lemma for line bundles fails for vector bundles of higher rank.

Sometimes rational singularities can be resolved in a family (cf. [1]) and one may ask whether this can be done in several ways. Hence conjecture 7.1 is related to the following conjecture. In its statement we adhere to the terminology of [1].

CONJECTURE 7.4. — *Let* $\widetilde{\mathrm{Res}}_s$, *resp.* $\widetilde{\mathrm{Def}}_s$, *be the henselization at the origin of the resolution space, resp. the deformation space, of a rational singularity. Then the local morphism*

$$\widetilde{\mathrm{Res}}_s \to \widetilde{\mathrm{Def}}_s$$

is a Galois covering and its group of automorphisms is the finite Coxeter group generated by the reflections corresponding to the nodal curves in the minimal resolution of the singularity operating on the vector space having as base all irreducible components of the minimal resolution of the singularity (the definition of those reflections may be found in 7.7) ([2]). This conjecture is true for rational double points (cf. [6]) and for the rational singularity defined by a cone over a rational space curve of degree r in $\mathbf{P}^n$ ($r \geqq 2$), cf. [1]. Closely related to conjecture 7.1 is theorem 1 in [13]. Since we'll need it later we give a (slightly simplified) statement and proof in the complex-analytic context:

LEMMA 7.5. — *Let* $p : X \to D$ *and* $p' : X' \to D$ *be two proper and smooth morphisms of complex-analytic varieties. Let*

$$\varphi : \quad X \times_D D^* \xrightarrow{\sim} X' \times_D D^*$$

be an isomorphism whose graph $\Gamma^* \subset (X \times_D X') \times_D D^*$ *extends to an analytic cycle* $\Gamma \subset X \times_D X'$. *If* X_0 *is not ruled* (*i. e., is not bimeromorphic to a* $\mathbf{P}^n$*-bundle over a variety of lower dimension*), *there is a unique component of multiplicity one of* Γ_0 *inducing a bimeromorphism between* X_0 *and* X'_0.

The following corollary is an immediate consequence of this lemma.

COROLLARY 7.6. — *With the notations of the above lemma, assume in addition that all fibres of* p, resp. p', *are absolutely minimal models of surfaces. Then* Γ_0 *is of the form*

$$\Gamma_0 = \Delta_0 + \sum a_{ij} C_i \times C'_j, \qquad a_{ij} \geqq 0,$$

where Δ_0 *defines an isomorphism between* X_0 *and* X'_0 *and where* C_i, *resp.* C'_j, *are curves on* X_0, *resp.* X'_0.

Proof of lemma 7.4. — The proof of lemma 5.4, shows that there is a unique irreducible component Z_0 of Γ_0 which projects onto X_0 by a morphism of degree 1.

It suffices to show that Z_0 projects *onto* X'_0 since, by the same argument as the one used in 5.4, the projection of Z_0 onto X'_0 will then be of degree 1 and the assertion will follow. Assume the contrary.

By Hironaka, there exists a blow-up (successively along non-singular centers over the origin) $\tilde{X}$ of X and a commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "\pi"] \arrow[dd] & & \\
& \Gamma \arrow[dl] \arrow[dr] & \\
X & & X'.
\end{tikzcd}$$

Let X_0 = proper transform of X_0 in X.

([2]) This has been independently conjectured by J. Wahl.

Then

$$\pi(\tilde{X}_0) = Z_0.$$

We may blow-up X′ to obtain $\tilde{X}'$ and the following commutative diagram :

$$\begin{tikzcd} \tilde{X} \arrow[d] \arrow[dr] & \tilde{X}' \arrow[l] \arrow[dl] \arrow[d, "\rho"] \\ & \Gamma \arrow[dl] \arrow[dr] \\ X & X' \end{tikzcd}$$

Let W_0 = proper transform of $\tilde{X}_0$ in $\tilde{X}'$.

Then $W_0 \subset \rho^{-1}(X'_0)$ and W_0 maps bimeromorphically to $\tilde{X}_0$. The morphism $\rho \mid W_0$ factors through $Z_0 \subset \Gamma_0$. Hence, by assumption,

$$\rho(W_0) \underset{\neq}{\subset} X'_0.$$

Decompose $\tilde{X}'_0$ into irreducible components :

$$\tilde{X}'_0 = Y_1 \cup \ldots \cup Y_r.$$

One of them, say Y_1, is the proper transform of X'_0 in $\tilde{X}'$. The other ones, being bimeromorphic to the total transform of a non-singular center of a blow-up of a smooth variety, are ruled. Since W_0 doesn't project *onto* X'_0, it cannot be Y_1. Hence W_0 is ruled. This contradicts the hypothesis made on X_0.

REMARK 7.7. — In the notation of corollary 7.5, we will always identify X_0 with X'_0 via the isomorphism defined by Δ_0.

REMARK 7.8. — Assume that X′ is obtained from X by an elementary operation corresponding to $C_0 \subset X_0$. Let

$$\Gamma \subset X \times_D X'$$

be the correspondence between X and X′; it is the graph of an isomorphism outside 0. Identifying X_0 with X'_0 (both being the minimal resolution of the singular surface $\overline{X}_0$),

$$\Gamma_0 = \Delta_0 + C_0 \times C_0, \qquad \Delta_0 = \text{diagonal}.$$

In particular

$$[\Gamma_0]_* : \quad H^2(X_0, \mathbf{Z}) \xrightarrow{\sim} H^2(X_0, \mathbf{Z})$$

is the reflection defined by C_0,

$$x \to x + (x.C_0).C_0.$$

Indeed, since X and X′ are naturally isomorphic outside C_0, Γ_0 has to be of the form

$$\Gamma_0 = \Delta_0 + n(C_0 \times C_0), \qquad n \geqq 0$$

and, since $[\Gamma_0]_* : H^2(X, Z) \xrightarrow{\sim} H^2(X', Z)$ is an isomorphism of lattices we conclude that $n = 0$ or $n = 1$. Since C_0 doesn't extend, Γ is not the graph of an isomorphism between X and X′, i. e. $n = 1$.

A similar argument shows that the identification of X_0 with X'_0 given by 7.6, coincides with the one obtained by regarding X_0 and X'_0 as minimal desingularizations of $\overline{X}_0$.

The main result of this paragraph is the following theorem :

THEOREM 2. — *Let*

$$p : X \to D \quad \textit{and} \quad p' : X' \to D$$

be two families of compact smooth surfaces of kählerian type whose fibres are absolutely minimal surfaces [in particular, no fibre is (birationally equivalent to) a ruled surface]. Assume that an isomorphism

$$\varphi : X \times_D D^* \xrightarrow{\sim} X' \times_D D^*$$

is given, whose graph

$$\Gamma^* \subset (X \times_D X') \times_D D^*$$

extends to an analytic cycle

$$\Gamma \subset X \times_D X'.$$

Then, locally around $0 \in D$, *the family* X′ *is obtained from* X *by a successive application of finitely many elementary operations.*

REMARK 7.9. — The assumption that Γ^* extend to Γ is verified if p, p' and φ come from a global algebraic situation, i. e. p, p' and φ come by restriction to a small neighborhood D (in the classical topology) of a point of a smooth algebraic curve C of

– proper smooth algebraic morphisms $p : X \to C$, $p' : X' \to C$;

– an isomorphism $\varphi : X \times_C (C \setminus \{0\}) \xrightarrow{\sim} X' \times_C (C \setminus \{0\})$.

This is the main case of interest.

On the other hand, this extension condition is indeed a non-trivial assumption : Let

$$X = \mathbf{C}^2/L, \qquad X' = \mathbf{C}^2/L$$

be two trivial families of tori and let

$$\varphi_t : X_t \xrightarrow{\sim} X'_t$$
$$(x, y) \bmod L \to (x + e^{1/t}, y) \bmod L$$

define an isomorphism between $X \times_D D^*$ and $X' \times_D D^*$. Its graph

$$\Gamma^* \subset (X \times_D X') \times_D D^*$$

doesn't extend to an analytic cycle in $X \times_D X'$.

Proof of theorem 2. — It suffices to show that there exists a finite succession of elementary operations leading from X to a family $p'' : X'' \to D$ such that, if $\Gamma \subset X' \times_D X''$ denotes the natural correspondence obtained,

$$[\Gamma_0]=[\Delta_0]\in H^4(X_0\times X_0, \mathbf{Z}).$$

Indeed, by corollary 7.5, Γ_0 is of the form

$$\Gamma_0=\Delta_0+\sum a_{ij}C_i\times C_j,\qquad a_{ij}\geqq 0.$$

Since $X_0\times X_0$ is of kählerian type, we infer that $\Gamma_0=\Delta_0$.

Hence the projections $\Gamma\to X$, resp. $\Gamma\to X'$, are isomorphisms fiberwise over D; since X and X′ are smooth, in particular normal, they are isomorphisms.

The proof of theorem 2 will proceed by checking each case in the classification of surfaces. We thus have to treat tori, hyperelliptic surfaces, K-3 surfaces, Enriques surfaces, elliptic surfaces and surfaces of general type (note that these classes are stable under deformations, cf. [10]). We note that the isomorphism

$$\varphi:\quad X\times_D D^*\xrightarrow{\sim}X'\times_D D^*$$

induces isomorphisms over D

$$\varphi^*:\quad R^i p'_*(\mathbf{Z})\xrightarrow{\sim}R^i p_*(\mathbf{Z}),\qquad i=0,\ldots,4,$$

between the (trivial) local systems of cohomology.

First case: tori.—This case is trivial: We may trivialize the universal covering space $\tilde{X}$ of X:

$$\tilde{X}\xrightarrow{\sim}\mathbf{C}^2\times S.$$

Then

$$X_s=\mathbf{C}^2/H_1(X_s,\mathbf{Z}).$$

Proceeding in the same way with X′ we see that the isomorphism

$$\varphi^*:\quad R^1 p'_*(\mathbf{Z})\xrightarrow{\sim}R^1 p_*(\mathbf{Z})$$

defines an isomorphism over D (extending the given one over D*) between X and X′.

[This conforms with the fact that the moduli space of tori (with trivialized cohomology) is an open subset of the vector space of 2×2—matrices—which is separated. Also note that a torus doesn't contain any curves with negative self-intersection.]

Second case: hyperelliptic surfaces.—These are the algebraic surfaces with $b_2=2$ and $p_g=0$. They are of the form

$$X=E\times E'/G,$$

where

- E and E′ are smooth curves of genus one;
- G is a finite group of automorphisms of $E\times E'$. Let $t\in D^*$ and let

$$X_t=E_t\times E'_t/G.$$

The coverings $\tilde{X}$, resp. $\tilde{X}'$, of X, resp. X′, corresponding to the factor group G of $\pi_1(X) = \pi_1(X')$ are two families of compact surfaces whose fibres over $t \in D^*$ are tori. Hence they both are families of tori. Since they are isomorphic over D^*, they are isomorphic (cf. first case). This implies that X and X′ are isomorphic. (Note that on a hyperelliptic surface there are no curves with negative self-intersection.)

Before proceeding in the proof of theorem 2, we insert the following remark.

REMARK 7.10.—Let

$$V = \{ x \in H_{\mathbf{R}}^{1,1}(X_0) \mid x^2 > 0 \}$$

and let

$$V^+ = \text{connected component of V containing a Kähler class } I$$

(cf. § 2).

Let

$$\Phi = \{ \delta_1, \ldots, \delta_n \}$$

be a finite set of nodal curves on X_0. Let W be the group generated by the reflections s_δ $(\delta \in \Phi)$ operating on V^+. Then

$$V_\Phi^+ = \{ x \in V^+ \mid (x \, . \, \delta_i) > 0, \delta \in \Phi \}$$

is a fundamental domain for the properly discontinuous action of W on V^+. The Kähler class I lies in V_Φ^+.

Let

$$\varphi : \ H^2(X_0, \mathbf{Z}) \xrightarrow{\sim} H^2(X_0, \mathbf{Z})$$

be an automorphism of lattices which preserves Hodge structures and such that $\varphi(V^+) = V^+$.

Assume that $\varphi(I)$ *doesn't lie on any hyperplane corresponding to a reflection in* W. Then

$$\varphi(I) \in w(V_\Phi^+)$$

for a uniquely determined element $w \in W$ and *we claim that* w *may be written as*

$$w = s_{\delta_r} \circ \ldots \circ s_{\delta_1}, \qquad \delta_i \in \Phi,$$

where $s_{\delta_i} \circ \ldots \circ s_{\delta_1}(\varphi(I))$ *lies on the negative side of the hyperplane*

$$H_{\delta_{i+1}} = \{ x \in H_{\mathbf{R}}^{1,1} \mid (x \, . \, \delta_{i+1}) = 0 \}$$

$(i = 1, \ldots, r)$.

The proof of this assertion is immediate by induction on the length of the representation of w as a product of reflections about hyperplanes H_{δ_i} bordering V_Φ^+. In particular, let

$$\varphi = [\Gamma_0]_* : \ H^2(X_0, \mathbf{Z}) \xrightarrow{\sim} H^2(X_0, \mathbf{Z}).$$

Then every reflection s_{δ_i} $(i = 1, \ldots v)$ in the presentation of w above may be realized by an elementary operation (recall, cf. remark 7.8, that an elementary operation defines a reflection on the cohomology). Indeed, assume the contrary and let i, $1 \leqq i \leqq r$, be the smallest index for which δ_i corresponds to a nodal curve C_i *which extends* to

$$p^{(i-1)}: \quad X^{(i-1)} \to D,$$

the result of a successive application of elementary operations realizing $s_{\delta_1}, \dots, s_{\delta_{i-1}}$. Let ω be a hermitian metric on X inducing a kählerian metric in every fiber of $p: X \to D$, and such that $[\omega_0] = l \in H_{\mathbf{R}}^{1,1}(X_0)$. Then, denoting by δ_{i_t} $(t \in D^*)$ an extension of δ_i,

$$0 < ([\Gamma_t]_* [\omega_t] . \delta_{i_t})_{X_t^{(i-1)}} = (s_{\delta_{i-1}} \circ \dots \circ s_{\delta_1}(\varphi(l)) . \delta_i)_{X_0} < 0.$$

This contradiction proves the contention.

The difficulty in applying the remark 7.10 in what follows is to verify the assumption above that $\varphi(l)$ doesn't lie on any hyperplane corresponding to a reflection in W. We now return to the proof of theorem 2.

Third case: K-3 *surfaces.* — An analytic cycle C with $C^2 = -2$ is either effective or anti-effective. Hence the image $[\Gamma_0]_*(l)$ of a Kähler class l doesn't lie on any hyperplane H_C. By remark 7.10, we may perform a finite number of elementary operations until $[\Gamma_0]_*(l)$ lies in the fundamental domain V_Φ^+ for the action of the group W generated by the reflections corresponding to the set Φ of nodal curves C_i appearing in the factor on the left in the expression for Γ_0,

$$\Gamma_0 = \Delta_0 + \sum a_{ij} . (C_i \times C_j).$$

(Note that, after applying an elementary transformation to X and replacing Γ_0 by the new correspondence, the set Φ cannot increase.)

So assume that $[\Gamma_0]_*(l) \in V_\Phi^+$. Then Γ_0 preserves effectivity of irreducible curves of self-intersection -2: This is clear for those $C_i \notin \Phi$, and has been so arranged for the $C_i \in \Phi$. By the Torelli theorem (theorem 1), $\Gamma_0 = \Delta_0$.

Fourth case: Enriques surfaces. — We will eventually use the Torelli theorem for K-3 surfaces. Let

$$\begin{tikzcd} \tilde{X} \arrow[d, "\tilde{p}"'] \arrow[dr, "\pi"] & \\ D & X, \arrow[l, "p"'] \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} \tilde{X}' \arrow[d, "\tilde{p}'"'] \arrow[dr, "\pi'"] & \\ D & X' \arrow[l, "p'"'] \end{tikzcd}$$

denote the universal covering space of X, resp. X'; then $\tilde{p}$, resp. $\tilde{p}'$, is a family of K-3 surfaces. Let σ, resp. σ', be the non-trivial covering involution on $\tilde{X}$, resp. $\tilde{X}'$.

We fix one of the (two) components

$$\tilde{\Gamma} \subset \tilde{X} \times_D \tilde{X}'$$

of $(\pi \times \pi')^{-1}(\Gamma)$.

We need the following

Addendum to remark 7.10.—We retain the notation of that remark except that, what was called X_0 there, will here be called $\tilde{X}_0$.

Assume that an orthogonal involution σ acts on $H^{1,1}_{\mathbf{R}}(\tilde{X}_0)$, preserving V^+ and commuting with φ. We assume that, if $\delta \in \Phi$ then also $\sigma(\delta) \in \Phi$, and that $\sigma(I) = I$ and $\sigma(\delta).\delta = 0$ for $\delta \in \Phi$. *Then, in the presentation of* $w \in W$,

$$w = s_{\delta_r} \circ \ldots \circ s_{\delta_1}, \qquad \delta_i \in \Phi,$$

we may choose

$$\delta_2 = \sigma(\delta_1),$$
$$\delta_4 = \sigma(\delta_3),$$
$$\delta_r = \sigma(\delta_{r-1}).$$

Indeed, proceeding by induction, assume that $\delta_2 = \sigma(\delta_1), \ldots, \delta_{2i} = \sigma(\delta_{2i-1})$. Then as is easily checked, the fact that $\varphi(I)$ is invariant under σ implies that

$$I_i \underset{\overline{Df}}{=} s_{\delta_{2i}} \circ \ldots \circ s_{\delta_1}(\varphi(I))$$

is invariant under σ. Hence from $(I_i.\delta_{2i+1}) < 0$ one obtains

$$\begin{aligned}(s_{\delta_{2i+1}}(I_i).\sigma(\delta_{2i+1})) &= (I_i + (I_i\,\delta_{2i+1})\,\delta_{2i+1}.\sigma(\delta_{2i+1})) \\ &= (I_i.\sigma(\delta_{2i+1})) = (\sigma(I_i).\delta_{2i+1}) < 0.\end{aligned}$$

Here we used the facts that $(\sigma(\delta_{2i+1}).\delta_{2i+1}) = 0$ and that σ is an orthogonal involution. This proves our contention.

In particular, assume that $\tilde{X}_0$ is a double covering of X_0 with covering involution σ and that

$$\varphi = [\tilde{\Gamma}_0]_* : \quad H^2(\tilde{X}_0, \mathbf{Z}) \tilde{\rightarrow} H^2(\tilde{X}_0, \mathbf{Z}).$$

Then for every pair of reflections as above, $s_{\delta_i} \circ s_{\sigma(\delta_i)}$ is induced in the following way from an elementary operation corresponding to a nodal curve $C \subset X_0$: *If*

$$\Gamma^{(i)} = \Delta_0 + C \times C \subset X_0 \times X_0^{(i)}$$

is the graph of the correspondence, then the connected component

$$\tilde{\Gamma}^{(i)} \subset \tilde{X} \times_D \tilde{X}^{(i)}$$

of $(\pi \times \pi^{(i)})^{-1}(\Gamma^{(i)})$ *which, over D^*, is the graph of the identity automorphism has as its special fibre*

$$\tilde{\Gamma}_0^{(i)} = \tilde{\Delta}_0 + \tilde{C} \times \tilde{C} + \sigma(\tilde{C}) \times \sigma(\tilde{C}).$$

Here

$$\pi^{-1}(C) = \tilde{C} + \sigma(\tilde{C}) \qquad \textit{with} \quad \tilde{C} \cap \sigma(\tilde{C}) = \varnothing.$$

This is proved in the same way as the similar remark in 7.10.

Let Φ be the set of nodal curves $\tilde{C}_i$ which appear in the expression for $\tilde{\Gamma}_0$:

$$\tilde{\Gamma}_0 = \tilde{\Delta}_0 + \sum \tilde{a}_{ij}\,\tilde{C}_i \times \tilde{C}_j$$

and which, in addition, verify

$$\tilde{C}_i \cap \sigma(\tilde{C}_i) = \varnothing.$$

Since

$$\sigma^* \circ [\tilde{\Gamma}_0]_* = [\tilde{\Gamma}_0]_* \circ \sigma^* : H^2(\tilde{X}_0, \mathbf{Z}) \to H^2(\tilde{X}_0, \mathbf{Z})$$

we infer that if $\tilde{C}_i \in \Phi$, then $\sigma(\tilde{C}_i) \in \Phi$.

Let l be a Kähler class invariant under σ. As remarked in the previous case with K-3 surfaces, $[\tilde{\Gamma}_0]_*(l)$ cannot lie on a hyperplane H_δ, $\delta^2 = -2$. By the addendum to remark 7.10 we may perform elementary operations *to* X until

$$[\tilde{\Gamma}_0]_*(l) \subset V_\Phi^+.$$

But then we claim that, for any nodal curve $\tilde{C}$ on $\tilde{X}_0$, $[\tilde{\Gamma}_0]_*(\tilde{C})$ is effective.

This is clear if $\tilde{C}$ isn't one of the $\tilde{C}_i$ appearing in the above expression for $\tilde{\Gamma}_0$ and has been so arranged for those $\tilde{C}_i$ which lie in Φ.

Hence, for any curve C on the Enriques surface X_0 one has

$$(\pi^*([\Gamma_0]_*(C)).l) < 0.$$

Let $\tilde{C} \subset X_0$ be a curve with $\tilde{C}^2 = -2$ and let $C = \pi(\tilde{C}) \subset X_0$. Then, by the commutativity of the following diagram

$$\begin{tikzcd} H^2(\tilde{X}_0, \mathbf{Z}) \arrow[r, "{[\tilde{\Gamma}_0]_*}"] & H^2(\tilde{X}_0, \mathbf{Z}) \\ H^2(X_0, \mathbf{Z}) \arrow[r, "{[\Gamma_0]_*}"] \arrow[u, "\pi^*"] & H^2(X_0, \mathbf{Z}), \arrow[u, "\pi^*"'] \end{tikzcd}$$

we infer that

$$0 < (\pi^*([\Gamma_0]_*(C)).l) = ([\tilde{\Gamma}_0]_*(\tilde{C}+\sigma(\tilde{C})).l) = 2([\tilde{\Gamma}_0]_*(\tilde{C}).l).$$

Here we used that σ is orthogonal and that l is invariant under σ.

By the Torelli theorem for K-3 surfaces, $\tilde{\Gamma}$ induces an isomorphism between $\tilde{X}$ and $\tilde{X}'$. Since this isomorphism commutes with the $\mathbf{Z}/2$-action on $\tilde{X}$, resp. $\tilde{X}'$, it induces an isomorphism between X and X′, coinciding over D^* with

$$\varphi : \quad X \times_D D^* \xrightarrow{\sim} X' \times_D D^*.$$

Fifth case: elliptic surfaces.—Let

$$[K_{X_t}] \in H^2(X_0, \mathbf{Z})$$

be the cohomology class of the canonical bundle of X_t $(t \in D)$. Then

$$[K_{X_t}] = -c_1(X_t);$$

hence

LEMMA 7.11:

$$[\Gamma_0]_*([K_{X_0}]) = [K_{X'_0}].$$

On the elliptic surface X_0, some positive multiple of the canonical bundle K_{X_0} is cohomologous to $m.\beta$ ($m > 0$), where β is the class of the general fibre of the elliptic fibration

$$f: \ X_0 \to B$$

over the non-singular complete algebraic curve B. Therefore, if C is an irreducible curve on X_0,

$$(K_{X_0}.C) \geqq 0$$

and inequality holds if and only if C is contained in a fibre of f.

From the equality

$$[K_{X_0}] = [\Gamma_0]_*([K_{X_0}]) = [K_{X_0}] + \sum a_{ij}(K_{X_0}.C_i).[C_j];$$

we conclude that

$$a_{ij} \neq 0 \quad \Rightarrow \quad C_i \text{ lies in a fibre of } f.$$

Analogously,

$$a_{ij} \neq 0 \quad \Rightarrow \quad C_j \text{ lies in a fibre of } f.$$

Let

E = union of the irreducible components of reducible fibres of f

and let

$$E = E_1 \cup \ldots \cup E_r$$

be the decomposition of E into connected components. We denote by

$$E_{i\mathbf{Z}} \quad (i = 1, \ldots, r)$$

the subgroup of $H^2(X_0, \mathbf{Z})$ generated by the irreducible components δ_{ij} of E_i ($i = 1, \ldots, r$). By remark 5.6,

$$[\Gamma_0]_*(E_{i\mathbf{Z}}) = E_{i\mathbf{Z}}.$$

Let Φ be the set of irreducible components δ_{ij} of E and let V_Φ^+ be the usual (cf. 7.10) fundamental domain for the action of the group W generated by the reflections about the hyperplanes $H_{\delta_{ij}}$.

Note, as in paragraph 5, that the irreducible components $\delta_{i1}, \ldots, \delta_{in_i}$ of E_i define, by the usual receipe, an extended Dynkin diagram.

We need the following lemma whose proof is given at the end of the proof of theorem 2.

LEMMA 7.12. — *With the notation introduced in paragraph 6, let $\tilde{V}_{\mathbf{Z}}$ be the lattice in $\tilde{V}$ generated by $\alpha_0, \ldots, \alpha_n$.*

Let

$$\delta \in \tilde{V}_{\mathbf{Z}} \qquad \textit{with} \quad \delta^2 = +2.$$

Then δ can be written as

$$\delta = \sum_{j=0}^{n} c_j.\alpha_j,$$

where either all $c_j \geqq 0$ or all $c_j \leqq 0$.

This lemma ensures that the image $[\Gamma_0]_*(l)$ of a Kähler class l doesn't lie on any hyperplane H_δ, $\delta \in \sum E_{i\mathbf{Z}}$, $\delta^2 = -2$. By remark 7.10, we may perform a finite number of elementary operations until

$$[\Gamma_0]_*(V_\Phi^+) \subset V_\Phi^+.$$

But then $[\Gamma_0]_* \mid E_{i\mathbf{Z}}$ verifies all hypotheses of lemma 6.2. Hence

$$[\Gamma_0] = [\Delta_0] + m.(\beta \times \beta).$$

Lemma 5.7 shows that $m = 0$.

Sixth case: surfaces of general type. — Let X_0 be a surface of general type. For any irreducible curve C on X_0:

$$(K_{X_0}.C) \geqq 0,$$

and equality holds if and only if C is one of the finitely many nodal curves which are contracted to a point by all pluricanonical systems on X_0. Let E be the union of those curves.

We proceed as in the fifth case. By lemma 7.11,

$$[K_{X_0}] = [\Gamma_0]_*([K_{X_0}]) = [K_{X_0}] + \sum a_{ij}(C_i.K_{X_0}).[C_j];$$

hence, since $a_{ij} \geqq 0$ and $(K_{X_0}.C_i) \geqq 0$, we infer that $(K_{X_0}.C_i) = 0$ and thus

$$a_{ij} \neq 0 \quad \Rightarrow \quad C_i \in E.$$

Analogously,

$$a_{ij} \neq 0 \quad \Rightarrow \quad C_j \in E.$$

Let

$$E = E_1 \cup E_2 \cup \ldots \cup E_r$$

be the decomposition of E into connected components. Let $E_{i\mathbf{Z}}$ be the subgroup of $H^2(X_0, \mathbf{Z})$ generated by the irreducible components of E_i ($i = 1, \ldots, r$). The irreducible components of E_i form, by the usual recipe (cf. § 5), a Dynkin diagram (cf. [11]).

The proof of the following lemma is postponed until the end of the proof of theorem 2.

LEMMA 7.13. — *Let* R *be a root system in a real vector space* V *with symmetric Cartan matrix; put on* V *the euclidean metric defined by the Cartan matrix* (cf. § 6) *so that* $\delta^2 = +2$ *for all* $\delta \in R$. *Let* $V_{\mathbf{Z}}$ *be the lattice in* V *generated by* R. *Then every* $\delta \in V_{\mathbf{Z}}$ *with* $\delta^2 = 2$ *is an element of* R.

In particular, if R *is irreducible and* $\{\delta_1, \ldots, \delta_n\}$ *is a fundamental system of simple roots, every* $\delta \in V_{\mathbf{Z}}$ *with* $\delta^2 = 2$ *may be written as*

$$\delta = \sum_{i=1}^{n} r_i.\delta_i$$

with all $r_i \geqq 0$ *or all* $r_i \leqq 0$.

This lemma implies that the image $[\Gamma_0]_*$ (I) of a Kähler class I doesn't lie on a hyperplane H_δ, $\delta \in \sum E_{i\mathbf{Z}}$, $\delta^2 = -2$. By remark 7.10, we may thus perform a finite succession of elementary operations until

$$[\Gamma_0]_*(V^+_\Phi) \subset V^+_\Phi = \{x \in V^+ \mid (x.\delta) > 0 \text{ for } \delta \in \sum E_{i\mathbf{Z}} \text{ with } \delta^2 = -2\}.$$

But then $[\Gamma_0]_* \mid E_{i\mathbf{Z}}$ verifies the hypotheses of lemma 6.1, hence $[\Gamma_0] = [\Delta_0]$.

Proof of lemma 7.13. — We may assume that R is irreducible. Let

$$R' = \{\delta \in V_{\mathbf{Z}} \mid \delta^2 = 2\}.$$

Then R′ is a root system. Indeed, R′ is finite, doesn't contain 0 and generates the vector space V; R′ is stable under the reflections s_δ ($\delta \in R'$) and for all $\delta, \delta' \in R'$ the product $(\delta.\delta')$ is integral.

The first part of the lemma which, by standard facts about roots implies the second statement, is now a consequence of the following fact.

LEMMA 7.14. — *Let* R *and* R′ *be two irreducible root systems in a euclidean vector space* V, *all of whose roots have equal length. Assume that*

$$R \subset R'$$

and that R *and* R′ *generate the same lattice* $V_{\mathbf{Z}}$ *in* V. *Then*

$$R = R'.$$

Proof. — The assumptions imply that index of connectivity (R) = discriminant ($V_{\mathbf{Z}}$) = index of connectivity (R′). Now a glance at the tables shows that R = R′.

Q. E. D.

Proof of lemma 7.12. — In the notation of paragraph 6, write δ as

$$\delta = m.\beta + \sum_{j=1}^{n} k_j.\alpha_j.$$

Then $\left(\sum_1^n k_j.\alpha_j\right)^2 = +2$. By lemma 7.13 we infer that $\sum_1^n k_j.\alpha_j$ lies in the root system R corresponding to $\{\alpha_1, \ldots . \alpha_n\} \subset V$. In particular, either $k_j \geqq 0$ ($j = 1, \ldots, n$) or $k_j \leqq 0$ ($j = 1, \ldots, n$). If $m = 0$, we are finished. But $\tilde{\alpha} = \sum_1^n n_i.\alpha_i$ is the highest positive root in R and $\beta = \alpha_0 + \tilde{\alpha}$. Hence, if $m \neq 0$, writting

$$\delta = \sum_{j=0}^{n} c_j.\alpha_j,$$

we infer that sign (c_j) = sign (m).

Q. E. D.

REMARK 7.15.—In the case of surfaces of general type we may blow down in the family $p: X \to D$ *all* nodal curves on X_0. We obtain $p': X' \to D$ by blowing up in a different way the singularities of the family thus obtained. A similar statement is wrong in the case of K-3 surfaces.

REFERENCES

[0] I. I. PIATETSKII-SHAPIRO and I. R. SHAFAREVITCH, *A Torelli Theorem for Algebraic Surfaces of Type* K-3 (*Izv. Akad. Nauk SSSR*, vol. 35, 1971; English translation, *Math. USSR Izv.*, vol. 5, 1971, p. 547-587).

[1] M. ARTIN, *Algebraic Construction of Brieskorn's Resolutions* (*J. Algebra*, vol, 29, 1974, p. 330-348).

[2] M. ARTIN, *On Isolated Rational Singularities of Surfaces* (*Amer. J. Math.*, vol. 88, 1966, p. 129-137).

[3] M. ATIYAH, *On Analytic Surfaces with Double Points* (*Proc. Roy. Soc.*, A 247, 1958, p. 237-244).

[4] E. BISHOP, *Conditions for the Analyticity of Certain Sets* (*Mich. Math. J.*, vol. 11, 1964, p. 289-304).

[5] N. BOURBAKI, *Groupes et algèbres de Lie*, Chap. IV, V, VI, Hermann, Paris, 1968.

[6] E. BRIESKORN, *Die Auflösung der rationalen Singularitäten holomorpher Abbildungen* (*Math. Ann.*, vol. 178, 1968, p. 255-270).

[7] P. GRIFFITHS, *Periods of Integrals on Algebraic Manifolds: Summary of Main Results and Open Problems* (*Bull. Amer. Math. Soc.*, vol. 76, 1970, p. 228-296).

[8] J. KING, *On the Currents Defined by Analytic Varieties* (*Acta Math.*, vol. 127, 1971, p. 185-220).

[9] K. KODAIRA, *On Compact Analytic Surfaces, I, II, III* (*Ann. of Math.*, vol. 77, 1963, p. 563-626 and vol. 78, 1963, p. 1-40).

[10] K. KODAIRA, *On the Structure of Compact Complex Analytic Surfaces, I, IV* (*Amer. J. Math.*, vol. 86, 1964, p. 751-798 and vol. 90, 1968, p. 1048-1066).

11] K. KODAIRA, *Pluricanonical Systems on Algebraic Surfaces of General Type* (*J. Math. Soc. of Japan.*, vol. 20, 1968, p. 170-192).

[12] K. KODAIRA and D. C. SPENCER, *On Deformations of Complex Analytic Structures, III* (*Ann. of Math.*, vol. 71, 1960, p. 43-76).

[13] T. MATSUSAKA and D. MUMFORD, *Two Fundamental Theorems on Deformations of Polarized Varieties* (*Amer. J. Math.*, vol. 86, 1964, p. 668-684).

[14] J. MORROW and K. KODAIRA, *Complex Manifolds*, New York, Holt, Rinehard and Winston, 1971.

[15] I. R. SHAFAREVITCH et al., *Algebraic Surfaces*, Proc. Steklov Inst. 75; English translation, Providence, A. M. S., 1967.

[16] E. B. VINBERG, *Discrete Groups Generated by Reflections in Lobatchevsky Spaces* (*Mat. Sbornik*, vol. 72, 1967; English translation, *Math. USSR Sbornik*, vol. 1, 1967, p. 429-444).

(Manuscrit reçu le 13 novembre 1974.)

D. BURNS,
Fine Hall,
Department of Mathematics,
Princeton University,
Princeton, N. J. 08540,
U. S. A.
and
M. RAPOPORT
I. H. E. S.,
35, route de Chartres,
91440 Bures-sur-Yvette,
France.