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ROBERT M. FOSSUM
PHILLIP A. GRIFFITH

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COMPLETE LOCAL FACTORIAL RINGS WHICH ARE NOT COHEN-MACAULAY IN CHARACTERISTIC p

By Robert M. FOSSUM and Phillip A. GRIFFITH (*)

The object of this paper is to provide the details of the proof of the theorem announced in our paper [FG] that Bertin's example [B] of a factorial local ring of dimension 4 which is not Cohen-Macaulay has a factorial completion. Subsequent to the announcement in [FG] we have found, for every prime integer p and every positive integer n, a complete local ring of dimension p^n which is factorial and not Cohen-Macaulay in case $p^n \geqq 4$. Consideration of the more general cases has led to a satisfying simplification of the original proof. Perhaps by way of introduction a bit of the history of related problems is in order.

Samuel, in his work on the divisor class group and factorial rings conducted in the late 1950's and into the 1960's, seemed to be bothered by the frequency of the occurrence of factorial rings which were Cohen-Macaulay. In his paper *On Unique Factorization Domains* [Sa] he states : "All the examples of UFD's I know are Macaulay rings. Is this true in general ?" Shortly after the appearance of these remarks, Murthy [M] showed that a factorial and Cohen-Macaulay homomorphic image of a regular (or Gorenstein) local ring is Gorenstein. Then Bertin [B] constructed an example of a factorial local ring which is not Cohen-Macaulay. This example is of dimension four. Hochster [H 2] has shown how, from a four dimensional factorial non-Cohen-Macaulay ring, one can derive a three dimensional example. Hochster and Roberts [HR] have noticed that Bertin's example is a special case, in characteristic 2, of a general class of examples of factorial and non-Cohen-Macaulay rings obtained by Serre [Se] already in 1958. One obtains, from Serre's examples, rings which are not Cohen-Macaulay. There remained only the problem to show they are factorial. This is done with the help of Samuel's application of Galois descent to the calculation of divisor class groups [SaL]. We use Serre's calculations to show that the completions of these local rings are not Cohen-Macaulay. Then we calculate the divisor class group by a limit argument to show that it is zero, and thus the completions of these examples are also factorial.

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Recently Raynaud (unpublished) and Boutot (to appear) have shown that a complete factorial ring of dimension 4 with an algebraically closed characteristic zero residue class field is Cohen-Macaulay. Using the Grauert-Riemenschneider vanishing theorem, Hartshorne and Ogus [HO] have given an independent proof of this result in case the local ring is the completion of a **C**-algebra of finite type. All of these results are obtained by showing, under the hypotheses mentioned above, that the ring satisfies the Serre (S_3) condition. Then they use the result : If A is factorial complete, (S_3) and dim A = 4, then A is Cohen-Macaulay (and therefore Gorenstein). Danilov has also demonstrated these results.

On the other hand Freitag and Kiehl [FK] have constructed an example of a factorial analytic local ring, a homomorphic image of a ring of convergent power series over **C**, which has depth 3, dimension 60, and is not Cohen-Macaulay. Hochster (private communication) has shown us that the completion is also factorial.

An excellent survey with extensive bibliographic references has been written by Lipman [L].

Our examples are easy to describe. Let p be a prime integer and n a positive integer. Suppose k is a perfect field of characteristic p. Let B be the ring of formal power series over k in the p^n variables $X_0, X_1, \ldots, X_{p^n-1}$, so

$$B = k[[X_0, X_1, \ldots, X_{p^n-1}]].$$

Let $\sigma : B \to B$ be the k-algebra automorphism induced by $\sigma(X_i) = X_{i+1}$ (i taken modulo p^n). Then σ induces an action of the group $\mathbf{Z}/p^n\mathbf{Z}$ on B as k-algebra automorphisms. Our claim is that the ring of invariants is a factorial ring (of dimension p^n) which is complete and has depth at most $p^{n-1}+2$, and is thus not Cohen-Macaulay (except possibly when $p^n = 4$, in which case Bertin has shown that the depth is at most 3).

In addition to providing examples, in characteristic $p \neq 0$, of complete local factorial rings which are not Cohen-Macaulay, these provide counter-examples (along with the examples of Freitag and Kiehl) of a conjecture (suggested by Example 5.9 in [H 1]) which states : If A is a complete noetherian domain, then some symbolic power of a prime ideal of height one is a maximal Cohen-Macaulay module. (For in the examples, each such symbolic power is a principal ideal and is therefore isomorphic to A.)

1. The ring of invariants is usually not Cohen-Macaulay

We fix a prime integer p, a positive integer n and an algebraically closed field k of characteristic p. We will be considering actions of $\mathbf{Z}/p^n\mathbf{Z}$ on various types of abelian groups. Thus we will sometimes denote this cyclic group by $\mathbf{Z}(p^n)$, for example when considering the group algebras $\mathbf{Z}[\mathbf{Z}(p^n)]$ and $k[\mathbf{Z}(p^n)]$.

If R is a commutative ring, then the group algebra $R[\mathbf{Z}(p^n)] \cong R[t]/(t^{p^n}-1)$, and in case char R = p, this algebra is the truncated polynomial algebra $R[t]/(t-1)^{p^n}$.

Let W be a non-zero indecomposable $k[\mathbf{Z}(p^n)]$-module. Then there is an integer i with $0 < i \leqq p^n$ such that $W \cong k[t]/(t-1)^i$, and $i = \dim_k W$. Furthermore each

$k[\mathbf{Z}(p^n)]$-module of finite type has a decomposition into indecomposable modules. Also $k[\mathbf{Z}(p^n)]$ is a quasi-Frobenius algebra. Therefore a module is free if and only if it is injective.

Let $V_n = k[\mathbf{Z}(p^n)]$. Then V_n has a k-basis which we denote by $e_0, e_1, \ldots, e_{p^n-1}$ and the action of $\mathbf{Z}/p^n\mathbf{Z}$ on V_n is given by $\sigma(e_i) = e_{i+1}$ (i taken modulo p^n), where

$$\sigma = 1 + p^n\mathbf{Z}.$$

For each integer m, denote by $S^m(V_n)$ the m-th symmetric power of V_n as a vector space over k. Each symmetric power is again a $k[\mathbf{Z}(p^n)]$-module and the dimension is given by

$$\dim_k S^m(V_n) = \binom{m+p^n-1}{m}.$$

The ring $k[X_0, \ldots, X_{p^n-1}]$ of polynomials in the p^n variables $X_0, \ldots, X_{p^n-1}$ is identified as the symmetric algebra $S(V_n)$. As such the group $\mathbf{Z}/p^n\mathbf{Z}$ becomes a group of k-algebra automorphisms of $S(V_n)$, automorphisms which are all of degree zero. Thus the ring of $\mathbf{Z}/p^n\mathbf{Z}$-invariants, in addition to being a k-algebra of finite type [F], and a normal domain, is also graded, with its m-forms being just the invariants of $S^m(V_n)$.

At this point we pause to consider the completions. Suppose $R = \coprod_{n \geq 0} R_n$ is a noetherian positively graded ring. Let $\mathfrak{a}_m = \coprod_{n \geq m} R_n$. The ideals $\{\mathfrak{a}_m\}_{m \geq 1}$ define a system of neighborhoods of 0 in R for a linear topology, which is separated. Call this the π-topology. The completion in the π-topology is denoted by R^π. The following lemma is an easy consequence of ([EGA], II, 2.1, 6 (vi)]).

LEMMA 1.1. — *The $\mathfrak{a}_1$-adic topology and the π-topology coincide. Consequently the $\mathfrak{a}_1$-adic completion coincides with R^π.*

As a result we conclude that the ring $\prod_{m \geq 0} S^m(V_n)$, which we denote by $\mathscr{S}(V_n)$, is isomorphic to the ring of formal power series $k[[X_0, \ldots, X_{p^n-1}]]$. Furthermore the ring of invariants $\mathscr{S}(V_n)^{\mathbf{Z}/p^n\mathbf{Z}}$ is just $\prod_{m \geq 0} (S^m(V_n)^{\mathbf{Z}/p^n\mathbf{Z}})$.

THEOREM 1.2. — *The ring of invariants of $\mathbf{Z}/p^n\mathbf{Z}$ acting on $k[[X_0, \ldots, X_{p^n-1}]]$ is the completion at the irrelevant maximal ideal of the ring of invariants of $\mathbf{Z}/p^n\mathbf{Z}$ acting on $k[X_0, \ldots, X_{p^n-1}]$.*

Let $B = k[X_0, \ldots, X_{p^n-1}]$ and set $A = B^{\mathbf{Z}/p^n\mathbf{Z}}$. Use $\hat{}$ to denote completions.

COROLLARY 1.3. — *The ring of invariants* A *is Cohen-Macaulay* (at the irrelevant maximal ideal) *if and only if the ring of invariants* $\hat{B}^{\mathbf{Z}/p^n\mathbf{Z}}$ *is Cohen-Macaulay.*

We now use the theory of Witt vectors and the calculations of cohomology in characteristic p as exposed in [Se].

If V is a finite dimensional vector space over k, denote by $\mathbf{P}(V)$ the associated projective space (of lines through the origin). If V affords a k-linear representation of a finite group G, then G acts also on $\mathbf{P}(V)$. Consider the orbit space $\mathbf{P}(V)/G$, which is a variety.

It is in fact a projective variety. For let $B = S(V)$ and $A = S(V)^G$. Then A is a graded k-algebra. So there is an integer d such that $A(d) = \coprod_m A_{md}$ is generated by its 1-forms (d-forms in A). If $\dim_k A_d = s+1$, then we have induced an embedding $\operatorname{Proj}(A) \to \mathbf{P}_k^s$. Now $\operatorname{Proj}(A) \cong \mathbf{P}(V)/G$. Denote this image by Z. Let Q be the set of points in $\mathbf{P}(V)$ whose orbits are not maximal. (That is the length is less than the order of G. This occurs for all points if the representation is not faithful. So we must assume that the representation is faithful.) Then $Q = \{ \mathbf{p} \in \mathbf{P}(V) \mid \text{there is } \tau \neq 1 \text{ in G such that } \tau(\mathbf{p}) = \mathbf{p} \}$. Let Q' be the image of Q in the orbit space Z. Suppose r is an integer with $1 \leqq r < \dim \mathbf{P}(V) - \dim Q$. Then there are $\dim V - r - 1$ forms in A of degree d [so they are linear forms in $A(d)$], say $f_1, \ldots, f_{n-r}$, whose zeros in $\mathbf{P}_k^s$ define a linear subspace L of dimension $s - (\dim V - 1 - r)$ which intersects Z transversally and hence does not meet Q'. Let $X = Z \cap L$ and let Y be the preimage of X in $\mathbf{P}(V)$. Then $Y \to X$ is a principal homogeneous G-space. The inequality $1 \leqq r < \dim \mathbf{P}(V) - \dim Q$ is the only restriction we have made and this implies that our representation is faithful.

Now let $G = \mathbf{Z}/p^n\mathbf{Z}$. Suppose V is an indecomposable $k[G]$-module. Then $\dim \mathbf{P}(V) = \dim_k V - 1$ and the representation is faithful if and only if $\dim_k V > p^{n-1}$. The following result gives the restriction on $\dim_k V$.

PROPOSITION 1.4. — *The dimension of the singular orbits* Q *is* $p^{n-1} - 1$.

Proof. — A point $\mathbf{p}$ in $\mathbf{P}(V)$ is represented by a nonzero element $v \in V$. Since the subgroups of $\mathbf{Z}/p^n\mathbf{Z}$ are linearly ordered, the element v is stabilized by a non-trivial element if and only if $\sigma^{p^{n-1}}(v) = v$ where σ is a generator of $\mathbf{Z}/p^n\mathbf{Z}$. It is assumed that V is indecomposable. Let $\dim_k V = d$. Then $V \cong k[t]/(t-1)^d$ with $d > p^{n-1}$ and

$$\operatorname{Ker}((t-1)^{p^{n-1}} \mid V) = k[t](t-1)^{d-p^{n-1}}/k[t](t-1)^d.$$

The space Q is the image in $\mathbf{P}(V)$ of this subspace of dimension p^{n-1}. Hence Q has dimension $p^{n-1} - 1$.

COROLLARY 1.5. — *The codimension of* Q *in* $\mathbf{P}(V)$ *is given by*

$$\operatorname{codim}_{\mathbf{P}(V)} Q = \dim V - p^{n-1}.$$

Thus we have $1 \leqq r < \dim V - p^{n-1}$ and hence $\dim V > 1 + p^{n-1}$ in order that we have a non-trivial $\mathbf{Z}/p^n\mathbf{Z}$ covering.

Remark. — The above material is found in Paragraph 20 of [Se].

Let $\mathscr{W}_m$ denote the abelian group scheme of Witt vectors of length m over $\operatorname{Spec} k$. Since X is a projective variety the group $\mathscr{W}_m(X) = \mathscr{W}_m(k)$. Let F denote the Frobenius on $\mathscr{W}_m$ while V and R denote the *Verschiebung* and restriction respectively. The constant group scheme $\mathbf{Z}/p^n\mathbf{Z}$ is the kernel of $F - I$ on $\mathscr{W}_n$. The principal homogeneous $\mathbf{Z}/p^n\mathbf{Z}$-space $Y \to X$ constructed above gives a non-zero element in the cohomology group $H^1(X, \mathbf{Z}/p^n\mathbf{Z})$ provided $\dim V - p^{n-1} > 1$. By ([Se] Prop. 13), this group is just the kernel of $F - I$ on $H^1(X, \mathscr{W}_n)$. In particular $H^1(X, \mathscr{W}_n) \neq 0$.

LEMMA 1.6. — *There is an integer* m *such that* $H^1(X, \mathscr{W}_m) \neq 0$ *if and only if* $H^1(X, \mathcal{O}_X) \neq 0$. *Therefore* $H^1(X, \mathcal{O}_X) \neq 0$.

Proof. — If $H^1(X, \mathcal{O}_X) = 0$, then we can show, by induction on m and use of the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{V^{m-1}} \mathscr{W}_m \xrightarrow{R} \mathscr{W}_{m-1} \to 0,$$

that $H^1(X, \mathscr{W}_m) = 0$ for all m.

Returning to the construction above, we take the maximal value of $r = \dim V - p^{n-1} - 1$, assuming that $\dim V - p^{n-1} - 1 \geqq 1$. Then we pick $\dim V - r$ forms of degree d in A. But

$$\dim V - r - 1 = \dim V - (\dim V - p^{n-1} - 1) - 1 = p^{n-1}.$$

So we have p^{n-1} forms $f_1, \ldots, f_{p^{n-1}}$ of degree d in A, which, when considered in S (V), form part of a system of parameters. Hence

$$(\operatorname{depth} A(d)/(f_1, \ldots, f_{p^{n-1}})) + p^{n-1} \geqq \operatorname{depth} A(d).$$

Since A is a finite graded $A(d)$-module containing $A(d)$ as a direct summand, we have

$$\operatorname{depth} A \leqq \operatorname{depth} A(d),$$

where the depths are computed at the irrelevant maximal ideals. Hence

$$\operatorname{depth} A \leqq p^{n-1} + \operatorname{depth} A(d)/(f_1, \ldots, f_{p^{n-1}}).$$

By ([EGA] III.2.1.4), the local cohomology group $H^2_m(A(d)/(f_1, \ldots, f_{p^{n-1}}))$ contains $H^1(X, \mathcal{O}_X)$ as a direct summand. Therefore

$$\operatorname{depth} A(d)/(f_1, \ldots, f_{p^{n-1}}) \leqq 2.$$

Thus we have almost demonstrated the next result.

PROPOSITION 1.7. — *If* $\dim_k V > p^{n-1} + 2$, *then the ring of invariants* $S(V)^{Z/p^nZ}$ *is not Cohen-Macaulay.*

All that remains to be remarked is that the Krull dimension $\dim S(V)^{Z/p^nZ} = \dim_k V$.

COROLLARY 1.8. — *If* V *is an indecomposable* $k[Z(p^n)]$*-module, then* $S(V)^{Z/p^nZ}$ *is a factorial ring which is not Cohen-Macaulay, provided* $\dim_k V > p^{n-1} + 2$.

Proof. — This follows from Proposition 1.7 and the theory of Galois descent in [SaL]. In [SaL] it is shown that the ideal class group of the ring of invariants is (a subgroup of) $H^1(Z/p^n Z, G_m(k))$. Since k is a field of characteristic p, this group of homomorphisms is zero.

Remark. — It does not follow that Bertin's example [B] is not Cohen-Macaulay, for in this example $V = k[Z/4Z]$, $p = 2$, $n = 2$, and therefore $\dim_k V = p^{n-1} + 2$.

(It should be noted that there is a misprint in [B]. In the definition of the element z, on page 656 the equation should read

$$z = u_2 x_1^2 + x x_1 + x_2^2 (x_2 + x_3 + x_4)(x_2 + x_4).$$

This misprint is carried over to page 88 of [Fo] and should also be corrected there. Many hours of calculations, using several hundred sheets of paper, have convinced the authors that the depth of Bertin's example is 3.)

CONJECTURE. — We conjecture that depth $A = p^{n-1} + 2$.

2. The ring of invariants is usually factorial

Using the theory of Galois descent we showed, in the previous section, that the ring of invariants of $\mathbf{Z}/p^n\,\mathbf{Z}$ acting on S (V) is factorial. It is not generally the case that completions of factorial rings are factorial, but it is natural to ask whether a given factorial (local) ring has a factorial completion. In this section we prove the following theorem.

THEOREM 2.1. — *Suppose $\mathbf{Z}/p^n\,\mathbf{Z}$ acts as a group of automorphisms on the formal power series $k[[X_0, \ldots, X_{p^n-1}]]$ by cyclically permuting the variables. Then the ring of invariants is factorial.*

The proof we give is a substantial simplification of our first proof which handled only the cases $p^n = 4$ and $n = 1$ when $p \geqq 5$. Let $\mathbf{G}_m$ denote the group of units functor and let $\mathscr{S}(V)$ be the π-completion of the symmetric algebra of V. Then, as before, the divisor class group of $\mathscr{S}(V)^{\mathbf{Z}/p^n\mathbf{Z}}$ is (a subgroup of) the cohomology group $H^1(\mathbf{Z}/p^n\,\mathbf{Z}, \mathbf{G}_m(\mathscr{S}(V)))$. In this case however, the group does not act trivially on the group $\mathbf{G}_m(\mathscr{S}(V))$, and a certain amount of arrow theoretical gymnastics is needed in order to show that this cohomology group is zero. Actually our techniques work only in case $V = k[\mathbf{Z}(p^n)]$ because then we are able to successfully compute the cohomology. (We wish to express our gratitude to Professor Reiner for his helpful comments involving some crucial steps in this computation.) We conjecture that in fact all the rings $\mathscr{S}(V)^{\mathbf{Z}/p^n\mathbf{Z}}$ are factorial.

For the benefit of the reader we recall the definition of the group $H^1(G, M)$ for a G-module M, where G is a finite cyclic group generated by an element σ. Let $\mathrm{tr} : M \to M$ be given by $\mathrm{tr}(m) = \sum_{i=1}^{\mathcal{O}} \sigma_i(m)$, where $\mathcal{O} = [G : 1]$. The composition $(\sigma - 1) \circ \mathrm{tr} = 0$ on M and $H^1(G, M) \cong \mathrm{Ker}(\mathrm{tr})/(\sigma - 1)M$. Note also that $M^G = \mathrm{Ker}(\sigma - 1)$ and that $H^2(G, M) = M^G/\mathrm{tr}\,M$. These are the only cohomology groups, for the cohomology is periodic of period two.

From this point on we assume that $V = k[\mathbf{Z}(p^n)]$. However, we are going to induct on n, so we write a subscript $V_n = k[\mathbf{Z}^n(p)]$. Then, for each i, we have $V_i = k[\mathbf{Z}(p^i)]$ and V_i becomes a $\mathbf{Z}/p^{i+j}\,\mathbf{Z}$-module for each $j \geqq 0$. We get a relation between $S(V_n)$ and $S(V_{n-1})$ which induces a similar relation between the completions $\mathscr{S}(V_n)$ and $\mathscr{S}(V_{n-1})$.

First we need to know the structure of the $k[\mathbf{Z}(p^n)]$-modules $S^m(V_n)$. This is given in the next result. But first a bit of notation. We suppose $S^1(V_n)$ has as k-basis elements $X_0, X_1, \ldots, X_{p^n-1}$. Then $S^m(V_n)$ is spanned by the monomials in the $\{X_i\}$ of degree m. In $S^p(V_n)$ there is a particularly nice set of monomials, namely the orbit of $X_0\, X_{p^{n-1}}\, X_{2p^{n-1}} \ldots X_{(p-1)p^{n-1}}$ [which is just the element $\prod_{r=0}^{p-1} \sigma^{rp^{n-1}}(X_0)$ and which we denote by Y_0] and which consists of the p^{n-1} elements $Y_0, \sigma(Y_0), \ldots, \sigma^{p^{n-1}-1}(Y_0)$. The k-space spanned by these elements is $\mathbf{Z}/p^n\,\mathbf{Z}$-invariant.

Now we use the fact that $\mathbf{Z}/p^n\,\mathbf{Z}$ maps a monomial to a monomial in order to compute the length of an orbit of a monomial.

In order to simplify notation, a monomial will be denoted only by its sequence of exponents. Thus a monomial in the X_i will be denoted by $X(e_0, \ldots, e_{p^n-1})$ and a monomial in the Y_j will be denoted by $Y(f_0, \ldots, f_{p^{n-1}-1})$ or $Y(\mathbf{f})$.

LEMMA 2.2. — *Suppose* $X(\mathbf{e})$ *is an monomial of degree* m. *If its orbit under* $\mathbf{Z}/p^n\,\mathbf{Z}$ *has length less than* p^n, *then* p *divides* m *and* $X(\mathbf{e}) = Y(e_0, \ldots, e_{p^{n-1}-1})$.

Proof. — As noted in Paragraph 1, the stabilizer subgroup of the monomial, if it is not the zero subgroup, contains $\sigma^{p^{n-1}}$. Hence the assumption that the orbit has length smaller than p^n implies that it is stabilized by $\sigma^{p^{n-1}}$. Therefore $e_{i+p^{n-1}} = e_i$ for each i. Now

$$m = \sum_{j=0}^{p^n-1} e_i = \sum_{i=0}^{p-1}\left(\sum_{j=0}^{p^{n-1}-1} e_j\right) = p\left(\sum_{j=0}^{p^{n-1}-1} e_j\right).$$

The last statement is also clear.

COROLLARY 2.3. — (*a*) *If* $p \nmid m$, *then* $S^m(V_n)$ *is a free* $k[\mathbf{Z}(p^n)]$-*module of rank* $(1/p^n)\binom{m+p^n-1}{m}$.

(*b*) *If* $p \mid m$, *say* $pm_0 = m$, *then the inclusion* $S^{m_0}(V_{n-1}) - S^{pm_0}(V_n)$ (with V_{n-1} spanned by $Y_0, \ldots, Y_{p^{n-1}-1}$) *of* $k[\mathbf{Z}(p^n)]$-*modules has a free cokernel of rank*

$$(1/p^n)\left\{\binom{p(m_0+p^{n-1})-1}{pm_0} - \binom{m_0+p^{n-1}-1}{m_0}\right\}.$$

Proof. — If the orbit of a monomial has length p^n, then the space spanned by the monomial is a free $k[\mathbf{Z}(p^n)]$-module. Since $k[\mathbf{Z}(p^n)]$ is quasi-Frobenius, a free module is also an injective module and hence splits off. The statements follows from these remarks and the lemma.

The group of units in $\mathscr{S}(V)$ has a filtration by $\mathbf{Z}/p^n\,\mathbf{Z}$-invariant subgroups. For each $m \geqq 1$, let $1+\mathscr{S}_m(V)$ denote the subgroup of $\mathbf{G}_m(\mathscr{S}(V))$ consisting of power series of the form $1+x$ with x in $\prod_{l \geqq m} S^l(V)$. [If we identify $\mathscr{S}(V)$ with the ring of formal power series $k[[X_0, \ldots, X_v]]$, then $1+\mathscr{S}_m(V)$ consists of power series of the form $1+f(X_0, \ldots, X_v)$ where $f(X_0, \ldots, X_v) \in (X_0, \ldots, X_v)^m$.] The adjacent factors in the filtration have the following form :

(0) $$\mathbf{G}_m(\mathscr{S}(V))/1+\mathscr{S}_1(V) \cong \mathbf{G}_m(k).$$

(m) $$1+\mathscr{S}_m(V)/1+\mathscr{S}_{m+1}(V) \cong S^m(V),$$

for all $m \geqq 1$.

Each of these isomorphisms is an isomorphism of $\mathbf{Z}/p^n\,\mathbf{Z}$-modules. [Note that the operation in $1+\mathscr{S}_m(\mathrm{V})$ is multiplication.]

The ring homomorphism (of degree p of graded rings) :

$$\mathrm{S}(\mathrm{V}_{n-1}) \to \mathrm{S}(\mathrm{V}_n)$$

induces a homomorphism

$$\mathscr{S}(\mathrm{V}_{n-1}) \to \mathscr{S}(\mathrm{V}_n)$$

which induces maps for each $m_0 \geqq 1$:

$$1+\mathscr{S}_{m_0}(\mathrm{V}_{n-1}) \to 1+\mathscr{S}_{pm_0}(\mathrm{V}_n).$$

The cokernel of this injection is

$$(\mathrm{S}^{pm_0}(\mathrm{V}_n)/\mathrm{S}^{m_0}(\mathrm{V}_{n-1})) \times \mathrm{S}^{pm_0+1}(\mathrm{V}_n) \times \ldots \times \mathrm{S}^{p(m_0+1)-1}(\mathrm{V}_n) \times (\mathrm{S}^{p(m_0+1)}(\mathrm{V}_n)/\mathrm{S}^{m_0+1}(\mathrm{V}_{n-1})) \times \ldots.$$

N. B. This product is *not* a product of $\mathbf{Z}/p^n\,\mathbf{Z}$-modules.

We are now prepared to state the main result of this section. The notation is as above.

PROPOSITION 2.4. — *The cohomology group*

$$\mathrm{H}^1(\mathbf{Z}/p^n\mathbf{Z},\, 1+\mathscr{S}_1(\mathrm{V}_n)) = \mathrm{H}^1(\mathbf{Z}/p^{n-1}\mathbf{Z},\, 1+\mathscr{S}_1(\mathrm{V}_{n-1})).$$

Therefore

$$\mathrm{H}^1(\mathbf{Z}/p^n\mathbf{Z},\, 1+\mathscr{S}_1(\mathrm{V}_n)) = 0.$$

Consequently

$$\mathrm{H}^1(\mathbf{Z}/p^n\mathbf{Z},\, \mathbf{G}_m(\mathscr{S}(\mathrm{V}_n))) = 0$$

and $\mathscr{S}(\mathrm{V}_n)^{\mathbf{Z}/p^n\mathbf{Z}}$ *is factorial.*

The group $1+\mathscr{S}_1(\mathrm{V}_n)/1+\mathscr{S}_1(\mathrm{V}_{n-1})$ has a filtration whose factor groups have trivial cohomology. Although this fact in itself is not usually enough to insure that there is an isomorphism of cohomology, there are extra conditions on the filtration which will allow this conclusion. The filtration is given by the filtration on $1+\mathscr{S}_1(\mathrm{V}_n)$ as follows.

The quotient

$$1+\mathscr{S}_1(\mathrm{V}_n)/1+\mathscr{S}_1(\mathrm{V}_{n-1}) = \mathrm{S}^1(\mathrm{V}_n) \times \ldots \times \mathrm{S}^{p-1}(\mathrm{V}_n) \times (\mathrm{S}^p(\mathrm{V}_n)/\mathrm{S}^1(\mathrm{V}_{n-1})) \times \ldots.$$

We get induced subgroups given by the image in $1+\mathscr{S}_1(\mathrm{V}_n)/1+\mathscr{S}_1(\mathrm{V}_{n-1})$ of the subgroups $1+\mathscr{S}_m(\mathrm{V}_n)$ for $m \geqq 1$. Let U_m denote this image. Then we get :

1. The quotient

$$\mathrm{U}_m/\mathrm{U}_{m+1} \cong \mathrm{S}^m(\mathrm{V}_n) \quad \text{if } p \nmid m$$

and

$$\mathrm{U}_m/\mathrm{U}_{m+1} \cong \mathrm{S}^m(\mathrm{V}_n)/\mathrm{S}^{m/p}(\mathrm{V}_{n-1}) \quad \text{if } p \mid m.$$

2. The group

$$1+\mathscr{S}_1(\mathrm{V}_n)/1+\mathscr{S}_1(\mathrm{V}_{n-1}) \cong \varprojlim_m \mathrm{U}_1/\mathrm{U}_m$$

as $\mathbf{Z}/p^n\,\mathbf{Z}$-modules.

3. For each $m > 1$, the homomorphism

$$(U_1/U_{m+1})^{\mathbf{Z}/p^n\mathbf{Z}} \to (U_1/U_m)^{\mathbf{Z}/p^n\mathbf{Z}}$$

is a surjection.

Since each factor U_m/U_{m+1} has been shown to be a free $k[\mathbf{Z}(p^n)]$-module, the groups $H^i(\mathbf{Z}/p^n\mathbf{Z}, U_m/U_{m+1})$ vanish for $i > 0$, and therefore $H^i(\mathbf{Z}/p^n\mathbf{Z}, U_1/U_m) = 0$ for $i > 0$ and for all $m > 1$.

Now consider the homomorphism

$$\prod_{m>1} (U_1/U_m) \xrightarrow{\varphi} \prod_{m>1} (U_1/U_m)$$

which is given on the element $\{u_m\}$ by $\varphi u_m = (u_m - \bar{u}_{m+1})$, where $^-$ denotes reduction from U_1/U_{m+1} to U_1/U_m. This map of $\mathbf{Z}/p^n\mathbf{Z}$-modules is a surjection (as is easily verified) and its kernel is just $\varprojlim_m U_1/U_m$. By 3 above, the homorphism of invariants induced by φ is also a surjection. Thus, from the long exact sequence of cohomology, we infer that

4. $H^i(\mathbf{Z}/p^n\mathbf{Z}, 1+\mathcal{S}_1(V_n)/1+\mathcal{S}_1(V_{n-1})) = 0$ for all $i > 0$ and therefore that

$$H^i(\mathbf{Z}/p^n\mathbf{Z}, 1+\mathcal{S}_1(V_{n-1})) \to H^i(\mathbf{Z}/p^n\mathbf{Z}, 1+\mathcal{S}_1(V_n))$$

is an isomorphism for all $i > 0$. (Here we are using the periodicity of the cohomology modules.)

The action of $\mathbf{Z}/p^n\mathbf{Z}$ on $1+\mathcal{S}_1(V_{n-1})$ is such that $\sigma^{p^{n-1}} = 1$. Therefore the multiplicative trace (which we denote by the symbol N_n-the norm) of $\mathbf{Z}/p^n\mathbf{Z}$ acting on $1+\mathcal{S}_1(V_{n-1})$ takes the form $N_n = F \circ N_{n-1} = N_{n-1} \circ F$, where F is the Frobenius $(F(x) = x^p)$, and N_{n-1} denotes the norm on the $\mathbf{Z}/p^{n-1}\mathbf{Z}$-module. Since F is a monomorphism, we get

$$\operatorname{Ker} N_n = \operatorname{Ker} N_{n-1}.$$

Therefore the induced map

$$H^1(\mathbf{Z}/p^{n-1}\mathbf{Z}, 1+\mathcal{S}_1(V_{n-1})) \to H^1(\mathbf{Z}/p^n\mathbf{Z}, 1+\mathcal{S}_1(V_{n-1}))$$

is a bijection. Thus we obtain a composition of isomorphisms

$$H^1(\mathbf{Z}/p^{n-1}\mathbf{Z}, 1+\mathcal{S}_1(V_{n-1})) \xrightarrow{\sim} H^1(\mathbf{Z}/p^n\mathbf{Z}, 1+\mathcal{S}_1(V_{n-1})) \xrightarrow{\sim} H^1(\mathbf{Z}/p^n\mathbf{Z}, 1+\mathcal{S}_1(V_n))$$

and therefore the first statement is established.

Since

$$H^1(\mathbf{Z}/p\mathbf{Z}, 1+\mathcal{S}_1(V_1)) \cong H^1(0, 1+\mathcal{S}_1(V_0)),$$

the second statement follows by induction. The third statement follows from Samuel [SaL].

We conclude this section with a curious result and a few remarks.

What is shown in the proof of Proposition 2.4, is that the cohomology groups

$$H^i(\mathbf{Z}/p^n\mathbf{Z}, 1+\mathcal{S}_1(V_n)/1+\mathcal{S}_1(V_{n-1}))$$

vanish for all $i > 0$. Hence we obtain

$$H^2(\mathbf{Z}/p^n\mathbf{Z}, 1+\mathcal{S}_1(V_n)) \cong H^2(\mathbf{Z}/p^n\mathbf{Z}, 1+\mathcal{S}_1(V_0)).$$

Now $\mathbf{Z}/p^n\mathbf{Z}$ acts trivially on $1+\mathcal{S}_1(V_0)$. In fact $\mathcal{S}(V_0) \cong k[[T]]$ where $T = X_0 X_1 \ldots X_{p^n-1}$, and $1+\mathcal{S}_1(V_0) = 1+Tk[[T]]$. Since $H^2(\mathbf{Z}/p^n\mathbf{Z}, \mathbf{G}_m(k)) = 0$, we obtain the isomorphism

$$H^2(\mathbf{Z}/p^n\mathbf{Z}, \mathbf{G}_m(\mathcal{S}(V_n))) \cong 1+T\,k[[T]]/1+T^{p^n}k[[T^{p^n}]].$$

This group has an underlying set which is the product of countably many copies of k. Applying first Theorem 6 and then Theorem 3 of Yuan's paper [Y] (*see* also [A]), we obtain the result

$$\bigcup_{\mathrm{ht}\,\mathfrak{p}=1} \mathrm{Br}(\mathcal{S}(V_n)_{\mathfrak{p}}^{\mathbf{Z}/p^n\mathbf{Z}}) = 1+T\,k[[T]]/1+T^{p^n}k[[T^{p^n}]].$$

Clearly $\mathrm{Br}(\mathcal{S}(V_n)^{\mathbf{Z}/p^n\mathbf{Z}}) = 0$ since $\mathcal{S}(V_n)^{\mathbf{Z}/p^n\mathbf{Z}}$ is a complete local ring with algebraically closed residue class field.

The reason Yuan's Theorem 6 can be applied is that the homomorphism from the twisted group ring

$$\mathcal{S}(V_n)[\mathbf{Z}(p^n)] \to \mathrm{Hom}_A(\mathcal{S}(V_n), \mathcal{S}(V_n))$$

is an isomorphism (where A is the ring of $\mathbf{Z}/p^n\mathbf{Z}$ invariants). This is an isomorphism because $\mathcal{S}(V_n)$ is reflexive as an A-module. The map

$$\mathcal{S}(V_n)^{\mathbf{Z}/p^n\mathbf{Z}} \to \mathcal{S}(V_n)$$

is unramified at height one prime ideals, since the different contains all elements of the form $\prod_{i \neq j}(X_i - X_j)$, the product taken over $0 \leqq j \leqq p^n-1$, and i running through all the possible indexes. Hence the map on the twisted group ring is an isomorphism at each prime ideal of A of height one.

PROPOSITION 2.5. — *The notation being as above, there is an isomorphism*

$$\bigcup_{\mathrm{ht}\,\mathfrak{p}=1} \mathrm{Br}((k[[X_0, \ldots, X_{p^n-1}]]^{\mathbf{Z}/p^n\mathbf{Z}})_{\mathfrak{p}}) \cong (1+T\,k[[T]])/(1+T\,k[[T]])^{p^n}.$$

As a final parting question, we ask : Suppose V is an indecomposible $k[\mathbf{Z}(p^n)]$-module. What is the decomposition of the symmetric powers $S^m(V)$?

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