M. ARTIN

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# SUPERSINGULAR K 3 SURFACES

BY M. ARTIN

Let X be a K 3 surface over an algebraically closed field $k$. It is known that if $k = \mathbf{C}$, the rank ρ of the Néron-Severi group of X is at most $20 = b_2 - 2h^{02}$. Igusa [8] showed that the weak inequality $\rho \leqq b_2$ continues to hold if the characteristic is non-zero, and in fact the stronger one does fail. There exist K 3 surfaces for which $\rho = 22$. The first example of one was given by Tate [19] : The Fermat surface

$$x^4 + y^4 + z^4 + w^4 = 0$$

has rank 22 if $p \equiv 3$ (modulo 4). More recently, Shioda [17], [18] has given other examples : the elliptic modular surface of level 4 if $p \equiv 3$ (modulo 4), and the Kummer surface associated to a product of supersingular elliptic curves if $p \neq 2$ (1). Examples in characteristic 2 also exist (*see* Section 2).

In our paper we propose to study these peculiar surfaces, using the *formal Brauer group* $\hat{\mathrm{Br}}\,\mathrm{X}$ [4]. If X is any surface whose Picard variety is smooth, $\hat{\mathrm{Br}}\,\mathrm{X}$ is a smooth formal group of dimension $h^{02} = \dim \mathrm{H}^2(\mathrm{X}, \mathcal{O})$, which pro-represents the following functor on the category of finite local $k$-algebras A with residue field $k$ :

$$\hat{\mathrm{Br}}(\mathrm{A}) = \ker(\mathrm{H}^2(\mathrm{X}_\mathrm{A}, \mathbf{G}_m) \to \mathrm{H}^2(\mathrm{X}, \mathbf{G}_m)),$$

where $\mathrm{X}_\mathrm{A} = \mathrm{X} \times \mathrm{Spec}\,\mathrm{A}$, and cohomology is etale cohomology. In joint work [4], Mazur and I have related this formal group to the rank ρ by the theorem :

THEOREM (0.1). — *Let* X *be a surface which lifts projectively to characteristic zero, and assume that* $\hat{\mathrm{Br}}\,\mathrm{X}$ *is a p-divisible formal group. Let h be the height* [10] *of* $\hat{\mathrm{Br}}$. *Then* $\rho \leqq b_2 - 2h$.

We also conjecture that if $\hat{\mathrm{Br}}$ is unipotent, i. e., is annihilated by some power of $p$, then in fact $\rho = b_2$.

In our case, where X is a K 3 surface, we have $\dim \hat{\mathrm{Br}} = h^{02} = 1$. Formal 1-parameter groups in characteristic $\neq 0$ are classified by their height $h$, which can take on any

(1) According to Shioda, these three examples are all related by correspondences.

value between 1 and $\infty$ [10]. The group having $h = \infty$ is the additive group $\hat{\mathbf{G}}_a$, and the groups with $h < \infty$ are $p$-divisible. Therefore, the above theorem implies that any projectively liftable K 3 surface X with $\rho = 22$ must have $h = \infty$. We prove the converse here for K 3 surfaces having a pencil of elliptic curves [Theorem (1.7)], and we conjecture that this includes every case :

CONJECTURE (0.2). — Every K 3 surface X in characteristic $p > 0$ with $h = \infty$ carries a pencil of elliptic curves.

Our result is some justification for the following :

DEFINITION (0.3). — A K 3 surface defined over a field of characteristic $p \neq 0$ is *supersingular* if $h = \infty$, i. e., if $\hat{\mathrm{Br}}\, \mathrm{X} \approx \hat{\mathbf{G}}_a$.

Thus elliptic K 3 surfaces which are projectively liftable have $\rho = 22$ if and only if they are supersingular. Note also that, as a consequence of (0.1) and (1.7), the case $\rho = 21$ can not arise. Swinnerton-Dyer (unpublished) has constructed examples of elliptic K 3 surfaces in characteristic $p \neq 0$ having rank $\rho = 19$, and it seems probable that all the remaining values $2 \leqq \rho \leqq 20$ occur. However, as Swinnerton-Dyer has remarked, it follows from Tate's conjecture ([19], [5]) that a surface with $\rho$ odd can not be defined over a finite field.

One reason for defining the notion supersingular in terms of the height $h$ is that $h \geqq r$ is an *algebraic* condition (*cf.* Section 2) contrary to what occurs for $\rho \geqq r$ (though $\rho = 22$ seems, a posteriori, to be algebraic after all). As we show here, the elliptic supersingular K 3 surfaces form a limited family, and depend on at least 9 moduli !

The later sections of this paper contain results which are still conjectural, since they depend on as yet unproven duality theorems for flat cohomology. The conjectures are stated in Section 3. We hope that they will prove to be accessible to presently available techniques. In the remaining sections, we use them to derive further properties of supersingular K 3 surfaces, analogous to properties of general K 3 surfaces over the complex numbers. Among other things, we define the periods of X, which form a map (4.10) :

$$\mathrm{N}^* \approx \mathbf{Z}^{22} \xrightarrow{\varphi} \mathbf{G}_a,$$

where N* is the dual lattice to the Néron-Severi group N. The kernel of $\varphi$ is N, i. e., is the set of " algebraic " vectors. So, although the rank of X is always 22, the group N can vary with X to the extent that a vector $v \in \mathrm{N}$ which is primitive on a generic supersingular surface may become divisible by $p$ on a specialization. This occurs when

$$p^{-1}\, v \in \mathrm{N}^*$$

and $\varphi(p^{-1}\, v)$ specializes to zero, and is reflected in a change in the discriminant $-p^{2\sigma_0}$ of N [*see* (4.6)]. We show in the last section that all values $1 \leqq \sigma_0 \leqq 10$ actually arise (7.8).

A number of obvious questions related to the Torelli theorem for K 3 surfaces [12] arise in connection with the period map. These remain to be investigated.

1. THE RANK OF A SUPERSINGULAR K 3 SURFACE. — Let $X \xrightarrow{\pi} S$ be a smooth family of surfaces, with connected parameter space S of characteristic $p \neq 0$.

THEOREM (1.1). — *Assume that* $Pic^{\tau}\, X/S$ *is smooth, and that the formal Brauer group* $\hat{Br}\, X_s$ *is unipotent for every geometric point* $s$ *of* S. *Then the rank* $\rho(X_s)$ *of the Néron-Severi group of* $X_s$ *is independent of* $s$.

Let $L_s$ denote the Néron-Severi group of $X_s$. If $\eta \in S$ is a generalization of $s$, then since X is smooth there is an injective specialization map

$$L_\eta \to L_s. \tag{1.2}$$

Thus $\rho(X_\eta) \leqq \rho(X_s)$. To prove the theorem we need to show that the opposite inequality holds. We may assume that $S = \operatorname{Spec} k[[t]]$, with $k$ algebraically closed, and that $X_\eta$ & $X_0 = X_{s_0}$ are the open and closed fibres. So the theorem will follow from this more precise assertion :

THEOREM (1.1 *a*). — *With the above notation, assume that* $Pic^{\tau}\, X/S$ *is smooth and that* $\hat{Br}\, X_\eta$ *is annihilated by* $p^{\nu}$. *Suppose that the torsion group* $Pic^{\tau}/Pic^{0}$ *has p-exponent* $\lambda$. *Then the cokernel of* (1.2) *is a finite group anninilated by* $p^{\nu+\lambda}$.

*Proof.* — We begin by reviewing the relative Brauer group $Br\, X/S = R^2 \pi_* \mathbf{G}_m$ [4]. The relative Picard scheme $Pic\, X/S$ will generally not be smooth, because of jumps in the Néron-Severi groups N. This will obstruct the pro-representability of $Br\, X/S$, though that functor has a Schlessinger hull at every point $\xi_0 \in H^2(X_0, \mathbf{G}_m)$. We work instead with the complex $\mathbf{G}_m[\infty] = [\mathbf{G}_m \to \mathbf{G}_m \otimes \mathbf{Q}]$. The etale cohomology of this complex is the same as flat cohomology of the sheaf $\mu = \bigcup_n \mu_n$ ([4], IV.1.7). Moreover, when $Pic^{\tau}\, X/S$ is smooth, the functor $R^2 \pi_* \mathbf{G}_m[\infty]$ is pro-representable at every point

$$\alpha_0 \in H^2(X_0, \mathbf{G}_m[\infty]) = H^2_{fl}(X_0, \mu),$$

and its tangent space it the same as that of $Br\, X/S$, i. e., is $H^2(X_0, \mathcal{O})$ ([4], IV.1.5).

Let $\hat{H}$ be the smooth formal group which pro-represents $R^2 \pi_* \mathbf{G}_m[\infty]$ at the origin. Then $\hat{H}$ is a hull for $Br\, X/S$ at 0, the closed fibre $\hat{H} \times_S s_0$ is the formal Brauer group $\hat{Br}\, X_0$, and the completion of $\hat{H}$ at the generic point of its 0-section is $\hat{Br}\, X_\eta$. Since this last group is annihilated by $p^{\nu}$, it follows that $\hat{H}$ is, too.

We now proceed with the proof of (1.1 *a*). Let $z_0 \in \operatorname{Pic} X_0$ be a given element, and let $n$ be an integer. Denote by $\alpha_0 \in H^2_{fl}(X_0, \mu_{p^n})$ the image of $z_0$ by the map $\delta$ of Kummer theory :

$$\operatorname{Pic} X_0 \xrightarrow{p^n} \operatorname{Pic} X_0 \xrightarrow{\delta} H^2_{fl}(X_0, \mu_{p^n}) \to \operatorname{Br} X_0 \xrightarrow{p^n} \operatorname{Br} X_0.$$

We try to extend $\alpha_0$ to a cohomology class on X/S. Let us denote by $\beta_0$ the image of $\alpha_0$ in $H^2_{fl}(X_0, \mu) = R^2 \pi_* \mathbf{G}_m[\infty](k)$. Since $R^2 \pi_* \mathbf{G}_m[\infty]$ is formally smooth, $\beta_0$ can be extended to a formal class $\beta$, i. e., to a sequence $\{ \beta_r \}$ of classes in

$$R^2 \pi_* \mathbf{G}_m [\infty] (S_r), \qquad S_r = \operatorname{Spec} k [[t]]/(t^r).$$

Since $p^n \alpha_0 = 0$, it follows that $p^n \beta = \bar{\beta}$ is a section of the formal group $\hat{H}$. Therefore $p^{n+\nu} \beta = p^\nu \bar{\beta} = 0$.

The sequence

$$0 \to \mathbf{G}_m [p^{n+\nu}] \to \mathbf{G}_m [\infty] \xrightarrow{p^{n+\nu}} \mathbf{G}_m [\infty] \to 0$$

shows that each $\beta_r$ can be represented by a class $\beta'_r \in R^2 \pi_* \mathbf{G}_m [p^{n+\nu}] (S_r)$, and moreover the image $\gamma_r$ of $\beta'_r$ in $R^2 \pi_* \mathbf{G}_m [N] (S_r)$ is unique, if $N = p^{n+\nu+\lambda}$. Thus we obtain a formal element $\gamma = \{ \gamma_r \}$ in $R^2 \pi_* \mathbf{G}_m [N]$ extending the image $\gamma_0$ of the given class $\alpha_0$. The image $\bar{\alpha}$ of $\gamma$ in $R^2 \pi_* \mathbf{G}_m [p^n]$ via multiplication by $p^{\nu+\lambda}$ determines an extension of the class $p^{\nu+\lambda} \alpha_0$.

Now by construction, $\alpha_0$ (and hence $\gamma_0$) maps to 0 in $\operatorname{Br} X_0$. So, the formal element $\gamma$ determines a formal deformation of 0 in *Br* X/S, which lifts to a section of the hull $\hat{H}$, and is therefore annihilated by $p^\nu$. It follows that the image of $\bar{\alpha}$ in *Br* X/S is zero.

Let $P_r = \operatorname{Pic} X \times_S S_r$. By Kummer theory, $\bar{\alpha}_r$ lifts to $P_r / p^n P_r$ for every $r$. If we denote by $C_r$ the cokernel of the map $P_r \to P_0$, then what we have shown implies that $C_r / p^n C_r$ is annihilated by $p^{\nu+\lambda}$ for every $n$ and $r$. On the other hand, $H^2 (X_0, \mathcal{O})$ is annihilated by $p$, and so the exact sequences

$$0 \to \mathcal{O}_{X_0} \to \mathcal{O}^\times_{X_r} \to \mathcal{O}^\times_{X_{r-1}} \to 0$$

show that $C_r$ is annihilated by $p^r$. Hence $C_r$ is in fact annihilated by $p^{\nu+\lambda}$ for every $r$. Thus our given element $z_0 \in \operatorname{Pic} X_0$ has the property that $p^{\nu+\lambda} z_0$ extends to a section of *Pic* X/S over $S_r$, for every $r$. By [1], it extends to a section over S. Therefore the cokernel of $\operatorname{Pic} X \to \operatorname{Pic} X_0$ is annihilated by $p^{\nu+\lambda}$. This completes the proof.

REMARK (1.2). — Obstructions annihilated by powers of $p$ certainly do arise, and are one of the interesting aspects of this theory (*see* Section 7).

We now return to the case of supersingular K 3 surfaces. They have no torsion (cf. Section 8), and $\hat{\operatorname{Br}} X \approx \hat{\mathbf{G}}_a$ is annihilated by $p$. So, theorem (1.1 *a*) reads

COROLLARY (1.3). — *Let* $S = \operatorname{Spec} k [[t]]$, *and let* X/S *be a smooth family of* K 3 *surfaces such that* $X_\eta$ *is supersingular. Then so is* $X_0$, *and the cokernel of the map* $N_\eta \to N_0$ *of Néron-Severi groups is an elementary p-group.*

PROPOSITION (1.4). — *Let* X/S *be a family of supersingular* K 3 *surfaces. The set of points* $s \in S$ *such that* $X_s$ *is elliptic is an open set.*

*Proof.* — It is easily seen that a pencil of elliptic curves on a general fibre $X_\eta$ specializes to a pencil of elliptic curves on all fibres of some open set. So, what has to be shown is that the property of being elliptic is preserved under generalization. We may therefore suppose that $S = \operatorname{Spec} k[[t]]$, and that a pencil $|C_0|$ of elliptic curves is given on the closed fibre $X_0$. In this situation, we will actually prove

PROPOSITION (1.5). — *With the above notation, either* $|C_0|$ *or* $|p\,C_0|$ *is the specialization of an irreducible pencil of elliptic curves* $|E_\eta|$ *on* $X_\eta$.

*Proof.* — Let $z_0 \in \operatorname{Pic} X_0 = L_0$ be the corresponding element. By the above Corollary (1.3), $pz_0$ is the specialization of a class, say $y_\eta$ on $X_\eta$. Let $D_\eta$ be a divisor in this class. Then $(D_\eta)^2 = 0$, and it follows from Riemann-Roch (8.1) on $X_\eta$ that either $|D_\eta|$ or $|-D_\eta|$ is a linear system of dimension $\geqq 1$. Since $y_\eta$ specializes to $pz_0$, it must be $|D_\eta|$. We claim that $|D_\eta|$ is composite with a pencil of elliptic curves : Otherwise, $|D_\eta| = |D'_\eta| + \Delta_\eta$, where $|D'_\eta|$ is variable and $\Delta_\eta \geqq 0$ is the fixed component. Necessarily, $(D'_\eta)^2 \geqq 0$ and $(\Delta_\eta)^2 < 0$. Specializing this to $X_0$, we find that $p\,C_0$ is linearly equivalent to a sum $D_0 = D'_0 + \Delta_0$ of positive divisors, with $(\Delta_0)^2 < 0$ and $(D_0)^2 \geqq 0$. This is impossible for $|p\,C_0|$, which is composite with the pencil of elliptic curves $|C_0|$. Thus $|D_\eta|$ is composite with a pencil of elliptic curves $|E_\eta|$, as was asserted. Since

$$|D_0| = |p\,C_0|,$$

it is clear that $|D_\eta| = |E_\eta|$ or $= |p\,E_\eta|$.

LEMMA (1.6). — *Let* X/S *be a limited family of supersingular* K 3 *surfaces. There is an integer* $n$ *such that any pencil of elliptic curves* $|C_0|$ *on a fibre* $|X_0|$ *has a multisection of degree* $\leqq n$.

*Proof.* — There is such an integer for every individual K 3 surface, by [5], Lemma (5.18). We choose $n$ to work for each one of the generic fibres X, and argue by specialization using Proposition (1.5).

THEOREM (1.7). — *Let* X *be an elliptic, supersingular* K 3 *surface. Then*

(i) $\rho(X) = 22$;

(ii) $\operatorname{Br} X = H^2(X, \mathbf{G}_m)$ *is a p-torsion group.*

*Proof.* — Since X is a K 3 surface, $H^3(X, \mu_l) = 0$ for all $l \neq p$. Therefore Kummer theory implies that Br X is divisible by $l$. Also, Pic X has no torsion. So all vertical arrows in the diagram below are surjective.

$$\begin{array}{ccccccccc} 0 \longrightarrow & \operatorname{Pic} X/l^r & \longrightarrow & H^2(X, \mu_{l^r}) & \longrightarrow & {}_{l^r}(\operatorname{Br} X) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \operatorname{Pic} X/l^{r-1} & \rightarrow & H^2(X, \mu_{l^{r-1}}) & \rightarrow & {}_{l^{r-1}}(\operatorname{Br} X) & \rightarrow 0 \end{array}$$

It follows that if ${}_l(\operatorname{Br} X) \neq 0$, we can construct an $l$-adic class, say $\alpha \in H^2(X, \mathbf{Z}_l(1))$, whose image $\alpha_r \in H^2(X, \mu_{l^r})$ determines an element of order $l^r$ in Br X. This $\alpha$ is a non-algebraic class, and so $\rho < b_2 = 22$. Thus (i) $\Rightarrow$ (ii), and it remains to prove (i).

Let $f : X \rightarrow Y$ be an elliptic fibration on X, and let $A^* \rightarrow Y$ be the minimal model of the associated Jacobian fibration, which is again a K 3 surface. Assume for a moment the following.

LEMMA (1.8). — $\rho(X) = \rho(A^*)$, *and* $\hat{Br}\, X \approx \hat{Br}\, A^*$.

Then we may replace X by A*, i. e., we may assume that the elliptic fibration has a section. Let A′ be the Weierstrass fibration [5] associated to A*. The groups $H^2(A^*, \mathbf{Z}_l(1))$ and $H^2(A', \mathbf{Z}_l(1))$ differ by algebraic classes ([5], 2.1), and so it suffices to prove that every class in $H^2(A', \mathbf{Z}_l(1))$ is $Q_l$-algebraic.

Suppose not. Then since cup product is nondegenerate on the image of Pic X = N, there is a class $\alpha$ orthogonal to N and such that $\alpha \cup \alpha \neq 0$. We now proceed as in the proof of [5], (5.2), to show that there are homogeneous spaces $X_\nu$ of A, of arbitrarily high order $l^{\nu-c}$, lying in a limited family F of K 3 surfaces. Since the condition

$$\hat{Br}\, X \approx \hat{\mathbf{G}}_a$$

is algebraic (Section 2), the supersingular surfaces in F form a closed subfamily S. Now Lemma (1.6) implies that the $X_\nu$ have multisections of bounded degree. This is a contradiction, and completes the proof of the theorem.

It remains to prove Lemma (1.8). The assertion of rank is well known and follows immediately from the formula

$$\rho(X) = r + \sum_y (m_y - 1) + 2,$$

where $r$ is the rank of the Jacobian of the generic fibre $X_y$, and $m_y$ is the number of components of the fibre $X_y$. This formula is elementary, and can be found in [16], (1.5), for the case that X has a section. All terms on the right side agree for X and for A*.

The assertion on $\hat{Br}$ is treated in the next section.

2. EXPLICIT CALCULATION OF THE HEIGHT. — Since the formal Brauer group is defined rather abstractly, it may be worthwhile to show how it can be computed in the case of an elliptic K 3 surface X/Y. Let A′/Y be the Jacobian Weierstrass fibration, and let A*/Y be the associated minimal model. We can compute the formal Brauer groups of these surfaces using their fibrations over $Y = \mathbf{P}^1$. Let us denote all the projections to Y by $f$, the projections to Spec $k$ by $\pi$, and the map $Y \to \operatorname{Spec} k$ by $g$. The terms

$$E_2^{pq} = R^p g_* R^q f_* \mathbf{G}_m$$

which may contribute to $R^2 \pi_* \mathbf{G}_m$ are $E_2^{20}$, $E_2^{11}$, and $E_2^{02}$. Since $f_* \mathbf{G}_m = \mathbf{G}_m$ and Y has dimension 1, $E_2^{20} = R^2 g_* (f_* \mathbf{G}_m)$ is discrete, i. e., all deformations of elements are trivial. Since $f$ has relative dimension 1, $R^2 f_* \mathbf{G}_m$ is discrete, and hence $E_2^{02}$ is, too. Thus the formal structure of $R^2 \pi_* \mathbf{G}_m$ is that of $R^1 g_* R^1 f_* \mathbf{G}_m = R^1 g_*$ *Pic* X/Y (respectively *Pic* A′/Y, *Pic* A*/Y). All three of these relative Picard groups differ by discrete group schemes from the fibre system of groups A/Y. Thus we have shown

PROPOSITION (2.1). — $\hat{\mathrm{Br}}\, X \approx \hat{\mathrm{Br}}\, A' \approx \hat{Br}\, A^*$ *is the formal group which pro-represents the functor* $R^1 g_* A$ *at the origin.*

In particular, this completes the proof of Lemma (1.8).

Denote by $\hat{A}$ the formal completion of A along its 0-section.

PROPOSITION (2.2). — $\hat{\mathrm{Br}}\, X(S) = H^1_{\mathrm{Zar}}(Y_S, \hat{A})$, *where this cohomology may be computed as Čech cohomology for any affine covering of* Y.

*Proof.* — Let L denote the conormal bundle of the 0-section in A, which is $R^1 f_* A'$.

$$\deg L = -2. \tag{2.3}$$

Let $S \subset S'$ be a length 1 extension of finite local schemes. Identifying the underlying spaces of $Y_{S'}$ and $Y_S$ with Y, one has a sequence

$$0 \to L \to \hat{A}_{S'} \to \hat{A}_S \to 0$$

($A_S = A \times S$), which is exact for the Zariski or etale topologies on Y. Consider the diagram

$$\begin{array}{ccccccccc} 0 \to & H^1_{\mathrm{Zar}}(Y, L) & \to & H^1_{\mathrm{Zar}}(Y, \hat{A}_{S'}) & \to & H^1_{\mathrm{Zar}}(Y, \hat{A}_S) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^1_{\mathrm{et}}(Y, L) & \to & H^1_{\mathrm{et}}(Y, \hat{A}_{S'}) & \to & H^1_{\mathrm{et}}(Y, \hat{A}_S) & \to 0. \end{array} \tag{2.4}$$

Ordinarily there would be a coboundary map $H^0(Y, \hat{A}_S) \to H^1(Y, L)$. But since deg $L < 0$, induction shows $H^0(Y, \hat{A}_S) = 0$. Therefore this coboundary is zero. In any case, the diagram and induction prove that the etale topology may be replaced by the Zariski topology, and that moreover it can be computed using any affine cover of Y. By [4], (II.1.7) and Proposition (2.1), $H^1_{\mathrm{et}}(Y, \hat{A}_S) \approx \hat{\mathrm{Br}}\, X(S)$.

We now ask to compute $\hat{\mathrm{Br}}\, X$. The surface A′ can be given in Weierstrass form over $\mathbf{P}^1$ as follows [5], (2.5) : Let the coordinate rings of the standard covering of $Y = \mathbf{P}^1$ be $k[\bar{t}]$, $k[t]$ with $t\bar{t} = 1$. Over $U = \operatorname{Spec} k[t]$, we can write the Weierstrass form for A′ as

$$Y^2 Z - a_1 XYZ - a_3 YZ^2 = X^3 + a_2 X^2 Z + a_4 XZ^2 + a_6 Z^3,$$

where $a_i \in k[t]$ are polynomials of degree $\leqq 2i$. We dehomogenise this equation to obtain

$$z = x^3 + a_1 xz + a_2 x^2 z + a_3 z^2 + a_4 xz^2 + a_6 z^3. \tag{2.5}$$

A similar form exists over $\bar{U} = \operatorname{Spec} k[\bar{t}]$ :

$$\bar{z} = \bar{x}^3 + \bar{a}_1 \overline{xz} + \bar{a}_2 \bar{x}^2 \bar{z} + \bar{a}_3 \bar{z}^2 + \bar{a}_4 \overline{xz}^2 + \bar{a}_6 \bar{z}^3,$$

the two being related by the equations

$$(2.6)\qquad \begin{cases} \bar{x} = t^2 x, \\ \bar{z} = t^6 z, \\ \bar{a}_i(\bar{t}) = t^{-2i}\, a_i(t). \end{cases}$$

Using equation (2.5), $z$ may be expressed as a series in $x$ :

$$(2.7)\qquad z = x^3 + a_1 x^4 + (a_1^2 + a_2) x^5 + (a_1^3 + 2 a_1 a_2 + a_3) x^6 + \ldots .$$

This series expresses $\hat{A}$ as a formal 1-parameter space over Y.

If $(x_1, z_1)$, $(x_2, z_2)$ are points of A on some Y-scheme Y′, their sum may be computed in the usual way, using addition of points on a plane cubic curve with equation (2.5). Substitution of the series (2.7) for $z_i$ in terms of variables $x_i$ gives the addition law of the formal group $\hat{A}$, which we write as

$$(2.8)\qquad x_1 \oplus x_2 = x_1 + x_2 + a_1 x_1 x_2 + \ldots .$$

Let us review the cohomology of $\mathcal{O}(-2)$. This is a free module on the affines U, $\bar{U}$, and we can choose bases $\{ u \}$, $\{ \bar{u} \}$ over these opens, related by the equation

$$\bar{t}^2 \bar{u} = u.$$

A 1-cocycle for this cover of Y is any section on V = Spec $k[t, \bar{t}]$, say $f(t, \bar{t})\,u$. The coboundaries are the sections of the form

$$g(t) u + \bar{g}(\bar{t}) \bar{u} = g(t) u + t^2 \bar{g}(\bar{t}) u.$$

Thus all terms of $f(t, \bar{t})\,u$ can be eliminated except for the monomial $ctu = c\bar{t}\bar{u}$. Such monomials represent the 1-dimensional cohomology of $\mathcal{O}(-2)$ in a unique way.

The line bundle L of (2.3) has sheaf of sections $\mathcal{O}(-2)$, and as scheme, it can be written as

$$\operatorname{Spec} k[t, x], \quad \operatorname{Spec} k[\bar{t}, \bar{x}]$$

over U, $\bar{U}$ respectively, with

$$\bar{x} = t^2 x.$$

The 1-cocycle which represents the universal cohomology class is given by the V-map Spec $k[t, \bar{t}][s] \to$ L ($s$ variable) :

$$x \mapsto \bar{t} s,$$
$$\bar{x} \mapsto ts.$$

Now for our group $\hat{A}$, a 1-cocycle parametrized by some artinian local ring R is any map

$$(2.9)\qquad k[t, \bar{t}][[x]] \to k[t, \bar{t}] \otimes R = R[t, \bar{t}]$$

sending $x$ to a polynomial $\equiv 0$ (modulo $\mathfrak{M}_R$). We know that there is a universal 1-parameter formal cohomology class, and since the first-order approximation to $\hat{A}$ is L, it must be represented by the formal 1-cocycle

$$k[t, \bar{t}][[x]] \to k[t, \bar{t}][[s]], \quad x \mapsto \bar{t}s. \tag{2.10}$$

The fact that this represents a universal cohomology class means that any 1-cocycle (2.9) is cohomologous to one obtained by a map $k[[s]] \to R$ from (2.10). In particular, this is true of the *sum* of two copies of (2.10) in A, which is the formal series $\bar{t}s_1 \oplus \bar{t}s_2$ obtained by substitution into the addition law (2.8). Therefore there is a map

$$\varphi : k[[s]] \to k[[s_1, s_2]]$$

(a power series $\varphi(s_1, s_2)$) such that

$$\bar{t}s_1 \oplus \bar{t}s_2 = \bar{t}\varphi(s_1, s_2) \oplus B,$$

where B is the image of $x$ under a coboundary map. This power series $\varphi$ is the formal group law on $k[[s]]$ giving $\hat{\mathrm{Br}}\, X$.

The coboundaries are sums in A′ of the two kinds of map

$$x \mapsto f(s_1, s_2) \in k[\bar{t}][[s_1, s_2]], \qquad f(0, 0) = 0$$

and

$$\bar{x} \mapsto \bar{f}(s_1, s_2) \in k[\bar{t}][[s_1, s_2]], \qquad \bar{f}(0, 0) = 0$$

or

$$x \mapsto t^2 \bar{f}(s_1, s_2).$$

So, $B = f(s_1, s_2) \oplus \bar{t}^2\bar{f}(s_1, s_2)$. Now the law $\oplus$ is ordinary addition, plus higher order terms. Thus we can eliminate all monomials of $\bar{t}s_1 \oplus \bar{t}s_2$ inductively using coboundaries except those of the form $\bar{t}s_1^i s_2^j$, thereby obtaining $\varphi$.

If we write each coefficient $a_i$ of (2.5) out as

$$a_i(t) = a_{i0} + a_{i1}t + \ldots + a_{i,2i}t^{2i}, \tag{2.11}$$

then the group law $\varphi(s_1, s_2)$ appears as a series whose coefficients are integral polynomials in $\{a_{ij}\}$.

The height $h$ of $\hat{\mathrm{Br}}\, X$ can be calculated, by using $\varphi$ to express multiplication by $p$ as a power series $f(s) = \sum c_i s^i$. Its first non-zero coefficient will have degree $p^h$ in $s$. Since the coefficients $c_i$ are integral polynomials in $\{a_{ij}\}$, the condition $h > i$, which is

$$c_p = c_{p^2} = \ldots = c_{p^i} = 0,$$

is exhibited as a closed condition of codimension $\leqq i$. We have worked out the height for low values in the case $p = 2$, and obtain the following result.

THEOREM (2.12). — *Let $p = 2$. Equation (2.5) can be chosen so that $a_2(t) = 0$ and $a_{12} = 1$. Assume this done. Then*

$$\begin{aligned} h &= 1 && \text{if } a_{11} \neq 0, \\ h &\geqq 2 && \text{if } a_{11} = 0, \\ h &\geqq 3 && \text{if } a_{11} = a_{33} = 0, \\ h &\geqq 4 && \text{if } a_{11} = a_{33} = a_{56} = 0. \end{aligned}$$

There are in general two supersingular elliptic curves occuring as fibres of A'/Y; they are given by $a_1(t) = 0$. Thus we have $h > 1$ if and only if these two supersingular values coincide. We do not know a geometric interpretation of the condition $h > 2$.

In the future, we hope to check out the conditions $h \geqq i$ for $i > 4$ by computer. This is not a completely trivial task since it involves computation of the series to degree $2^{10}$. For the moment, we have only an idea about the condition $h = \infty$.

PROPOSITION (2.13). — ($p = 2$). *Assume that all odd degree coefficients $a_{ij}$ ($j$ odd) of (2.5) vanish. Then X is supersingular.*

In fact, the vanishing of the odd degree coefficients implies that the Weierstrass fibration A'/Y "depends only on $t^2$", i. e., is obtained by pull-back of some other fibration $B'/\mathbf{P}^1$ via the Frobenius map $\mathbf{P}^1 \xrightarrow{F} \mathbf{P}^1$. Moreover, B' will have a Weierstrass form (2.5) in which deg $a_i \leqq i$. Except for degenerate cases, this implies that B' is a rational surface. Thus every A' in an open set of such Weierstrass fibrations is purely inseparable over a rational surface B'. Mumford (unpublished) has shown that every such surface has $\rho = b_2$. On the other hand, it is not hard to see that the generic such A', at least, lifts to characteristic zero. Hence it is supersingular, and so the same is true of any specialization.

The surfaces A' with all odd degree coefficients zero form a family depending on the expected number of moduli, which is 8. So it is probable that every elliptic supersingular X with section is one of these ([2]). Note that since A' is purely inseparable over a rational surface B', it is *unirational*, namely it has the purely inseparable covering $B'^{1/p}$.

Recently Shioda has proved that the Fermat surface is also unirational if $p \equiv 3$ (modulo 4). These examples give some indication that perhaps all supersingular surfaces may be unirational.

3. STATEMENT OF A CONJECTURAL FLAT DUALITY. — Let X be a smooth surface over an algebraically closed field $k$. Then there is a canonical isomorphism

$$H^4_{et}(X, \mu_n \otimes \mu_n) = \mathbf{Z}/n$$

([2]) Those having no section should depend on one more modulus, making 9 parameters in all [cf. (7.7) (iii)]. This is because there is a continuous family of homogeneous spaces of a given A/Y, coming from the family of elements in Br A* given by the map (4.3), (ii). The unusual phenomenon of continuous families of homogeneous spaces occurs only for supersingular surfaces.

([3], exposé 18) for $n$ prime to $p$, and cup product into this group defines "Poincaré duality", a perfect duality

$$H^p(X, \mu_n) \otimes H^{4-p}(X, \mu_n) \to \mathbf{Z}/n.$$

We want to state a conjectural extension ot this to general $n$, for flat cohomology. It is analogous to certain conjectures of Grothendieck concerning flat cohomology of curves.

We replace $H^q_{fl}(X, \mu_n)$ by the functors $R^q_{fl}\pi^* \mu_n$ on the big flat (*fppf*) site, where

$$\pi : X \to \operatorname{Spec} k$$

denotes the structure map, and we drop the assumption that $k$ is algebraically closed.

THEOREM (3.1). — *The functors $R^q_{fl}\pi_* \mu_n$ are represented by finite type group schemes over $k$.*

The proof of this theorem will be published elsewhere. The conjectural duality concerns these groups, but it is not complete as we do not know, even conjecturally, how to retain control of their infinitesimal parts. So we pass to the associated quasi-algebraic groups of Serre [14]. Let us denote the quasi-algebraic group associated to $R^q_{fl}\pi_* \mu_n$ by $H^q(X, \mu_n)$. Since $\mu_n$ is torsion, so is $\underline{H}^q(X, \mu_n)$. Therefore this is a quasi-*unipotent* group. We put it into an exact sequence

$$0 \to \underline{U}^q(X, \mu_n) \to \underline{H}^q(X, \mu_n) \to \underline{D}^q(X, \mu_n) \to 0 \tag{3.2}$$

where $\underline{U}^q(X, \mu_n)$ denotes the connected component, which is unipotent, and where $\underline{D}^q(X, \mu_n)$ is a finite discrete group scheme. It is easily seen that $\underline{U}^q(X, \mu_n) = 0$ if $q = 0, 1$. We write $U^q(X, \mu_n)$, $D^q(X, \mu_n)$ for the points of the corresponding groups with values in the ground field $k$, provided $k$ is perfect.

The conjectures are based on the observation of Grothendieck that $\mathbf{Q}/\mathbf{Z}$ is a dualizing ind-object in the category (QU) of quasi-unipotent, quasi-algebraic groups, i. e., that $A^{\cdot} \approx A^{\cdot DD}$ in the derived category, where $A^{\cdot D} = \operatorname{RHom}(A^{\cdot}, \mathbf{Q}/\mathbf{Z})$. For discrete A,

$$\operatorname{Hom}(A, \mathbf{Q}/\mathbf{Z}) = A^*,$$

the Pontryagin dual, and $\operatorname{Ext}^q(A, \mathbf{Q}/\mathbf{Z}) = 0$ if $q \neq 0$. For $A = \mathbf{G}_a$, we have

$$\operatorname{Ext}^1(\mathbf{G}_a, \mathbf{Q}/\mathbf{Z}) \approx \mathbf{G}_a$$

and $\operatorname{Ext}^q(\mathbf{G}_a, \mathbf{Q}/\mathbf{Z}) = 0$ if $q \neq 1$. These facts can be shown easily, using the results of Oort [11].

CONJECTURE (3.3). — The functors $\underline{H}^q(X, \mu_n)$ are cohomology of a canonical complex $\underline{H}^{\cdot}(\mu_n)$ in the derived category of (QU), and there is an isomorphism

$$\underline{H}^{\cdot}(X, \mu_n) \to \operatorname{RHom}(\underline{H}^{\cdot}(X, \mu_n), \mathbf{Q}/\mathbf{Z})$$

of degree $-4$, functorial in $n$.

Using the exact sequences (3.2), this assertion for given $q$ has two parts :

$$(3.4)\qquad \underline{D}^q \approx \underline{D}^{4-q*} \text{ (Pontryagin dual)},$$
$$\underline{U}^q \approx \operatorname{Ext}^1(U^{5-q}, \mathbf{Q}/\mathbf{Z}).$$

So the possibly non-zero unipotent groups are $\underline{U}^2$ and $\underline{U}^3$, and these must be dual via $\operatorname{Ext}^1(\cdot, \mathbf{Q}/\mathbf{Z})$. Of course, the conjecture does imply the existence of a pairing

$$H^q_{fl}(X, \mu_n) \times H^{4-q}_{fl}(X, \mu_n) \to \mathbf{Q}/\mathbf{Z}.$$

We will denote it by $\alpha \cup \beta$. If $k$ is algebraically closed, then the null space of this pairing is just $U^q(X, \mu_n)$.

We will need to assume two naturality properties of their duality. First, the symbol $\alpha \cup \beta$ should be compatible with specialization in the obvious sense : if $\pi : X \to S$ is a smooth family of surfaces over a connected scheme S, and if $\alpha$, $\beta$ are classes in $R^q \pi_* \mu_n$ and $R^{4-q} \pi_* \mu_n$ respectively, then $\alpha \cup \beta$ is defined at each fibre. It shou'd be constant. Second, assume $\alpha, \beta \in H^2_{fl}(X, \mu_n)$ are classes represented by divisors D, E on X. Then we should have

$$\alpha \cup \beta \equiv \frac{1}{n}(D.E) \qquad (\text{modulo } \mathbf{Z}).$$

There are several formal consequences of (3.3), of which we will mention two : Consider the inverse system of sequences (3.2) filtered by divisibility of $n$. It follows immediately from Kummer theory that dim $\underline{U}^q(X, \mu_n)$ is bounded by $h^{0q-1} + h^{0q}$ for all $n$. Therefore, since $\underline{U}^1 = 0$, the maps $\underline{U}^2(X, \mu_n) \to U^2(X, \mu_{mn})$ are surjective for large enough $m$, $n$. Hence the left side of (3.2) is an essentially zero inverse system for $q = 2$, and so if we set

$$(3.5)\qquad H^q(X, T_p(\mu)) \underset{\text{def}}{=} \varprojlim_{\nu} H^q_{fl}(X, \mu_{p^\nu}),$$

we have

$$H^2(X, T_p(\mu)) = \varprojlim D^2(X, \mu_{p^\nu}).$$

The pairing on $\underline{D}^q$ therefore induces a non-degenerate pairing on $H^2(X, T_p(\mu))$/torsion.

Now suppose $k$ is a finite field. The functor (QU) $\to$ (sets) taking $A \to A(\bar{k})$ has the derived functors $H^q(G, A(\bar{k}))$, where G is the Galois group of $\bar{k}$ over $k$. It is easily checked that duality for Galois cohomology extends to (QU) to a duality

$$(3.6)\qquad H^{\cdot}(G, A(\bar{k})) \otimes H^{\cdot}(G, A^D(\bar{k})) \to H^1(G, \mathbf{Q}/\mathbf{Z}) \approx \mathbf{Q}/\mathbf{Z}.$$

Combining this with (3.3) leads to

COROLLARY (3.7). — *Assuming* (3.3), *let* X *be defined over the finite field* $k$. Then there is a perfect pairing

$$H^q_{fl}(X, \mu_n) \otimes H^{5-q}_{fl}(X, \mu_n) \to H^1_{fl}(G, \mathbf{Q}/\mathbf{Z}).$$

4. CONSEQUENCES OF DUALITY FOR SUPERSINGULAR SURFACES. — The remaining parts of this paper depend on the conjectural duality of Section 3. So the results are valid under

HYPOTHESIS (4.1). — A duality formalism as in Section 3 holds for K 3 surfaces.

PROPOSITION (4.2). — *Let* X *be an elliptic supersingular* K 3 *surface over an algebraically closed field k, and assume* (4.1). *Then*

$$\underline{H}^q(X, \mu_{p^\nu}) = 0 \qquad \text{for} \quad q = 0, 1, 4,$$
$$\underline{H}^3(X, \mu_{p^\nu}) \approx \mathbf{G}_a,$$
$$\underline{U}^2(X, \mu_{p^\nu}) \approx \mathbf{G}_a.$$

*Proof.* — The assertion $\underline{H}^0 = 0$ is trivial, and $\underline{H}^4 = 0$ follows by duality. The fact (*cf.* Section 8) that Pic X has no torsion implies $\underline{H}^1 = 0$. Again by duality, it follows that $\underline{D}^3 = 0$, i. e., that $\underline{H}^3 = \underline{U}^3$ is connected. Now by assumption, $\widehat{\mathrm{Br}}\, X \approx \widehat{\mathbf{G}}_a$. So by Kummer theory, the formal groups of $\underline{H}^2$ and $\underline{H}^3$ are isomorphic to $\widehat{\mathbf{G}}_a$. The only connected algebraic group with this formal structure is $\mathbf{G}_a$. This proves the remaining assertions.

THEOREM (4.3). — *With the assumptions of* (4.2), *we have*

(i) *The rank of* $H^2(X, T_p(\mu))$ *is* 22;

(ii) $H^2(X, \mathbf{G}_m) = \mathrm{Br}\, X$ *is annihilated by p, and in fact the map* $H^2_{fl}(X, \mu_p) \to \mathrm{Br}\, X$ *induces a surjection* $U^2(X, \mu_p) \to \mathrm{Br}\, X$.

This theorem is proved in Section 5.

COROLLARY (4.4). — *With the assumptions of* (4.2), *the following diagram is exact:*

$$\begin{array}{ccccccc}
p^\nu N^* & \xrightarrow{\varphi_\nu} & U^2(X, \mu_{p^\nu}) & \to & \mathrm{Br}\, X & \to & 0 \\
\downarrow & & \downarrow & & \| & & \\
N & \longrightarrow & H^2_{fl}(X, \mu_{p^\nu}) & \to & \mathrm{Br}\, X & \to & 0 \\
\downarrow & & \downarrow & & & & \\
N/p^\nu N^* & \tilde{\to} & D^2(X, \mu_{p^\nu}) & & & &
\end{array}$$

where N* denotes the dual lattice of the Néron-Severi group N.

Here the middle row is induced by Kummer theory, and the middle vertical is (3.2). We know $U^2(X, \mu_p)$ maps onto Br X, from which it follows immediately that $U^2(X, \mu_{p^\nu})$ does too. This implies that N maps onto $D^2(X, \mu_{p^\nu})$. The duality pairing (3.4) on $D^2$ is therefore induced by the pairing on N (reduced modulo $p^\nu$). Since it is non-degenerate on $D^2$, the kernel of $N \to D^2$ is the set of vectors $v \in N$ such that $(v.w) \equiv 0$ (modulo $p^\nu$) for all $w \in N$, i. e., is $p^\nu N^*$. The remaining assertions of the diagram are now clear.

Since $\rho = 22$ on X, we have $N \otimes \mathbf{Z}_l \approx H^2(X, \mathbf{Z}_l(1))$ for every $l \neq p$. Therefore the discriminant of the quadratic form on N is a unit at all $l \neq p$, i. e., is $\pm 1$ times a power of $p$. Its signature is $(+1, -21)$ by the Hodge index theorem, and so the sign is $-1$. We will show in the next section (6.7) that it is always an *even* power of $p$ which occurs, provided $p \neq 2$. So, we introduce the notation

$$\sigma+\sigma_0 = 11, \tag{4.5}$$

where

$$\operatorname{discr} \mathrm{N} = -p^{2\sigma_0} \qquad (p \neq 2). \tag{4.6}$$

By Corollary (4.4), $\mathrm{N}^* \supset \mathrm{N} \supset p\,\mathrm{N}^*$. Hence $\mathrm{N}^*/\mathrm{N}$ and $\mathrm{N}/p\,\mathrm{N}^*$ are elementary $p$-groups, of ranks $2\sigma_0$ and $2\sigma$ respectively, and in particular

$$\mathrm{D}^2(\mathrm{X}, \mu_p) \approx (\mathbf{Z}/p)^{2\sigma} \qquad (p \neq 2). \tag{4.7}$$

The case $\sigma_0 = 0$ would correspond to N unimodular. But the form on N is even (*cf.* Section 8) and there is no even form with signature $(+1, -21)$ ([15], p. 91, for instance). Thus this can not occur. Similarly, the case $\sigma_0 = 11$ would correspond to $\mathrm{N} = p\,\mathrm{N}^*$, i. e., that the form on N were divisible by $p$ :

$$\langle v.w \rangle = p^{-1}(v.w) \in \mathbf{Z}, \quad \text{all } v, w \in \mathrm{N}.$$

The form $\langle v.w \rangle$ would again be unimodular and even, which is impossible. Thus

$$1 \leqq \sigma, \sigma_0 \leqq 10 \qquad (p \neq 2). \tag{4.8}$$

NOTE (4.9). — The discriminant is also an even power of $p$ in the characteristic 2 examples of Section 2, and the inequalities (4.8) hold for these examples.

Using the isomorphism $\underline{\mathrm{U}}^2(\mathrm{X}, \mu_{p^\nu}) \approx \mathbf{G}_a$, the map $\varphi_\nu$ defines a map $\varphi = p^{-\nu}\varphi_\nu$ :

$$\mathrm{N}^* \approx \mathbf{Z}^{22} \xrightarrow{\varphi} \mathbf{G}_a(k) \tag{4.10}$$

determined up to multiplication by a non-zero scalar. It is easily seen that $\varphi$ is independent of $\nu$. We call it the *periods* of X. A choice of basis for N determines via $\varphi$ a unique point in $\mathbf{P}^{21}(k)$.

PROPOSITION (4.11). — *The kernel of* $\varphi$ *is* N. *In other words, a vector* $x \in \mathrm{N}^*$ *is represented by a divisor on* X *if and only if its period* $\varphi(\mathrm{X})$ *is zero.*

*Proof.* — The map $\mathrm{N} \to \mathrm{H}^2_{fl}(\mathrm{X}, \mu_p)$ of (4.4) is given by Kummer theory, and so a vector $v \in \mathrm{N}$ maps to zero if and only if $v$ is divisible by $p$ in N. This map induces $\varphi_1$. Hence $\varphi(x) = \varphi_1(px) = 0$ if and only if $x \in \mathrm{N}$.

There are several analogies with K 3 surfaces in the classical case which we have listed below.

TABLE OF ANALOGIES (4.12)

| Algebraic, $k = \mathbf{C}$ | Supersingular |
|---|---|
| Periods | Periods |
| Period vanishes iff. class is algebraic | Same |
| $H^2(X, \mathcal{O}_{an}^{\times}) \approx \mathbf{C}/\text{im } H^2(X, \mathbf{Z})$ | $H^2(X, \mathbf{G}_m) \approx \mathbf{G}_a/\text{im } N^*$ (3) |
| $\rho + \rho_0 = 22$ | $\sigma + \sigma_0 = 11$ |
| $\rho_0 \geqq 2$ | $\sigma_0 \geqq 1$ |
| $\rho > 0$ | $\sigma > 0$ |

We propose the study of the variation of the period map as an interesting problem.

5. PROOF OF THEOREM (4.3). — The first part (i) of (4.3) is equivalent with the assertion that Br X is annihilated by some power of $p$. This follows by passing to the limit over the exact sequences

$$0 \to N/p^{\nu} \to H^2_{fl}(X, \mu_{p^{\nu}}) \to {}_{p^{\nu}}(\text{Br } X) \to 0,$$

in which all terms satisfy the Mittag-Leffler condition.

Let $X \xrightarrow{g} Y = \mathbf{P}^1$ be a pencil of elliptic curves on X, and let $A^* \xrightarrow{f} Y$ be the minimal model of its Jacobian fibration. It suffices to prove (i) for A*, which is supersingular by (1.8). For, we have $R^2 g_* \mathbf{G}_m = 0$ ([8], p. 98, or (5.1)) and hence

$$H^2(X, \mathbf{G}_m) \approx H^1(Y, \mathit{Pic}\, X/Y).$$

Similarly, $H^2(A^*, \mathbf{G}_m) \approx H^1(Y, \mathit{Pic}\, A^*/Y)$. There is a canonical exact sequence of groups on Y :

$$0 \to \mathit{Pic}^0 X/Y \to \mathit{Pic}\, X/Y \to \mathbf{Z} \to 0,$$

where $\mathbf{Z}$ measures *total degree* on the fibres, and it is easy to see that

$$\mathit{Pic}^0 X/Y = \mathit{Pic}^0 A^*/Y.$$

This sequence shows that $H^2(X, \mathbf{G}_m)$ differs from $H^1(Y, \mathit{Pic}^0 X/Y)$ by a finite group, and hence from $H^2(A^*, \mathbf{G}_m)$ by a finite group. So, Br X has bounded order if and only if Br A* does.

We now work with the fibration $A^* \xrightarrow{f} Y$, and use notation similar to that of [5]. Let A′/Y be the associated Weierstrass fibration obtained by contracting all components of fibres of A*/Y except the identity component. Let $\pi : A^* \to A'$ be the structure map.

LEMMA (5.1). — *Let* $\pi : X \to S$ *be a proper map. Assume* X *regular of dimension* 2, *that the fibres of* $\pi$ *are of dimension* $\leqq 1$, *and that* S *is of finite type over an excellent dedekind scheme. Then the sheaf* $R^q_{et}\, \pi_* \mathbf{G}_m$ *is zero for* $q \geqq 2$.

(3) *See* Remark (6.5).

*Proof.* — This is similar to [7] (p. 98), and the proof in the case $q \geqq 3$ is identical to the one given there. The proof for $q = 2$ is also roughly the same, the only change being that Lemma (3.3) of [7] has to be proved without the flatness assumption. We proceed as in that proof, up to formula (3.3) on p. 103. Since the problem of giving a locally free sheaf V and an isomorphism $A \approx \operatorname{End} V$ is clearly limit preserving (locally of finite presentation), we may apply [1] to complete the proof.

This lemma applies to our map $\pi : A^* \to A'$. Also, it is clear that $R^1_{et}\pi_* \mathbf{G}_m$ is concentrated at the singular points $p$ of the fibres of A'/Y, and at such a point it is

$$\Delta_p^* = \operatorname{Hom}(\Delta_p, \mathbf{Z}),$$

where $\Delta$ is the free abelian group on the exceptional curves for $\pi$ lying over $p$. Let

$$\Delta = \sum_p \Delta_p.$$

Then composition of functors yields an exact triangle.

$$\mathbf{H}^{\cdot}(A', \mathbf{G}_m) \to \mathbf{H}^{\cdot}(A^*, \mathbf{G}_m) \to (\Delta^*)_{+1} \tag{5.2}$$

in the derived category. Interpreting $\mu_n$ cohomology as etale cohomology of the complex $\mathbf{G}_m \xrightarrow{n} \mathbf{G}_m$, we obtain an exact sequence

$$0 \to H^2_{fl}(A', \mu_n) \to H^2_{fl}(A^*, \mu_n) \to \Delta^*/n \tag{5.3}$$

valid for all $n$, where the right-hand arrow is of course restriction of cohomology to the exceptional curves. Hence

LEMMA (5.4). — *$H^2_{fl}(A', \mu_n)$ is isomorphic to the subgroup of $H^2(A^*, \mu_n)$ orthogonal to the exceptional curves for $\pi$.*

LEMMA (5.5). — *Let A/Y denote the group over Y of smooth points of A', and let A (n) denote the complex $A \xrightarrow{n} A$. Then $H^1(Y, A(n))$ identifies canonically with the subgroup of $H^2_{fl}(A^*, \mu_n)$ of elements orthogonal to the zero section and to the components of the fibres of A*/Y.*

We omit the proof, which is like that of [5] (1.4). Note also that elements of

$$H^1(Y, A(n))$$

may be interpreted, as in [5], as pairs (X, D) consisting of a principal homogeneous space X under A and an $n$-fold multi-section D of X'/Y.

LEMMA (5.6). — *Let A'/Y be a Weierstrass fibration such that the associated minimal model A* is a K 3 surface. Let (X, D) represent $\alpha \in H^2_{fl}(A^*, \mu_{p^\nu})$. Then*

$$(D)^2 \equiv P(\alpha) \qquad (\text{modulo } 2p^\nu),$$

where if $p \neq 2$, $P(\alpha) \in \mathbf{Z}/2p^{\nu} = \mathbf{Z}/2 \oplus \mathbf{Z}/p^{\nu}$ is $\alpha \cup \alpha$ in the second summand. If $p = 2$, we define $P(\alpha)$ only for those $\alpha$ which lift to a class $\bar{\alpha} \in H^2(A^*, T_p(\mu))$, setting $P(\alpha) =$ residue of $\bar{\alpha} \cup \alpha$ (modulo $2^{\nu+1}$).

Assume the lemma for now. Then we run through the argument of [5], Section 5, once more : Assuming rank $H^2(A^*, T_p(\mu)) > 22$, we can choose a class $\alpha$ orthogonal to algebraic classes, with $\alpha \cup \alpha \neq 0$. As in [5], this leads to surfaces $X_{\nu}$, which are homogeneous spaces under A of large order $p^{\nu-c}$, lying in a limited family. Moreover, the $X_{\nu}^*$ are supersingular by (1.8), and hence lie in a limited family of supersingular K 3 surfaces. This contradicts Lemma (1.6) and completes the proof of part (i) of the theorem.

Consider the assertion of (ii). By (i), we know Br X is annihilated by some $p^{\nu}$. Thus Kummer theory yields

$$(5.7) \qquad H^2_{fl}(X, \mu_{p^{\nu}}) \to \operatorname{Br} X \xrightarrow{0} \operatorname{Br} X \to H^3_{fl}(X, \mu_{p^{\nu}}).$$

Therefore Br X is isomorphic to a subgroup of $H^3_{fl}(X, \mu_{p^{\nu}})$, which is $\mathbf{G}_a(k)$ by (4.2). So, Br X is annihilated by $p$. Thus we can take $\nu = 1$ in the sequence (5.7). Combining the left and right arrows gives a map $H^2_{fl}(X, \mu_p) \to H^3_{fl}(X, \mu_p)$ which is easily seen to the Bockstein map $\delta$ :

$$\underline{H}^2(X, \mu_p) \xrightarrow{i} \underline{H}^2(X, \mu_{p^2}) \to \underline{H}^2(X, \mu_p) \xrightarrow{\delta} \underline{H}^3(X, \mu_p).$$

Clearly $i$ induces an isomorphism on $U^2$. Hence the kernel of $\delta$ is finite, and so $\delta$ induces a surjection $\underline{U}^2(X, \mu_p) \to \underline{H}^3(X, \mu_p) = \mathbf{G}_a$. Thus $\operatorname{Br} X \xrightarrow{\sim} H^3_{fl}(X, \mu_p)$ in (5.7), and $U^2(X, \mu_p) \to \operatorname{Br} X$ is surjective.

*Proof of Lemma* (5.6). — Our proof is rather brutal. We try to lift the pair (X, D) to characteristic zero, compatibly with a lifting of the Weierstrass fibration A′/Y. Let $A'_T$, $(X_T, D_T)$ be such a lifting, where say $T = \operatorname{Spec} R$ and R is some discrete valuation ring with residue field $k$. By [2] there is a Brieskorn resolution $\pi_T : A^*_T \to A'_T$, if T is replaced by some ramified extension. The pair $(X_T, D_T)$ induces a class $\alpha_T$ in

$$H^2_{fl}(A^*_T, \mu_{p^{\nu}}),$$

and by our hypothesis on the duality formalism, cup product is constant on the family $A^*_T/T$. The number $(D)^2$ will also be constant, and so we will be able to apply the known result [5] (2.3) in characteristic zero, if $p \neq 2$. If $p = 2$, we divide the class by $2^r$ before lifting.

For the moment, let $A'_S$ be any family of Weierstrass fibrations over $Y_S = \mathbf{P}^1_S$, parametrized by some scheme S. Let $A' = A'_0$ be the fibre at $s_0 \in S$, and let $X_0$ be a homogeneous space under $A_0$. Consider deformations of $X_0$, i. e., the functor from pointed schemes over $(S, s_0)$ to sets defined by

(5.8) $(T, t_0) \mapsto$ (isom. cl. of homog. spaces $X_T$ under $A_T$ with fibre $X_0$ at $t_0$).

The formal properties of this functor are easily described. Say that $R' \to R$ is a surjective map of local $\mathcal{O}_S$-algebras whose kernel I satisfies $I^2 = 0$. Let $X_R$ be a homogeneous space of $A_R$ over $Y_R$. It is given by a class in $H^1(Y_R, A_R)$, and so extensions to R′ are explained by the cohomology of the exact sequence of group sheaves on $Y_R$ for etale topology

$$0 \to L \otimes I/I^2 \to A_{R'} \to A_R \to 0,$$

where L is the bundle of tangents to the fibres of $A_R$. The sheaf $L \otimes I/I^2$ is coherent. Hence $H^2(Y_R, L \otimes I/I^2) = 0$, and so every homogeneous space $X_R$ extends to Spec R′.

COROLLARY (5.9). — *The functor* (5.8) *has a formal hull, say* $\bar{T}$, *smooth over* $\bar{S} = \operatorname{Spec} \mathcal{O}_{S, s_0}$ *and of relative dimension* $h^1(Y_0, L_0)$.

Now let S be the parameter space of a family $A'_S$ of Weierstrass fibrations which is a versal deformation of our given surface $A' = A'_0$ over the ring of Witt vectors W (*k*). S is smooth over W (*k*). The formal space $\bar{T}$ classifies deformations of $X = X_0$, and we now ask to extend the multi-section D. By Proposition (8.5), extension of the line bundle $\mathcal{O}_0(D) = L_0$ is possible above a closed subset $\bar{Z} \subset \bar{T}$. Adding fibres to D if necessary, we may suppose $L_0$ ample. Then $L_{\bar{Z}}$ is an ample sheaf on the formal deformation on $X_{\bar{Z}}$, which is therefore projective and so is induced by a scheme over $\bar{Z}$. This gives the required lifting, provided $\bar{Z}$ contains a point of characteristic zero. So, we are done unless $\bar{Z}$ contains points of characteristic *p* only. But by (8.5) $\bar{Z}$ is defined in the smooth scheme $\bar{T}$ by one equation. So, this bad case can occur only if $\bar{Z}$ is the whole scheme $\bar{T} \times \operatorname{Spec}(W(k)/p^\nu)$ for some ν. In that case we can at least replace the pair $(X_0, D_0)$ by a generic deformation in characteristic *p*, i. e., we can assume $A'_0$ is generic. Then $A'_0$ is smooth : $A'_0 = A^*_0$, and deformations of $(X_0, D_0)$ are in one-one correspondence with deformations of the associated class $\alpha_0 \in H^2_{fl}(A'_0, \mu_{p^\nu})$. By [4], such a class can be lifted to characteristic zero provided $\widehat{\mathrm{Br}}\, A'_0$ is *p*-divisible. Therefore we are done by the final lemma :

LEMMA (5.10). — *A generic Weierstrass fibration* A′/Y *which is a* K 3 *surface is not supersingular.*

*Proof.* — Since the condition $h = \infty$ is closed, it suffices to show that some Weierstrass fibration A′/Y has minimal model A* which is not supersingular. By (1.7), this will be the case if A* is projectively liftable and has $\rho < 22$. If $p = 2$, we made the required calculation (2.12). If $p \neq 2$ we can use the Kummer surface associated to $E \times E'$ where E, E′ are non-isogenous elliptic curves. This has rank 18 [18].

6. THE CASE OF A FINITE GROUND FIELD. — We continue to derive consequences of the conjectures of Section 3. As in the previous section, hypothesis (4.1) is in force here.

Consider the case that an elliptic supersingular K 3 surface $X = X_k$ is defined over the finite field $k = \mathbf{F}_q$, and that *k* is large enough so that $G = \operatorname{Gal}(\bar{k}/k)$ acts trivially on the Néron-Severi group N of $X_{\bar{k}}$. The Hochschild-Serre spectral sequences

$$E_2^{pq} = H^p(G, H^q_{fl}(X_{\bar{k}}, F)) \Rightarrow H^{p+q}_{fl}(X_k, F)$$

show that

$$\text{(6.1)} \qquad \operatorname{Br} X_k = (\operatorname{Br} X_{\bar{k}})^G,$$
$$H^2_{fl}(X_k, \mu_{p^\nu}) = H^2_{fl}(X_{\bar{k}}, \mu_{p^\nu})^G.$$

An analysis of diagram (4.4) shows that its G-invariants again form an exact diagram

$$\text{(6.2)} \qquad \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ p^\nu N^* & \longrightarrow & G_a(k) & \longrightarrow & \operatorname{Br} X_k \to 0 \\ \downarrow & & \downarrow & & \| \\ N & \longrightarrow & H^2_{fl}(X_k, \mu_{p^\nu}) & \to & \operatorname{Br} X_k \to 0 \\ \downarrow & & \downarrow & & \\ N/p^\nu N^* & = & N/p^\nu N^*. & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

The top row of this diagram for $\nu = 1$, together with (4.7), shows that Br $X_k$ is finite and gives its order :

$$\text{(6.3)} \qquad |\operatorname{Br} X_k| = q\,|N^*/N|^{-1}.$$

COROLLARY (6.4). — *The conjectural formula* (C) *of Tate* [20], *Section* 4, *is a consequence of* (4.1), *for elliptic supersingular* K 3 *surfaces.*

In fact, since $\rho = 22$, the polynomial $P_2(X, q^{-S})$ appearing in the zeta function of X is

$$P_2(X, q^{-S}) = (1-q^{1-S})^{22}.$$

The torsion on X is trivial, and $\alpha(X) = 1$. Thus Tate's formula (C) reads

$$\text{(6.4)} \qquad |\operatorname{Br} X|.|\operatorname{discr} N| = q,$$

which follows from (6.3) and (4.6).

REMARK (6.5). — The surjective map $U^2(X_k, \mu_p) = \mathbf{G}_a(k) \to \operatorname{Br} X$ of (6.2) has finite kernel $N^*/N$ independent of $k$, and it may be tempting to identify Br X with the quotient *variety* $\mathbf{G}_a/\operatorname{im} N^*$, which is of course isomorphic to $\mathbf{G}_a$ again. As the formula (6.4) shows, this is not correct functorially. The *functor* $\operatorname{Br} X = R^2\pi_* \mathbf{G}_m$ is not representable, but is the "presheaf quotient" $\operatorname{Br} X(S) = \mathbf{G}_a(S)/\operatorname{im} N^*$ of $\mathbf{G}_a$ by the *discrete group* im $N^*$.

Tate [20], (5.1) has shown that arithmetic Poincaré duality defines a skew-symmetric pairing on Br $X_k$, provided that group is finite. His proof carries over without change to the case of $p$-torsion groups :

(6.6) There is a non-degenerate skew-symmetric form on Br $X_k$ $(p \neq 2)$.

Thus $|\mathrm{Br}\, X_k|$ is an even power of $p$. This has two consequences : First of all, if $k$ contains $\mathbf{F}_{p^2}$, then $q$ is an even power of $p$. Hence by (6.4) :

(6.7) $$|\mathrm{discr}\, N| \text{ is an even power of } p \qquad (p \neq 2).$$

A specialization argument shows that this is true for any elliptic supersingular K 3 surface, not necessarily defined over a finite field. Thus (4.6) follows.

Secondly, now that we know that $|\mathrm{Br}\, X_k|$ and $|\mathrm{discr}\, N|$ are even powers of $p$, it follows that $q$ is, too :

(6.8) Suppose $p \neq 2$. If X is defined over $\mathbf{F}_q$ and G acts trivially on N, then $\mathbf{F}_q \supset \mathbf{F}_{p^2}$.

REMARK (6.9). — Going back to the case of an algebraically closed field $k$, it is conceivable that some subgroup of finite index $\Gamma \subset H^2(X, T_p(\mu))$ can be lifted to characteristic zero over a lifting $X_\eta$ of X. If so, then the compatibility of specialization with cup product would imply that $|\mathrm{discr}\, N|$ is an even power of $p$. For, this evenness is preserved if we pass to the subgroup $\Gamma$, and the lifting of $\Gamma$ would be of finite index in $H^2(X_{\bar{\eta}}, T_p(\mu))$, which is unimodular. However, the problem of deforming $\mu$ cohomology classes when X is supersingular seems quite delicate, and we have no results on that beyond lifting of individual classes done in Section 5.

## 7. A FILTRATION ON THE MODULI SPACE.

— Hypothesis (4.1) remains in force here.

Consider a versal family M of polarized, projectively liftable K 3 surfaces in characteristic $p$. As we have remarked, the conditions height $(\hat{\mathrm{Br}}\, X) = h \geq i$ are closed algebraic conditions on M. We define a decreasing filtration by these conditions :

(7.1) $$M_i : \quad h \geq i,$$

so that in particular $M = M_1$. Since $\rho \geq 1$, Theorem (0.1) implies that $M_{11}$ and $M_\infty$ have the same reduced structure. Moreover, $M_{i+1}$ is defined locally be one equation in $M_i$ (cf. Section 2).

Now consider the family of supersingular surfaces $M_\infty$ and suppose $p \neq 2$. We restrict attention to the open set (1.4) of elliptic members. So, let us replace M by an open set, on which $M_\infty$ contains only elliptic surfaces. It follows from (1.1 *a*) that the vector spaces $V = N \otimes \mathbf{Q}$ defined at each point form a local system on M. Let us further replace M by an etale extension on which this local system can be trivialized, and choose a trivialization.

One sees easily that only finitely many lattices $L \subset V$ arise as Néron-Severi groups of fibres of $M_\infty$. For each such lattice L, we define a closed subscheme $\Sigma_L \subset M_\infty$ by the condition

(7.2) $$\Sigma_L : \quad L \subseteq N.$$

Clearly this is defined scheme-theoretically, and not only as a closed set. If $L' \supset L$, then $\Sigma_{L'} \subset \Sigma_L$. Suppose that moreover $L'/L \approx \mathbf{Z}/p$. Let $v \in L' - L$ be arbitrary. Then $\Sigma_{L'}$ is defined in $\Sigma_L$ by the condition $v \in N$, which is given by one equation (8.5) :

(7.3) If $L'/L = \mathbf{Z}/p$, then $\Sigma_{L'}$ is cut out locally by one 1 equation in $\Sigma_L$.

Define $\sigma(L)$ and $\sigma_0(L)$ by $\sigma+\sigma_0 = 11$ and discr $L = -p^{2\sigma_0}$. We set

$$\Sigma_i = \bigcup_{\sigma(L)\geqq i} \Sigma_L, \tag{7.4}$$

so that $\Sigma_i$ is a decreasing filtration on $M_\infty$, and the reduced structures of $M_\infty$ and $\Sigma_1$ are equal (4.8). Obviously, the $\Sigma_i$ can be defined without reference to a trivialization of V.

Now suppose $p \equiv 3$ (modulo 4), and let $X_0$ be the elliptic modular surface of level 4. This is certainly liftable, and it has $\rho = 22$ by Shioda [18]. Hence it is supersingular.

PROPOSITION (7.5). — *Let $X_0$ be the elliptic modular surface of level* 4 *in characteristic* $\neq 2$. *Then*

(i) $H^0(X_0, \Theta) = 0$;

(ii) *Any infinitesimal deformation $X_S$ of $X_0$ over* $S = \operatorname{Spec} k[t]/(t^2)$, *such that the Néron-Severi group* N *of $X_0$ extends to $X_S$, is trivial.*

We defer the proof, and look at some consequences when $p \equiv 3$ (modulo 4).

Denote by $\tilde{M}$ the formal versal space for deformations of $X_0$ as unpolarized surface of characteristic $p$. This is a smooth 20-dimensional formal scheme (8.4). Let $M \subset \tilde{M}$ be the closed set defined by some polarization which is a specialization from the modular surface in characteristic zero. By Grothendieck's existence theorem [6], we can view M as parameter space of an actual family of K 3 surfaces.

Let $L_0 = N$ denote the Néron-Severi group of $X_0$, and say $\sigma(L_0) = i$. Then the proposition implies that $\Sigma_{L_0} = \Sigma_i$ is the "origin" $m_0 \in \tilde{M}$, scheme-theoretically.

Consider the filtration defined above :

$$\tilde{M} \supset M = M_1 \supset \ldots \supset M_{11} \supseteq M_\infty \supseteq \Sigma_1 \supset \ldots \supset \Sigma_{10}. \tag{7.6}$$

If we work only with reduced structure for the moment, then we have $M_{11} = M_\infty = \Sigma_1$, and so this chain contains 21 members, each having codimension $\leqq 1$ in its predecessor [cf. (8.5) for the first inclusion]. Since $\tilde{M}$ is irreducible and 20-dimensional while $\dim \Sigma_i = 0$, it follows that $i = 10$ and that all terms of (7.6) are equi-dimensional, of the expected dimension.

COROLLARY (7.7). — *Assume* (4.1), *and that* $p \equiv 3$ (*modulo* 4). *Let $X_0$ be the elliptic modular surface of level* 4. *Then*

(i) *The dimension of $M_i$ at $X_0$ is* $20-i$. *In particular,*

(ii) *All possible values* $1 \leqq h \leqq 10$ *for the height are taken on by* K 3 *surfaces in characteristic* $p$;

(iii) *The family $M_\infty$ of supersingular surfaces is of dimension* 9 *at* $X_0$;

(iv) discr $N = -p^2$ *for* $X_0$.

Now consider the $\Sigma$ filtration on $M_\infty$, scheme-theoretically. Let $L_\eta$ be the Néron-Severi lattice of a generic deformation in $M_\infty$. Then dim $\Sigma_{L_\eta}$ is of dimension 9. Clearly $L_\eta \subset L_0$. We can refine this inclusion to a composition series for $L_0/L_\eta$ whose successive quotients are $\mathbf{Z}/p$. Therefore (7.3) and the fact that $\Sigma_{L_0}$ consists of the origin alone imply $\sigma(L_\eta) = 1$, and

COROLLARY (7.8). — *With the assumptions of* (7.7), *we have*

(i) *For any* $L \subset L_0$ *which is the Néron-Severi lattice of a generalization of* $X_0$, $\Sigma_L$ *is smooth at* $X_0$, *of the expected dimensions* $\sigma_0(L)-1$.

(ii) *If* L *is a Néron-Severi lattice as in* (i), *so is any* L′ *between* L *and* $L_0$.

(iii) $\Sigma_i$ *is a union of smooth varieties* $\Sigma_L$ *of dimension* $(10-i)$, *at* $X_0$.

*Proof of Proposition* (7.5). — Let $X_S$ be a deformation, as in the lemma. Let C be an irreducible curve on $X = X_0$, and $L = \mathcal{O}(C)$. Then $h^1(X, L) = 0$. This implies that the sections of L extend to the (unique) invertible sheaf $L_S$ over $X_S$ inducing L, which exists by hypothesis. Hence C is induced by a Cartier divisor $C_S$ on $X_S$. This reasoning applies in particular to the elliptic fibration of X over Y, to the 16 sections $\Gamma_i$ corresponding to points of order 4, and to the components of the reducible fibres. So, $X_S$ is an elliptic fibre system over $Y_S = \mathbf{P}^1 \times S$ with given 0-section $\Gamma_{0S}$, and 15 other sections $\Gamma_{iS}$ extending the $\Gamma_i$.

LEMMA (7.9). — *The sections* $\Gamma_{iS}$ *are of order* 4 *in the group* $A_S/Y_S$ *of smooth points of* $X_S/Y_S$.

*Proof.* — The fact that $\Gamma_i$ is of order 4 can be expressed by the assertion.

$$4\Gamma_i - 4\Gamma_0 \sim 0 \qquad \text{(modulo components of fibres)},$$

i. e., that a certain canonically constructed line bundle is trivial. Its extension to $X_S$ is unique, and hence is also trivial. Therefore $\Gamma_{iS}$ is of order 4, too.

We now claim that there is a cartesian diagram

(7.10)
$$\begin{array}{ccc} X_S & \rightarrow & X \\ \downarrow & & \downarrow \\ Y_S & \rightarrow & Y \end{array}$$

compatible with the inclusion of X/Y into $X_S/Y_S$. It will be automatic that $Y_S \rightarrow Y$ commutes with Spec $S \rightarrow$ Spec $k$, and so this will show that $X_S$ is a trivial deformation.

Consider the problem of constructing (7.10) locally on Y, to begin with. Let U be the open set of Y above which X is smooth, i. e., where $j \neq \infty$. This U represents the functor of elliptic curves with level 4 structure, X being the universal element. Since $X_S/Y_S$ comes with sections $\Gamma_{iS}$ of order 4, there is a unique diagram (7.10) over U, and it is unique when pulled back to any U′ lying over U. To construct (7.10) globally, it suffices to do so locally, say for the etale topology, at the points of Y which are poles of *j*. The uniqueness will imply global existence by descent.

Let $X_y$ be a fibre at which $j = \infty$. It will have Kodaira's form $I_4$ ([9] and [16], 4.2), i. e., will be $X_y = C_0 + C_1 + C_2 + C_3$, the curves forming a quadrilateral. We are given $X_y$ together with a group law on the smooth locus $A_y$, and the 16 points $\gamma_i$ of $A_y$ of order 4 are fixed. Let $G = (\mathbf{Z}/4)^2$ be the group underlying this point set, acting on $X_y$ by translation. We denote the element of G corresponding to $\gamma_i$ by $g_i$.

We consider flat deformations of the structure $(X_y, \{\gamma_i\}, G)$ consisting of the curve $X_y$, the 16 points $\gamma_i$, and the action of G on $X_y$. It is obvious that the structure $(X_y, \{\gamma_i\})$ has no infinitesimal automorphisms. This implies that the group action will extend uniquely, if at all, to any deformation of this structure. It follows that the deformations of $(X_y, \{\gamma_i\}, G)$ form a closed subfunctor of the deformations of $(X_y, \{\gamma_i\})$. Since this last space has a hull Z, so does the first, say $Z' \subset Z$.

The dimensions of Z and of Z′ are easily computed. To determine a deformation of $X_y$, we have to assign local deformations at each of the 4 singular points $p_1, \ldots, p_4$, and then to choose the 16 deformations of the $\gamma_i$. (There are "no" locally trivial deformations of $X_y$.) The 1-parameter group of automorphisms of each $C_j$ as subscheme of $X_y$ allows any two deformations of one point $\gamma_i$ to be equalized uniquely. Thus Z is a smooth space of dimension $4+16-4 = 16$.

When does a deformation lie in Z′ ? The group G acts transitively on the set $\{p_\nu\}$. This means that the local deformations at the $p_\nu$ must all be isomorphic via the action. Then, once one point $\gamma_i$ has been extended, the action of G determines extensions of the remaining $\gamma_i$. Thus at most one parameter remains, namely the choice of a local deformation, say at $p_1$. On the other hand, the fibration $X \to Y$ furnishes us with a 1-parameter deformation which is locally versal at $p_1$. So, it is a versal deformation of the structure $(X_y, \{\gamma_i\}, G)$. Since $X_S \to Y_S$ also has such a structure, this fibration is obtained formally at $y$ by pull-back. In other words, a diagram (7.10) exists formally, and versality insures that it can be chosen compatibly with the inclusion $X/Y \subset X_S/Y_S$. By [1] the formal diagram may be approximated by one locally for the etale topology, as required.

The assertion $H^0(X, \Theta) = 0$ is proved easily using the universal property of X over U.

## 8. Appendix: Notational conventions and background material.

— All schemes or algebraic spaces occurring are understood to be noetherian. Algebraic spaces occur only incidentally, as total spaces of families of smooth surfaces.

Cohomology means etale cohomology except when otherwise stated. But when working with smooth coefficient groups such as $\mathbf{G}_m$, we often pass informally to the flat topology, invoking Grothendieck's theorem ([7], p. 171) that it yields the same cohomology.

We use the notation Br X for the Brauer group of a smooth algebraic surface. This is the same as the cohomological one $H^2(X, \mathbf{G}_m)$ ([7], p. 76). The symbol *Br* X denotes the functor $R^2\pi_* \mathbf{G}_m$, where $\pi : X \to \operatorname{Spec} k$ is the structure map.

The Néron-Severi group of a surface X is denoted by $N = N(X)$.

Here is a brief review of the invariants of a K 3 surface X. By definition, we have $p_a(X) = 1$, i. e., $\chi(\mathcal{O}_X) = 2$, and $\Omega^2_X \approx \mathcal{O}_X$. This means $h^{01} = 0$ and $h^{02} = 1$. It follows that Pic X is discrete : Pic X = N.

If C is a divisor on X, its genus is $p(C) = (1/2)(C)^2+1$, an integer. Hence the intersection form on N is *even*, and the Riemann-Roch formula is

$$\chi(\mathcal{O}(C)) = p(C)+1 = \frac{1}{2}(C)^2+2. \tag{8.1}$$

It is known that $h^q(\mathcal{O}(C)) = 0$ for $q \neq 0$ if C is an irreducible curve.

A consequence of (8.1) is that N has no torsion. For, let $L = \mathcal{O}(C)$ represent a non-trivial torsion class in N. Then $h^0(L) = h^2(L) = 0$. By (8.1), we get $\chi(L) = 2$, hence $h^1(L) = -2$, which is absurd. Another consequence is that a pencil of elliptic curves on X can have no multiple member.

The Hirzebruch Riemann-Roch formula reads $12\chi = c_1^2+c_2$, and $c_1^2 = 0$. Hence $c_2 = 24$. It is known that $c_2$ can be calculated as the $l$-adic Euler characteristic

$$c_2 = \sum(-1)^q b_q, \tag{8.2}$$

where $b_q = \text{rank } H^q(X, \mathbf{Z}_l)$, for any $l \neq p = \text{char } k$. (This follows from Igusa [8], and [13].) Since Pic X is discrete, we have $b_1 = 0$. So, rank $H^2(X, \mathbf{Z}_l) = b_2 = 22$, for all $l \neq p$. Since N has no torsion, neither do $H^2(X, \mathbf{Z}_l)$ and $H^3(X, \mathbf{Z}_l)$. Since $\Omega^2 \approx \mathcal{O}$, the pairing $\Omega^1 \otimes \Omega^1 \to \Omega^2$ shows that $\Omega^1 \approx \Theta$. Serre duality therefore gives

$$h^0(X, \Theta) = h^2(X, \Theta), \tag{8.3}$$

while $\chi(\Theta) = -20$. It is not known whether K 3 surfaces in characteristic $p \neq 0$ can have vector fields. This is an interesting question. We can prove only that any one having a vector field must be unirational. But for a surface without vector fields, we have

(8.4) If $h^0(X, \Theta) = 0$, then $h^2(X, \Theta) = 0$, and $h^1(X, \Theta) = 20$. Moreover, the versal deformation of X is unobstructed, of dimension 20.

We will make frequent use of the following fact.

PROPOSITION (8.5). — *Let* $X_S/S$ *be a local family, possibly formal, of* K 3 *surfaces. Let* $L_0$ *be an invertible sheaf on the closed fibre* $X_0$ *at* $s_0$. *There is a closed subset* $T \subset S$ *such that* $L_0$ *extends to an invertible sheaf* $L_T$ *on* $X_T$, *and such that* T *is universal with this property. Moreover,* T *is defined in* S *by one equation.*

*Proof.* — The existence of T is standard, and we omit it. Let us show that T is defined by one equation. Say that $S = \text{Spec } R$, let I be the ideal defining T, and let $T' = \text{Spec } R/I^2$. The truncated exponential sequence

$$0 \to \mathcal{O}_{X_T} \otimes I/I^2 \to \mathcal{O}^{\times}_{X_{T'}} \to \mathcal{O}^{\times}_{X_T} \to 0$$

defines an obstruction $\mathfrak{o} \in H^2(X, \mathcal{O}_{X_T} \otimes I/I^2)$ to the extension of $L_T$ to $X_{T'}$. Since $X_T$ is a family of K 3 surfaces, $H^2(X_T, \mathcal{O} \otimes I/I^2) \approx I/I^2$. By definition of T, $L_T$ can be extended no further. This means that if J is any ideal between $I^2$ and I and $J \neq I$, then the obstruction to extending $L_T$ to Spec R/J, which is the image of $\mathfrak{o}$ in I/J, is not zero. Since this is true for all J, the element $\mathfrak{o}$ is a generator for $I/I^2$. Any representative in I will generate I.

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M. ARTIN,
Research Institute for Mathematical Sciences,
Kyoto University,
Kyoto, Japan.