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GERSTEN'S CONJECTURE AND THE HOMOLOGY OF SCHEMES

BY SPENCER BLOCH AND ARTHUR OGUS

0. Introduction

Let X be a smooth algebraic variety over a field k. The deepest conjectures in algebraic geometry (Weil, Hodge, Tate) are attempts to calculate the " arithmetic filtration " [7] on a suitable cohomology group $H^i(X)$. Recall that this filtration is given by

$$N^p H^i(X) = \cup \operatorname{Ker}\{ H^i(X) \to H^i(X-Z) : Z \subset X \text{ is closed of codimension } p \};$$

it is called the filtration by " coniveau " in [5]. These conjectures assert that this mysterious filtration is equal to (or contained in) another filtration which can " actually be computed ".

The filtration by coniveau is the filtration of a natural spectral sequence, whose E_1 term was written down by Grothendieck [4]; one has $E_1^{pq} = \bigoplus_{x \in Z^p} H^{q-p}(k(x))$; the direct sum being taken over points of codimension p. Our main result is an expression for the E_2-term. Namely we can regard $H^{q-p}(k(x))$ as a constant sheaf on $\{x\}^-$ and extend it by zero to X. Then the differentials of the spectral sequence furnish us with a complex of sheaves on X :

$$(0.1) \qquad 0 \to \mathscr{H}^q \to \bigoplus_{Z^0} H^q(k(x)) \to \bigoplus_{Z^1} H^{q-1}(k(x)) \to \dots \bigoplus_{Z^q} H^0(k(x)) \to 0,$$

where $\mathscr{H}^q$ is the sheaf associated to the presheaf $U \mapsto H^q(U)$. Our theorem asserts that the above sequence is *exact*. Some consequences :

(0.2) The $E_2^{p,q}$ term of the spectral sequence of coniveau is $H^p(X, \mathscr{H}^q)$.

(0.3) $$H^p(X, \mathscr{H}^q) = 0 \quad \text{if} \quad p > q.$$

(0.4) In the case of de Rham cohomology over a field of characteristic zero, the coniveau spectral sequence coincides, from E_2 on, with the second spectral sequence of hypercohomology. In particular the two filtrations are the same, as conjectured by Washnitzer.

(0.5) One has $H^p(X, \mathscr{H}^p) \cong A^p(X) \otimes H^0(pt)$, in the étale and de Rham theories, where $A^p(X)$ is the group of cycles mod *algebraic* equivalence.

(0.6) $H^0(X, \mathscr{H}^q)$ can be identified with the space of cohomology classes of the second kind in the sense of Lefschetz. We can then see (using Griffiths' famous example) that this is *not* equivalent with the notion used by Atiyah and Hodge, contrary to some claims in the literature [16]. We are grateful to W. Messing for bringing this issue to our attention.

Our paper is organized as follows : In paragraph 1 we describe the axioms that a cohomology theory must satisfy for our proof to go through. The main tool is a suitable notion of (Borel-Moore) homology which is a covariant functor for proper maps. Paragraph 2 establishes these properties for étale cohomology, de Rham cohomology, and " singular cohomology of associated analytic space ". In paragraph 3 we review the general fomalism of the spectral sequence and the expression for its E_1-term.

The fourth section contains the statement of the main result (Theorem 4.2), which is the analogue of Gersten's conjecture in K-theory [2]. This section also contains the first steps in the proof, notably employing a trick used by Quillen [10] in his proof of Gersten's conjecture. We finish the proof in the next section, and give the applications in the last three.

We would like to emphasize our intellectual debt to Gersten and Quillen. Essentially, the purpose of this paper is to apply their ideas to various homology and cohomology theories other than K-theory.

1. **Poincaré duality with supports**

Let k be a fixed ground field, and $\mathscr{V}$ be a category of schemes of finite type over k, containing all quasi-projective k-schemes. If $X \in \mathrm{Ob}\,\mathscr{V}$ and $Y \subseteq X$ is locally closed, we assume $Y \in \mathrm{Ob}\,\mathscr{V}$. We shall here describe the axioms a cohomology functor on $\mathscr{V}$ must satisfy in order to have a reasonable theory of " coniveau " as described by Grothendieck in [5]. These are consequences of a satisfactory theory of $f^!$ and $f_!$, but we have found it more convenient to work with these consequences than with the derived categories themselves. Our main tool is the notion of a suitable " Borel Moore homology " [1]. Notice that what we call Poincaré duality is not a duality theorem at all, since no pairings occur. More precisely, we use the existence of the functor $f^!$ and of the trace map $R f_! f^! \to \mathrm{id}$ for proper f, but not the " duality theorem " itself.

(1.1) DEFINITION. — Let $\mathscr{V}$ be a category of algebraic k-schemes, as above. Then $\mathscr{V}^*$ is the category whose objects are closed immersions $Y \subsetneq X$ and whose morphisms are Cartesian squares :

$$f_Y \subsetneq f_X : \ (Y \subsetneq X) \to (Y' \subsetneq X') : \quad \begin{array}{ccc} Y & \subsetneq & X \\ {\scriptstyle f_Y}\downarrow & & \downarrow{\scriptstyle f_X} \\ Y' & \subsetneq & X' \end{array}$$

A *twisted cohomology theory with supports* is a sequence (indexed by $n \in \mathbf{Z}$) of contravariant functors $\mathscr{V}^* \to$ (graded abelian groups), written $(Y \subsetneq X) \mapsto \bigoplus_i H^i_Y(X, n)$.

For $X \in \mathscr{V}$, we write $H^i(X, n)$ in place of $H^i_X(X, n)$. The n is included in the notation to keep track of " Tate twist " in the étale theory.

We assume the following axioms :

(1.1.1) For $Z \subseteq Y \subseteq X$, there is a long exact sequence :

$$\ldots \to H^i_Z(X, n) \to H^i_Y(X, n) \to H^i_{Y-Z}(X-Z, n) \to H^{i+1}_Z(X, n) \to \ldots.$$

(1.1.2) If

$$f: \quad (Y \subsetneq X) \to (Y' \subsetneq X')$$

and

$$g: \quad (Z \subsetneq Y) \to (Z' \subsetneq Y')$$

are arrows in $\mathscr{V}^*$, and k is the induced arrow $(Y-Z \subsetneq X-Z) \to (Y'-Z' \subsetneq X'-Z')$, then the arrows $H^*(h)$, $H^*(f)$, and $H^*(k)$ fit together to form a commutative ladder of the long exact sequences for $Z \subsetneq Y \subsetneq X$ and $Z' \subsetneq Y' \subsetneq X'$. Here $h: (Z \subsetneq X) \to (Z' \subsetneq X')$.

(1.1.3) If $Z \subsetneq X \in \operatorname{Ob} \mathscr{V}^*$ and if $U \subsetneq X$ is open in X and contains Z, the map $H^i_Z(X, n) \to H^i_Z(U, n)$ is an isomorphism.

(1.2) DEFINITION. – Let $\mathscr{V}_*$ be the category with $\operatorname{Ob} \mathscr{V}_* = \operatorname{Ob} \mathscr{V}$ but whose arrows consist only of proper morphisms. A *twisted homology theory* is a sequence of covariant functors $\mathscr{V}_* \to$ (graded abelian groups), written $H_i(X, n)$ for $X \in \mathscr{V}_*$. We assume the following axioms :

(1.2.1) H_* is a presheaf in the étale topology, namely :

If $\alpha : X' \to X$ is étale, there is a functorial map

$$\alpha^*: \quad H_i(X, n) \to H_i(X', n).$$

(1.2.2) If the diagram below on the left is Cartesian, with proper vertical arrows and étale horizontal arrows, then the diagram on the right commutes.

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\alpha} & Y \end{array} \qquad \begin{array}{ccc} H_i(X', n) & \xleftarrow{\beta^*} & H_i(X, n) \\ {\scriptstyle H_i(g, n)}\downarrow & & \downarrow{\scriptstyle H_i(f, n)} \\ H_i(Y', n) & \xleftarrow{\alpha^*} & H_i(Y, n) \end{array}$$

(1.2.3) Let $i : Y \subsetneq X$ be a closed immersion in $\mathscr{V}$, and let $\alpha : (X-Y) \subsetneq X$ be the corresponding open immersion. Then there is a long exact sequence :

$$\ldots \to H_i(Y, n) \xrightarrow{i^*} H_i(X, n) \xrightarrow{\alpha^*} H_i(X-Y, n) \to H_{i-1}(Y, n) \to \ldots$$

(here we have written i_* for $H_i(i, n)$).

(1.2.4) Let $f : X' \to X$ be a proper morphism in $\mathscr{V}$, let $Z = f(Z')$, and let $\alpha : X' - f^{-1}(Z) \subsetneq X' - Z'$. Then the diagram below commutes :

$$\begin{array}{ccccccccc}
\ldots \to & H_i(Z', n) & \to & H_i(X', n) & \to & H_i(X'-Z', n) & \to & H_{i-1}(Z', n) & \to \ldots \\
 & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \alpha^* & & \downarrow f_* & \\
\ldots \to & H_i(Z, n) & \to & H_i(X, n) & \to & H_i(X-Z, n) & \to & H_{i-1}(Z, n) & \to \ldots
\end{array}$$

(1.3) DEFINITION. — A *Poincaré duality theory with supports* is a twisted cohomology theory H^*, together with the following structure :

(1.3.1) (Cap product with supports). — For any $Y \subsetneq X$ in $\mathrm{Ob}\, \mathscr{V}^*$, a pairing :

$$H_i(X, m) \otimes H^j_Y(X, n) \to H_{i-j}(Y, m-n).$$

(1.3.2) (Compatibility of cap product with restriction). — If $Y \subsetneq X \in \mathrm{Ob}\, \mathscr{V}^*$ and if

$$(\beta \subsetneq \alpha) : \quad (Y' \subsetneq X') \to (Y \subsetneq X) \in \mathrm{Arr}\, \mathscr{V}^*$$

and is étale, then for $a \in H^j_Y(X, n)$ and $z \in H_i(X, m)$, $\alpha^*(a) \cap \alpha^*(z) = \beta^*(a \cap z)$.

(1.3.3) (Projection formula). — If f is a proper morphism in $\mathscr{V}^*$,

$$f : \quad (Y_1 \subsetneq X_1) \to (Y_2 \subsetneq X_2),$$

then for $a \in H^j_{Y_2}(X_2, n)$ and $z \in H_i(X_1, m)$, $H_i(f_X)(z) \cap a = H_i(f_Y)(z \cap H^j(f)(a))$.

(1.3.4) (Fundamental class). — If $X \in \mathrm{Ob}\, \mathscr{V}$ is irreducible and of dimension d, then there is a global section η_X of $H_{2d}(X, d)$: thus if $\alpha : X' \to X$ is étale, $\alpha^* \eta_X = \eta_{X'}$.

(1.3.5) (Poincaré duality). — If $X \in \mathrm{Ob}\, \mathscr{V}$ is smooth of dimension d and if $Y \subsetneq X$ is a closed immersion, then cap-product induces an isomorphism : $\eta_X \cap : H^{2d-i}_Y(X, d-n) \to H_i(Y, n)$.

For future convenience we shall record here a compatibility which is a consequence of the above axioms and which will be an important tool in the proof of our main theorem.

(1.4) LEMMA. — *Suppose we are given a Poincaré duality theory with supports satisfying the above axioms. Suppose that the square on the left is Cartesian and that f_X is étale. Then the square on the right commutes :*

(1.4.1)
$$\begin{array}{ccc}
Z' & \subsetneq & X' \\
{\scriptstyle f_Z}\downarrow & & \downarrow{\scriptstyle f_X} \\
Z & \subsetneq & X
\end{array}
\qquad
\begin{array}{ccc}
H^i_{Z'}(X', n) & \xrightarrow{\cap \eta_{X'}} & H_{2d-i}(Z', d-n) \\
{\scriptstyle H^*(f)}\uparrow & & \uparrow{\scriptstyle f^*_Z} \\
H^i_Z(X, n) & \xrightarrow{\cap \eta_X} & H_{2d-i}(Z, d-n)
\end{array}$$

(1.4.2) *Remark.* — When we apply this result, we will know more, namely that X is smooth and f_Z is an open immersion. This is the only application we shall make of the fact that homology is a presheaf in the étale topology (instead of just the Zariski topology). In practice it may be easier in specific cases to verify compatibility (1.4) with these hypotheses than to construct α^* for étale maps in general. All the results in this paper apply whenever this can be done.

(1.4.3) *Remark.* — The reader is warned against making too flippant a use of Poincaré duality. In particular if $f : (X', Z') \to (X, Z)$ is a morphism in $\mathscr{V}^*$ with X and X′ smooth, it does not follow that we get a map $f_Z^* : H_* (Z) \to H_* (Z')$. Of course such a map exists, but it depends on f_X, in general, not just on f_Z, unless f is étale.

Finally, in order to prove our main result, we need the following local triviality property, which would follow from a theory of Chern classes :

(1.5) (Principal triviality). — Let $i : W \subsetneq X$ be a smooth principal divisor in the smooth scheme X. Then $i_* \eta_W = 0$.

2. Examples

(2.1) Example. — Let l be a positive integer prime to char k, and let ν be a fixed positive integer. Let μ denote the étale sheaf of l^ν-th roots of unity on Spec (k) and let

$$\mu^n = \mu \otimes \ldots \otimes \mu, \qquad \mu^{-n} = \mathrm{Hom}(\mu^n, Z/l^\nu Z).$$

For $X \in \mathrm{Ob}\, \mathscr{V}$, let $\pi_X : X \to \mathrm{Spec}\,(k)$ denote the structure map. Define

$$H_Y^i(X, n) = H_Y^i(X, \pi_X^* \mu^n),$$
$$H_i\,(Y, n) = H^{-i}(Y, \pi_Y^! \mu^{-n}).$$

We shall sketch the proofs of some of the properties in paragraph 1 above. For an explanation of the Grothendieck-Verdier style duality, including definitions of $f^!$ and $Rf_!$, the most concise references are [11], [12], [13]; more details are given in [8] and [6].

First of all, if $f : Y_1 \to Y_2$ is proper, $f_! \cong f_*$. Since $\pi_{Y_1}^! \mu^n = f^! \pi_{Y_2}^! \mu^{-n}$, there is a trace map $\varphi_f : Rf_! \pi_{Y_1}^! \mu^{-n} \to \pi_{Y_2}^! \mu^{-n}$, hence $Rf_* \pi_{Y_1}^! \mu^{-n} \to \pi_{Y_2}^! \mu^{-n}$. Thus we obtain the functoriality of homology by composing

$$H^{-i}(Y_1, \pi_{Y_1}^! \mu^{-n}) \to H^{-i}(Y_2, Rf_* \pi_{Y_1}^! \mu^{-n}) \to H^{-i}(Y_2, \pi_{Y_2}^! \mu^{-n}).$$

To obtain the restriction maps in the étale topology, use the fact that if $\alpha : X' \to X$ is étale, $\alpha^! \cong \alpha^*$ ([6], 3.1.8). Then the natural maps :

$$H^{-i}(X, \pi_X^! \mu^{-n}) \to H^{-i}(X, R\alpha_* \alpha^* \pi_X^! \mu^{-n}) \xrightarrow{\cong} H^{-i}(X', \pi_{X'}^! \mu^{-n})$$

define α^*. To verify the compatibility (1.2.2) we use the compatability of the trace map $\varphi_f : Rf_! f^! \to \mathrm{id}$ and of the adjoint map $\theta_f : \mathrm{id} \to Rf_* f^*$ with base change. Namely if we start with the Cartesian diagram in (1.2.2) (with étale horizontal arrows and proper vertical arrows), we have commutative diagrams ([8], p. 207) :

$$\begin{array}{ccc} \alpha^* Rf_* f^! & \overset{\cong}{\longleftrightarrow} & R g_* \beta^* f^! \\ {\scriptstyle \alpha^*(\varphi_f)} \downarrow & & \downarrow {\scriptstyle \cong} \\ \alpha^* & \underset{\varphi_g}{\longleftarrow} & R g_* g^! \alpha^* \end{array} \qquad \begin{array}{ccc} Rf_* & \xrightarrow{Rh_*(\theta_\beta)} & Rf_* R\beta_* \beta^* \\ {\scriptstyle \theta_\alpha} \downarrow & & \| \wr \\ R\alpha_* \alpha^* Rf_* & \overset{\sim}{=\!=} & R\alpha_* R g_* \beta^* \end{array}$$

Compose (on the left) with the functor $R\,\alpha_*$ in the square on the left, and compose (on the right) with the functor $f^!$ in the square on the right. Then fit the square on the right on top of the one on the left and fill in the diagram (using naturality of θ_α) to obtain :

$$\begin{array}{ccccc}
 & & Rf_*f^! & \longrightarrow & Rf_*R\beta_*\beta^*f^! \\
 & & \downarrow & & \| \wr \\
Rf_*f^! & \to & R\alpha_*\alpha^*Rf_*f^! & \cong & R\alpha_*Rg_*\beta^*f^! \\
\varphi_f \downarrow & & \downarrow {\scriptstyle R\alpha_*\alpha^*(\varphi_f)} & & \| \wr \\
\mathrm{id} & \xrightarrow[\theta_\alpha]{} & R\alpha_*\alpha^* & \longleftarrow & R\alpha_*Rg_*g^!\alpha^*
\end{array}$$

Thus we get the commutative square on the left

$$\begin{array}{ccc}
Rf_*f^! & \to & Rf_*R\beta_*\beta^*f^! \\
\downarrow & & \downarrow \\
\mathrm{id} & \longrightarrow & R\alpha_*\alpha^*
\end{array}
\qquad
\begin{array}{ccc}
H_i(X, n) & \to & H_i(X', n) \\
\downarrow & & \downarrow \\
H_i(Y, n) & \to & H_i(Y'', n)
\end{array}$$

Apply this to the sheaf $\pi_Y^!\,\mu^{-n}$ and recall that $\alpha^* \cong \alpha^!$. The square then becomes, after applying $H^{-i}(Y, \ )$, the square on the right.

For the long exact sequence (1.2.3) we use the isomorphism $i^! \cong \underline{\Gamma}_Y$ in the derived category of sheaves of $\mathbf{Z}/l^\nu\,\mathbf{Z}$-modules, where $i : Y \to X$ is a closed immersion ([6], 3.1.8). This shows in fact that there is a canonical isomorphism :

(2.1.1) $H^{2d-i}(X, d-n) \to H_i(Y, n)$ if X is smooth of dimension d. (This isomorphism will be Poincaré duality, once we identify the fundamental class.)

Cap product with supports comes from the pairing, $\pi^*\,\mu^m \otimes \pi^!\,\mu^n \to \pi^!\,\mu^{m+n}$, which is compatible with the trace and restriction maps.

For the fundamental class, we first need :

(2.1.2) LEMMA. – *Suppose that* $\dim X \leqq d$. *Then* $H_i(X, n) = 0$ *for* $i > 2d$.

Proof. – We may assume that k is perfect, since the étale theory and dim X are independent of purely inseparable base extension. We shall fix this assumption for the rest of this example. We may also assume that X is reduced, since the étale cohomology of X and X_{red} are isomorphic.

We proceed by induction on d. If $d = 0$, X is smooth over k by the assumptions above, hence $H_i(X, n) \cong H^{-i}(X, -n)$ by (1.3.5), which vanishes for $i > 0$. Assuming the result for all Y with dimension $< d$, we observe that an X of dimension $\leqq d$ is generically smooth over k by our assumptions, hence its singular locus Σ has $H_i(\Sigma, n) = 0$ for $i > 2d-2$. Then the long exact sequence (1.2.3) gives us that $H_i(X, n) \cong H_i(X-\Sigma, n)$ for $i \geqq 2d$. Since $X-\Sigma$ is smooth (say of pure dimension $= d$) we get $H_i(X-\Sigma, n) \cong H^{2d-i}(X-\Sigma, d-n)$ which vanishes for $i > 2d$.

Note that the lemma gives, for any irreducible X of dimension d, an isomorphism :

$$H_{2d}(X, d) \cong H_{2d}(X-\Sigma, d) \cong H^0(X-\Sigma, 0).$$

The latter has a natural global section, namely 1, and hence we get $\eta_X \in H_{2d}(X, d)$ satisfying (1.3.4).

(2.2) EXAMPLE. — Let k have characteristic zero and let $\mathscr{V}$ be the category of all schemes embeddable in a smooth scheme over k. Then Hartshorne has written a detailed exposition of a Poincaré duality theory satisfying the axioms of paragraph 1, based on algebraic de Rham cohomology. The only thing missing is the construction of α^* for α étale, so we will get away with Remark (1.4.2). With the hypotheses stated there, it is easy to verify (1.4). Indeed, if we let $X'' = X-(Z-Z')$, then $f'' : X' \to X''$ is an étale neighborhood of Z' in which Z' is its own inverse image, and the map

$$df'' : \quad \Omega^{\bullet}_{X''/k} \to f''_* \Omega^{\bullet}_{X''/k} \cong f''_* f''^* \Omega^{\bullet}_{X''/k}$$

induces the inverse of the trace map $f''_* \Gamma_{Z'} E'_{X''/k} \to \underline{\Gamma}_{Z'} E^{\bullet}_{X/k}$, where $E^{\bullet}$ is the canonical resolution of the de Rham complex [9]. We leave the details to the reader.

(2.3) EXAMPLE. — Let $k = \mathbf{C}$ and let $\mathscr{V}$ be the category of algebraic varieties of finite type over $\mathbf{C}$. For each $X \in \mathscr{V}$, let X_{an} be the corresponding complex analytic variety, and let $H^*(X) = H^*(X_{an}, \mathbf{Z})$ and $H_*(X) = H_*^{B.M.}(X_{an}, \mathbf{Z})$ - the Borel-Moore homology of X_{an}. Again all the properties are standard except α^* for étale maps. One can argue either as in (2.1) using Verdier's duality for paracompact spaces [13], or as in (2.2). Of course one can take any ring of coefficients in place of $\mathbf{Z}$.

3. Filtration by niveau and coniveau

Fix a ground field k and a category $\mathscr{V}$ as above. If $X \in \operatorname{Ob} \mathscr{V}$, let $Z_d = Z_d(X)$ denote the set of all closed subsets $Z \subset X$ of dimension $\leqq d$, ordered by inclusion. Let Z_d/Z_{d-1} denote the ordered set of pairs $(Z, Z') \in Z_d \times Z_{d-1}$ such that $Z' \subset Z$, with the ordering

$$(Z, Z') \geqq (Z_1, Z'_1) \qquad \text{if} \quad Z \supset Z_1 \quad \text{and} \quad Z' \supseteq Z'_1.$$

Suppose now we are given an homology theory as above. We can form

$$H_i(Z_d(X), n) \underset{\text{def}}{=} \varinjlim_{Z \in Z_d} H_i(Z, n), \tag{3.1}$$

$$H_i(Z_d/Z_{d-1}, n) \underset{\text{def}}{=} \varinjlim_{(Z, Z') \in Z_d/Z_{d-1}} H_i(Z-Z', n). \tag{3.2}$$

Notice that $(Z, Z') < (Z_1, Z'_1)$ gives

$$Z-Z' \underset{\text{closed}}{\overset{u}{\hookrightarrow}} Z_1-Z' \underset{\text{open}}{\overset{v}{\hookleftarrow}} Z_1-Z'_1.$$

The transition maps in (3.2) are $v^* u_*$.

If $f : X \to Y$ is proper, there are maps

$$H_i(Z_d(X), n) \to H_i(Z_d(Y), n),$$
$$H_i(Z_d/Z_{d-1}(X), n) \to H_i(Z_d/Z_{d-1}(Y), n).$$

The filtration by *niveau* is the ascending filtration $N_d H_i(X, n)$ on $H_i(X, n)$:

$$N_d H_i(X, n) = \operatorname{Im}(H_i(Z_d(X), n) \to H_i(X, n)). \tag{3.3}$$

Equivalenty, $N_d H_i(X, n)$ is the subgroup of $H_i(X, n)$ generated by all images $f_* : H_i(W, n) \to H_i(X, n)$ where $W \in \operatorname{Ob} \mathscr{V}$, $f : W \to X$ is proper, and $\dim f(W) \leqq d$.

Suppose now that $(Z, Z') \in Z_d/Z_{d-1}$. There is a long exact sequence

$$\ldots \to H_m(Z', n) \to H_m(Z, n) \to H_m(Z-Z', n) \to H_{m-1}(Z', n) \to \ldots$$

Taking the limit over such pairs gives

$$(3.4) \qquad \ldots \to H_m(Z_{d-1}, n) \xrightarrow{i} H_m(Z_d, n) \xrightarrow{j} H_m(Z_d/Z_{d-1}, n) \xrightarrow{k} H_{m-1}(Z_{d-1}, n) \to \ldots$$

Form an *exact couple* as follows

$$D = \bigoplus_{m, d=-\infty}^{\infty} H_m(Z_d, n) = \bigoplus_{p, q} D_{p, q}\,; \qquad D_{p, q} = H_{p+q}(Z_p, n),$$

$$E = \bigoplus_{m, d=-\infty}^{\infty} H_m(Z_d/Z_{d-1}, n) = \bigoplus_{p, q} E_{p, q}\,; \qquad E_{p, q} = H_{p+q}(Z_p/Z_{p-1}, n).$$

There is an exact triangle

$$\begin{array}{ccc} D & \xrightarrow{\ i\ } & D \\ & {\scriptstyle k}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & E & \end{array}$$

where i, j, k are obtained from the maps in (3.4). These maps are homogeneous of degrees $(1, -1)$, $(0, 0)$, and $(-1, 0)$ respectively, so there is an associated spectral sequence

$$(3.5) \qquad E^1_{p, q} = H_{p+q}(Z_p/Z_{p-1}, n) \quad \Rightarrow \quad N.H_{p+q}(X, n).$$

This construction is entirely analogous to the construction of the spectral sequence of a simplicial complex by p-skeletons (*cf.* [14] for example).

For $x \in X$, we write $x \in Z_d$ instead of $\overline{\{x\}} \in Z_d$. Given $x \in Z_d$, define

$$H_m(x, n) = \varinjlim_{U \subseteq \{x\}^-} H_m(U, n),$$

the limit being taken over all non-empty U which are open in $\overline{\{x\}}$. Clearly

$$(3.6) \qquad H_m(Z_p/Z_{p-1}, n) \cong \bigoplus_{x \in Z_p/Z_{p-1}} H_m(x, n).$$

Combining (3.5) and (3.6) we have shown :

(3.7) PROPOSITION. — *Let* H_* *be an homology theory on* $\mathscr{V}$ *as in paragraph* 1. *For any* $X \in \operatorname{Ob} \mathscr{V}$, *there is a spectral sequence*

$$E^1_{p, q} = \bigoplus_{x \in Z_p/Z_{p-1}} H_{p+q}(x, n) \quad \Rightarrow \quad N.H_{p+q}(X, n).$$

This spectral sequence is covariant with respect to proper morphisms, and contravariant with respect to étale maps.

Now suppose given a Poincaré duality theory with supports, H^*.

$$\text{Let } Z^p = \{ Z \subset X \text{ closed, } \operatorname{codim}_X Z \geqq p \}.$$

Define the filtration by *coniveau*

$$(3.8) \qquad N^p H^*(X, n) = \operatorname{Ker}(H^*(X, n)) \to \varinjlim_{Z \in Z^p} H^*(X-Z, n)$$

$$= \operatorname{Im}(\varinjlim_{Z \in Z^p} H^*_Z(X, n) \to H^*(X, n)).$$

(3.9) PROPOSITION. — *With notation as above, assume the ground field k is perfect, and that X is smooth over k. For $x \in Z^p/Z^{p+1}$ (i.e. $\overline{\{x\}} \in Z^p$, $\overline{\{x\}} \notin Z^{p+1}$) define* $H^*(x, n) = \varinjlim_{U \subseteq \overline{\{x\}}} H^*(U, n)$. *Then there is a cohomological spectral sequence*

$$E_1^{p,q} = \bigoplus_{x \in Z^p/Z^{p+1}} H^{q-p}(x, n-p) \Rightarrow N^{\cdot} H^{q+p}(X, n).$$

Proof. — Because k is perfect, there exists for $x \in Z^p/Z^{p+1}$ an open $U \subset \overline{\{x\}}$ smooth over k. Poincaré duality (1.3.4) gives isomorphisms

$$H_{p+q}(x, n) \xrightarrow{(\cap \eta_x)^{-1}} H^{p-q}(x, p-n), \qquad \dim \overline{\{x\}} = p,$$

$$H_{p+q}(X, n) \xrightarrow{(\cap \eta_X)^{-1}} H^{2d-p-q}(x, d-n), \qquad \dim X = d.$$

If we take $p' = d-p$, $q' = d-q$, $n' = d-n$, we get

$$H_{p+q}(x, n) \cong H^{q'-p'}(x, n'-p'),$$
$$H_{p+q}(X, n) \cong H^{q'+p'}(X, n')$$

and the desired spectral sequence follows easily from (3.7).

Q. E. D.

(3.10) REMARKS. — " Dual " to (3.1), (3.2) one can define $H^*_{Z^p}(X, n)$, $H^*_{Z^p/Z^{p+1}}(X, n)$, and there is a spectral sequence, generalizing (3.9) :

$$(3.11) \qquad E_1^{p,q} = H^{p+q}_{Z^p/Z^{p-1}}(X, n) \Rightarrow N.H^{p+q}(X, n).$$

If $f : Y \to X$ is a flat morphism and $Z \in Z^p(X)$, we have $f^{-1}(Z) \in Z^p(Y)$. Together with the assumed contravariant functoriality of cohomology with supports (1.1), this implies the spectral sequences (3.9) and (3.11) are contravariant with respect to flat morphisms.

4. The arithmetic resolution

Fix a perfect field k, an $X \in \operatorname{Ob} \mathscr{V}$ with X smooth over k, and a Poincaré duality theory with supports (H^*, H_*). Define sheaves $\mathscr{H}_*(n)$, $\mathscr{H}^*(n)$ for the Zariski topology on X by sheafifying the presheaves

$$U \mapsto H_*(U, n); \qquad U \mapsto H^*(U, n).$$

If A is an abelian group and $x \in X$, let $i_x A$ denote the constant sheaf A on $\overline{\{x\}}$, extended by zero to all of X. With this notation, there is an obvious way to " sheafify " the spectral sequences (3.7), (3.9), (3.11). For example :

(4.1) PROPOSITION. — *Assume k perfect and X smooth over k. Then there is a spectral sequence of sheaves*

$$\mathscr{E}_1^{p,q} = \bigoplus_{x \in Z^p/Z^{p+1}} i_x H^{q-p}(x, n-p) \Rightarrow \mathscr{H}^{p+q}(n).$$

Our main result is

(4.2) THEOREM. — *Let k be a perfect field, $X \in \text{Ob}\, \mathscr{V}$ smooth over k, and H^*, H_* a Poincaré duality theory with supports* (§ 1). *Then*

(4.2.1) *The spectral sequence* (4.1) *is degenerate at $\mathscr{E}_2$, in fact $\mathscr{E}_2^{p,q} = (0)$ for $p > 0$.*

(4.2.2) *The complex of sheaves*

$$0 \to \mathscr{H}^q(n) \to \bigoplus_{x \in Z^0/Z^1} i_x H^q(x, n) \to \bigoplus_{x \in Z^1/Z^2} i_x H^{q-1}(x, n-1) \to \dots$$
$$\to \bigoplus_{x \in Z^q/Z^{q+1}} i_x H^0(x, n-q) \to 0$$

is exact.

(4.2.3) *Let $\mathscr{H}^q_{Z^p}(n)$ be the sheaf associated to the presheaf*

$$U \mapsto \varinjlim_{Z \in Z^p} H^q_{Z \cap U}(U, n).$$

The natural map

$$\mathscr{H}^q_{Z^{p+1}}(n) \to \mathscr{H}^q_{Z^p}(n)$$

is zero for all p, q, n.

When the conditions of the theorem are fulfilled, the complex (4.2.2) will be called the *arithmetic resolution of $\mathscr{H}^q(n)$*.

Notice that (4.2.3) implies the other statements. Indeed, there is an exact sequence

$$(4.3) \qquad \dots \to \mathscr{H}^q_{Z^p/Z^{p+1}}(n) \to \mathscr{H}^{q+1}_{Z^{p+1}}(n) \underset{0}{\to} \mathscr{H}^{q+1}_{Z^p}(n) \to \mathscr{H}^{q+1}_{Z^p/Z^{p+1}}(n) \to \dots$$

with

$$\mathscr{H}^q_{Z^p/Z^{p-1}}(n) \cong \bigoplus_{x \in Z^p/Z^{p+1}} i_x H^{q-p}(x, n-p).$$

The differential in (4.2.2) is obtained by composing

$$\mathscr{H}^q_{Z^p/Z^{p+1}}(n) \to \mathscr{H}^{q+1}_{Z^{p+1}}(n),$$

with

$$\mathscr{H}^{q+1}_{Z^{p+1}}(n) \to \mathscr{H}^{q+1}_{Z^{p+1}/Z^{p+2}}(n).$$

Comparing with (4.3), one sees that (4.2.2) is exact. This remark motivates the following :

(4.4) DEFINITION. — Let $f : Z_1 \to Z_2$ be a morphism in $\mathscr{V}_*$, and let $S \subseteq Z_2$ be a finite set. We shall say that f is *homologically effaceable at* S iff there is an open neignborhood

$U \overset{\alpha}{\subsetneq} Z_2$ of S such that the composition :

$$H_*(Z_1) \overset{f_*}{\to} H_*(Z_2) \overset{\alpha^*}{\to} H_*(U) \quad \text{is zero.}$$

The main technical point in the proof of (4.2) is the following :

(4.5) PROPOSITION. — *Let $i : Z_1 \to Z_2$ be a closed immersion of affine schemes, and suppose there is a morphism $\pi : Z_2 \to Z_1$ with $\pi \circ i = \mathrm{id}$ and π smooth of relative dimension 1 at the points of $S \subsetneq Z_2$. Then i is homologically effaceable at S.*

Assuming (4.5), the proof of (4.2) goes as follows : for $x \in X$ we must show the stalk of the map

$$\mathscr{H}^q_{Z^{p+1}}(n) \to \mathscr{H}^q_{Z^p}(n)$$

at x is zero. Using the expression of these sheaves as direct limits together with (1.3.5), the problem reduces to proving :

CLAIM. — *Given $Z' \in Z^{p+1}$, $x \in Z'$, there exists a $Z \in Z^p$ containing Z' and an affine neighborhood U of x in X such that the map $Z' \cap U \to Z \cap U$ is locally homologically effaceable at x.*

Proof of Claim. — We use a trick of Quillen. Find a $Y \in Z^1$ containing Z'; say $\dim Y = d$. Then shrinking X around x, there exists a *finite* morphism $f : Y \to A^d_k$ (affine d-space over k) and a lifting $g : X \to A^d_k$ with g smooth at x [10]. Let $X' = X \times_{A^d} Y$ and $Z'' = Z' \times_Y Y'$.

$$\begin{array}{ccccc} Z'' & \subsetneq & X' & \overset{f'}{\to} & X \\ {\scriptstyle i}\circlearrowleft\downarrow{\scriptstyle g'} & & \downarrow & & \downarrow{\scriptstyle g} \\ Z' & \subsetneq & Y & \overset{f}{\to} & A^d_k \end{array}$$

In the Cartesian diagram above, g' is smooth of relative dimension 1 at the points of $S' = f'^{-1}(x)$, i is the natural section and f' is finite. Hence if $Z = f'(Z'')$, $Z \in Z^p(X)$, and $Z \supseteq Z'$. Moreover by (4.5), we can find an open neighborhood U'' of S' in Z'' such that the composite $H_i(Z', n) \to H_i(Z'', n) \to H_i(U'', n)$ is zero. Since $Z'' \to Z$ is finite and $f'^{-1}(x) \subset U''$, we can find a neighborhood U of x such that $U' = f'^{-1}(U) \subseteq U''$. Then the result follows by the commutativity of the diagram below :

$$\begin{array}{ccccccc} H_i(Z', n) & \overset{i_*}{\to} & H_i(Z'', n) & \to & H_i(U'', n) & \to & H_i(U', n) \\ \| & & \downarrow{\scriptstyle f'_*} & & & \swarrow & \\ H_i(Z', n) & \to & H_i(Z, n) & \longrightarrow & H_i(U, n) & & \end{array}$$

(4.6) REMARK. — The claim above makes sense on a singular scheme, but it is false. In fact there is a 3-dimensional cone X whose vertex p is an isolated singularity, and a closed subset $i : Y \subset X$ of dimension 2, such that $(i_* \eta_Y)|_U \neq 0$ for any open neighborhood of p, in algebraic de Rham cohomology.

Namely let $X = \operatorname{Spec} k[t_1, t_2, t_3, t_4]/(t_1 t_2 - t_3 t_4)$ and let Y be defined by $t_1 = t_3 = 0$. Let $\hat{X}$ be the completion of X at p; it is even true that $i_* \eta_Y$ is nonzero in $H_4(\hat{X})$. In fact X is the cone over $X_0 = \mathbf{P}^1 \times \mathbf{P}^1 \subsetneq \mathbf{P}^3$ and Y is the cone over one of the rulings Y_0, so we have an exact sequence [9] :

$$\begin{array}{ccccccc} 0 \to H_4(X_0) & \xrightarrow{\cap \xi} & H_2(X_0) & \to & H_4(\hat{X}) & \to & 0 \\ & & \uparrow {\scriptstyle i_{0*}} & & \uparrow & & \\ & 0 \to & H_2(Y_0) & \to & H_4(\hat{Y}) & & \end{array}$$

where $\xi = c_1 \mathcal{O}_{X_0}(1)$. Since $i_{0*}(\eta_{Y_0}) \notin \operatorname{Im} \xi$, the claim is clear.

(4.7) REMARK. — When $\mathscr{H}^*$ is the étale theory described in paragraph 2, the assumption that the ground field k is perfect can be suppressed. Indeed all expressions in (4.1) and (4.2) are invariant under purely inseparable base extension.

5. **Proof of** (4.5)

(5.1) LEMMA. — *Suppose given a commutative square of schemes in* $\mathscr{V}$:

$$\begin{array}{ccc} Z_2 & \overset{i_2}{\subsetneq} & X_2 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z_1 & \underset{i_1}{\subsetneq} & X_1 \end{array}$$

with i_1, i_2 closed immersions and f, g smooth. Let $S \subset X_2$ be a finite set of points contained in an affine. Then after replacing Z_2 and X_2 by neighborhoods of S, *there exists a closed subscheme $X_2' \subsetneq X_2$ containing Z_2 such that the induced morphism $f' : X_2' \to X_1$ is still smooth and the square*

(5.1.1)
$$\begin{array}{ccc} Z_2 & \overset{i_2'}{\subsetneq} & X_2' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ Z_1 & \underset{i_1}{\subsetneq} & X_1 \end{array}$$

is Cartesian.

Proof. — Let $Y = f^{-1}(Z_1) \subset X_2$ and let I (resp. $\bar{I}$) be the ideal of Z_2 in X_2 (resp. in Y). Since Y and Z_2 are both smooth over Z_1 we have an exact sequence of locally free sheaves on Z_2 :

$$0 \to \bar{I}/\bar{I}^2 \to \Omega^1_{Y/Z_1} \otimes \mathcal{O}_{Z_2} \to \Omega^1_{Z_2/Z_1} \to 0.$$

After replacing X_2 by a neighborhood of S if necessary, we can find sections $f_1, \ldots, f_r$ of I whose images form a basis for $\bar{I}/\bar{I}^2$. Take X_2' to be the scheme defined by $f_1 = f_2 = \ldots = f_r = 0$.

Since $\Omega^1_{Y/Z_1} \cong \Omega^1_{X_2/X_1} \otimes \mathcal{O}_{Z_1}$, the differentials $df_1, \ldots, df_r \in \Omega^1_{X_2/X_1}$ are independent in some neighborhood of S, so that after shrinking, X_2' is smooth over X_1. Moreover it follows from Nakayama's lemma that $\bar{f}_1, \ldots, \bar{f}_r$ generate $\bar{I}$ in some neighborhood of S, and hence that I is generated by $f_1, \ldots, f_r$ together with the ideal I_Y of Y in X_2. This implies (5.1.1) is Cartesian.

Q. E. D.

(5.2) REMARK. — The relative dimension of f' in (5.1.1) is the same as that of g. In particular, if g is an open immersion, f' is étale.

Recall that we wish to prove :

(4.5) PROPOSITION. — *Let : $Z_1 \hookrightarrow Z_2$ be a closed immersion of affine schemes in $\mathscr{V}$ and let $\pi : Z_2 \to Z_1$ be a section of i, smooth of relative dimension* 1 *at a finite set* S *of points in* Z_2. *Then i is homologically effaceable at* S.

STEP 1. — Find a commutative diagram

(5.3)
$$\begin{array}{ccc} Z_2 & \overset{i_2}{\hookrightarrow} & X_2 \\ \pi \downarrow \circlearrowright i & & \downarrow f \\ Z_1 & \overset{i_1}{\hookrightarrow} & X_1 \end{array}$$

with X_1, X_2, and f smooth and i_1, i_2 closed immersions. This is easy since Z_1 and Z_2 are affine.

STEP 2. — Since π is smooth in a neighborhood of S, we may apply Lemma (5.1) to the square (5.3). We find a neighborhood U of S in X_2 and a closed scheme $X'_2 \subset U \cap X_2$ containing $Z'_2 = Z_2 \cap U$, such that $f' : X'_2 \to X_1$ is smooth of relative dimension 1, $S \subset Z'_2$ and $Z'_2 = Z_1 \times_{X_1} X'_1$. If we let $Z'_1 = i^{-1}(Z'_2)$, the map $\alpha : Z'_1 \to Z_1$, induced by $\pi' : Z'_2 \to Z_1$ is an open immersion :

(5.4)
$$\begin{array}{ccccc} Z'_1 & \overset{i'}{\hookrightarrow} & Z'_2 & \hookrightarrow & X'_2 \\ & \searrow^{\alpha} & \downarrow \pi' & & \downarrow f' \\ & & Z_1 & \hookrightarrow & X_1 \end{array}$$

STEP 3. — Apply (5.1) to the square below on the left to get the Cartesian square on the right, again shrinking X'_2 in some neighborhood of S :

$$\begin{array}{ccc} Z'_1 & \hookrightarrow & X'_2 \\ \alpha \downarrow & & \downarrow f' \\ Z_1 & \hookrightarrow & X_1 \end{array} \qquad \begin{array}{ccccc} Z'_1 & \hookrightarrow & W & \overset{h}{\hookrightarrow} & X'_2 \\ \alpha \downarrow & & g \downarrow & \swarrow_{f'} & \\ Z_1 & \hookrightarrow & X_1 & & \end{array}$$

Note that W is a smooth divisor in the smooth X'_2, hence after again shrinking to a neighborhood of S, we may assume W is principal.

STEP 4. — We have constructed a pair of morphisms in $\mathscr{V}^*$:

$$h : (Z'_1 \hookrightarrow W) \to (Z'_2 \hookrightarrow X'_2),$$
$$f' : (Z'_2 \hookrightarrow X'_2) \to (Z_1 \hookrightarrow X_1)$$

such that in the composition $f \circ h = \alpha \subsetneq g$:

$$\alpha \subsetneq g : \quad (Z'_1 \subsetneq W) \to (Z_1 \subsetneq X_1),$$

g is étale and α is an open immersion. It follows that the diagram below commutes [*cf.* (1.4)] :

$$\begin{array}{ccccc}
H^i_{Z_1}(X_1, n) & & \xrightarrow{\quad f^* \quad} & & H^i_{Z'_2}(X'_2, n)\\
\Big\downarrow{\scriptstyle \cap \eta_{X_1}} & \searrow^{g^*} & & \swarrow^{h^*} & \Big\downarrow{\scriptstyle \cap h_* \eta_W}\\
 & & H^i_{Z'_1}(W, n) & & \\
H_{2d-i}(Z_1, d-n) & & \Big\downarrow{\scriptstyle \cap \eta_W} & & H_{2d-i}(Z'_2, d-n)\\
 & \searrow^{\alpha^*} & & \nearrow_{i'_*} & \\
 & & H_{2d-i}(Z'_1, d-n) & &
\end{array}$$

By the principal triviality axiom, $h_* \eta_W = 0$, and by Poincaré duality $\cap\, \eta_{X_1}$ is an isomorphism. It follows that the composition $i'_* \alpha^*$ is the zero map. But since $Z'_1 = Z_1 \cap Z'_2$ where $Z'_2 \overset{\beta}{\subsetneq} Z_2$ is open, $\beta^* i_* = i'_* \alpha^*$, and we have proved the result.

6. **Applications : the filtration by coniveau**

Let k be a perfect [but *cf.* (4.7)] field, $X \in \mathrm{Ob}\, \mathscr{V}$ smooth over k, and H^*, H_* a Poincaré duality theory with supports. The arithmetic resolution (4.2.2) is a resolution of $\mathscr{H}^*(n)$ by flasque sheaves in the Zariski topology. As a consequence, we have

(6.1) THEOREM :

$$H^p(X, \mathscr{H}^q(n)) \cong \frac{\operatorname{Ker}\Big(\bigoplus_{x \in Z^p/Z^{p+1}} H^{q-p}(x, n-p) \to \bigoplus_{x \in Z^{p+1}/Z^{p+2}} H^{q-p-1}(x, n-p-1)\Big)}{\operatorname{Im}\Big(\bigoplus_{x \in Z^{p-1}/Z^{p}} H^{q-p+1}(x, n-p+1) \to \bigoplus_{x \in Z^{p}/Z^{p+1}} H^{q-p}(x, n-p)\Big)}.$$

(6.2) COROLLARY :

$$H^p(X, \mathscr{H}^q(n)) = (0) \qquad \textit{for} \quad p > q.$$

(6.3) COROLLARY. — *The* E_2 *term of the spectral sequence* (3.8) :

$$E_1^{p,q} = \bigoplus_{x \in Z^p/Z^{p+1}} H^{q-p}(x, n-p) \Rightarrow N^{\bullet} H^{q+p}(X, n)$$

is given by

$$E_2^{p,q} = H^p(X, \mathscr{H}^q(n)).$$

(6.4) REMARK. — If as in (2.1) and (2.3) we have a " fine " topology X_a on X and if the cohomology theory is given by $H^*(X, n) = H^*(X_a, \mu^n)$ for some sheaf μ^n on X_a, we have $\mathscr{H}^*(n) \cong R^* \alpha_* (\mu^n)$, where $\alpha : X_a \to X_{\text{Zariski}}$ is the " continuous map. " Thus the Leray spectral sequence for $\Gamma \circ \alpha$ has the same E_2 terms as the " coniveau spectral sequence " (3.8). It is tempting to suppose that these two coincide from E_2 onward, but we can only prove this for the de Rham theory [1].

(6.4) PROPOSITION. — *Suppose* $X \in \mathrm{Ob}\,\mathscr{V}$ *is smooth and the field k has characteristic zero. Let* $H^*_{DR}(X)$ *be the de Rham theory,*

$$H^*_{DR}(X) = \mathbf{H}^*(X, \Omega^{\cdot}_X) \quad [(2.2)].$$

The coniveau spectral sequence

$$(6.4.1) \qquad E_1^{p,q} = \bigoplus_{x \in Z^p/Z^{p+1}} H^{q-p}_{DR}(x) \;\Rightarrow\; N^{\cdot} H^{p+q}_{DR}(X)$$

is isomorphic from E_2 *on with the " second spectral sequence of hypercohomology " associated to the complex* $\Omega^{\cdot}_X$,

$$(6.4.2) \qquad E_2^{p,q} = H^p(X, \mathscr{H}^q_{DR}) \;\Rightarrow\; H^{p+q}_{DR}(X).$$

If, moreover, $k = \mathbf{C}$, $X_{an} = X$ *with the classical topology, and* $\alpha : X_{an} \to X$ *is the canonical map, then both coincide with the Leray spectral sequence*

$$(6.4.3) \qquad E_2^{p,q} = H^p(X, R^q \alpha_*(\mathbf{C}_{X_{an}})) \;\Rightarrow\; H^{p+q}(X_{an}, \mathbf{C}).$$

Proof. — The fact that (6.4.2) and (6.4.3) coincide is a consequence of Grothendieck's theorem calculating de Rham cohomology with algebraic differentials [4]. Namely the holomorphic de Rham complex $\Omega^{\cdot}_{X_{an}}$ is a resolution of $\mathbf{C}_{X_{an}}$ by α-acyclic sheaves, so (6.4.3) is the second spectral sequence of hypercohomology associated to the complex $\alpha_* \Omega^{\cdot}_{X_{an}}$. Since $\mathscr{H}^q_{DR} \cong R^q_{\alpha_*}(\mathbf{C}_{X_{an}})$ by Grothendieck's theorem, the map $\Omega^{\cdot}_X \to \alpha_* \Omega^{\cdot}_{X_{an}}$ induces an isomorphism between (6.4.2) and (6.4.3).

To compare (6.4.1) and (6.4.2) we use Hartshorne's canonical resolution of $\Omega^{\cdot}_X$. Recall if F is any abelian sheaf on X, there is a complex (Cousin complex) of sheaves

$$(6.5) \qquad F \to \bigoplus_{x \in Z^0/Z^1} i_x H^0_x(F) \to \bigoplus_{x \in Z^1/Z^2} i_x H^1_x(F) \to \dots,$$

where $H^i_x(F)$ denote the i-th local cohomology group of F with supports at the point x (for details, *see* [8], chapter IV). When X is regular and F is a locally free sheaf of $\mathscr{O}_X$-modules, (6.5) gives a resolution of F (*op. cit.* prop. 2.6).

If we replace F by the complex $\Omega^{\cdot}_X$ in (6.5), we get a double complex $C^{\cdot\cdot}$ with $C^{\cdot q}$ a resolution of Ω^q_X for all q (assuming X smooth). Let $\Gamma C^{\cdot}$ be the total complex of $\Gamma C^{\cdot\cdot}$:

$$\Gamma C^r = \bigoplus_{p+q=r} \Gamma C^{p,q}$$

and filter $\Gamma C^{\cdot}$ by

$$F^S \Gamma C^r = \bigoplus_{\substack{p+q=r \\ p \geqq S}} \Gamma C^{p,q} = \bigoplus_{\substack{p+q=r \\ p \geqq S}} \bigoplus_{x \in Z^p/Z^{p+1}} H^p_x(X, \Omega^q).$$

(1) P. Deligne has kindly supplied us with a general proof of the hoped for coincidence.

Notice $F^S \Gamma C^{\cdot} = \Gamma_{Z^S} C^{\cdot}$, where Γ_{Z^S} means sections with codimension of support $\geqq S$. Thus the two spectral sequences

$$(6.6) \qquad E_1^{p,q} = H^{p+q}(F^p \Gamma C^{\cdot}/F^{p+1}\Gamma C^{\cdot}) \Rightarrow H^{p+q}(\Gamma C^{\cdot}),$$

$$(6.7) \qquad E_1^{p,q} = H^{p+q}(\Gamma_{Z^p/Z^{p+1}} C^{\cdot}) \Rightarrow H^{p+q}(\Gamma C^{\cdot})$$

coincide. On the one hand, since $C^{\cdot}$ is a flasque resolution of $\Omega^{\cdot}$, (6.7) is the de Rham version of (3.11). On the other hand, if $\Omega^{\cdot} \to E^{\cdot\cdot}$ is a Cartan Eilenberg resolution and $C^{\cdot\cdot} \to E^{\cdot\cdot}$ is a map, we get a map of spectral sequences

$$(6.8) \qquad E_1^{p,q}(C^{\cdot\cdot}) \to E_1^{p,q}(E^{\cdot\cdot}) = H^{p+q}(F^p \Gamma E^{\cdot}/F^{p+1}\Gamma E^{\cdot}).$$

Since $E^{\cdot\cdot}$ is a Cartan Eilenberg resolution, the right hand side of (6.8) is $\Gamma(\underline{H}_{II}^{p,q})$, where $\underline{H}_{II}^{\cdot q}$ is an injective resolution of $\mathscr{H}^q_{DR}$. On the E_2 level, this gives a map of spectral sequences

$$E_2 \text{ level of } (6.4.1) \to (6.4.2).$$

The fact that this map is an isomorphism can be seen for example by sheafifying (6.8) and noting that both sides give resolutions of $\mathscr{H}^q_{DR}$.

Q. E. D.

(6.9) COROLLARY (Conjecture of Washnitzer). — *The filtration arising from the second spectral sequence of hypercohomology*

$$E_2^{p,q} = H^p(X, \mathscr{H}^q_{DR}) \Rightarrow H^{p+q}(X)$$

is the filtration by coniveau.

7. Algebraic cycles

Throughout this section we assume k is algebraically closed, and that our Poincaré duality theory takes values in the category of R-modules, with $R = H^0(\underline{\operatorname{Spec}} k, 0)$. We need the following axioms :

(7.1.1) If $\dim X \leqq d$, $H_i(X, n) = 0$ if $i > 2d$.

(7.1.2) If $f : X \to Y$ is a proper map between varieties of the same dimension, then $f_* \eta_X = r.\eta_Y$, where r is the degree of $k(X)$ over $k(Y)$.

These hold for all the cohomology theories in paragraph 2. One can deduce (7.1.2) from the other axioms for maps which are generically étale, and (7.1.1) from an alternate version : $H^i = 0$ if $i < 0$.

For any $X \in \operatorname{Ob} \mathscr{V}$ we denote by $\mathscr{Z}_p(X)$ the free R-module generated by all irreducible $Z \subseteq X$ of dimension p. Recall that $\mathscr{Z}_p$ is made into a functor for proper morphisms f according to the following rule : If $f(Z)$ has dimension $< p$ then $\mathscr{Z}_p(f)(Z) = 0$; if $f(Z)$ has dimension p then $\mathscr{Z}_p(f)(Z) = d.f(Z)$ where d is the degree of $k(Z)$ over $k(f(Z))$. It is a tautology that the map $\zeta : \mathscr{Z}_p(X) \to H_{2p}(X, p)$ determined by $Z \to i_* \eta_Z$, where $i : Z \to X$ is the inclusion, is a natural transformation.

(7.2) TAUTOLOGY :

(7.2.1) There are natural transformations $\zeta^r : \mathscr{Z}_p \to E^r_{pp}(p)$ for all r, compatible with the (surjective) edge homomorphisms $E^r_{pp}(p) \to E^{r+1}_{pp}(p)$ and the (inje tive) edge homomorphisms $E^{\infty}_{pp}(p) \to H_{2p}(\ , p)$, where $E^{\cdot}_{\cdot\cdot}$ is the niveau spectral sequence (3.7).

(7.2.2) ζ^1_X is injective, and $\operatorname{Ker}(\zeta^r_X) = \{ Z :$ there is a $Y \in Z_{p+r}(X)$ such that the support of Z is contained in Y and $\zeta_Y(Z) = 0$ in $H_{2p}(Y, p) \}$.

Proof. — Notice that

$$E^1_{pq}(n) = H_{q+p}(Z_p/Z_{p-1}, n) = 0 \qquad \text{if} \quad q > p,$$

by (7.1.1), hence the assertions about the edge homomorphisms. Note that

$$E^1_{pp}(p) = H_{2p}(Z_p/Z_{p-1}, p) \cong H_{2p}(Z_p, p)$$

and it is clear that we can factor ζ through $E^1_{pp}(p)$.

For the next statement, observe first that the map $\pi^*_X : R \to H^0(X, 0)$ induced by the structure map of X is injective (find a point on X), and hence, by Poincaré duality on the smooth part of X, the map $R \to H_{2d}(X, d)$ given by $\alpha \mapsto \pi^*_X(\alpha) \cap \eta_X$ is injective. It follows immediately that ζ is injective. The determination of $\operatorname{Ker}(\zeta_X)$ follows from the construction of the spectral sequence.

As an example, suppose X is smooth and projective over the complex numbers, and let $H_i(X) = H_i(X, \mathbf{Z})$ be classical integral homology.

(7.3) THEOREM. — *With assumptions as above, the kernel of* $\zeta^2 : \mathscr{Z}_p \to E^2_{p,p}$ *is the group* $\mathscr{Z}^a_p$ *of p-cycles algebraically equivalent to zero.*

Proof. — Recall $\mathscr{Z}^a_p$ is generated by cycles

$$Z = \mathscr{Z}_p(f)(Y_1) - \mathscr{Z}_p(f)(Y_2),$$

where $f : Y \to X$ is proper, and where Y_1 and Y_2 are two fibers (counting multiplicity) of a flat map $\pi : Y \to C$ with C a smooth, connected, complete curve.

STEP 1. — $\mathscr{Z}^a_p(X) \subset \operatorname{Ker}\zeta^2$. Indeed, let Y, Z be as above. It suffices by naturality to show $\zeta^2_Y(Y_1 - Y_2) = 0$. Suppose $Y_i = \pi^{-1}(p_i)$ with $p_i \in C$. If $L = \mathscr{O}_C(p_1 - p_2)$, $Y_1 - Y_2$ is a Cartier divisor associated to $\pi^*(L)$. By compatibility of the cycle class with the Chern class, we deduce

$$\zeta^2(Y_1 - Y_2) = \eta_Y \cap c_1(\pi^* L) = \eta_Y \cap \pi^* c_1(L).$$

But $c_1(L) = 0$, since $i_*(p_1) = i_*(p_2)$ in $H_0(C, \mathbf{Z})$.

STEP 2. — $\operatorname{Ker}(\zeta^2_X) \subseteq \mathscr{Z}^a_p(X)$, i. e. :

$$d^1 : E^1_{p+1,p} = \bigoplus_{y \in Z_{p+1}/Z_p} H_{2p+1}(y) \to E^1_{p,p} = \mathscr{Z}_p$$

factors through $\mathscr{Z}_p^a$. Indeed, let Y be the closure of $\{y\}$ and let $f: W \to Y$ be a projective desingularization of Y. If $\omega \in W$ is the generic point, we have an isomorphism $f_* : H_{2p+1}(w) \to H_{2p+1}(y)$. Since $\mathscr{Z}_p(f)$ preserves $\mathscr{Z}_p^a$, it suffices to show that $d^1(\alpha) \in \mathscr{Z}_p^a(W)$ for $\alpha \in H_{2p+1}(\omega)$. In other words, we are reduced to the case of divisors on a smooth variety W of dimension $p+1$. Since $E_{p,p}^2(W) = E_{p,p}^{\infty}(W)$, the assertion amounts to the well-known fact that homological and algebraic equivalence coincide for divisors on a smooth schemes [easily proved from the exponential sequence

$$H^1(W, \mathcal{O}_W) \to H^1(W, \mathcal{O}_W^*) \to H^2(W, \mathbf{Z})].$$

(7.4) COROLLARY. — *Let $\mathscr{H}^p$ be the Zariski sheaf on* X *associated to the presheaf* $U \to H^p(U, \mathbf{Z})$. *Then there is a natural isomorphism*

$$A^p(X) \cong H^p(X, \mathscr{H}^p),$$

where $A^p(X)$ *is the group of cycles of codimension p modulo algebraic equivalence.*

Proof. — It follows from (7.2) and (7.3) that $A^p(X) \cong E_{n-p,n-p}^2$ where $n = \dim X$. By (3.9) and (6.1) we have $E_{n-p,n-p}^2 \cong H^p(X, \mathscr{H}^p)$ as claimed.

(7.5) EXAMPLE. — Let $B^p(X) \subset H^{2p}(X, \mathbf{Z})$ be the subgroup generated by algebraic cycles. When $p = 2$ we get an exact sequence

$$H^3(X, \mathbf{Z}) \to \Gamma(X, \mathscr{H}^3) \xrightarrow{\delta} A^2(X) \to B^2(X) \to 0.$$

Thus $A^2(X)$ is a finitely generated abelian group if and only if $\Gamma(X, \mathscr{H}^3)$ is. Note that in general $\delta \neq 0$ even mod torsion (Griffith's counterexample [3]).

(7.6) REMARK. — When the ground field k is any field of characteristic zero, one can replace the sheaf $\mathscr{H}^p$ by the p-th cohomology sheaf $\mathscr{H}_{DR}^p$ of the de Rham complex $\Omega_{X/k}^{\bullet}$. One gets an isomorphism $A^p(X) \otimes_{\mathbf{Z}} k \cong H^p(X, \mathscr{H}_{DR}^p)$.

One has analogues of (7.4) in other cohomology theories. For example, let char k be arbitrary and fix an integer r prime to char k. Let $\mathscr{H}_{\text{ét}}^*(n)$ be the Zariski sheaf on X associated to the presheaf

$$U \mapsto H_{\text{ét}}^*(U, \mu^{\otimes n}),$$

where $\mu = \mu_r$ is the étale sheaf of r-th roots of 1.

(7.7) THEOREM :

$$H^p(X, \mathscr{H}_{\text{ét}}^p(p)) \cong A^p(X) \otimes \mathbf{Z}/r\mathbf{Z}.$$

Proof. — We have

$$(7.8) \qquad \bigoplus_{x \in Z^{p-1}/Z^p} H_{\text{ét}}^1(x, 1) \to \bigoplus_{x \in Z^p/Z^{p+1}} \mathbf{Z}/r\mathbf{Z} \to H^p(X, \mathscr{H}_{\text{ét}}^p(p)) \to 0.$$

From Hilbert's theorem 90, we have

$$H_{\text{ét}}^1(x, 1) = H_{\text{Galois}}^1(k(x), \mu) \cong k(x)^*/k(x)^{*r}$$

so (7.8) can be identified with the bottom row of the diagram

$$(7.9)\qquad \begin{array}{ccccccc} \bigoplus_{x \in Z^{p-1}/Z^p} k(x)^* & \rightarrow & \bigoplus_{x \in Z^p/Z^{p+1}} \mathbf{Z} & \longrightarrow & H^p(X, \underline{K}_p) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus k(x)^*/k(x)^{*r} & \longrightarrow & \bigoplus \mathbf{Z}/r\,\mathbf{Z} & \longrightarrow & H^p(X, \mathcal{H}^p_{\text{ét}}(p)) & \rightarrow & 0 \end{array}$$

The top row in (7.9) is the K-theoretic analogue [10]. In particular, $H^p(X, \underline{K}_p) \cong CH^p(X)$, the group of cycles modulo rational equivalence (*op. cit.* 5.14). It follows that $H^p(X, \mathcal{H}^p_{\text{ét}}(p)) \cong CH^p(X) \otimes \mathbf{Z}/r\,\mathbf{Z}$. Define R^p by the sequence

$$0 \rightarrow R^p \rightarrow CH^p(X) \rightarrow A^p(X) \rightarrow 0.$$

(7.10) LEMMA. — *R^p is a divisible group.*

Proof. — R^p is generated by cycles which come *via* an algebraic correspondence from a difference of two points on a smooth curve. Divisibility for R^p follows from divisibility for the Jacobian of the curve.

As a consequence we get $CH^p(X) \otimes \mathbf{Z}/r\,\mathbf{Z} \cong A^p(X) \otimes \mathbf{Z}/r\,\mathbf{Z}$ proving (7.6).

8. Differentials of the second kind

Suppose X/k is smooth and connected, where k is a field of characteristic zero. Let η be the generic point of X, $K = \mathcal{O}_{X,\eta}$ the fraction field of X, and $\Omega^{\cdot}_{K/k}$ the stalk of the de Rham complex $\Omega^{\cdot}_{X/k}$ at η. Recall that a (closed) form $\omega \in \Omega^q_{K/k}$ is called (in the classical language) a form " of the second kind " iff for each $x \in X$ there is a $\varphi \in \Omega^{q-1}_{K/k}$ such that $\omega - d\varphi$ is regular at x, i. e. belongs to $\Omega^q_{X/k,x}$. Equivalently, ω is of the second kind iff its image $\bar{\omega}$ in $H^q(\Omega^{\cdot}_{K/k})$ lies in the image of the map :

$$H^q(\Omega^{\cdot}_{X/k,x}) = \mathcal{H}^q_x \rightarrow H^q(\Omega^{\cdot}_{K/k}) = \mathcal{H}^q_\eta,$$

for every x in X. We shall then say that $\bar{\omega}$ is a " meromorphic cohomology class of the second kind ", or, for emphasis, " ...locally of the second kind ".

Notice that a special case of our Main Theorem asserts that the map $\mathcal{H}^q_x \rightarrow \mathcal{H}^q_\eta$ is *injective*. (Concretely, this says that if $\omega \in \Omega^q_{K/k}$ is regular at x and is $d\varphi$ for some $\varphi \in \Omega^{q-1}_{K/k}$, then also $\omega = d\varphi'$ with some φ' which is regular at x.) This observation makes it possible to give a classical interpretation of the mysterious terms $H^0(X, \mathcal{H}^q)$:

(8.1) THEOREM. — *There is a natural isomorphism between* $H^0(X, \mathcal{H}^q)$ *and the space of meromorphic cohomology classes of the second kind, i. e. the space* { *differential forms of the second kind* }/{ *exact ones* }.

Proof. — The map is just the map $H^0(X, \mathcal{H}^q) \rightarrow \mathcal{H}^q_\eta = H^q(\Omega^{\cdot}_{K/k})$. This map is injective, and clearly its image is contained in the space of classes of the second kind.

To prove the reverse inclusion, let $\omega \in \Omega^q_{K/k}$ be of the second kind. For each $x \in X$, let $\varphi^x \in \Omega^{q-1}_{K/k}$ be such that $\omega - d\varphi^x = \omega^x$ is regular at x, say in the affine neighborhood U^x of x. Then $d\omega^x = 0$, and hence ω^x defines an element $\bar{\omega}^x$ of $H^q(U^x, \Omega^{\cdot}_{X/k})$, and hence a section γ^x of $\mathcal{H}^q$ over U^x. To see that the sections agree on $U^x \cap U^y$ and so define a global section of $\mathscr{H}^q$, it is enough to see that they have same stalks, i. e. that if $y \in U^x$, the image of γ^x in $\mathscr{H}^q_y$ is equal to γ^y_y. But since $\mathscr{H}^q_y \subsetneq \mathscr{H}^q_\eta$, it suffices to take the case $y = \eta$, and this case is obvious from the definition.

Recall that Atiyah and Hodge [15] and later Grothendieck [5] found a different, *a priori* smaller space of " differential of the second kind " more convenient, namely the image of the map : $H^q(X) \to H^q_\eta$, and they asked if the two notions are equivalent. In terms of our spectral sequence, this questions asks if all the maps $d_r^{0,q}$ vanish for $r \geqq 2$, which is trivially true for $q \leqq 2$. For $q = 3$, this distinction turns out to be equivalent to the distinction between algebraic and homological equivalence. In fact, we have an exact sequence :

$$H^3(X) \xrightarrow{e} H^0(X, \mathscr{H}^3) \xrightarrow{d} H^2(X, \mathscr{H}^2) \xrightarrow{c} H^4(X). \tag{8.2}$$

Using the interpretation of $H^2(X, \mathscr{H}^2)$ as cycles mod algebraic equivalence, we see that, thanks to Griffiths, c is not injective, so d is not zero, so e is not surjective. In fact we have an isomorphism between the kernel of c and the cokernel of e.

(8.3) COROLLARY. — *For cycles of codimension* 2, *the k-vector space of cycles mod algebraic equivalence is finite dimensional iff the space of cohomology classes locally of the second kind is.*

Actually it is quite easy to construct an example of the distinction directly, at least in Griffiths example. In that case, X is a 3-fold containing two disjoint smooth curves Z_1 and Z_2, and the cycle $z = [Z_1] - [Z_2]$ is homologically but not algebraically equivalent to zero. The discussion of the previous section shows that if $Z = Z_1 \cup Z_2$, $H^4_Z(X) = k \oplus k$, with basis Z_1, Z_2, and $z \in H^4_Z(X)$ maps to zero in $H^4(X)$ but is nonzero in $H^4_D(X)$ for every divisor D containing Z. From the exact sequence $H^3(X-Z) \to H^4_Z(X) \to H^4(X)$ we see that there is a (nonzero) $\omega \in H^3(X-Z)$ which maps to z. It follows from the main theorem that ω_η is of the second kind, because for any point x in X we can find a neighborhood U and a divisor D containing Z such that the map $H^4_Z(X) \to H^4_{D\cap U}(U)$ is zero. In fact, since Z is smooth one can do this very easily, without recourse to our results. Hence from the diagram

$$\begin{array}{ccccc} & & H^3(X-Z) & \to & H^4_Z(X) \\ & & \downarrow & & \downarrow \\ H^3(U) & \to & H^3(U-D) & \to & H^4_{D\cap U}(U) \end{array}$$

we see that $\omega|_{U-D} = \omega'|_{U-D}$ for some $\omega' \in H^3(U)$. On the other hand, if we had $\omega_\eta = \omega''_\eta$ for some $\omega'' \in H^3(X)$, then for some divisor D we would have $\omega|_{X-D} = \omega''|_{X-D}$, and the diagram below would then show that z would vanish in $H^4_D(X)$, a contradiction,

$$\begin{array}{ccccc} & & H^3(X-Z) & \to & H^4_Z(X) \\ & & \downarrow & & \downarrow \\ H^3(X) & \to & H^3(X-D) & \to & H^4_D(X) \end{array}$$

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