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SPHERICAL FUNCTIONS ARE FOURIER TRANSFORMS OF L_1-FUNCTIONS

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In this brief note we apply a result of Kostant, (4.1) in [2], to prove the following. (All notation is as in Kostant's paper).

THEOREM 1. — *Let* (G, K) *be an irreducible Riemannian symmetric pair of non-compact type. Fix an Iwasawa decomposition* $G = KAN$. *For each* $b \in A$, $b \neq 1$, *let* $\mu_b \in M^1(\mathfrak{a})$ *be the finite measure on* $\mathfrak{a}$ *such that*

$$\int_K f(\log a(bv))\, dv = \int_{\mathfrak{a}} f(x)\, d\mu_b(x) \qquad f \in \mathcal{K}(\mathfrak{a}).$$

Then $\mu_b \in L_1(\mathfrak{a})$ *and* supp μ_b *is the compact set* $\mathfrak{a}(\log b)$.

REMARK 2. — T. H. Koornwinder has proved this in the rank 1 case by explicitly computing μ_b (*see* [1]).

This result has an immediate application to the spherical functions on G. If we write $\hat{\mu}(\tau) = \int_{\mathfrak{a}} e^{-i\tau(x)}\, d\mu(x)$, $\tau \in \mathfrak{a}^*$, for the Fourier Stieltjes transform on $\mathfrak{a}$, then we have

COROLLARY 3. — *For* $b \neq 1$, $b \in A$ *and* $\nu = \sigma - i\tau \in \mathfrak{a}^* + i\mathfrak{a}^*$, *the spherical function* $\varphi_\nu(b) = \int_K e^{\langle \nu, \log a(bv) \rangle}\, dv$ *is, as a function of* τ, *the Fourier transform of the compactly supported measure* $e^\sigma \mu_b \in L_1(\mathfrak{a})$. *Hence, for any tube* $T = C + i\mathfrak{a}^*$ *with* C *compact in* $\mathfrak{a}^*$, $\varphi_\nu(b) \to 0$ *as* $\nu \to \infty$ *in* T.

REMARK 4. — The second sentence generalizes (3.13) in [3].

Proof of Theorem 1. — The map $g_b : K \to \mathfrak{a}$ with $g_b(v) = \log a(bv)$, $v \in K$, is real analytic and, for $S \subseteq \mathfrak{a}$, $\mu_b(S) = m_K(g_b^{-1}(S))$ where m_K is Haar measure on K. We must show $\mu_b(S) = 0$ when S has Lebesgue measure zero. We claim that it suffices to show that g_b has rank equal to dim $\mathfrak{a}$ at some point of K. For if this is so then g_b has rank equal to dim $\mathfrak{a}$ except on a proper real analytic subvariety U of K since K is

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connected. But then $\dim U < \dim K$ and hence $m_K(U) = 0$. Now, on $K - U$, g_b, in appropriate coordinates, is just an orthogonal projection between Euclidean spaces. So since m_K is equivalent to Lebesgue measure in any coordinate patch, Fubini's theorem shows

$$m_K(g_b^{-1}(S)) = m_K(g_b^{-1}(S) \cap K - U) = 0,$$

when S has Lebesgue mesure zero.

Now to see g_b has rank equal to $\dim \mathfrak{a}$ at some point, it suffices by Sard's theorem (or the theorem on functional dependence) to show that the range of g_b has interior points in $\mathfrak{a}$. Now Kostant shows in (4.1) of [2] that $g_b(K) = \mathfrak{a}(\log b) = \text{co}(W \cdot \log b)$, in particular, $\mathfrak{a}(\log b)$ is a non-trivial convex W-invariant set. So by the irreducibility of the action of W on $\mathfrak{a}$, $0 \in \mathfrak{a}(\log b)$ and $\text{span}(\mathfrak{a}(\log b)) = \mathfrak{a}$. Thus $\mathfrak{a}(\log b)$ must have interior.

It is clear that $\text{supp}\ \mu_b = g_b(K) = \mathfrak{a}(\log b)$ and so is compact. $\square$

REMARK 5. — The same proof holds for non-irreducible (G, K) provided $\mathfrak{a}(\log b)$ has interior in $\mathfrak{a}$. For instance if b is regular or more generally if b has non-zero coordinate in each irreducible factor.

Proof of Corollary 3. — The first statement follows from the definition of μ_b. For the second note that if $C = \{\sigma\}$, then the Riemann-Lebesgue lemma says $\varphi_{\sigma+i\tau}(b) = (e^\sigma \mu_b)^\wedge(\tau) \to 0$ as $\tau \to \infty$. In general, $\sigma \to e^\sigma \mu_b$ is a continuous function from $\mathfrak{a}^*$ to $L_1(\mathfrak{a})$ since μ_b has compact support. So it is uniformly continuous on the compact set C from which the result follows as

$$|\varphi_{\sigma+i\tau}(b) - \varphi_{\sigma'+i\tau}(b)| \leq \| e^\sigma \mu_b - e^{\sigma'} \mu_b \|_{L_1(\mathfrak{a})}. \quad \square$$

One would like to have more precise asymptotic information on φ_ν as $\nu \to \infty$, but that does not seem to be obtainable by our simple methods.

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