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SPHERICAL FUNCTIONS
ARE FOURIER TRANSFORMS OF L1-FUNCTIONS

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In this brief note we apply a result of Kostant, (4.1) in [2], to prove the following. (All notation is as in Kostant's paper).

**Theorem 1.** — Let \((G, K)\) be an irreducible Riemannian symmetric pair of non-compact type. Fix an Iwasawa decomposition \(G = KAN\). For each \(b \in A, b \neq 1\), let \(\mu_b \in M^1 (a)\) be the finite measure on \(a\) such that

\[
\int_k f (\log a (bv)) \, dv = \int_a f(x) \, d\mu_b (x) \quad f \in \mathcal{K} (a).
\]

Then \(\mu_b \in L_1 (a)\) and \(\text{supp} \, \mu_b\) is the compact set \(a (\log b)\).

**Remark 2.** — T. H. Koornwinder has proved this in the rank 1 case by explicitly computing \(\mu_b\) (see [1]).

This result has an immediate application to the spherical functions on \(G\). If we write \(\hat{\varphi} (\tau) = \int_a e^{-i\tau(x)} \, d\mu (x), \tau \in a^*\), for the Fourier Stieltjes transform on \(a\), then we have

**Corollary 3.** — For \(b \neq 1, b \in A\) and \(\gamma = \tau - i \tau \in a^* + i a^*\), the spherical function \(\varphi_\gamma (b) = \int e^{\gamma \log a (b \nu)} \, dv\) is, as a function of \(\tau\), the Fourier transform of the compactly supported measure \(e^{\gamma} \mu_b \in L_1 (a)\). Hence, for any tube \(T = C + i a^*\) with \(C\) compact in \(a^*\), \(\varphi_\gamma (b) \to 0\) as \(\tau \to \infty\) in \(T\).

**Remark 4.** — The second sentence generalizes (3.13) in [3].

**Proof of Theorem 1.** — The map \(g_\nu : K \to a\) with \(g_\nu (v) = \log a (bv)\), \(v \in K, K\), is real analytic and, for \(S \subseteq a, \mu_b (S) = m_k (g_\nu (S))\) where \(m_k\) is Haar measure on \(K\). We must show \(\mu_b (S) = 0\) when \(S\) has Lebesgue measure zero. We claim that it suffices to show that \(g_\nu\) has rank equal to \(\dim a\) at some point of \(K\). For if this is so then \(g_\nu\) has rank equal to \(\dim a\) except on a proper real analytic subvariety \(U\) of \(K\) since \(K\) is

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connected. But then dim \( U \subset \dim K \) and hence \( m_k(U) = 0 \). Now,
on \( K - U \), \( g_o \), in appropriate coordinates, is just an orthogonal projection
between Euclidean spaces. So since \( m_k \) is equivalent to Lebesgue mea-
sure in any coordinate patch, Fubini's theorem shows
\[
m_k(g_o^{-1}(S)) = m_k(g_o^{-1}(S) \cap K - U) = 0,
\]
when \( S \) has Lebesgue measure zero.

Now to see \( g_o \) has rank equal to \( \dim a \) at some point, it suffices by
Sard's theorem (or the theorem on functional dependence) to show that
the range of \( g_o \) has interior points in \( a \). Now Kostant shows in (4.1)
of [2] that \( g_o(K) = a(\log b) = \co(W \cdot \log b) \), in particular, \( a(\log b) \) is a
non-trivial convex \( W \)-invariant set. So by the irreducibility of the
action of \( W \) on \( a \), \( \emptyset \in a(\log b) \) and \( \text{span}(a(\log b)) = a \). Thus \( a(\log b) \)
must have interior.

It is clear that \( \text{supp } \mu_o = g_o(K) = a(\log b) \) and so is compact. \( \square \)

**Remark 5.** — The same proof holds for non-irreducible \((G, K)\)
provided \( a(\log b) \) has interior in \( a \). For instance if \( b \) is regular or more
generally if \( b \) has non-zero coordinate in each irreducible factor.

**Proof of Corollary 3.** — The first statement follows from the definition
of \( \mu_o \). For the second note that if \( C = \{ \sigma \} \), then the Riemann-Lebesgue
lemma says \( \varphi_{\sigma+\tau}(b) = (e^{\sigma} \mu_o)(\tau) \to 0 \) as \( \tau \to \infty \). In general, \( \sigma \to e^{\sigma} \mu_o \)
is a continuous function from \( a^* \) to \( L_1(a) \) since \( \mu_o \) has compact support.
So it is uniformly continuous on the compact set \( C \) from which the result
follows as
\[
\| \varphi_{\sigma+\tau}(b) - \varphi_{\tau}(b) \| \leq \| e^{\sigma} \mu_o - e^{\sigma} \mu_o \|_{L_1(a)}. \quad \square
\]

One would like to have more precise asymptotic information on \( \varphi \).
as \( \nu \to \infty \), but that does not seem to be obtainable by our simple methods.

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