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THE PLANCHEREL FORMULA FOR GROUP EXTENSIONS II

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1. INTRODUCTION. — In our previous work [7] we discussed the problem of computing the Plancherel measure of a group in terms of the Plancherel measures of a normal subgroup and the corresponding little groups. In this paper we want to complete those results in several respects.

First, the methods we adopted in [7] led us in a natural way to consider non-unimodular groups. For those, only the class of Plancherel measure and not the measure itself, is uniquely determined. Though we were able to compute the exact Plancherel measure within the class for *certain* unimodular group extensions, we were not able to do so in general. We remedy that here in Theorem 2.3 with the *precise* Plancherel measure for a general unimodular extension.

It is worth nothing that the proof of Theorem 2.3 (although somewhat disguised) is actually quite similar to the general argument presented in [7, Theorem 10.2]. The main difference is the following : instead of relying on unitary equivalences between the *representations* that occur at various stages, we keep careful track of the *functions* and *traces* which arise in the extension procedure. The key fact that makes it possible to do this is the unimodularity of almost all the little groups (*see* Lemma 2.2); the main tool for actually carrying out the analysis is the character formula [7, Theorem 3.2].

In paragraph 3, we use Theorem 2.3 to compute the Plancherel measure of several different types of unimodular groups : namely, the inhomogeneous Lorentz groups; the Cartan motion groups; a semidirect product of semisimple and nilpotent Lie groups (in which the Weil representation

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arises); and certain kinds of type R solvable Lie groups. We also give the Plancherel measure class of a non-unimodular, non-type I group which was first described by Mackey.

The second topic we discuss is a natural consequence of ideas arising from the non-unimodular Plancherel theorem itself. Specifically, we shall describe the orthogonality relations for non-unimodular groups. In addition, we obtain necessary and sufficient conditions for an irreducible representation to be in the discrete series of a non-unimodular group. In the spirit of [7, § 6] we work as far as possible with quasi-Hilbert algebras. To a large extent (at least in the initial stages), our results are an adaptation of Rieffel's results [11] on Hilbert algebras to the case of a quasi-Hilbert algebra. However at the end, the results on groups (*see* Theorems 4.8, 4.9) present some interesting differences from the unimodular case. First, an unbounded operator, which appears in the Fourier transform (*see* [7, Theorem 6.4]), also occurs in the orthogonality relations. Second, not all the coefficients of a discrete series representation need be square-integrable.

Unless mentioned otherwise, all notation and terminology will be the same as that established in [7]. However, we only assume our groups are separable in paragraphs 2 and 3; it is not necessary for the results of paragraph 4.

2. THE UNIMODULAR CASE. — We begin by generalizing [7, Lemma 4.3] to the case of a unimodular group with multiplier. Let ω be a normalized multiplier on G. Consider the twisted group algebra $L_1(G, \omega)$, that is the space $L_1(G)$ with the usual involution and multiplication

$$f \star_\omega h(x) = \int_G \overline{\omega}(x, y^{-1}) f(xy^{-1}) h(y)\, dy, \qquad x \in G, \quad f, h \in L_1(G).$$

[This is a change of notation from [7, § 8] where we denoted this algebra by $L_1(G, \overline{\omega})$.] The ω-representations of G correspond in the usual fashion to non-degenerate $\star$-representations of $L_1(G, \omega)$.

Now let $G(\omega)$ be the extension of $\mathbf{T}$ by G defined by ω (*cf.* [7, § 7]). For each $f \in L_1(G, \omega)$, we put $f^\#(t, x) = t^{-1} f(x)$, $t \in \mathbf{T}$, $x \in G$. It is easily checked that $f \to f^\#$ is an isometric $\star$-isomorphism of $L_1(G, \omega)$ into $L_1(G(\omega))$.

We denote by $C^*(G, \omega)$ and $C^*(G(\omega))$ the enveloping C^*-algebras of these algebras. They are respectively the completion of $L_1(G, \omega)$ for the norm

$$f \to \sup_{\sigma \in \hat{G}^\omega} \| \sigma(f) \|$$

and the completion of $L_1(G(\omega))$ for the norm

$$h \to \sup_{\pi \in \widehat{G(\omega)}} \| \pi(h) \|.$$

If σ is an ω-representation of G, then σ^0 defined by $\sigma^0(t, x) = t\sigma(x)$, $t \in \mathbf{T}$, $x \in G$, is an ordinary representation of $G(\omega)$. The map $\sigma \to \sigma^0$ is an injection of $\hat{G}^\omega$ into $\widehat{G(\omega)}$ which is an isomorphism for the Mackey and topological Borel structures.

Since $\mathbf{T}$ is a central subgroup of $G(\omega)$, each $\pi \in \widehat{G(\omega)}$ reduces on $\mathbf{T}$ to a multiple of a character, i. e., there is $n \in \mathbf{Z}$ such that $\pi(t, x) = t^n \pi(1, x)$, $t \in \mathbf{T}$, $x \in G$. Let ${}^n\widehat{G(\omega)} = \{ \pi \in \widehat{G(\omega)} : \pi(t, x) = t^n \pi(1, x) \}$. Then $\widehat{G(\omega)}$ is the disjoint union of the sets ${}^n\widehat{G(\omega)}$, and the image of $\hat{G}^\omega$ under the map $\sigma \to \sigma^0$ is precisely ${}^1\widehat{G(\omega)}$.

For all $\sigma \in \hat{G}^\omega$ and $f \in L_1(G, \omega)$, we have

$$\sigma^0(f^\#) = \int f^\#(t, x)\, \sigma^0(t, x)\, dt\, dx = \sigma(f).$$

On the other hand, if $\pi \in {}^n\widehat{G(\omega)}$, $n \neq 1$, then

$$\pi(f^\#) = \int t^{n-1}\, \pi(1, x)\, f(x)\, dt\, dx = 0.$$

It follows from this that the map $f \to f^\#$ is also an isometry for the norms induced by $C^*(G, \omega)$ and $C^*(G(\omega))$. Because the closure of an ideal is an ideal, we may identify $C^*(G, \omega)$ (by means of the extension to the enveloping algebras of the map $f \to f^\#$) with an ideal in $C^*(G(\omega))$.

Let $\delta_{G(\omega)}$ be the trace on $C^*(G(\omega))^+$ defined by the point measure at the identity. By restriction this defines a trace $\delta_{G,\omega}$ on $C^*(G, \omega)^+$; more precisely

$$\delta_{G,\omega}(f) = \delta_{G(\omega)}(f^\#), \qquad f \in C^*(G, \omega)^+.$$

We need an alternate description of this trace and for this it is convenient to introduce the following terminology. A function f on G will be called *ω-continuous* if $f^\#$ is continuous on $G(\omega)$; and f will be called *ω-positive-definite* if $f^\#$ is positive-definite on $G(\omega)$. Let $P(G, \omega)$ denote the space of ω-continuous and ω-positive-definite functions on G.

LEMMA 2.1. — (i) *$\delta_{G(\omega)}$ is a lower semicontinuous, semifinite trace on* $C^*(G, \omega)^+$.

(ii) *If* $\psi \in L_1(G, \omega)$ *is an* ω*-continuous function such that for every* ω*-representation* σ *of* G, $\sigma(\psi)$ *is a positive operator, then* $\psi \in P(G, \omega) \cap C^*(G, \omega)^+$ *and* $\delta_{G,\omega}(\psi) = \psi(e)$.

Proof. — (i) Since $\delta_{G(\omega)}$ is lower semicontinuous, so is the trace $\delta_{G,\omega}$. To show it is semifinite, it is enough to show it is densely defined [4, 6.1.3]. Let $P : L_1(G(\omega)) \to L_1(G, \omega)$ be the projection defined by

$$P f(t, x) = t^{-1} \int_T s f(x, s)\, ds.$$

P is an algebra homomorphism and may be extended to the projection of $C^*(G(\omega))$ onto $C^*(G, \omega)$. One sees easily that if $\mathfrak{m}$ is the ideal of definition of $\delta_{G(\omega)}$, then $P\mathfrak{m}$ is the ideal of definition of $\delta_{G,\omega}$. Since $\mathfrak{m}$ is dense in $C^*(G(\omega))$, $P\mathfrak{m}$ is dense in $C^*(G, \omega)$.

(ii) By the hypotheses, $\psi^\# \in L_1(G(\omega))$ and is continuous. If $\pi \in {}^1\widehat{G(\omega)}$, then $\pi = \sigma^0$ for some $\sigma \in \hat{G}^\omega$ and $\pi(\psi^\#) = \sigma(\psi)$ is a positive operator. If $\pi \in {}^n\widehat{G(\omega)}$, $n \neq 1$, then $\pi(\psi^\#) = 0$. Hence $\sigma(\psi^\#)$ is positive for all $\pi \in \widehat{G(\omega)}$. But any representation of $G(\omega)$ may be expressed (possibly not uniquely) as a direct integral of irreducibles; hence $\pi(\psi^\#)$ is a positive operator for every representation π. By [7, Lemma 4.3], $\psi^\# \in P(G(\omega)) \cap C^*(G(\omega))^+$ and $\delta_{G(\omega)}(\psi^\#) = \psi^\#(1, e) = \psi(e)$. It follows that $\psi \in P(G, \omega) \cap C^*(G, \omega)^+$ and $\delta_{G,\omega}(\psi) = \delta_{G(\omega)}(\psi^\#) = \psi(e)$.

Remark. — Recall that the projective Plancherel Theorem [7, Theorem 7.1] provides a decomposition of the trace $\delta_{G,\omega}$ into " characters ". It says that under a certain type I assumption (which can be weakened — *see* the comments after the proof of Theorem 2.3), there exists a unique Plancherel measure $\mu_{G,\omega}$ on $\hat{G}^\omega$ such that

$$\delta_{G,\omega} = \int_{\hat{G}^\omega} \operatorname{Tr} \sigma \, d\mu_{G,\omega}(\sigma).$$

Now we pass to a brief discussion of moduli of automorphisms and relatively invariant measures. Let α be an automorphism of G. In the following we define all " dual " automorphisms in a contragredient fashion, and continue (*see* [7, § 1]) to write actions on the right. So $(\pi\alpha)(x) = \pi(x\alpha^{-1})$, $\pi \in \hat{G}$, $(f\alpha)(x) = f(x\alpha^{-1})$, $f \in L_1(G)$, $\langle \mu.\alpha, f \rangle = \langle \mu, f\alpha^{-1} \rangle$, μ a measure, etc.

If α is an automorphism of G, we denote by $\Delta_G(\alpha)$ its modulus, that is

(2.1) $$\Delta_G(\alpha) \int_G (f\alpha)(x)\, dx = \int_G f(x)\, dx, \qquad f \in L_1(G),$$

or $\nu . \alpha = \Delta_G(\alpha)\, \nu$ for Haar measure ν. If H is a closed subgroup of G such that $\alpha|_H$ is an automorphism of H, we write $\Delta_H(\alpha)$ for $\Delta_H(\alpha|_H)$. Suppose that G/H has an invariant measure. Then we may choose Haar measures dx, dh on G, H and an invariant measure $d\bar{x}$ on G/H such that

$$\int_G f(x)\, dx = \int_{G/H} \int_H f(hx)\, dh\, d\bar{x}, \qquad f \in L_1(G). \tag{2.2}$$

Let $\bar{\alpha}$ be the homeomorphism of G/H defined by passage to the quotient. Let $\Delta_{G/H}(\bar{\alpha})$ be the modulus of $\bar{\alpha}$,

$$\Delta_{G/H}(\bar{\alpha}) \int_{G/H} f \bar{\alpha}(\bar{x})\, d\bar{x} = \int_{G/H} f(\bar{x})\, d\bar{x}, \qquad f \in L_1(G/H). \tag{2.3}$$

It is a simple matter to verify from (2.1), (2.2), and (2.3) that

$$\Delta_{G/H}(\bar{\alpha}) = \Delta_G(\alpha)\, \Delta_H(\alpha)^{-1}.$$

In particular, we consider the special case where H is a closed normal subgroup N and α is an inner automorphism i_x, $i_x(y) = x^{-1} yx$, $x, y \in G$. As we observed in [7, § 10], $\Delta_G(i_x) = \Delta_G(x)$. Also $\Delta_{G/N}(\bar{i_x}) = \Delta_{G/N}(\bar{x})$. Writing $\Delta_N(x)$ for $\Delta_N(i_x)$, we have

$$\Delta_{G/\hat{N}}(\bar{x}) = \Delta_G(x)\, \Delta_N(x)^{-1}, \qquad x \in G. \tag{2.4}$$

Assume now that N is unimodular and has a type I regular representation. Let α be an automorphism of N. Since α acts on $\hat{N}$, it also acts on the Plancherel measure μ_N. We show μ_N is relatively invariant. First if $f \in L_1(N)$, we have

$$\begin{aligned}(f\alpha)^\wedge(\gamma) = \gamma(f\alpha) &= \int_N \gamma(n)\, f(n\alpha^{-1})\, dn \\ &= \Delta_N(\alpha)^{-1} \int_N \gamma(n\alpha)\, f(n)\, dn = \Delta_N(\alpha)^{-1} \hat{f}(\gamma\alpha^{-1}) = \Delta_N(\alpha)^{-1} \hat{f}\alpha(\gamma).\end{aligned}$$

That is

$$(f\alpha)^\wedge = \Delta_N(\alpha)^{-1} \hat{f}\alpha.$$

But then for $f \in C_0(N)$,

$$\begin{aligned}\int_\wedge \| \hat{f}(\gamma) \|_2^2\, d\mu_N(\gamma) &= \int_N | f(n) |^2\, dn = \Delta_N(\alpha) \int_N | f\alpha(n) |^2\, dn \\ &= \Delta_N(\alpha) \int_{\hat{N}} \| (f\alpha)^\wedge(\gamma) \|_2^2\, d\mu_N(\gamma) = \Delta_N(\alpha)^{-1} \int_{\hat{N}} \| \hat{f}\alpha(\gamma) \|_2^2\, d\mu_N(\gamma).\end{aligned}$$

That is

$$\mu_N . \alpha = \Delta_N(\alpha)^{-1} \mu_N. \tag{2.5}$$

We can use this now to prove

LEMMA 2.2. — *Let* N *be a closed normal subgroup of a unimodular group* G. *Suppose that* N *has a type* I *regular representation and that* $\bar{\mu}_N$ *is countably separated. Then for* μ_N*-almost all* γ, G_γ/N *is unimodular.*

Proof. — $\bar{\mu}_N$ denotes a pseudo-image of μ_N on $\hat{N}/G$. It is unique up to equivalence and the countable separability is independent of the representative in the equivalence class. Then for μ_N-almost all γ, there exist a quasi-invariant measure ν_γ on the homogeneous space G/G_γ such that

$$\int_{\hat{N}} f(\gamma)\, d\mu_N(\gamma) = \int_{\hat{N}/G} \int_{G/G_\gamma} f(\gamma . g)\, d\nu_\gamma(\bar{g})\, d\bar{\mu}_N(\bar{\gamma})$$

(*see* [7, Theorem 2.1]; as usual $\bar{g}$ and $\bar{\gamma}$ denote respectively the images of g and γ in G/G_γ and $\hat{N}/G$). Now μ_N is relatively invariant under the action of G, in fact by (2.5) :

$$\mu_N . g = \Delta_N(g)^{-1} \mu_N.$$

Combining this with (2.4), we obtain

$$\mu_N . g = \Delta_{G/N}(\bar{g})\, \mu_N.$$

But then exactly as in [7, Theorem 2.1], we use arguments analogous to [8, Lemmas 11.4, 11.5] to conclude that μ_N-almost all the measures ν_γ are relatively invariant with modulus $\Delta_{G/N}(\bar{g})$,

(2.6) $$\nu_\gamma . g = \Delta_{G/N}(\bar{g})\, \nu_\gamma, \quad \text{a.a. } \gamma, \qquad g \in G.$$

Finally it is a simple and well-known observation that the homogeneous space G/H can carry a relatively invariant measure ν if and only if the map $h \rightarrow \Delta_G(h)\, \Delta_H(h)^{-1}$ extends from H to G as a group homomorphism. In such a case $\nu . g = \Delta_G(g)\, \Delta_H(g)^{-1}\, \nu$, $g \in G$. Combining this fact with (2.6) and the identification

$$G/G_\gamma = \frac{G/N}{G_\gamma/N},$$

we obtain $\Delta_{G/N}(\bar{g}) = \Delta_{G/N}(\bar{g})\, \Delta_{G_\gamma/N}(\bar{g})^{-1}$, $g \in G$, a. a. γ. In particular, for μ_N-almost all γ, the modular function $\Delta_{G_\gamma/N}$ is identically one. That completes the proof of the lemma.

Remark. — It is not true in general that every G_γ/N is unimodular. For instance, the stability group of the trivial representation, namely G/N, need not be unimodular.

We come now to the principal result of this section.

THEOREM 2.3. — *Let* N *be a closed normal* (*unimodular*) *subgroup of the unimodular group* G. *Assume that* N *has a type* I *regular representation,* *that $\bar{\mu}_N$ is countably separated, and that G_γ/N has a type I regular $\bar{\omega}_\gamma$-representation for almost all γ. Then G has a type I regular representation and the Plancherel formula is determined as follows : Fix Haar measure dg on G. Let $\bar{\mu}_N$ be any pseudo-image of Plancherel measure on $\hat{N}$. Then for μ_N-almost all γ, there is a uniquely determined Plancherel measure $\mu_\gamma = \mu_{G_\gamma/N, \bar{\omega}_\gamma}$ on $(\widehat{G_\gamma/N})^{\bar{\omega}_\gamma}$ such that*

$$(2.7)\qquad \int_G |\varphi(g)|^2\, dg = \int_{\hat{N}/G} \int_{(\widehat{G_\gamma/N})^{\bar{\omega}_\gamma}} \|\pi_{\gamma,\sigma}(\varphi)\|_2^2\, d\mu_\gamma(\sigma)\, d\bar{\mu}_N(\bar{\gamma}), \qquad \varphi \in C_0(G).$$

Proof. — We shall first prove (2.7) and then deduce that G has a type I regular representation. First let *dn* be any Haar measure on N and let $\bar{\mu}_N$ be a pseudo-image of Plancherel measure on $\hat{N}/G$. We begin the computation exactly as in the special cases treated in [7, § 4, 5, 8]. Let $\varphi \in C_0(G)$ and put $\psi = \varphi \star \varphi^*$, $\theta = \psi|_N$. Then by [7, Lemma 4.3 (ii)], $\theta \in C^*(N)^+$ and

$$(2.8)\qquad \int_G |\varphi(g)|^2\, dg = \psi(e) = \theta(e) = \delta_N(\theta)$$
$$= \int_{\hat{N}} \mathrm{Tr}\, \gamma(\theta)\, d\mu_N(\gamma) = \int_{\hat{N}/G} \int_{G/G_\gamma} \mathrm{Tr}\, (\gamma \cdot g)(\theta)\, d\nu_\gamma(\bar{g})\, d\bar{\mu}_N(\bar{\gamma}).$$

The last equality is justified by [7, Theorem 2.1]. Moreover, having specified the choice of $\bar{\mu}_N$, the relatively-invariant measures ν_γ are uniquely determined for almost all γ. There is then a unique choice of Haar measure on G_γ so that

$$(2.9)\qquad \int_G f(x)\, \Delta_N(x)\, dx = \int_{G/G_\gamma} \int_{G_\gamma} f(hx)\, dh\, d\nu_\gamma(\bar{x})$$

for almost all γ. Next, since *dn* is already chosen, this specifies a choice of Haar measure on G_γ/N. By Lemma 2.2 almost all the groups G_γ/N are unimodular. Therefore a choice of Plancherel measure μ_γ on $(\widehat{G_\gamma/N})^{\bar{\omega}}$ is fixed as well. We shall maintain this choice of measures throughout the rest of the proof.

Comparing (2.7) and (2.8), we see it is enough to show

$$(2.10)\qquad \int_{(\widehat{G_\gamma/N})^{\bar{\omega}_\gamma}} \mathrm{Tr}\, \pi_{\gamma,\sigma}(\psi)\, d\mu_\gamma(\sigma) = \int_{G/G_\gamma} \mathrm{Tr}\, (\gamma \cdot g)(\theta)\, d\nu_\gamma(\bar{g})$$

for all γ such that ν_γ is relatively invariant and G_γ/N is unimodular. Taking such a γ, we apply the character formula [7, Theorem 3.2] to the representation $\pi_{\gamma,\sigma}$. As we have observed already in (2.9), the q function corresponding to the relatively invariant measure ν_γ is Δ_N and so [7, Theorem 3.2] yields

$$\operatorname{Tr} \pi_{\gamma,\sigma}(\psi) = \int_{G/G_\gamma} \Delta_N(g)^{-1} \operatorname{Tr}\left(\int_{G_\gamma} \psi(g^{-1}hg)(\gamma' \otimes \sigma'')(h)\,\Delta_{G_\gamma}(h)^{-1/2}\,dh\right) d\nu_\gamma(\bar{g}).$$

We denote by Λ_σ the integrand in this expression

$$\Lambda_\sigma(g) = \Delta_N(g)^{-1} \operatorname{Tr}\left(\int_{G_\gamma} \psi(g^{-1}hg)(\gamma' \otimes \sigma'')(h)\,\Delta_{G_\gamma}(h)^{-1/2}\,dh\right).$$

From the proof of the character formula [7, Theorem 3.2], we know that Λ_σ is a non-negative function, possibly assuming the value $+\infty$, which is measurable as a function of $\bar{g}$. In fact, with the notation as in the proof of [7, Theorem 3.2], we have

$$\Lambda_\sigma(\bar{g}) = \operatorname{Tr} \Phi_\psi(\bar{g}, \bar{g}),$$

where

$$\Phi_\psi(\bar{g}, \bar{g}) = \int_{G/G_\gamma} \Phi_\varphi(\bar{g}, \bar{g}_1)\,\Phi_\varphi(\bar{g}, \bar{g}_1)^*\,d\nu_\gamma(\bar{g}_1),$$

and Φ_φ is the measurable operator-valued function

$$\Phi_\varphi(\bar{g}, \bar{g}_1) = \Delta_N(g)^{-1/2} \int_{G_\gamma} \varphi(g^{-1}hg_1)(\gamma' \otimes \sigma'')(h)\,\Delta_N(hg_1)^{-1/2}\,dh.$$

By Tonelli's theorem, we may write the left-hand side of (2.10) as

$$\int_{(\widehat{G_\gamma/N})^{\varpi_\gamma}} \int_{G/G_\gamma} \Lambda_\sigma(g)\,d\nu_\gamma(\bar{g})\,d\mu_\gamma(\sigma) = \int_{G/G_\gamma} \int_{(\widehat{G_\gamma/N})^{\varpi_\gamma}} \Lambda_\sigma(g)\,d\mu_\gamma(\sigma)\,d\nu_\gamma(\bar{g}).$$

Therefore our proof is reduced to showing

$$\int_{(\widehat{G_\gamma/N})^{\varpi_\gamma}} \Lambda_\sigma(g)\,d\mu_\gamma(\sigma) = \operatorname{Tr}(\gamma . g)(\theta), \qquad g \in G.$$

Now let $\{\xi_i\}$ be an orthonormal basis for $\mathcal{H}_\gamma$, and $\{\eta_j^\sigma\}$ an orthonormal basis for $\mathcal{H}_\sigma$. Then

$$\begin{aligned}\Lambda_\sigma(g) &= \Delta_N(g)^{-1} \sum_{i,j} \int_{G_\gamma} \psi(g^{-1}hg)\,((\gamma' \otimes \sigma'')(h)\,\xi_i \otimes \eta_j^\sigma, \xi_i \otimes \eta_j^\sigma)\,\Delta_{G_\gamma}(h)^{-1/2}\,dh \\ &= \Delta_N(g)^{-1} \sum_{i,j} \int_{G_\gamma/N} \int_N \psi(g^{-1}nhg)\,(\gamma'(nh)\,\xi_i, \xi_i)\,(\sigma''(nh)\,\eta_j^\sigma, \eta_j^\sigma)\,\Delta_{G_\gamma}(nh)^{-1/2}\,dn\,d\bar{h}.\end{aligned}$$

Let

$$\Omega_{i,g}(\bar{h}) = \Delta_N(g)^{-1} \int_N \psi(g^{-1}nhg)\,(\gamma'(nh)\,\xi_i, \xi_i)\,\Delta_{G_\gamma}(nh)^{-1/2}\,dn.$$

Then we have

$$\int_{(\widehat{G_\gamma/N})^{\bar\omega_\gamma}} \Lambda_\sigma(g)\, d\mu_\gamma(\sigma) = \int_{(\widehat{G_\gamma/N})^{\bar\omega_\gamma}} \sum_{i,j} \int_{G_\gamma/N} \Omega_{i,g}(\bar h)\,(\sigma(\bar h)\,\eta_j^\sigma, \eta_j^\sigma)\, d\bar h\, d\mu_\gamma(\sigma).$$

Since

$$\sum_j \int_{G_\gamma/N} \Omega_{i,g}(\bar h)\,(\sigma(\bar h)\,\eta_j^\sigma, \eta_j^\sigma)\, d\bar h = \operatorname{Tr}\sigma(\Omega_{i,g})$$

we then have

$$\int_{(\widehat{G_\gamma/N})^{\bar\omega_\gamma}} \Lambda_\sigma(g)\, d\mu_\gamma(\sigma) = \int_{(\widehat{G_\gamma/N})^{\bar\omega_\gamma}} \sum_i \operatorname{Tr}\sigma(\Omega_{i,g})\, d\mu_\gamma(\sigma).$$

Thus it remains to prove

$$\int_{(\widehat{G_\gamma/N})^{\bar\omega_\gamma}} \sum_i \operatorname{Tr}\sigma(\Omega_{i,g})\, d\mu_\gamma(\sigma) = \operatorname{Tr}(\gamma.g)(\theta).$$

Suppose for the moment that $\Omega_{i,g}$ satisfies the hypotheses of Lemma 2.1, that is $\Omega_{i,g} \in L_1(G_\gamma/N, \bar\omega_\gamma)$ and is an $\bar\omega_\gamma$-continuous function such that $\sigma_1(\Omega_{i,g})$ is a positive operator for every $\bar\omega_\gamma$-representation σ_1 of G_γ/N. Then by that lemma, and by the projective Plancherel theorem [7, Theorem 7.1], we would have

$$\begin{aligned}\int_{(\widehat{G_\gamma/N})^{\bar\omega_\gamma}} \sum_i \operatorname{Tr}\sigma(\Omega_{i,g})\, d\mu_\gamma(\sigma) &= \sum_i \int_{(\widehat{G_\gamma/N})^{\bar\omega_\gamma}} \operatorname{Tr}\sigma(\Omega_{i,g})\, d\mu_\gamma(\sigma) = \sum_i \Omega_{i,g}(e)\\ &= \sum_i \Delta_N(g)^{-1} \int_N \psi(g^{-1} ng)\,(\gamma(n)\,\xi_i, \xi_i)\, dn\\ &= \sum_i \int_N \psi(n)\,(\gamma(gng^{-1})\,\xi_i, \xi_i)\, dn\\ &= \operatorname{Tr}\int_N \theta(n)\,\gamma(n.i_g^{-1})\, dn = \operatorname{Tr}(\gamma.g)(\theta).\end{aligned}$$

It remains to show that $\Omega_{i,g}$ satisfies the conditions of Lemma 2.1. Using the notation employed in the discussion preceding that lemma, we have

$$\begin{aligned}\Omega^\#_{i,g}(t, \bar h) &= \Delta_N(g)^{-1}\, t^{-1} \int_N \psi(g^{-1} nhg)\,(\gamma'(nh)\,\xi_i, \xi_i)\,\Delta_{G_\gamma}(nh)^{-1/2}\, dn\\ &= \Delta_N(g)^{-1} \int_N \psi(g^{-1} nhg)\,(\gamma'^{00}(t, nh), \xi_i, \xi_i)\,\Delta_{G_\gamma}(nh)^{-1/2}\, dn,\end{aligned}$$

where

$$\gamma'^{00}(t, h) = t^{-1}\,\gamma'(h).$$

Note that since γ' is an ω_γ-representation of G_γ, γ'^{00} is an ordinary representation of $G_\gamma(\overline{\omega}_\gamma)$. In particular, the matrix coefficients of γ'^{00} are continuous functions on $G_\gamma(\overline{\omega}_\gamma)$. Thus it follows that $\Omega^{\#}_{i,g}$ is a continuous function on $G_\gamma(\overline{\omega}_\gamma)$, constant on the N-cosets, and compactly supported modulo N, i. *e.*,

$$\Omega^{\#}_{i,g} \in C_0[(G_\gamma/N)(\overline{\omega}_\gamma)].$$

Therefore $\Omega_{i,g} \in L_1(G_\gamma/N, \overline{\omega}_\gamma)$ and it is $\overline{\omega}_\gamma$-continuous.

Finally, let σ_1 be any $\overline{\omega}_\gamma$-representation of G_γ/N, $\eta \in \mathcal{H}_{\sigma_1}$. Then

$$\begin{aligned}(\sigma_1(\Omega_{i,g})\eta, \eta) &= \int_{G_\gamma/N} \Omega_{i,g}(\bar{h})(\sigma_1(\bar{h})\eta, \eta)\, d\bar{h} \\ &= \int_{G_\gamma/N} \Delta_N(g)^{-1} \int_N \psi(g^{-1}ngh)(\gamma'(nh)\xi_i, \xi_i)\Delta_{G_\gamma}(nh)^{-1/2}\, dn\, (\sigma_1(h)\eta, \eta)\, d\bar{h} \\ &= \Delta_N(g)^{-1} \int_{G_\gamma} \psi(g^{-1}hg)((\gamma' \otimes \sigma''_1)(h)\xi_i \otimes \eta, \xi_i \otimes \eta)\Delta_{G_\gamma}(h)^{-1/2}\, dh.\end{aligned}$$

But as we have seen

$$\Delta_N(g)^{-1} \int_{G_\gamma} \psi(g^{-1}hg)(\gamma' \otimes \sigma''_1)(h)\Delta_{G_\gamma}(h)^{-1/2}\, dh$$

is precisely the kernel $\Phi_\psi(\bar{g}, \bar{g})$ of the positive operator

$$\pi_{\gamma,\sigma_1}(\psi) = \operatorname{Ind}^G_{G_\gamma}(\gamma' \otimes \sigma''_1)(\psi), \qquad \psi = \varphi \star \varphi^*.$$

Hence $\sigma_1(\Omega_{i,g})$ is a positive operator on $\mathcal{H}_{\sigma_1}$. That completes the proof of formula (2.7).

Let μ_G denote the image of the measure $\int \mu_\gamma\, d\bar{\mu}_N(\bar{\gamma})$ under the map $(\gamma, \sigma) \to \pi_{\gamma,\sigma}$. Then μ_G is a measure on $\hat{G}$ and we may rewrite (2.7) as

$$\int_G |\varphi(g)|^2\, dg = \int_{\hat{G}} \|\pi(\varphi)\|_2^2\, d\mu_G(\pi).$$

From this, it follows that μ_G is concentrated in the set $\hat{G}_l$. Indeed for fixed non-zero $\varphi \in C_0(G)$, the set $\{\pi \in \hat{G} : \|\pi(\varphi)\|_2 < \infty\}$ has complement of measure zero. But by [4, 6.7.2], $\|\pi(\varphi)\|_2 < \infty$ guarantees that $\pi \in \hat{G}_l$. Therefore μ_G is canonical [7, Lemma 6.3], and the corresponding decomposition of the left regular representation

$$\lambda_G = \int_{\hat{G}}^{\oplus} \pi \otimes 1_\pi\, d\mu_G(\pi) = \int_{\hat{N}/G}^{\oplus} \int_{(\widehat{G_\gamma/N})^{\overline{\omega}_\gamma}}^{\oplus} \pi_{\gamma,\sigma} \otimes 1_{\gamma,\sigma}\, d\mu_\gamma(\sigma)\, d\bar{\mu}_N(\bar{\gamma})$$

is the central decomposition. Hence λ_G is type I. This finishes the proof of Theorem 2.3.

Remarks 1. — It is clear from the proof that if $\bar{\mu}_N$ is replaced by an equivalent pseudo-image $c(\bar{\gamma})\, d\bar{\mu}_N(\bar{\gamma})$, $c(\bar{\gamma}) > 0$, then almost every μ_γ is altered by $c(\bar{\gamma})^{-1}$. Thus the Plancherel measure $\mu_G = \int_{\hat{N}/G} \mu_{\bar{\gamma}}\, d\mu_N(\bar{\gamma})$ is uniquely specified as a " fibered measure ".

2. We would like to comment here on the questions raised in [7, end of § 10]. Theorem 2.3 provides the precise Plancherel measure in the case of a unimodular group extension. This settles completely question 5. Theorem 2.3 also settles question 3 in the unimodular case. A somewhat similar result addressed to question 3 has already been stated by Mackey in [10, p. 323]. E. Carlton has shown that the assumption that G (ω) has a type I regular representation in the projective Plancherel theorem can be replaced by the assumption that G has a type I regular ω-representation. In fact we have used that in Theorem 2.3. This handles question 2. Question 1 is still open; but there is increasing evidence that in fact a CCR group must be unimodular. Finally we have not considered question 4 any further.

3. It often happens that there is a finite disjoint collection $\{\Omega_1, \ldots, \Omega_n\}$ of μ_N-measurable, G-invariant subsets of $\hat{N}$ such that $\bigcup_j \Omega_j$ has complement of measure zero, and sections $s_j : \Omega_j/G \to \Omega_j$ along which the stability groups and multipliers are constant, $G_{s_j(\bar{\gamma})} = H_j$, $\omega_{s_j(\bar{\gamma})} = \omega_j$, $\bar{\gamma} \in \Omega_j/G$. Then in the disintegration of measures, we can write

$$\int_{\hat{N}} f(\gamma)\, d\mu_N(\gamma) = \sum_j \int_{\Omega_j/G} \int_{G/H_j} f(s_j(\bar{\gamma})\cdot g)\, d\nu_j(\bar{g})\, d\bar{\mu}_N(\bar{\gamma})$$

because we can absorb any difference in the relatively invariant measures $\nu_{\bar{\gamma}}$, $\bar{\gamma} \in \Omega_j/G$, into the pseudo-image $\bar{\mu}_N$. Thus we also have that the projective Plancherel measures $\mu_{\bar{\gamma}}$, $\bar{\gamma} \in \Omega_j/G$, are constant along the section, $\mu_{\bar{\gamma}} = \mu_j$ say. Then the Plancherel formula takes the form

$$\int_G |\varphi(g)|^2\, dg = \sum_j \int_{\Omega_j/G} \int_{(\widehat{H_j/N})^{\bar{\omega}_\gamma}} \| \pi_{\gamma,\sigma}(\varphi) \|_2^2\, d\mu_j(\sigma)\, d\bar{\mu}_N(\bar{\gamma}).$$

For each j, the pseudo-image $\bar{\mu}_N|_{\Omega_j/G}$ is now uniquely determined to within a constant.

3. Examples. — In this section we use the results of paragraph 2 to compute explicitly the Plancherel measure of several different kinds of unimodular groups. We shall also discuss a non-unimodular example

which we feel is interesting, even though the discussion does not depend on paragraph 2.

(*a*) *The inhomogeneous Lorentz groups.* We computed this example in [7, § 10], although at that time, we could only specify the equivalence class of Plancherel measure. Recall that (in the notation of that section) the irreducible representations of $G_n = \mathbf{R}^{n+1}.H_n$ fall into four classes :

(i) $\pi_{0,\sigma}$, $\sigma \in \hat{H}_n$, the representations which are trivial on $\mathbf{R}^{n+1}$;
(ii) $\pi_{+,\sigma}$, $\sigma \in (MN)\hat{}$,
$\pi_{-,\sigma}$, $\sigma \in (MV)\hat{}$;
(iii) $\pi_{\pm r,\sigma}$, $\sigma \in \hat{K}$, $r > 0$;
(iv) $\pi_{\rho,\tau}$, $\tau \in \hat{H}_{n-1}$, $\rho > 0$.

In particular, within each class the stability groups are constant. The first two classes have Plancherel measure zero. It follows from the computations in [7, § 10, Example 2], Theorem 2.3, and Remark 3 of paragraph 2 that the Plancherel formula for G_n is

$$\int_{G_n} |\varphi(g)|^2 \, dg = c_1 \int_0^\infty \sum_{\sigma \in \hat{K}} (\| \pi_{+r,\sigma}(\varphi) \|_2^2 + \| \pi_{-r,\sigma}(\varphi) \|_2^2) \dim \sigma \, r^n \, dr$$
$$+ c_2 \int_0^\infty \int_{\hat{H}_{n-1}} \| \pi_{\tau,\rho}(\varphi) \|_2^2 \, d\mu_{H_{n-1}}(\tau) \, \rho^n \, d\rho.$$

The constants c_1, c_2 depend only on the dimension n and the normalization of Haar measure dg.

(*b*) *Cartan motion groups.* Let G be a connected semisimple Lie group having finite center. Fix a maximal compact subgroup K and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra. Then K acts on $\mathfrak{p}$ via the adjoint representation. Considering $\mathfrak{p}$ as a vector group, we define the Cartan motion group to be the semidirect product $H = \mathfrak{p}.K$. We apply the extension theory to the pair $(H, \mathfrak{p})$. First identify $\mathfrak{p}$ with its dual via the Killing form. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Choose a system of restricted roots Δ for $(\mathfrak{g}, \mathfrak{a})$ and let Σ denote a choice of positive roots. Finally, let $\mathfrak{a}^+$ denote the corresponding positive Weyl chamber $\mathfrak{a}^+ = \{ Y \in \mathfrak{a} : \alpha(Y) > 0, \text{ all } \alpha \in \Sigma \}$. Then it is known that $\bar{\mathfrak{a}}^+ = \{ Y \in \mathfrak{a} : \alpha(Y) \geq 0, \text{ all } \alpha \in \Sigma \}$ is a fundamental domain for the action of K on $\mathfrak{p}$. Moreover we have the integral formula

$$\int_{\mathfrak{p}} f(X) \, dX = \frac{1}{w} \int_{K/M} \int_{\mathfrak{a}^+} f(\operatorname{Ad} k \, Y) \, | \Pi_{\alpha \in \Sigma} \, \alpha(Y) | \, d\bar{k} \, dY, \tag{3.1}$$

where M is the centralizer of $\mathfrak{a}$ in K (M is also the centralizer in K of any $Y \in \mathfrak{a}^+$), dX, dY, $d\bar{k}$ are invariant measures chosen in a canonical fashion, and w is the order of the Weyl group $W = [\text{Norm}(\mathfrak{a}) \cap K]/M$. The reader can find this proven in [6, p. 380, 381]. By Theorem 2.3, the Plancherel formula for H becomes

$$\int_{H} |\varphi(h)|^2\, dh = c \int_{\mathfrak{a}^+} \sum_{\hat{\mathfrak{M}}} \| \pi_{Y,\sigma}(\varphi) \|_2^2 \dim \sigma \, | \Pi_{\alpha \in \Sigma}\, \alpha(Y) | \, dY$$

where c is a fixed constant computable in terms of the normalizations of dh and the measures in (3.1). Actually this result could have been obtained from [7, Theorem 4.4] which is of course a special case of Theorem 2.3. We apologize for omitting it at that time and we thank Cary Rader for bringing it to our attention. Note also that it is another specific example of the general situation considered in paragraph 2, Remark 3.

(c) *Weil representations.* Let $\mathbf{H} = \mathbf{H}_m$ be the group of $(m+2) \times (m+2)$ matrices of the form

$$\begin{bmatrix} 1 & x_1 & \dots & x_m & z \\ & 1 & & 0 & y_1 \\ & & \ddots & & \vdots \\ & 0 & & 1 & y_m \\ & & & & 1 \end{bmatrix}$$

with real entries. $\mathbf{H}$ is a two-step nilpotent group called the m^{th} order Heisenberg group. The center of $\mathbf{H}$ is the subgroup Z in which all entries except the z-component are zero. If Z is identified to $\mathbf{R}$, then the commutator operation induces a symplectic form B on $\mathbf{H}/Z$. Using this form, $\mathbf{H}$ may be realized as $\mathbf{R}^{2m} \times \mathbf{R}$ with multiplication

$$(v_1, z_1).(v_2, z_2) = (v_1 + v_2, z_1 + z_2 + \frac{1}{2} B(v_1, v_2)).$$

Now consider Sp (B), that is the group of linear transformations of $\mathbf{R}^{2m}$ leaving B invariant. This is a connected semisimple Lie group. Sp (B) also acts as a group of automorphisms of $\mathbf{H} \cong \mathbf{R}^{2m} \times \mathbf{R}$ by leaving fixed the second component, that is the center of $\mathbf{H}$. We wish to apply the Mackey theory and Theorem 2.3 to the group $G = \mathbf{H}.\text{Sp}(B)$.

It is well-known that the irreducible representations of $\mathbf{H}$ fall into two classes : those trivial on Z, i. e., the characters of $\mathbf{H}/Z \cong \mathbf{R}^{2m}$; and a one-parameter family π_γ, $\gamma \in \mathbf{R}^*$. π_γ is the unique infinite-dimensional irreducible representation whose restriction to Z acts via the non-trivial character $z \to e^{i\gamma z}$. Note that since Sp (B) leaves Z invariant, it follows that $\pi_\gamma . h$ and π_γ are equivalent for $h \in \mathrm{Sp}(\mathrm{B})$. In [5] ([2]) Duflo shows explicitly how to construct the extension π'_γ of π_γ to G. It is a projective representation [whose restriction to Sp (B) is the so-called Weil representation], and the order of the multiplier is 2. In fact, let Mp (B) be the metaplectic group, that is, the two-fold covering of Sp (B). Let $\psi : \mathrm{Mp}(\mathrm{B}) \to \mathrm{Sp}(\mathrm{B})$ denote the covering projection, $\mathrm{D} = \ker \psi$, $\# (\mathrm{D}) = 2$, and denote by $\psi^{-1} : \mathrm{Sp}(\mathrm{B}) \to \mathrm{Mp}(\mathrm{B})$ a Borel cross-section. Then Duflo constructs an ordinary representation Π_γ of Mp (B), non-trivial on D so that $\pi'_\gamma (h) = \Pi_\gamma (\psi^{-1}(h))$, $h \in \mathrm{Sp}(\mathrm{B})$. Let τ be any irreducible representation of Mp (B) which is not trivial on D. Then the collection of all irreducible representations of G whose restriction to **H** gives a multiple of π_γ is precisely

$$\pi_{\gamma, \sigma} = \pi'_\gamma \otimes \sigma''$$

where $\sigma''(nh) = \sigma(h)$, $n \in \mathbf{H}$, $h \in \mathrm{Sp}(\mathrm{B})$ and $\sigma(h) = \tau(\psi^{-1}(h))$, $h \in \mathrm{Sp}(\mathrm{B})$. The Plancherel formula is then

$$\int_{\mathrm{G}} |\varphi(g)|^2 \, dg = \int_{-\infty}^{\infty} \int_{\widehat{\mathrm{Mp}(\mathrm{B})'}} \| \pi_{\gamma, \sigma}(\varphi) \|_2^2 \, d\mu'(\sigma) \, |\gamma|^m \, d\gamma$$

where $\widehat{\mathrm{Mp}(\mathrm{B})}' = \{ \tau \in \mathrm{Mp}(\mathrm{B})^\wedge : \tau|_{\mathrm{D}} \neq 1 \}$ and μ' is the restriction of Plancherel measure on $\mathrm{Mp}(\mathrm{B})^\wedge$ to $\widehat{\mathrm{Mp}(\mathrm{B})}'$. Note that the irreducible representations of G trivial on Z, i. e., the representations of $\mathbf{R}^{2m}.\mathrm{Sp}(\mathrm{B})$ form a set of Plancherel measure zero here. Finally, we note that this example can be generalized to a great extent to the case of other locally compact fields.

(*d*) *Certain semidirect products.* We give a result which has application to certain kinds of solvable Lie groups.

THEOREM 3.1. — *Suppose* G *is a semidirect product* G = N.H *where :*

(1) N *is normal, unimodular, and type* I;

(2) H *is abelian*;

(3) H → Aut (N) *has co-compact kernel.*

Then N *is regularly embedded in* G, G *is unimodular and type* I (*actually* CCR *if* N *is* CCR) *and the Plancherel formula is*

$$\int_{\mathrm{N}} |\varphi(g)|^2 \, dg = \int_{\hat{\mathrm{N}}/\mathrm{H}} \int_{\hat{\mathrm{H}}_\gamma^{\bar{\omega}_\gamma}} \| \pi_{\gamma, \sigma}(\varphi) \|_2^2 \, d\mu_\gamma(\sigma) \, d\mu_{\mathrm{N}}(\gamma) \tag{3.2}$$

([2]) The referee has reminded us that Duflo's work is a generalization of [12], especially in this context.

where $\bar{\mu}_N$ *may be taken to be the image of* μ_N *and the projective Plancherel measures* μ_γ *can be specified explicitly.*

Proof. — Let D be the kernel of the homomorphism $H \to \mathrm{Aut}(N)$. By assumption H/D is compact. The group $G_1 = N.(H/D)$ is well defined and is an extension of N by a compact abelian group. Thus N is regularly embedded in G_1. Since $\hat{N}/G = \hat{N}/G_1$, it follows that N is also regularly embedded in G.

It is obvious that $(H/D)_\gamma = H_\gamma/D$. Let ω_γ be the Mackey obstruction to extending γ to a representation of $N.(H_\gamma/D)$. We also write ω_γ for the lift of ω_γ from H_γ/D to H_γ via the canonical projection $p : H_\gamma \to H_\gamma/D$. Now let γ_1' be the extension of γ to $N.(H_\gamma/D)$ as an ω_γ-representation. Setting $\gamma'(nh) = \gamma_1'(np(h))$, $n \in N$, $h \in H_\gamma$, we see that γ' is an ω_γ-representation of $N.H_\gamma$. Applying [9, Theorem 9.3] to the group extension $D \subseteqq H$ with multiplier $\bar{\omega}_\gamma$, we see (because the little groups are compact and $\hat{D}^{\bar{\omega}_\gamma} = \hat{D}$) that H_γ must have a type I $\bar{\omega}_\gamma$-dual. Therefore G is a type I group (again by [9, Theorem 9.3] applied to $N \subseteqq G$).

Regarding the unimodularity, we have by (2.4) :

$$\Delta_G(nh) = \Delta_H(h)\,\Delta_N(nh) = \Delta_H(h)\,\Delta_N(n)\,\Delta_N(h) = \Delta_N(h).$$

But the homomorphism $h \to \Delta_N(h)$, $H \to R_+^*$ has a kernel at least as big as D. Hence $\Delta_N(H) = \Delta_N(H/D)$ is compact, and so trivial. Therefore G is unimodular.

The irreducible representations of G are given by

$$\pi_{\gamma,\sigma} = \mathrm{Ind}_{N.H_\gamma}^{G}\, \gamma' \otimes \sigma'', \qquad \gamma \in \hat{N}/H, \quad \sigma \in \hat{H}_\gamma^{\bar{\omega}_\gamma}.$$

Since $D \subseteqq H_\gamma$, $G/N.H_\gamma \cong H/H_\gamma$ is compact. If $f \in C_0(N.H_\gamma)$, then

$$(\gamma' \otimes \sigma'')(f) = \int_{NH_\gamma} \gamma'(nh)\,\sigma(h)\,f(nh)\,dn\,dh = \int_{H_\gamma}\int_N \gamma(n)\,f(nh)\,dn\,\gamma'(h)\,\sigma(h)\,dh.$$

If γ is CCR, then $K_f(h) = \int_N \gamma(n)\,f(nh)\,dn$ is a compact operator. Then $h \to K_f(h)\,\gamma'(h)\,\sigma(h)$ is a continuous compact-operator-valued function on H_γ having compact support. Therefore $(\gamma' \otimes \sigma'')(f)$ is a compact operator. Since inducing from a co-compact subgroup preserves the CCR property, we see that N CCR ensures that G is CCR.

The Plancherel formula (3.2) follows immediately from Theorem 2.3. In addition, Theorem 2.3 and [7, Theorem 4.4] show that $\bar{\mu}_N$ is the image of μ_N and the measures μ_γ are obtained as follows. Fix Haar measure dy on D.

For each γ, choose normalized Haar measure on H_γ/D. Then there is a unique Haar measure ν_γ on H_γ such that

$$\int_{H_\gamma} f\, d\nu_\gamma = \int_{H_\gamma/D} \int_D f(yx)\, dy\, d\dot{x}.$$

This choice of ν_γ uniquely specifies the projective Plancherel measure μ_γ.

Remarks. — (1) This theorem applies in particular to the collection of simply connected almost algebraic solvable Lie groups which are type R and whose nilradical is regularly embedded. If the stability groups H_γ happen to be connected, the obstructions are trivial — since then the compact connected abelian groups H_γ/D can have no multipliers. In the general case, the obstructions are computed in [1].

(2) An interesting solvable Lie group considered by many is the so-called oscillator group [13]. We note that using (*d*), or a slight variant of (*c*), the Plancherel measure of this group can be computed easily. We leave the details to the interested reader.

Lastly we come to the non-unimodular example.

(*e*) *A non-type* I *group*. Mackey [10] has given an example of a non-type I group with a type I regular representation. We shall compute the Plancherel measure class of that group. Let H be the Mautner group, $H = \mathbf{C}^2.\mathbf{R}$ where $\mathbf{R}$ acts on $\mathbf{C}^2$ by

$$(z, w).x = (e^{2\pi i x} z, e^{2\pi i \lambda x} w), \qquad x \in \mathbf{R}, \quad z, w \in \mathbf{C},$$

λ a fixed irrational number. H is a simply connected type R solvable Lie group in which $\mathbf{C}^2$ is not regularly embedded (and so not type I [1, p. 129]). Let $\theta : H \to H/\mathbf{C}^2 = \mathbf{R}$ be the canonical projection.

Now H acts on the real line by : $y.h = e^{\theta(h)} y$, $y \in \mathbf{R}$, $h \in H$. Let $G = \mathbf{R}.H$ be the corresponding semidirect product. G is not unimodular and not type I, while the normal subgroup $\mathbf{R}$ is of course type I. There are three orbits for the action of H on $\hat{\mathbf{R}} \cong \mathbf{R}$, namely $\{(-\infty, 0), \{0\}, (0, \infty)\}$, so $\mathbf{R}$ is regularly embedded in G. At this point we choose a pseudo-image of $\mu_{\mathbf{R}}$ on $\hat{\mathbf{R}}/H$ be assigning measure 1 to the two intervals and 0 to the remaining one-point orbit. For the stability groups, we have

$$H_0 = H, \qquad H_{\pm 1} = \operatorname{Ker} \theta = \mathbf{C}^2,$$

and the multipliers are trivial. The first above occurs on a set of $\bar{\mu}_{\mathbf{R}}$-measure zero. Hence G has a type I regular representation and [7, Theorem 10.2] gives

$$\lambda_G = \int_{\hat{\mathbf{C}}^2}^{\oplus} \pi_{1,\sigma} \otimes 1_\sigma\, d\sigma \oplus \int_{\hat{\mathbf{C}}^2}^{\oplus} \pi_{-1,\sigma} \otimes 1_\sigma\, d\sigma$$

where 1_σ is the identity on $\mathcal{H}_{\pm 1, \sigma}$ = space of the induced representation $\mathrm{Ind}^{G}_{R.H_{\pm 1}} \pm 1\, \sigma$, and $d\sigma$ is Haar measure on $\hat{\mathfrak{C}}^2$.

4. ORTHOGONALITY RELATIONS. — All unexplained terminology and notation in the following can be found in [7, § 6]. However we observe that the results on quasi-Hilbert algebras from [7] which we use here are valid without any assumption of separability.

Let $\mathfrak{A}$ be a quasi-Hilbert algebra with a semi-finite left ring. Let t be a faithful normal semi-finite trace on $\mathcal{U}(\mathfrak{A})$; set $\mathfrak{n} = \mathfrak{n}_t$, the Hilbert-Schmidt operators with respect to t; and let M be the corresponding tracing operator. $\mathfrak{n}$ is a full Hilbert algebra [3. I, § 6, Theorem 1]. We denote by $\mathcal{D}^1_M$ the set of left bounded elements a in the domain of M for which $U_a \in \mathfrak{n}$. Then M is the closure of its restriction to $\mathcal{D}^1_M$ and the map $M\, a \to U_a$ of $\mathcal{D}^1_M$ into $\mathfrak{n}$ extends to a unitary map Υ of $\mathcal{H}_{\mathfrak{A}}$ onto $\mathcal{H}_{\mathfrak{n}}$ which carries $\mathcal{U}(\mathfrak{A})$ onto $\mathcal{U}(\mathfrak{n})$.

In the following, if $\mathcal{K}$ is a subspace of $\mathcal{H}_{\mathfrak{A}}$ (or of $\mathcal{H}_{\mathfrak{n}}$), then $\mathcal{K}^{\ell}$ and $\mathcal{K}^{r}$ will denote respectively the left and right bounded elements in $\mathcal{K}$. Our first order of business is to show that $\mathcal{D}^1_M = \mathcal{D}^{\ell}_M$.

LEMMA 4.1. — $\mathcal{D}^{\ell}_M = \mathcal{D}^1_M$; *that is, if* $a \in \mathcal{D}_M$ *is left bounded then* $U_a \in \mathfrak{n}$.

Proof. — Let $a \in \mathcal{D}^{\ell}_M$. Assume for the moment that we can find $a_n \in \mathcal{D}^1_M$ such that $a_n \to a$, $M\, a_n \to M\, a$ and $\sup_n \| U_{a_n} \| < \infty$. Suppose that $t(U^*_a U_a) = +\infty$. Since $M\, a_n \to M\, a$, it must be true that $\Upsilon\, M\, a_n \to \Upsilon\, M\, a$ in $\mathcal{H}_{\mathfrak{n}}$. This implies in particular that there is a constant $c > 0$ such that $t(U^*_{a_n} U_{a_n}) = \| \Upsilon\, M\, a_n \|^2 < c$, all n. Now by [3, Proposition 2, Corollary I, § 6], we can find $\{ x_i \}_{i \in I}$ in $\mathcal{H}_{\mathfrak{A}}$ such that

$$t(T^* T) = \sum_{i \in I} \| T\, x_i \|^2, \qquad T \in \mathcal{U}(\mathfrak{A}).$$

Hence there is a finite subset $I_0 \subsetneqq I$ such that

$$t(U^*_a U_a) \geqq \sum_{i \in I_0} \| U_a\, x_i \|^2 > c.$$

Next if $y \in \mathcal{H}^{r}_{\mathfrak{A}}$, then

$$U_{a_n} y = V_y\, a_n \to V_y\, a = U_a\, y.$$

By the uniform boundedness of the U_{a_n}, we conclude that $U_{a_n} \to U_a$ strongly on $\mathcal{H}_{\mathfrak{A}}$. Hence from the finiteness of I_0, it follows that for all sufficiently large n,

$$\sum_{i \in I_0} \| U_{a_n}\, x_i \|^2 > c.$$

This is a contradiction.

It remains to show that for $a \in \mathcal{D}_M^{\mathcal{E}}$ there exists $a_n \in \mathcal{D}_M^1$ such that $a_n \to a$, $M a_n \to M a$ and $\sup_n \| U_{a_n} \| < \infty$. Consider the set

$$\mathfrak{s} = \{ U_a : a \in \mathcal{D}_M^1 \cap \mathcal{D}_{\Lambda^{-1}} \}.$$

While $\mathfrak{s}$ is not an algebra, it does follow from [2, Lemma 6] that $\mathfrak{s} = \mathfrak{s}^*$. Next consider the algebra

$$\mathfrak{q} = \{ U_a : a \in \mathcal{D}_M^1 \}.$$

Letting $\mathfrak{r}$ be the subalgebra of $\mathfrak{u}$ generated by $\mathfrak{s}$, we see that $\mathfrak{s} \subseteq \mathfrak{r} \subseteq \mathfrak{q}$ and so $\mathfrak{r}$ is a Hilbert subalgebra of $\mathfrak{u}$. By [2, Lemma 18], $\mathfrak{r}$ is dense. Hence by [2, p. 299], $\mathfrak{r}$ contains a bounded approximate identity $\{ E_i \}_{i \in I}$. Then $E_i = U_{e_i}$, $e_i \in \mathcal{D}_M^1$, and we set $a_i = U_{e_i} a$. Clearly $a_i \to a$. Also, since M is affiliated with $\mathcal{V}(\mathfrak{A})$, we have $M a_i = U_{e_i} M a \to M a$. Finally

$$\| U_{a_i} \| = \| U_{e_i} U_a \| \leq \sup_i \| E_i \| . \| U_a \| < \infty.$$

This completes the proof.

Remark. — This lemma (together with the fact that $\Delta^{-1/2} f \in L_1$ guarantees that f is left bounded [7, 6.2 *a*]) is required for the inclusion $\mathcal{D}_M^2 \subseteq \mathcal{D}_M^1$ that appears in the proof of [7, Theorem 6.4].

We continue the discussion with a

Definition. — *A net* $\{ e_i \}$ *in* $\mathfrak{A}$ *is called a traceable approximate identity if :*

(1) $U_{e_i} \to 1$, $V_{\Lambda^{-1} e_i} \to 1$ *strongly;*

(2) $\sup \| U_{e_i} \| < \infty$, $\sup \| V_{\Lambda^{-1} e_i} \| < \infty$;

(3) $\{ e_i \} \subseteq \mathcal{D}_M$.

Lemma 4.2. — (i) $\mathfrak{A}$ *has a traceable approximate identity if and only if* $\mathcal{D}_M \cap \mathfrak{A}$ *is dense in* $\mathcal{H}_{\mathfrak{A}}$.

(ii) *If* $\{ e_i \}$ *is a traceable approximate identity, then* $\{ U_{e_i} \}$ *is a bounded approximate identity for* $\mathfrak{u}$ (*in the sense of* [11, 2.11]).

Proof. — We first observe that $\mathcal{D}_M \cap \mathfrak{A} \subseteq \mathcal{D}_M^{\mathcal{E}} = \mathcal{D}_M^1$. Now suppose that $\mathcal{D}_M \cap \mathfrak{A}$ is dense. Because M is invertible and self-adjoint, it follows that $\{ M a : a \in \mathcal{D}_M \cap \mathfrak{A} \}$ is dense. Applying Υ, we see that

$$\mathfrak{p} = \{ U_a : a \in \mathcal{D}_M \cap \mathfrak{A} \}$$

is dense in $\mathcal{H}_{\mathfrak{u}}$. If $U_a, U_b \in \mathfrak{p}$, then $U_{ab} = U_a U_b \in \mathfrak{u}$ which implies $ab \in \mathcal{D}_M$. Thus $\mathfrak{p}$ is an algebra. Also $U_a^* = U_{\Lambda J a} \in \mathfrak{u}$ and $\Lambda J a \in \mathfrak{A}$, hence $U_a^* \in \mathfrak{p}$. Therefore $\mathfrak{p}$ is a dense Hilbert subalgebra of $\mathfrak{u}$. By [2, p. 299], $\mathfrak{p}$ contains a bounded approximate identity; that is, a net $\{ E_i \}$ such that $E_i T \to T$ and $T E_i \to T$, for all $T \in \mathfrak{p}$, and such that $\sup \| U(E_i) \| < \infty$,

$\sup \| V(E_i) \| < \infty$. From the uniform bounds on the norms of $U(E_i)$ and $V(E_i)$, we conclude that $U(E_i) \to 1$, $V(E_i) \to 1$ strongly on $\mathcal{H}_{\mathfrak{n}}$.

Now we may write $E_i = U_{e_i}$, for some $e_i \in \mathcal{D}_M \cap \mathfrak{A} \subseteq \mathcal{D}_M^e$. By [7, Lemma 6.2 (i)], we have $U(U_{e_i}) = \Upsilon U_{e_i} \Upsilon^{-1}$. Hence $U_{e_i} \to 1$ strongly and $\sup \| U_{e_i} \| = \sup \| U(E_i) \| < \infty$. By [7, Lemma 6.2 (ii)] and the equations $U_{e_i}^* = U_{\Lambda J e_i}$, $J \Lambda J = \Lambda^{-1}$, we have

$$V(U_{e_i}) = V(U^*_{\Lambda J e_i}) = \Upsilon (V_{J \Lambda J e_i}) \Upsilon^{-1} = \Upsilon V_{\Lambda^{-1} e_i} \Upsilon^{-1}.$$

Hence $V_{\Lambda^{-1} e_i} \to 1$ strongly and $\sup \| V_{\Lambda^{-1} e_i} \| = \sup \| V(E_i) \| < \infty$. The conclusion is that $\{ e_i \}$ is a traceable approximate identity.

Conversely, suppose $\{ e_i \}$ is a traceable approximate identity. Then for all $a \in \mathfrak{A}$, $e_i a \to a$. Since $U_{e_i a} = U_{e_i} U_a \in \mathfrak{n}$, we conclude that $e_i a \in \mathcal{D}_M$. Therefore $\mathcal{D}_M \cap \mathfrak{A}$ is dense in $\mathfrak{A}$, and so also in $\mathcal{H}_{\mathfrak{A}}$. That completes the proof of (i). In the course of the proof we saw that if $\{ e_i \}$ is a traceable approximate identity, then by setting $E_i = U_{e_i}$ we have

$$U(E_i) \to 1, \quad V(E_i) \to 1 \quad \text{strongly on } \mathcal{H}_n$$

and $\sup \| U(E_i) \| < \infty$, $\sup \| V(E_i) \| < \infty$. Since $E_i \in \mathfrak{p} \subseteq \mathfrak{n}$, this says in particular that $\{ E_i \}$ is a bounded approximate identity for $\mathfrak{p}$ or $\mathfrak{n}$.

In what follows we shall always assume that $\mathfrak{A}$ has a traceable approximate identity. If $\mathfrak{A}$ is full, that is automatically the case [7, § 6].

Definition. — *A representation* π *of* $\mathfrak{A}$ *is an algebra homomorphism of* $\mathfrak{A}$ *into* $\mathcal{L}(\mathcal{H}_\pi)$ *such that :*

(1) $\pi((x^*)^\vee) = \pi(x)^*$, $x \in \mathfrak{A}$;

(2) *there is a traceable approximate identity* $\{ e_i \}$ *such that* $\pi(e_i) \to 1$ *strongly.*

Set $\mathfrak{A}_1 = \mathcal{D}_M \cap \mathfrak{A}$ and $\mathfrak{p} = \{ U_a : a \in \mathfrak{A}_1 \}$ as above. If π is a representation of $\mathfrak{A}$, then we set

$$\pi^0(U_a) = \pi(a), \qquad a \in \mathfrak{A}_1.$$

It follows from Lemma 4.2 and the above definition that π^0 is a representation of the Hilbert algebra $\mathfrak{p}$ (in the sense of [11, 4.1]).

Lemma 4.3. — *Let* π_1, π_2 *be representations of* $\mathfrak{A}$ *which map the same traceable approximate identity into a net converging strongly to* 1. *Then an operator* T *intertwines* π_1 *and* π_2 *of and only if it intertwines* π_1^0 *and* π_2^0. *In particular :*

(i) π_1 *is irreducible if and only if* π_1^0 *is irreducible;*

(ii) $\pi_1 \simeq \pi_2$ *if and only if* $\pi_1^0 \simeq \pi_2^0$;

(iii) π_1 *is a subrepresentation of the left regular representation* $a \to U_a$ *of* $\mathfrak{A}$ *if and only if* π_1^0 *is a subrepresentation of the left regular representation of* $\mathfrak{p}$.

Proof. — Note that $\mathfrak{A}_1$ is a right (even two-sided) ideal in $\mathfrak{A}$. Let $\{e_i\}$ be a traceable approximate identity such that $\pi_j(e_i) \to 1$, $j = 1, 2$. If $a \in \mathfrak{A}$, then $\pi_j(e_i a) = \pi_j(e_i)\pi_j(a) \to \pi_j(a)$, strongly for $j = 1, 2$.

Now suppose T intertwines π_1^0 and π_2^0, $T\pi_1^0(U_a) = \pi_2^0(U_a)T$, $a \in \mathfrak{A}_1$. Then

$$\begin{aligned} T\pi_1(a) &= \lim T\pi_1(e_i a) = \lim T\pi_1^0(U_{e_i a}) \\ &= \lim \pi_2^0(U_{e_i a})T = \lim \pi_2(e_i a)T = \pi_2(a)T, \qquad a \in \mathfrak{A}. \end{aligned}$$

Conversely, if T intertwines π_1 and π_2, then

$$T\pi_1^0(U_a) = T\pi_1(a) = \pi_2(a)T = \pi_2^0(U_a)T, \qquad a \in \mathfrak{A}_1.$$

Statements (i) and (ii) follow immediately from the preceding. Statement (iii) follows once we observe that if U is the left regular representation of $\mathfrak{A}$, then U^0 is equivalent to the left regular representation of $\mathfrak{p}$. This is because, $U^0(U_a) = U_a = \Upsilon^{-1}U(U_a)\Upsilon$, $a \in \mathfrak{A}_1$ [7, Lemma 6.2 (i)].

Let π be a representation of $\mathfrak{A}$. To continue with the program of adopting Rieffel's results [11] to quasi-Hilbert algebras, we introduce the following notion.

Definition. — $\xi \in \mathcal{H}_\pi$ *is called quasi square-integrable if there exists* $r_\xi \in \mathfrak{A}_{\mathfrak{A}}$ *such that* $(\pi(a)\xi, \xi) = (M a, r_\xi)$, $a \in \mathfrak{A}_1 = \mathcal{D}_M \cap \mathfrak{A}$.

By the definitions of π^0 and Υ, we see that $\xi \in \mathcal{H}_\pi$ is quasi square-integrable for π if and only if

$$(\pi^0(U_a)\xi, \xi) = (\pi(a)\xi, \xi) = (M a, r_\xi) = (U_a, \Upsilon r_\xi), \qquad U_a \in \mathfrak{p},$$

that is, if and only if ξ is square-integrable for π^0 [11, 4.3].

We shall also say that π is in the *discrete series* of $\mathfrak{A}$ if it is equivalent to an irreducible subrepresentation of the left regular representation of $\mathfrak{A}$.

Lemma 4.4. — (i) *Let* π *be a cyclic representation of* $\mathfrak{A}$. *Then* π *is equivalent to a subrepresentation of the left regular representation if and only if it has a quasi square-integrable cyclic vector.*

(ii) *Let* π *be an irreducible representation of* $\mathfrak{A}$. *The following are equivalent :*

(*a*) π *is in the discrete series;*

(*b*) π *has a non-zero quasi square-integrable vector;*

(*c*) *every vector in* $\mathcal{H}_\pi$ *is quasi square-integrable.*

Proof. — Because there exists an approximate identity in $\mathfrak{A}_1 = \mathcal{D}_M \cap \mathfrak{A}$, it follows as in the proof of Lemma 4.3 that $\pi(\mathfrak{A})' = \pi(\mathfrak{A}_1)'$. Therefore we obtain $\pi(\mathfrak{A})'' = \pi(\mathfrak{A}_1)'' = \pi^0(\mathfrak{p})''$. This implies that a cyclic vector for π is a cyclic vector for π^0, and conversely. Therefore (i) is a consequence of Lemma 4.3 and [11, 4.6]. Statement (ii) follows from Rieffel's result [11, 5.6] applied to π^0 and then transfered to π by Lemma 4.3.

Consider next a representation π in the discrete series of $\mathfrak{A}$. Then π^0 is in the discrete series of $\mathfrak{p}$. Rieffel [11, 6.3, 6.4] has shown that for each ξ, $\eta \in \mathcal{H}_\pi$, there exists a representative element $R_{\xi,\eta} \in \mathcal{H}_\mathfrak{u}$ such that

$$(\pi^0(U_a)\xi, \eta) = (U_a, R_{\xi,\eta}), \qquad a \in \mathfrak{A}_1.$$

Let $r_{\xi,\eta} = \Upsilon^{-1} R_{\xi,\eta}$. Then we have

$$(\pi(a)\xi, \eta) = (\pi^0(U_a)\xi, \eta) = (U_a, R_{\xi,\eta}) = (M a, r_{\xi,\eta}), \qquad a \in \mathfrak{A}_1.$$

The $r_{\xi,\eta}$ are called *representative elements;* note $r_{\xi,\xi} = r_\xi$ defined previously.

LEMMA 4.5. — (i) *If r and r' are representative elements belonging to inequivalent discrete series representations, then* $(r, r') = 0$.

(ii) *If $r_{\xi,\eta}$ and $r_{\xi',\eta'}$ are representative elements belonging to the same discrete series representation π, then*

$$(r_{\xi,\eta}, r_{\xi',\eta'}) = \frac{1}{d}\,(\overline{\xi, \xi'})\,(\eta, \eta')$$

where $d > 0$ is a constant depending only on π (and on M).

In view of Lemma 4.3 and the definition of representative elements, this follows immediately from [11, 6.6 and 6.8].

Although we shall not have explicit use for it later, we feel it is of interest to see more precisely how the « formal degree » d depends on M. The following result is addressed to that question.

LEMMA 4.6. — *Let π be a discrete series representation. Then there exists a minimal closed left invariant subspace $\mathcal{K} \subseteqq \mathcal{H}_\mathfrak{A}$ and an element $e \in \mathcal{D}_M^c$ such that :*

(i) *π is equivalent to the subrepresentation of the left regular representation on $\mathcal{K}$;*

(ii) *V_{e*} is the projection of $\mathcal{H}_\mathfrak{A}$ onto $\mathcal{K}$;*

(iii) $d = \| M e \|^2$.

Proof. — By definition we know that there is a minimal closed left invariant subspace $\mathcal{K}_0$ of $\mathcal{H}_\mathfrak{A}$ such that π is equivalent to the subrepresentation of the left regular representation on $\mathcal{K}_0$. Let $\mathcal{B}$ be the closed bi-invariant subspace of $\mathcal{H}_\mathfrak{A}$ generated by $\mathcal{K}_0$. Because Υ maps $\mathfrak{U}(\mathfrak{A})$

onto $\mathcal{U}(\mathfrak{n})$ and $\mathcal{V}(\mathfrak{A})$ onto $\mathcal{V}(\mathfrak{n})$, it follows that $\Upsilon\,\mathcal{K}_0$ (respectively $\Upsilon\,\mathcal{B}$) is a minimal closed left (respectively bi-) invariant subspace of $\mathcal{H}_{\mathfrak{n}}$. In fact $\Upsilon\,\mathcal{K}_0$ is a minimal left ideal in $\mathfrak{n}$ [11, 5.2 and 5.5] and $\Upsilon\,\mathcal{B}$ is a minimal two-sided ideal in $\mathfrak{n}$ [11, 5.7 and 5.14].

Define $\mathfrak{q} \subseteqq \mathfrak{n}$ by

$$\mathfrak{q} = \{ U_a : a \in \mathcal{D}_M^{\mathcal{C}} \}.$$

It follows from [2, p. 293, 294] that $\mathfrak{q}$ is a left ideal in $\mathfrak{n}$. Now let P be the projection of $\mathcal{H}_{\mathfrak{n}}$ onto $\Upsilon\,\mathcal{B}$. Clearly, $P\,\mathfrak{n} \subseteqq \mathfrak{n}$. In fact, we have $P\,\mathfrak{q} \subseteqq \mathfrak{q}$. For if $a \in \mathcal{D}_M^{\mathcal{C}}$, then $P(U_a) = P\,\Upsilon\,M\,a = \Upsilon\,P_1\,M\,a$, where P_1 is the projection of $\mathcal{H}_{\mathfrak{A}}$ onto $\mathcal{B}$. Since $P_1 \in \mathcal{U}(\mathfrak{A})$ and M is affiliated with $\mathcal{V}(\mathfrak{A})$, $P_1\,a \in \mathcal{D}_M$ and $P_1\,M\,a = MP_1\,a$. Also, by [2, p. 293, 294], $b = P_1\,a \in \mathcal{D}_M^{\mathcal{C}}$. Therefore $P(U_a) = \Upsilon\,M\,b = U_b \in \mathfrak{A}$.

Next we claim that $P\,\mathfrak{q}$ is a left ideal in $\mathfrak{n}$. This follows also from [2, p. 293, 294], for if $T \in \mathfrak{n}$, $a \in \mathcal{D}_M^{\mathcal{C}}$, then $TP(U_a) = TU_b = U_{Tb} \in \mathfrak{q}$, $b = P_1\,a \in \mathcal{D}_M^{\mathcal{C}}$. By [11, 2.3], $P\,\mathfrak{q}$ contains a non-zero self-adjoint idempotent E of $\mathfrak{n}$. However since $P\,\mathfrak{q}$ is contained in the minimal two-sided ideal $\Upsilon\,\mathcal{B}$ [11, 5.9] guarantees that E is a finite orthogonal sum of minimal self-adjoint idempotents, $E = E_1 + \ldots$. Then $E_1 = E_1\,E \in P\,\mathfrak{q}$. The minimal left ideal containing E_1, namely $\mathfrak{n}\,E_1$ is therefore contained in $P\,\mathfrak{q} \subseteqq \mathfrak{q}$. Hence $E_1 = U_e$, $e \in \mathcal{D}_M^{\mathcal{C}}$.

Let $\mathcal{K}$ be the minimal closed left invariant subspace of $\mathcal{H}_{\mathfrak{A}}$ such that $\Upsilon\,\mathcal{K} = \mathfrak{n}\,E_1$. Since $\Upsilon\,\mathcal{K}_0$ and $\mathfrak{n}\,E_1$ are minimal left ideals of $\mathfrak{n}$ contained in the same minimal two-sided ideal, the corresponding subrepresentations of the left regular representation of $\mathfrak{n}$ are equivalent. Statement (i) follows easily from this. Next according to [11, 6.1], we have $d = \| E_1 \|^2 = \| U_e \|^2 = \| \Upsilon\,M\,e \|^2 = \| M\,e \|^2$, thus proving (iii). Finally, $V(E_1)$ is the projection of $\mathcal{H}_{\mathfrak{n}}$ to $\mathfrak{n}\,E_1$. Therefore by [7, Lemma 6.2 ii] $\Upsilon^{-1}\,V(E_1)\,\Upsilon = \Upsilon^{-1}\,V(E_1^*)\,\Upsilon = V_{e*}$ is the projection of $\mathcal{H}_{\mathfrak{A}}$ onto $\mathcal{K}$. This proves (ii).

Remark. — We know that M is uniquely determined up to multiplication by a positive invertible self-adjoint operator affiliated with $\mathcal{U}(\mathfrak{A}) \cap \mathcal{V}(\mathfrak{A})$. Therefore M is uniquely determined up to a scalar on $\mathcal{B}$ (and so on $\mathcal{K}$). Lemma 4.7 shows that $\left.\frac{M}{\sqrt{d}}\right|_{\mathcal{B}}$ is uniquely specified (independent of the choice of M).

In order to proceed, we now fix a minimal closed left invariant subspace $\mathcal{K}$ of $\mathcal{H}_{\mathfrak{A}}$ and the minimal closed bi-invariant subspace $\mathcal{B}$ it generates. For x, $y \in \mathcal{H}$, we have seen that there is a unique (representative) element $r_{x,y} \in \mathcal{H}_{\mathfrak{A}}$ satisfying

$$(\mathrm{U}_a\, x, y) = (\mathrm{M}\, a, r_{x,y}), \qquad a \in \mathfrak{A}_1 = \mathcal{D}_{\mathrm{M}} \cap \mathfrak{A}.$$

But the left side of this equation makes sense for $x, y \in \mathcal{H}_{\mathfrak{A}}$; and so we wish to see if $r_{x,y}$ can be defined for more general x, y. Let $x \in \mathcal{H}_{\mathfrak{A}}$ be such that $\Upsilon\, x \in \mathfrak{u}$ (for example if $x \in \mathcal{B}$). Since $\mathfrak{u}$ is full, $\Upsilon\, x$ is then right bounded. Then for $y \in \mathcal{H}_{\mathfrak{A}}$, $a \in \mathcal{D}_{\mathrm{M}}^{\mathcal{C}}$, we can compute.

$$\begin{aligned}(\mathrm{U}_a\, x. y) &= (\Upsilon\, \mathrm{U}_a\, x, y) = (\mathrm{U}\, (\mathrm{U}_a)\, \Upsilon\, x, \Upsilon\, y) \\ &= (\mathrm{V}\, (\Upsilon\, x)\, \mathrm{U}_a, \Upsilon\, y) = (\mathrm{U}_a, \mathrm{V}\, (\Upsilon\, x)^*\, \Upsilon\, y) \\ &= (\Upsilon\, \mathrm{M}\, a, \mathrm{V}\, (\Upsilon\, x)^*\, \Upsilon\, y) = (\mathrm{M}\, a, \Upsilon^{-1}\, (\mathrm{V}\, (\Upsilon\, x)^*\, \Upsilon\, y)).\end{aligned}$$

Thus for x, $y \in \mathcal{H}_{\mathfrak{A}}$, $\Upsilon\, x \in \mathfrak{u}$, if we set

$$r_{x,y} = \Upsilon^{-1}\, (\mathrm{V}\, (\Upsilon\, x)^*\, \Upsilon\, y), \tag{4.1}$$

then we have

$$(\mathrm{U}_a\, x, y) = (\mathrm{M}\, a, r_{x,y}), \qquad a \in \mathcal{D}_{\mathrm{M}}^{\mathcal{C}}.$$

Now let $x \in \mathcal{B}$, $y \in \mathcal{H}_{\mathfrak{A}}$. Then $\Upsilon\, x \in \mathfrak{u}$ and so $r_{x,y}$ is defined. Note also that since M′ is affiliated with $\mathcal{U}\,(\mathfrak{A})$, M′ commutes with the projection onto $\mathcal{B}$. Hence $\mathcal{D}_{\mathrm{M}'} \cap \mathcal{B}$ is dense in $\mathcal{B}$ and $\mathrm{M}'\, (\mathcal{D}_{\mathrm{M}'} \cap \mathcal{B}) \subseteq \mathcal{B}$. With this in mind, we state

Lemma 4.7. — *Let* $x \in \mathcal{B}$, $y \in \mathcal{H}_{\mathfrak{A}}$.

(i) *If x is right bounded, then $r_{x,y} \in \mathcal{D}_{\mathrm{M}}$ and* $\mathrm{M}\, r_{x,y} = \mathrm{V}_x^*\, y$.

(ii) *If* $x \in \mathcal{D}_{\mathrm{M}'}^{r}$ *then*

$$r_{\mathrm{M}' x, y} = \mathrm{M}\, r_{x,y}.$$

Proof. — (i) If x is right bounded, then by [2, Lemma 24] we have

$$(\mathrm{M}\, a, r_{x,y}) = (\mathrm{U}_a\, x, y) = (\mathrm{V}_x\, a, y) = (a, \mathrm{V}_x^*\, y), \qquad a \in \mathcal{D}_{\mathrm{M}}^{\mathcal{C}}.$$

Since M is self-adjoint and equal to the closure of its restriction to $\mathcal{D}_{\mathrm{M}}^{\mathcal{C}}$, the conclusion of (i) follows.

(ii) This is somewhat more delicate. First of all, by the comments prior to the statement of the lemma, $r_{\mathrm{M}' x, y}$ exists. To begin, we assume in addition to $x \in \mathcal{D}^r_{M'}$ that $x \in \mathcal{D}_\Lambda$, $M' x$ is right bounded and y is left bounded. By (i) $M\, r_{M'x,y} = V^*_{M'x}\, y$. Then by [2, Lemmas 6 and 24];

$$\begin{aligned} r_{M'x,y} &= M^{-1} V^*_{M'x}\, y \\ &= M^{-1} V_{J\Lambda M'x}\, y \\ &= M^{-1} U_y\, J\, \Lambda\, M' x \\ &= M^{-1} U_y\, MJ\, \Lambda\, x \\ &= U_y\, J\, \Lambda\, x \\ &= V_{J\Lambda x}\, y \\ &= V^*_x\, y \\ &= M\, r_{x,y}. \end{aligned}$$

We have also used here the facts that M' and Λ commute, $JM'J = M$, and that M [which is affiliated with $\mathcal{V}(\mathfrak{A})$] commutes with U_y.

In order to finish we have to remove the restrictions : y left bounded, $x \in \mathcal{D}_\Lambda$ and $M' x$ right bounded. If y is arbitrary, we choose y_n left bounded such that $y_n \to y$. It is obvious from the definition (4.1) that $r_{M'x,y_n} \to r_{M'x,y}$. Moreover since $M\, r_{x,y_n} = V^*_x\, y_n$ and $M\, r_{x,y} = V^*_x\, y$, it is clear that $M\, r_{x,y_n} \to M\, r_{x,y}$. Hence we have $r_{M'x,y} = M\, r_{x,y}$ for any $y \in \mathcal{H}_{\mathfrak{A}}$. Next let

$$\mathcal{H}' = \{ x \in \mathcal{D}^r_{M'} \cap \mathcal{D}_\Lambda : M' x \text{ is right bounded} \}. \tag{4.2}$$

Suppose we knew that M' is the closure of its restriction to $\mathcal{H}'$. Then for $x \in \mathcal{D}^r_{M'}$ we could choose $x_n \in \mathcal{H}'$ such that $x_n \to x$ and $M' x_n \to M' x$. Then

$$(M\, a, r_{M'x,y}) = (U_a\, M' x, y) = \lim (U_a\, M' x_n, y) = \lim (M\, a, r_{M'x_n,y}), \qquad a \in \mathcal{D}^{\ell}_{M}.$$

Since the set $M\mathcal{D}^{\ell}_{M}$ is dense in $\mathcal{H}_{\mathfrak{A}}$, we conclude that $r_{M'x_n,y} \to r_{M'x,y}$. Also

$$\begin{aligned} (a, M\, r_{x,y}) &= (M\, a, r_{x,y}) = (U_a\, x, y) = \lim (U_a\, x_n, y) \\ &= \lim (M\, a, r_{x_n,y}) = \lim (a, M\, r_{x_n,y}), \qquad a \in \mathcal{D}^{\ell}_{M}. \end{aligned}$$

Hence it also follows that $M\, r_{x_n,y} \to M\, r_{x,y}$. Since for $x_n \in \mathcal{H}'$ we know that

$$r_{M'x_n,y} = M\, r_{x_n,y}$$

the proof would be complete.

It remains to show that M' is the closure of its restriction to $\mathcal{H}'$. Now by [2, Lemma 20], M is the closure of its restriction to $\mathcal{D}^{\ell}_{M} \cap \mathcal{D}_{\Lambda^{-1}}$. Therefore $M' = JMJ$ is the closure of its restriction to $J(\mathcal{D}^{\ell}_{M} \cap \mathcal{D}_{\Lambda^{-1}}) = \mathcal{D}^r_{M'} \cap \mathcal{D}_\Lambda$ [2, Lemma 5]. That is, given $x \in \mathcal{D}_{M'}$ we can find $x_n \in \mathcal{D}^r_{M'} \cap \mathcal{D}_\Lambda$ such that $x_n \to x$ and $M' x_n \to M' x$.

Finally let $M' = \int_0^\infty \alpha \, dF_\alpha$ be the spectral resolution of M'. Consider $y_n = F_n x_n$. Since F_α commutes with M' and Λ, we have that $y_n \in \mathcal{D}_{M'} \cap \mathcal{D}_\Lambda$. By [2, Lemma 7 b] the elements y_n and $M' y_n$ are right bounded. That is $y_n \in \mathcal{H}'$. Also $y_n \to x$ and $M' y_n \to M' x$. This completes the proof.

At last we are going to apply our results to groups. Let G be a locally compact group having a semifinite left regular representation $\lambda = \lambda_G$. Recall that $C_0(G)$ is a quasi-Hilbert algebra with the operations

$$(f \star h)(x) = \int_G f(xy^{-1})\, h(y)\, dy,$$

$$f^*(x) = \Delta(x)^{-1/2} \bar{f}(x^{-1}),$$

$$f^\vee(x) = \Delta(x)^{-1/2} f(x)$$

and inner product

$$(f, h) = \int_G f(x)\, \bar{h}(x)\, dx.$$

Let $\mathfrak{A}_f(G)$ be the fulfillment of $C_0(G)$; $\mathfrak{A}_f(G) = \{ f \in L_2(G) : \Delta^{n/2} f \in L_2(G)^\mathcal{L}$ for all $n \in \mathbf{Z} \}$. Since it is full, $\mathfrak{A}_f(G)$ has a traceable approximate identity. It is clear that the left (respectively right) G-invariant subspaces of $L_2(G)$ correspond precisely to the left (respectively right) $\mathfrak{A}_f(G)$-invariant subspaces of $L_2(G)$. Moreover if $\mathcal{K}$ is a (minimal) closed left invariant subspace of $L_2(G)$, then the subrepresentation of the left regular representation of $\mathfrak{A}_f(G)$ on $\mathcal{K}$ is a representation of the quasi-Hilbert algebra in the sense that we have defined.

THEOREM 4.8. — (i) *Let $\mathcal{K}$ be a minimal closed left invariant subspace of $L_2(G)$. Then there is a constant $d > 0$ (depending only on $\mathcal{K}$ and M) such that for $f_i \in \mathcal{K}$, $h_i \in \mathcal{K} \cap \mathcal{D}_{M'}$, $i = 1, 2$,*

$$\int_G (\lambda(x) f_1, h_1)\, \overline{(\lambda(x) f_2, h_2)}\, dx = \frac{1}{d} (f_1, f_2)\, \overline{(M' h_1, M' h_2)}.$$

(ii) *Let $\mathcal{K}_1$, $\mathcal{K}_2$ be two minimal closed left invariant subspaces of $L_2(G)$ which generate distinct bi-invariant subspaces. Then for $f_i \in \mathcal{K}_i$, $h_i \in \mathcal{K}_i \cap \mathcal{D}_{M'}$, $i = 1, 2$,*

$$\int_G (\lambda(x) f_1, h_1)\, \overline{(\lambda(x) f_2, h_2)}\, dx = 0.$$

Proof. — We first remark that since M' is affiliated with $\mathcal{U}(\mathfrak{A})$, M' commutes with the projection onto a minimal closed left invariant subspace $\mathcal{K}$. Hence $\mathcal{K} \cap \mathcal{D}_{M'}$ is dense in $\mathcal{K}$ and $M'(\mathcal{K} \cap \mathcal{D}_{M'}) \subseteq \mathcal{K}$. Suppose $f \in L_2(G)$ and $h \in \mathcal{B} \cap \mathcal{D}'_{M'}$ (here $\mathcal{B}$ is the bi-invariant space generated by $\mathcal{K}$.) Then by Lemma 4.7 :

$$r_{M'h,f} = V_h^* f.$$

But in an addition $h \in \mathcal{D}_\Lambda$, then

$$V_h^* f = f \star (h^\vee)^*$$

and

$$[f \star (h^\vee)^*](x) = \int f(xy^{-1})(h^\vee)^*(y)\,dy = \int f(xy^{-1})\overline{h^\vee(y^{-1})}\,\Delta(y)^{-1/2}\,dy$$
$$= \int f(xy^{-1})\overline{h(y^{-1})}\,\Delta(y)^{-1}\,dy = \int f(xy)\overline{h(y)}\,dy$$
$$= \Delta(x)^{-1/2}(\lambda(x^{-1})f, h).$$

Thus for $f \in L_2(G)$, $h \in \mathcal{B} \cap \mathcal{D}''_{M'} \cap \mathcal{D}_\Lambda$, we have

$$r_{M'h,f}(x) = \Delta(x)^{-1/2}(\lambda(x^{-1})f, h).$$

We have already seen that M′ is the closure of its restriction to $\mathcal{H}'$ (*see* 4.2). Since the projection onto $\mathcal{B}$ commutes with M′ and Λ (the latter by [2, Proposition 1]) and since that projection preserves right boundedness, it follows that $M'|_{\mathcal{B}}$ is the closure of its restriction to $\mathcal{B} \cap \mathcal{D}''_{M'} \cap \mathcal{D}_\Lambda$. That is, for $h \in \mathcal{B} \cap \mathcal{D}_{M'}$ there exists $h_n \in \mathcal{B} \cap \mathcal{D}''_{M'} \cap \mathcal{D}_\Lambda$ such that $h_n \to h$ and $M'h_n \to M'h$. Reasoning exactly as in the proof of Lemma 4.7, we conclude that

$$r_{M'h_n,f} \to r_{M'h,f} \quad \text{in } L_2(G).$$

Replacing h_n by a subsequence if necessary, we may also assume that

$$r_{M'h_n,f} \to r_{M'h,f} \quad \text{pointwise a. e. on G.}$$

But since

$$r_{M'h_n,f}(x) = \Delta(x)^{-1/2}(\lambda(x^{-1})f, h_n)$$

clearly converges pointwise to

$$\Delta(x)^{-1/2}(\lambda(x^{-1})f, h),$$

we conclude that

$$(4.3) \qquad r_{M'h,f}(x) = \Delta(x)^{-1/2}(\lambda(x^{-1})f, h), \qquad f \in L_2(G), \quad h \in \mathcal{B} \cap \mathcal{D}_{M'},$$

We can now prove the statements of the theorem rather easily.

(i) If $f_i \in \mathcal{K}_i$ and $h_i \in \mathcal{K} \cap \mathcal{D}_{M'}$, then we know that the corresponding representative elements satisfy the orthogonality relations (ii) of

Lemma 4.5. Therefore

$$\frac{1}{d}(f_1, f_2)\,(\overline{M' h_1, M' h_2}) = (r_{M' h_1, f_1}, r_{M' h_2, f_2})$$
$$= \int_G \Delta(x)^{-1}\,(\lambda(x^{-1}) f_1, h_1)\,(\overline{\lambda(x^{-1}) f_2, h_2})\,dx$$
$$= \int_G (\lambda(x) f_1, h_1)\,(\overline{\lambda(x) f_2, h_2})\,dx.$$

(ii) This is an immediate consequence of Lemma 4.5 (i), equation (4.3) and essentially the same computation as in (i) above.

For our final result, we will characterize the irreducible subrepresentations of the regular representation of G, that is the discrete series. The algebra $\mathfrak{A}_f(G)$ has a traceable approximate identity; so we could perhaps make use of our earlier results (e. g., Lemma 4.4) if we knew the relation between representations of G and representations of $\mathfrak{A}_f(G)$. Failing this, we consider the smaller algebra

$$\mathfrak{A}_1(G) = \{ f \in L_2(G) : \Delta^{n/2} f \in L_1(G) \cap L_2(G), \text{ all } n \in \mathbf{Z} \}.$$

$\mathfrak{A}_1(G)$ is a sub quasi-Hilbert algebra of $\mathfrak{A}_f(G)$.

Now let π be a representation of G. For $f \in \mathfrak{A}_1(G)$ we put

$$\pi'(f) = \pi(\Delta^{-1/2} f).$$

It follows easily from $\Delta^{-1/2}(f \star h) = (\Delta^{-1/2} f) \star (\Delta^{-1/2} h)$ that π' is multiplicative, $\pi'(f \star h) = \pi'(f)\,\pi'(h), f, h \in \mathfrak{A}_1(G)$. Next set

$$f^{\dagger}(x) = \Delta(x)^{-1}\,\overline{f(x^{-1})},$$

the usual involution on $L_1(G)$. Then using the computation

$$(\Delta^{-1/2} f)^{\dagger}(x) = \Delta(x)^{-1}\,\Delta^{-1/2}(x^{-1})\,\overline{f(x^{-1})} = \Delta(x)^{-1/2}\,\overline{f(x^{-1})} = f^*(x),$$

we see that

$$\pi'((f^*)^{\vee}) = \pi'(\Delta^{1/2} f^*) = \pi(f^*) = \pi((\Delta^{-1/2} f)^{\dagger}) = \pi(\Delta^{-1/2} f)^* = \pi'(f)^*.$$

Summarizing, we have shown that π' satisfies property (i) of the definition of a representation of the quasi-Hilbert algebra $\mathfrak{A}_1(G)$.

We assume now that $\mathfrak{A}_1(G)$ has a traceable approximate identity. By Lemma 4.2 that is tantamount to assuming that $\mathcal{D}_M \cap \mathfrak{A}_1(G)$ is dense in $L_2(G)$. We cite two instances where this is the case. If G is unimodular, $\mathfrak{A}_1(G) = L_1(G) \cap L_2(G)$ and it suffices to take a net which is an approximate identity for both $L_1(G)$ and $L_2(G)$. If G is a connected solvable Lie group, then $\mathcal{D}_M \cap C_0(G)$ is dense in $L_2(G)$ and so there is a traceable approximate identity.

Now let $\mathfrak{A}_1(G)$ have the traceable approximate identity $\{e_i\}$ and suppose π is a representation of G. We say π is *extendible* if $\pi'(e_i) \to 1$ strongly. π' is then a representation of $\mathfrak{A}_1(G)$. In the two cases mentioned previously, every representation is extendible.

Before stating the theorem we make one more observation. If $\lambda = \lambda_G$ is the left regular representation of G, then for $f \in \mathfrak{A}_1(G)$, $h \in L_2(G)$,

$$\lambda'(f)\, h = \lambda(\Delta^{-1/2} f)\, h = f \star h$$

[7, equation (6.2 *a*)]. Thus λ' is the left regular representation of $\mathfrak{A}_1(G)$.

THEOREM 4.9. — *Let* G *be a locally compact group with a semi-finite regular representation. Assume that* $\mathfrak{A}_1(G)$ *has a traceable approximate identity and let* π *be an irreducible extendible representation of* G. *Then* π *is in the discrete series of* G *if and only if there exists a non-zero vector* $\xi \in \mathcal{H}_\pi$ *such that* $x \to (\pi(x^{-1})\,\xi, \xi)$ *is in the space* $\mathcal{D}_{M'} \cap \mathcal{D}_{M'^{-1}}$. *If this is the case there is a positive invertible self-adjoint operator* M'_π *on* $\mathcal{H}_\pi$ *such that for all* $\xi \in \mathcal{H}_\pi$, $\eta \in \mathcal{D}_{M'_\pi}$ *the matrix coefficient* $x \to (\pi(x)\,\xi, \eta)$ *is in* $L_2(G)$. *For all* $\xi_i \in \mathcal{H}_\pi$, $\eta_i \in \mathcal{D}_{M'_\pi}$, $i = 1, 2$, *we have*

$$\int_G (\pi(x)\,\xi_1, \eta_1)\,\overline{(\pi(x)\,\xi_2, \eta_2)}\, dx = \frac{1}{d}(\xi_1, \xi_2)\,\overline{(M'_\pi\,\eta_1, M'_\pi\,\eta_2)}$$

where $d > 0$ *is a constant depending only on* π *and* M'_π.

Proof. — Suppose π is in the discrete series. Then π is equivalent, by a unitary operator T, to the subrepresentation of λ on a minimal closed left-invariant subspace $\mathcal{K}$ of $L_2(G)$. We claim that there exists a non-zero element $f \in \mathcal{K}$ such that $f \in \mathcal{D}^r_{M'}$ and $M'f$ is right bounded. The argument is reminiscent of the proof of Lemma 4.7; it goes as follows. Set

$$\mathcal{H}'' = \{ f \in \mathcal{D}^r_{M'} : M'f \text{ is right bounded} \}.$$

As a consequence of what we proved in Lemma 4.7, $\mathcal{H}''$ is dense in $L_2(G)$. Let P be the projection onto $\mathcal{K}$. It suffices to show that $P\,\mathcal{H}'' \subseteqq \mathcal{H}''$. But this follows immediately from the fact that P commutes with M′ and [2, Lemma 1]. Thus the set of vectors f in $\mathcal{K}$ which are in $\mathcal{D}^r_{M'}$ and for which $M'f$ is right bounded is actually a dense subspace of $\mathcal{K}$. Let $f \neq 0$ be any such vector. Then by Lemma 4.7, $r_{M'f,f} \in \mathcal{D}_M$ and $r_{M'f,f} = M\, r_{f,f} \in \mathcal{D}_{M^{-1}}$. However by the proof of Theorem 4.8, we know that

$$r_{M'f,f}(x) = \Delta(x)^{-1/2}\,(\lambda(x^{-1})\,f, f) = \Delta(x)^{-1/2}\,\overline{(f, \lambda(x^{-1})\,f)}.$$

Set $c_f(x) = (f, \lambda(x) f)$. Then $c_f^* = r_{M', f, f}$. This implies in turn that $c_f \in J(\mathcal{D}_M \cap \mathcal{D}_{M^{-1}}) = \mathcal{D}_{M'} \cap \mathcal{D}_{M'^{-1}}$. Finally if $\xi \in \mathcal{H}_\pi$ is chosen so that $f = T\xi$, then

$$c_f(x) = (\lambda(x^{-1}) f, f) = (\pi(x^{-1}) \xi, \xi).$$

The operator M'_π is just $T^{-1} M' PT$ and the assertions about the orthogonality relations follow from Theorem 4.8.

Conversely, suppose there is a non-zero vector $\xi \in \mathcal{H}_\pi$ such that $x \to (\pi(x^{-1}) \xi, \xi)$ is in $\mathcal{D}_{M'} \cap \mathcal{D}_{M'^{-1}}$. We shall show that ξ is a quasi square-integrable vector for the representation π'. Then by Lemma 4.4, π' is equivalent to a subrepresentation of λ', and so π is equivalent to a subrepresentation of λ. In fact, for any $f \in \mathfrak{A}_1(G)$, we have

$$\begin{aligned}(\pi'(f)\xi, \xi) &= (\pi(\Delta^{-1/2} f)\xi, \xi) \\ &= \int_G \Delta(x)^{-1/2} f(x) (\pi(x)\xi, \xi)\, dx = \int_G \Delta(x)^{-1/2} f(x) \overline{(\pi(x^{-1})\xi, \xi)}\, dx.\end{aligned}$$

The function $c_\xi(x) = (\pi(x^{-1})\xi, \xi)$ is by assumption in $\mathcal{D}_{M'^{-1}}$. Therefore $c_\xi^* \in \mathcal{D}_{M^{-1}}$. But

$$c_\xi^*(x) = \Delta(x)^{-1/2} \overline{c_\xi(x^{-1})} = \Delta(x)^{-1/2} \overline{(\pi(x)\xi, \xi)} = \Delta(x)^{-1/2} (\pi(x^{-1})\xi, \xi).$$

Since $c_\xi^* \in \mathcal{D}_{M^{-1}}$, there exists $r_\xi \in \mathcal{D}_M$ such that $c_\xi^* = M r_\xi$. But then

$$(\pi'(f)\xi, \xi) = (f, M r_\xi) = (M f, r_\xi), \qquad f \in \mathcal{D}_M \cap \mathfrak{A}_1(G).$$

This says precisely that ξ is quasi square-integrable.

Remarks. — (1) Suppose G is the « $ax + b$ » group (*see* [7, § 10, Example 1]). G has a single irreducible infinite-dimensional representation π and by [7] it is in the discrete series of G. π may be realized in $L_2(\mathbf{R}^*)$ and in that realization M'_π is given by

$$M'_\pi f(x) = |x|^{-1/2} f(x).$$

It can be shown fairly easily that for $f \in L_2(\mathbf{R}^*)$, the function $x \to (\pi(x) f, f)$ is square-integrable *if and only if* $f \in \mathcal{D}_{M'^{1/2}_\pi}$.

(2) It seems likely that the operator M'_π is uniquely determined up to a scalar, although we have not been able to substantiate that.

(3) It is interesting to surmise as to whether the conditions of Theorem 4.9 are best possible. Is it enough for $x \to (\pi(x^{-1})\xi, \xi)$ to be in $L_2(G)$ in order to guarantee that π is in the discrete series; or do there exist representations π which have square-integrable matrix coefficients but which are not subrepresentations of the regular representation ? At this point we do not know the answer.

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