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THE PLANCHEREL FORMULA FOR GROUP EXTENSIONS

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1. INTRODUCTION. — In this paper we consider the problem of computing the Plancherel measure of a locally compact group G in terms of the Plancherel measure of a closed normal subgroup N. The results we obtain require also that we know the Plancherel measures of certain subgroups (little groups) of G/N. This is of course quite reasonable in light of Mackey's theory which describes the irreducible representations of G in terms of corresponding objects for N and the little groups.

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More explicitly, let G be a separable locally compact group, N a closed normal subgroup of G. Assume that N is type I. The space $\hat{N}$ has a natural Borel structure and is a standard Borel G-space under the action $(\gamma . g)(n) = \gamma(gng^{-1})$, $n \in N$, $g \in G$, $\gamma \in \hat{N}$. The stability subgroups $G_\gamma = \{ g \in G : \gamma . g$ and γ are equivalent$\}$ are closed subgroups. Moreover there exists a multiplier ω_γ on G_γ/N with the following properties : if we denote again by ω_γ the lift to G_γ, then γ extends to an ω_γ-representation γ' of G_γ; ω_γ is unique up to similarity; ω_γ may be chosen normalized, i. e., $\omega_\gamma(x, x^{-1}) = 1$. Next let σ be an irreducible $\overline{\omega}_\gamma$-representation of G_γ/N. Denote by σ'' its lift to G_γ. Then $\gamma' \otimes \sigma''$ is an ordinary representation of G_γ and $\pi_{\gamma,\sigma} = \mathrm{Ind}_{G_\gamma}^{G} \gamma' \otimes \sigma''$ is an irreducible representation of G. Moreover, if the orbit space $\hat{N}/G$ is countably separated, we obtain all irreducible representations of G as γ varies over $\hat{N}/G$ and σ varies over the irreducible $\overline{\omega}_\gamma$-representations of G_γ/N.

Suppose that G is unimodular and type I. Then there is a unique standard Borel measure class $\{\mu_G\}$ on $\hat{G}$ such that

$$\lambda_G = \int_{\hat{G}}^{\oplus} \pi \otimes 1_\pi \, d\mu_G(\pi),$$

where λ_G is the left regular representation of G; and a unique measure μ_G (up to a positive constant depending on the normalization of Haar measure on G) in that class such that

$$(1.1) \qquad \int_G |\varphi(g)|^2 dg = \int_{\hat{G}} \mathrm{Tr}(\pi(\varphi)\pi(\varphi)^*) \, d\mu_G(\pi), \qquad \varphi \in L_1(G) \cap L_2(G).$$

Our goal in this work is to describe (under reasonable assumptions) the measure μ_G in terms of corresponding objects for N and G_γ/N.

Several difficulties are immediately evident. Since ω_γ-representations (rather than ordinary representations) occur, we will need an analogue of the Plancherel theorem for projective representations (§ 7). A more serious problem is that the groups G_γ/N need not be unimodular. Thus it becomes necessary to develop a Plancherel theorem for non-unimodular groups (§ 6). We are able to overcome these and other obstacles in order to obtain the desired result, which can be described roughly as follows : in the above we have seen how $\hat{G}$ is a " fibre space " with base $\hat{N}/G$ and fibres $(\widehat{G_\gamma/N})^{\overline{\omega}_\gamma}$; we shall show that the Plancherel measure μ_G is also fibred — on the base it is a pseudo-image of μ_N, and on the fibres it is the projective Plancherel measure of G_γ/N.

We have assumed N is type I for ease of presentation here; in fact, we need only assume that N has a type I regular representation. It turns out that under reasonable assumptions the Plancherel measure is concentrated in the set $\hat{N}_t$ of traceable irreducible representations. This set, one easily deduces from recent work of Effros, Davies and Guichardet, is a well-behaved space in which none of the pathological behavior associated with non-type I groups can occur. Furthermore, Mackey's theory can be localized to $\hat{N}_t$ and remains true in unaltered form. With these facts, we can carry out our analysis without requiring that the group N be type I.

We do not assume G is unimodular in this paper. Therefore, because of the nature of the Plancherel theorem for non-unimodular groups, we are only able to compute the measure class of μ_G in our main theorem (§ 10). Even if G is unimodular, the fact that G_γ/N may be non-unimodular plus the built-in ambiguity of pseudo-images prevents us from specifying the precise measure in general ([5]). However, under various additional assumptions — namely, if G/N is compact (§ 4), or if G is μ_N-transitive on $\hat{N}$ (§ 5), or if N = Cent G (§ 8) — we can compute the specific Plancherel measure. This enables us to write down the Plancherel measure for several kinds of groups for which it was not previously known — for example, the group of rigid motions of Euclidean space, Moore groups, the semidirect product of SL (n, F) and F^n, F a local field (the latter providing, we believe, the first example of a specific determination of μ_G for a class of type I, non-CCR groups). We also obtain as special cases Plancherel measures which were previously known — central groups, certain kinds of nilpotent groups, and others. Among the examples where we compute only the measure class of μ_G are the " $ax + b$ " group and the inhomogeneous Lorentz groups.

Other results which play auxiliary roles in the solution of the general extension problem, but which may have independent interest are : two results on disintegration of measures (§ 2) — these are certainly well-known, but they exist in the literature only with (unnecessary) topological assumptions; a general formula for the character of an induced representation (§ 3); and a long section on measure-theoretic considerations (§ 9), which is necessitated partly by our inability to describe completely the Borel structure of $\hat{G}$ in terms of those for $\hat{N}$ and $\left(\widehat{G_\gamma/N}\right)^{\overline{\omega}_\gamma}$. We conclude the paper (§ 11) with some brief comments on the relations between the discrete series of G and those of N and G_γ/N.

We shall adhere to the notation established in the second paragraph of this introduction for describing the ingredients of the representation theory of a group extension. *All* groups throughout are assumed to be *separable.* With the exception of paragraph 9, we shall not distinguish between equivalent representations, i. e., $\pi_1 = \pi_2$ means they are equivalent representations. All of our coset spaces will be on the right and written $G/H = \{ Hg : g \in G \}$. We apologize in advance if that offends anyone — blame it on the fact that we are right-handed, but read from left to right. Finally we refer the reader to [1], [10], [23] and [24] for all unexplained terminology and results relating to Borel spaces and representation theory.

The authors wish to thank Arlan Ramsey for his help with certain measure-theoretic questions. Thanks are also due to L. Pukanszky for several helpful observations, in particular a remark which caused us to look more closely at $\hat{G}_t$.

2. Disintegration of measures. — Let X and Y be Borel spaces and $p : X \to Y$ a Borel map. Given any positive Borel measure μ on X, let $p^* \mu$ be its image on Y; that is, $p^* \mu (E) = \mu (p^{-1} E)$ for Borel subsets E of Y. Then $p^* \mu$ is a Borel measure on Y (possibly taking only the values $0, \infty$). Moreover, for all positive Borel functions f on Y, we have

$$\int_Y f \, dp^* \mu = \int_X F \, d\mu$$

where $F = f \circ p$. If X is a Borel space, R an equivalence relation on X (e. g., the orbits of a group action), $Y = X/R$, $p : X \to Y$ the canonical projection, we write $\bar{\mu}$ for $p^* \mu$ and call $\bar{\mu}$ the *image* of μ. By a *pseudo-image* of μ we mean the image of a finite measure equivalent to μ. We shall also write $\bar{\mu}$ for pseudo-images, being careful to indicate when $\bar{\mu}$ is the image or a pseudo-image of μ.

Theorem 2.1. — *Let* G *be a locally compact group. Let* X *be a right Borel* G*-space and* μ *a quasi-invariant* σ*-finite positive Borel measure on* X. *Assume that there is a* μ*-null set* X_0 *such that* X_0 *is* G*-invariant and* $X - X_0$ *is standard. Let* $\bar{\mu}$ *be a pseudo-image of* μ. *Assume finally that* $\bar{\mu}$ *is countably separated. Then for all* $x \in X - X_0$, *the orbit* $x.G$ *is Borel isomorphic to* G/G_x *under the natural mapping, and there is a quasi-invariant measure* μ_x *concentrated on* $x.G$ *such that for all* $f \in L_1(X, \mu)$,

$$\int_X f(x) \, d\mu(x) = \int_{(X - X_0)/G} \int_{G/G_x} f(x.g) \, d\mu_x(\bar{g}) \, d\bar{\mu}(\bar{x}), \tag{2.1}$$

where G_x *is the stability group at* x, $\bar{g}$ *is the image of* g *in* G/G_x, *and* $\bar{x}$ *is the image of* x *in* $(X - X_0)/G$.

Proof. — Replacing X by $X - X_0$, we may suppose that X is standard. It follows immediately from [1, Proposition 3.7] that G_x is closed and that $G_x\, g \to x.g$ effects a Borel isomorphism of G/G_x onto $x.G$. Since $\bar{\mu}$ is countably separated, there is a $\bar{\mu}$-null set Y in X/G such that $(X/G) - Y$ is countably separated. It follows from [23, Theorem 6.2] that (enlarging Y by another $\bar{\mu}$-null set if necessary) we may assume $(X/G) - Y$ is standard. The inverse image X_1 of Y in G is a μ-null, G-invariant set. Replacing X by $X - X_1$ we may thus assume that X and X/G are standard. But then there are (locally) compact separable metric topologies which generate the Borel structures of X and X/G. We may therefore apply [4, § 3, théorème 2] to the projection $X \to X/G$ to obtain the disintegration (2.1). Finally, the arguments used by Mackey in [22, Lemmas 11.4, 11.5] may be applied here to show that all the μ_x are quasi-invariant.

We will use Theorem 2.1 in a key step in our proof of the general Plancherel formula for group extensions (§ 10). However, we will need a more precise result than Theorem 2.1 for use in the special case of a compact extension (§ 4). That is contained in our next theorem.

THEOREM 2.2. — *With the assumptions of Theorem* 2.1, *suppose also that* G *is compact and that* μ *is invariant. Then the ordinary image* $\bar{\mu}$ *of* μ *is* σ*-finite and for all* $f \in L_1(X, \mu)$:

$$\int_X f(x)\, d\mu(x) = \int_{(X - X_0)/G} \int_G f(x.g)\, dg\, d\bar{\mu}(\bar{x}),$$

where dg denotes normalized Haar measure on G.

Proof. — Again we may assume $X_0 = \emptyset$. Let h be a strictly positive, bounded, μ-integrable Borel function on X. Set

$$h'(x) = \int_G h(x.g)\, dg.$$

The integral exists since $g \to h(x.g)$ is a bounded Borel function. Furthermore, $(x, g) \to h(x.g)$ is Borel on $X \times G$ and

$$\begin{aligned}\int_X h'(x)\, d\mu(x) &= \int_X \int_G h(x.g)\, dg\, d\mu(x)\\ &= \int_G \int_X h(x.g)\, d\mu(x)\, dg\\ &= \int_G \int_X h(x)\, d\mu(x)\, dg\\ &= \int_X h(x)\, d\mu(x) < \infty,\end{aligned}$$

Note the switch of integration is valid by Tonelli's theorem. It follows that h' is finite-valued a. e. and μ-integrable. It is obvious that h' is G-invariant. We may also assume that h' is strictly positive. Indeed by Varadarajan's theorem [32, Theorem 3.2], X may be given a separable metric topology which generates the Borel structure and with respect to which X is a topological G-space. In that case, we may take h strictly positive, μ-integrable, and *continuous*. The compactness of the orbits then guarantees that h' is strictly positive.

Now set $\nu = h'\,\mu$, a finite G-invariant Borel measure on X. Let $\bar{\nu}$ be the image of ν on X/G as usual. Then by théorème 1, § 3 of [4], there exists for almost all x a positive measure ν_x of norm 1 concentrated on $x.\mathrm{G}$ such that

$$\int_{\mathrm{X}} f(x)\, d\nu(x) = \int_{\mathrm{X/G}} \int_{\mathrm{G/G}_x} f(x.g)\, d\nu_x(\bar{g})\, d\bar{\mu}(\bar{x}), \qquad f \in \mathrm{L}_1(\mathrm{X}, \nu).$$

Note that théorème 1 applies since $\mathrm{X} \to \mathrm{X/G}$ is ν-*propre* [3, remarque 1, p. 74].

Since ν is invariant, it follows again from the arguments of [12, Lemmas 11.4, 11.5] that almost all ν_x are invariant. It follows immediately that we can write

$$\int_{\mathrm{X}} f(x)\, d\nu(x) = \int_{\mathrm{X/G}} \int_{\mathrm{G}} f(x.g)\, dg\, d\bar{\mu}(x), \qquad f \in \mathrm{L}_1(\mathrm{X}, \nu).$$

Note that the proof is finished if μ is finite (take $h' = h = 1$).

In general, we can say the following : for any G-invariant Borel function on X, we have

$$\int_{\mathrm{X}} f(x)\, d\mu(x) = \int_{\mathrm{X/G}} f(\bar{x})\, d\bar{\mu}(\bar{x}),$$

$$\int_{\mathrm{X}} f(x)\, d\nu(x) = \int_{\mathrm{X/G}} f(\bar{x})\, d\bar{\nu}(\bar{x});$$

and therefore

$$\begin{aligned} \int f(\bar{x})\, h'(\bar{x})\, d\bar{\mu}(\bar{x}) &= \int f(x)\, h'(x)\, d\mu(x) \\ &= \int f(x)\, d\nu(x) \\ &= \int f(\bar{x})\, d\bar{\nu}(\bar{x}). \end{aligned}$$

It follows that $h'\,\bar{\mu} = \bar{\nu}$. In particular $\bar{\mu}$ and $\bar{\nu}$ are equivalent. Since ν

is finite, $\bar{\mu}$ must be σ-finite. Finally, we compute for $f \in L_1(X, \mu)$:

$$\begin{aligned}\int_X f(x)\, d\mu(x) &= \int_X [f(x)/h'(x)]\, h'(x)\, d\mu(x)\\ &= \int_X [f(x)/h'(x)]\, d\nu(x)\\ &= \int_{X/G}\int_G [f(x.g)/h'(x.g)]\, dg\, d\bar{\nu}(\bar{x})\\ &= \int_{X/G}\left(\int_G f(x.g)\, dg\right) h'(\bar{x})^{-1}\, d\bar{\nu}(\bar{x})\\ &= \int_{X/G}\int_G f(x.g)\, dg\, d\bar{\mu}(\bar{x}).\end{aligned}$$

3. A CHARACTER FORMULA. — Let K be a separable locally compact Hausdorff space with a positive, σ-finite, regular Borel measure μ. In this section it will be convenient to suppress the μ and write dk. If $\mathcal{H}$ is a separable Hilbert space, we denote by $L_2(K; \mathcal{H})$ the Hilbert space of measurable, $\mathcal{H}$-valued, square-integrable functions on K. We use $\mathcal{L}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$ to denote the bounded operators and Hilbert-Schmidt operators on $\mathcal{H}$ respectively. We shall use $\|.\|_2$ for both the Hilbert-Schmidt norm in $\mathcal{S}(\mathcal{H})$ and the norm on $L_2(K; \mathcal{H})$; but the precise meaning will be clear from the context.

Let $\Phi : K \times K \to \mathcal{L}(\mathcal{H})$ be a (weakly) measurable operator-valued function. We say that Φ determines a kernel operator T_Φ whenever there is a constant C such that

$$\iint |\langle \Phi(k, \varkappa) f(\varkappa), f'(k)\rangle|\, d\varkappa\, dk \leq C\, \|f\|_2\, \|f'\|_2, \qquad f, f' \in L_2(K; \mathcal{H}). \tag{3.1}$$

When (3.1) holds, it is clear that T_Φ is given by

$$T_\Phi f(k) = \int_K \Phi(k, \varkappa) f(\varkappa)\, d\varkappa, \qquad f \in L_2(K; \mathcal{H}).$$

There are many sufficient conditions on Φ to guarantee equation (3.1). For example :

(i) K = locally compact group, dk = Haar measure, $\Phi(k, \varkappa) = \Omega(k\varkappa^{-1})$ where the operator norm $\|\Omega(.)\|$ is an integrable function on K;

(ii) $\|\Phi(.,.)\|_2 \in L_2(K \times K)$.

The latter example will be important for us. It is fairly well-known and not hard to see [from the identifications $L_2(K; \mathcal{H}) = L_2(K) \otimes \mathcal{H}$

and $\mathcal{S}(\mathcal{H}) = \mathcal{H} \otimes \mathcal{H}^{\sim}$], that the Hilbert-Schmidt operators on $L_2(K; \mathcal{H})$ are *precisely* the kernel operators T_Φ where $\Phi \in L_2(K \times K; \mathcal{S}(\mathcal{H}))$ and

$$\| T_\Phi \|_2^2 = \int_{K \times K} \| \Phi(k, x) \|_2^2 \, dk \, dx.$$

It is straightforward to check that $T_\Phi^* = T_{\Phi^*}$ where

$$\Phi^*(k, x) = \Phi(x, k)^*.$$

Given a kernel Φ, let us set

$$\Psi(k) = \int_K \Phi(k, x) \Phi(k, x)^* \, dx.$$

For any $k \in K$, $\Psi(k)$ is a (perhaps unbounded) linear operator on $\mathcal{H}$ $\left[\text{any vector } \xi \in \mathcal{H} \text{ having the property that } \int_K \| \Phi(k, x) \Phi(k, x)^* \xi \| \, dx < \infty \text{ will certainly be in the domain of } \Psi(k)\right]$. Given $\xi \in \mathcal{H}$, then $(\Psi(k)\xi, \xi)$ is either a finite non-negative number [if $\xi \in$ domain of $\Psi(k)$] or $+\infty$. Moreover $\operatorname{Tr} \Psi(k) = \Sigma(\Psi(k)\xi_i, \xi_i)$, $\{\xi_i\}$ an orthonormal basis of $\mathcal{H}$, is well-defined (independent of the choice of $\{\xi_i\}$) and will be finite if and only if $\Psi(k)$ is a bounded positive trace-class operator on $\mathcal{H}$. It is easily seen that $k \to \operatorname{Tr} \Psi(k)$ is a non-negative measurable function (perhaps taking the value $+\infty$).

LEMMA 3.1. — $\operatorname{Tr}(T_\Phi T_\Phi^*) = \int_K \operatorname{Tr} \Psi(k) \, dk$ *in the sense that if* T_Φ *is a Hilbert-Schmidt operator both sides are finite and equal, while if* T_Φ *is not Hilbert-Schmidt then both sides are equal to* $+\infty$.

Proof. — We compute that

$$\begin{aligned}
\int_K \operatorname{Tr} \Psi(k) \, dk &= \int_K \Sigma (\Psi(k) \xi_i, \xi_i) \, dk \\
&= \int_K \Sigma \int_K (\Phi(k, x) \Phi(k, x)^* \xi_i, \xi_i) \, dx \, dk \\
&= \int_K \Sigma \int_K \| \Phi(k, x)^* \xi_i \|^2 \, dx \, dk \\
&= \int_K \int_K \Sigma \| \Phi(k, x)^* \xi_i \|^2 \, dx \, dk \\
&= \int_{K \times K} \| \Phi(k, x) \|_2^2 \, dk \, dx.
\end{aligned}$$

But T_Φ Hilbert-Schmidt $\Rightarrow \int_{K\times K} \| \Phi(k, \varkappa) \|_2^2 \, dk \, d\varkappa < \infty$ and

$$\mathrm{Tr}(T_\Phi T_\Phi^*) = \| T_\Phi \|^2 = \int_{K\times K} \| \Phi(k, \varkappa) \|_2^2 \, dk \, d\varkappa.$$

On the other hand T_Φ not Hilbert-Schmidt [i. e., $\mathrm{Tr}\,(T_\Phi T_\Phi^*) = +\infty$] $\Rightarrow \int_{K\times K} \| \Phi(k, \varkappa) \|_2^2 \, dk \, d\varkappa = +\infty$, and so $\int \mathrm{Tr}\, \Psi^{*}(k) \, dk = +\infty$ also.

We use Lemma 3.1 to obtain a formula for the character of an induced representation that will be useful several times in the following. Let G be locally compact and let N be a closed (not necessarily normal) subgroup. Set $K = G/N$, the homogeneous space of right cosets. If we choose right Haar measures dg, dn on G, N then we may find a strictly positive continuous function q on G satisfying

$$q(e) = 1,$$
$$q(nx) = \Delta_H(n)\, \Delta_G(n)^{-1}\, q(x), \qquad n \in N, \quad x \in G, \tag{3.2}$$

where Δ_N, Δ_G denote the modular functions on N, G (*see* [5, p. 102, 103]). q defines a quasi-invariant measure dk on K as follows. For $f \in C_0(G)$, the continuous functions of compact support, put $f'(\bar{x}) = \int_N f(nx)\, dn$, $\bar{x} = N\,x$. Then dk (wich we sometimes write $d\bar{g}$) is defined by

$$\int_{G/N} f'(\bar{g})\, d\bar{g} = \int_G f(x)\, q(x)\, dx.$$

Let γ be a unitary representation of N and set $\pi = \mathrm{Ind}_N^G \gamma$. We write down a formula for the character of π in terms of that for γ. First we want to realize π on the homogeneous space K. Let $\mathcal{H}_\gamma$ be the space of γ. Then π acts on the space $\mathcal{H}_\pi$ of functions $f : G \to \mathcal{H}_\gamma$ satisfying

$$f(ng) = \gamma(n)\, f(g) \qquad \text{and} \qquad \int_K \| f(\bar{g}) \|^2 \, d\bar{g} < \infty.$$

The action of π is

$$\pi(g)\, f(x) = f(xg)\, [q(xg)/q(x)]^{1/2}.$$

We transfer to the Hilbert space $L_2(K; \mathcal{H}_\gamma)$ as follows. Choose a Borel cross-section $s : K \to G$ such that $s(\bar{e}) = e$. Then we have unitaries

$$\begin{gathered} F(k) \to f(g) = f(ns(k)) = \gamma(n)\, F(k), \\ L_2(K; \mathcal{H}_\gamma) \to \mathcal{H}_\pi; \\ f(g) \to F(k) = f(s(k)), \\ \mathcal{H}_\pi \to L_2(K; \mathcal{H}_\gamma), \end{gathered}$$

and these are inverses of each other. It is easy to compute the action of π on $L_2(K; \mathcal{H}_\gamma)$:

$$\pi(g)\,F(k) = \gamma(\beta(k, g))\,F(\overline{s(k)\,g})\,[q(s(k)\,g)/q(s(k))]^{1/2}, \qquad F \in L_2(K; \mathcal{H}_\gamma)$$

where for $g \in G$, $k \in K$, we write

$$\beta(k, g) = s(k)\,gs(\overline{s(k)\,g})^{-1} \in N.$$

Now let $\varphi \in C_0(G)$. Then

$$\begin{aligned}
\pi(\varphi)\,F(k) &= \int_G \varphi(g)\,\pi(g)\,F(k)\,dg \\
&= \int_G \varphi(g)\,\gamma(\beta(k, g))\,F(\overline{s(k)\,g})\,[q(s(k)\,g)/q(s(k))]^{1/2}\,dg \\
&= \int_G \Delta_G(s(k))^{-1}\,q(s(k))^{-1/2}\,\varphi(s(k)^{-1}\,g)\,\gamma(\beta(\bar{e}, g))\,F(\bar{g})\,q(g)^{1/2}\,dg \\
&= \Delta_G(s(k))^{-1}\,q(s(k))^{-1/2}\int_{G/N}\int_N \varphi(s(k)^{-1}\,ng)\,\gamma(\beta(\bar{e}, ng))\,F(\bar{g})\,q(ng)^{-1/2}\,dn\,dg \\
&= \int \Phi_\varphi(k, \bar{g})\,F(\bar{g})\,d\bar{g}
\end{aligned}$$

where

$$\begin{aligned}
\Phi_\varphi(k, \bar{g}) &= \Delta_G(s(k))^{-1}\,q(s(k))^{-1/2}\int_N \varphi(s(k)^{-1}\,ng)\,\gamma(\beta(\bar{e}, ng))\,q(ng)^{-1/2}\,dn \\
&= \Delta_G(s(k))^{-1}\,q(s(k))^{-1/2}\int_N \varphi(s(k)^{-1}\,ns(\bar{g}))\,\gamma(n)\,q(ns(\bar{g}))^{-1/2}\,dn.
\end{aligned}$$

We know *a priori* that $\pi(\varphi)$ is a bounded operator. Thus $\pi(\varphi)$ is in fact a kernel operator on $L_2(K; \mathcal{H}_\gamma)$ with kernel Φ_φ.

THEOREM 3.2. — *Let* $\varphi \in C_0(G)$, $\varphi^*(g) = \overline{\varphi(g^{-1})}\,\Delta_G(g^{-1})$, *and set* $\psi = \varphi \star_G \varphi^*$. *Then*

(3.3) $$\mathrm{Tr}(\mathrm{Ind}_N^G\,\gamma)(\psi) = \int_{G/N} \Delta_G(g)^{-1}\,q(g)^{-1}\,\mathrm{Tr}\left[\int_N \psi(g^{-1}\,ng)\,\gamma(n)\,\Delta_G(n)^{1/2}\,\Delta_H(n)^{-1/2}\,dn\right]d\bar{g},$$

in the sense that both sides are finite and equal, or both $= +\infty$.

Proof. — On the left side of (3.3), we have

$$\mathrm{Tr}(\mathrm{Ind}_N^G\,\gamma)(\psi) = \mathrm{Tr}\,\pi(\varphi \star \varphi^*) = \mathrm{Tr}\,\pi(\varphi)\,\pi(\varphi)^* = \mathrm{Tr}(T_{\Phi_\varphi}\,T^*_{\Phi_\varphi}).$$

As for the right, using the substitution $g \to n_1 g$ and equation (3.2), it is easily checked that the integrand is N-invariant. Claim : with the notation of Lemma 3.1, we have $\Psi(k) = \Phi_\psi(k, k)$. Postponing the proof of this momentarily, we can then invoke Lemma 3.1 to verify

$$\begin{aligned}\operatorname{Tr}\pi(\psi) &= \operatorname{Tr}(T_{\Phi_\varphi} T^*_{\Phi_\varphi}) = \int_K \operatorname{Tr}\Psi(k)\,dk = \int_K \operatorname{Tr}\Phi_\psi(k,k)\,dk \\ &= \int_K \operatorname{Tr}\left[\Delta_G(s(k))^{-1} q(s(k))^{-1/2} \int_N \psi(s(k)^{-1} ns(k))\,\gamma(n)\,q(ns(k))^{-1/2}\,dn\right] dk \\ &= \int_{G/N} \Delta_G(g)^{-1} q(g)^{-1} \operatorname{Tr}\left[\int_N \psi(g^{-1}ng)\,\gamma(n)\,\Delta_G(n)^{1/2}\,\Delta_H(n)^{-1/2}\,dn\right] d\bar{g}.\end{aligned}$$

It remains to prove the claim. According to the definition of Ψ,

$$\begin{aligned}\Psi(k) &= \int_K \Phi(k,x)\,\Phi(k,x)^*\,dx \\ &= \int_K \left(\int_N \Delta_G(s(k))^{-1} q(s(k))^{-1/2} \varphi(s(k)^{-1} n_1 s(x))\,\gamma(n_1)\,q(n_1 s(x))^{-1/2}\,dn_1\right) \\ &\quad\times\left(\int_N \Delta_G(s(k))^{-1} q(s(k))^{-1/2} \varphi(s(k)^{-1} ns(x))\,\gamma(n)\,q(ns(x))^{-1/2}\,dn\right)^* dx \\ &= \int_K\int_N\int_N \Delta_G(s(k))^{-2} q(s(k))^{-1} \varphi(s(k)^{-1} n_1 s(x))\,\bar{\varphi}(s(k)^{-1} ns(x)) \\ &\quad\times\gamma(n_1)\,\gamma(n)^{-1} q(n_1 s(x))^{-1/2} q(ns(x))^{-1/2}\,dn_1\,dn\,dx.\end{aligned}$$

On the other hand

$$\begin{aligned}\Phi_\psi(k,k) &= \int_N \Delta_G(s(k))^{-1} q(s(k))^{-1/2} \psi(s(k)^{-1} n_1 s(k))\,\gamma(n_1)\,q(n_1 s(k))^{-1/2}\,dn_1 \\ &= \int_N \Delta_G(s(k))^{-1} q(s(k))^{-1/2} \int_G \varphi(s(k)^{-1} n_1 s(k) g^{-1})\,\varphi^*(g)\,dg \\ &\quad\times\gamma(n_1)\,q(n_1 s(k))^{-1/2}\,dn_1 \\ &= \int_N \Delta_G(s(k))^{-1} q(s(k))^{-1/2} \int_G \varphi(s(k)^{-1} n_1 g^{-1})\,\varphi^*(gs(k))\,dg \\ &\quad\times\gamma(n_1)\,q(n_1 s(k))^{-1/2}\,dn_1 \\ &= \int_N \Delta_G(s(k))^{-2} q(s(k))^{-1} \int_G \varphi(s(k)^{-1} n_1 g)\,\bar{\varphi}(s(k)^{-1} g)\,dg \\ &\quad\times\gamma(n_1)\,q(n_1)^{-1/2}\,dn_1 \\ &= \int_N \Delta_G(s(k))^{-2} q(s(k))^{-1} \int_{G/N}\int_N \varphi(s(k)^{-1} n_1 ns(x)) \\ &\quad\times\bar{\varphi}(s(k)^{-1} ns(x))\,q(ns(x))^{-1}\,dn\,d\bar{g}\,\gamma(n_1)\,q(n_1)^{-1/2}\,dn_1 \\ &= \int_{G/N}\int_N\int_N \Delta_G(s(k))^{-2} q(s(k))^{-1} \varphi(s(k)^{-1} n_1 s(x)) \\ &\quad\times\bar{\varphi}(s(k)^{-1} ns(x))\,\gamma(n_1 n^{-1})\,q(ns(x))^{-1} q(n_1 n^{-1})^{-1/2}\,dn_1\,dn\,d\bar{g} \\ &= \int_{G/N}\int_N\int_N \Delta_G(s(k))^{-2} q(s(k))^{-1} \varphi(s(k)^{-1} n_1 s(x)) \\ &\quad\times\bar{\varphi}(s(k)^{-1} ns(x))\,\gamma(n_1 n^{-1})\,q(ns(x))^{-1/2} q(n_1 s(x))^{-1/2}\,dn_1\,dn\,d\bar{g}.\end{aligned}$$

Remark. — All of our applications will be in the unimodular case $\Delta_G = \Delta_N = 1$. In that event the character formula becomes more simply

$$\operatorname{Tr}(\operatorname{Ind}_N^G \gamma)(\psi) = \int_{G/N} \operatorname{Tr}\left[\int_N \psi(g^{-1} ng)\, \gamma(n)\, dn\right] d\bar{g}.$$

Still we feel it is worthwhile to present the most general case (3.3); in fact, we shall have use for it in a subsequent paper.

4. Compact extensions. — We return now to the situation (and notation) described in the introduction, $N \subset G$ closed and normal. We suppose in addition that N is unimodular and type I, and that G/N is compact. It follows that G is unimodular and also by [18, Theorem 1] G is type I.

Recall that the irreducible representations of G are described by $\pi = \pi_{\gamma,\sigma} = \operatorname{Ind}_{G_\gamma}^G \gamma' \otimes \sigma''$, γ' is the extension of $\gamma \in \hat{N}$ to an ω_γ-representation of G_γ, σ'' is the lift to G_γ of an irreducible $\overline{\omega}_\gamma$-representation of G_γ/N. Let us write $\check{G}_\gamma = \{\tau \in \hat{G}_\gamma : \tau|_N \text{ is a multiple of } \gamma\}$. Every $\tau \in \check{G}_\gamma$ is of the form $\tau = \gamma' \otimes \sigma''$; we write $n_\gamma(\tau) =$ the number of times $\tau|_N$ contains γ. Since G_γ/N is compact, it is clear that $n_\gamma(\tau) = \dim \sigma < \infty$. Next, let ρ_ω be the right regular $\overline{\omega}_\gamma$-representation of G_γ/N and ρ''_ω its lift to G_γ.

Lemma 4.1 : $\rho''_\omega = \bigoplus_{\sigma \in (\widehat{G_\gamma/N})^{\overline{\omega}_\gamma}} (\dim \sigma)\, \sigma''$.

Proof. — It is obviously enough to show that $\rho_\omega = \bigoplus_\sigma (\dim \sigma)\, \sigma$. But this is precisely the Peter-Weyl theorem for compact groups with a multiplier. The details are essentially all contained in [2, p. 286]. This result is also a special case of the results of paragraph 7, and so we omit the details.

Lemma 4.2 : $\operatorname{Ind}_N^G \gamma = \bigoplus_{\tau \in \check{G}_\gamma} n_\gamma(\tau) \operatorname{Ind}_{G_\gamma}^G \tau$, $\gamma \in \hat{N}$.

Proof. — We have $\operatorname{Ind}_N^G \gamma = \operatorname{Ind}_{G_\gamma}^G \operatorname{Ind}_N^{G_\gamma} \gamma$, while on the other hand

$$\bigoplus_\tau n_\gamma(\tau) \operatorname{Ind}_{G_\gamma}^G \tau = \operatorname{Ind}_{G_\gamma}^G [\bigoplus_\tau n_\gamma(\tau)\, \tau].$$

Thus it suffices to prove that $\operatorname{Ind}_N^{G_\gamma} \gamma = \bigoplus_{\tau \in \check{G}_\gamma} n_\gamma(\tau)\, \tau$; that is, we may assume $G = G_\gamma$. But then Baggett [2, p. 283] has shown that $\operatorname{Ind}_N^G \gamma = \gamma' \otimes \rho''_\omega$. However, by Lemma 4.1,

$$\begin{aligned} \bigoplus_\tau n_\gamma(\tau)\, \tau &= \bigoplus_\sigma (\dim \sigma)(\gamma' \otimes \sigma'') = \bigoplus_\sigma [\gamma' \otimes (\dim \sigma)\, \sigma''] \\ &= \gamma' \otimes [\bigoplus_\sigma (\dim \sigma)\, \sigma''] = \gamma' \otimes \rho''_\omega. \end{aligned}$$

Q. E. D.

We need one more lemma before giving the main result of this section. It will be useful for later applications as well; so we state it in greater generality than is needed here.

Let G be unimodular and type I. Denote by δ_G the Dirac measure, $\delta_G : C_0(G) \to \mathbf{C}$, $\delta_G(f) = f(e)$. δ_G is positive and central and thus induces a trace on $C^*(G)^+$ which we also denote by δ_G. Then an alternate form of the Plancherel formula is

$$\delta_G(\psi) = \int_{\hat{G}} \operatorname{Tr} \pi(\psi)\, d\mu_G(\pi), \qquad \psi \in C^*(G)^+ \tag{4.1}$$

(*see* e. g., [10, 18.8.1 (ii)]). If $\psi = f \star f^*$, $f \in L_1(G) \cap L_2(G)$, we obtain the Plancherel formula (1.1). Conversely (4.1) is a consequence of (1.1) and [10, 6.5.3].

In a sense, this scenario is really taking place on the reduced dual of G. Let $C_r^*(G)$ be the uniform closure of $\{\lambda_G(f) : f \in L_1(G)\}$. Then the Dirac measure also generates a trace on $C_r^*(G)^+$, which by a further abuse of notation we also denote δ_G. This is legitimate since $C_r^*(G)$ is a quotient algebra of $C^*(G)$ and the canonical projection $p : C^*(G) \to C_r^*(G)$ has the property $\delta_G(p(\psi)) = \delta_G(\psi)$, $\psi \in C^*(G)^+$, as is easily checked.

LEMMA 4.3. — (i) *Let* $\psi \in L_1(G)$ *be continuous. Suppose that for any unitary representation* π *of* G, $\pi(\psi)$ *is a positive operator. Then* $\psi \in P(G) \cap C^*(G)^+$, $P(G)$ = *continuous positive definite functions.*

(ii) *Let* $\psi \in L_1(G) \cap P(G)$. *Then* $\psi \in C_r^*(G)^+$ *and* $\delta_G(\psi) = \psi(e)$ (2).

Proof. — (i) For any unitary representation π of G and any vector ξ in the space of π, we are given that

$$(\pi(\psi)\xi, \xi) = \int (\pi(g)\xi, \xi)\, \psi(g)\, dg \geqq 0.$$

In particular

$$\int h(g)\, \psi(g)\, dg \geqq 0, \qquad h \in P(G).$$

Therefore $\psi \in P(G)$ [10, 13.4.4]. It also follows immediately from [10, 2.6.2] that $\psi \in C^*(G)^+$.

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(2) Originally, we stated this result with the conclusion $\psi \in C^*(G)^+$ in part (ii). We thank R. Mosak for pointing out that the proper assertion is $\psi \in C_r^*(G)^+$. In addition, we have noted since writing the paper that the proof of (ii) could be simplified by using the factorization theorem on p. 277 in M. RIEFFEL, *Square-integrable representations of Hilbert algebras* (*J. Funct. Anal.*, vol. 3, 1969, p. 265-300).

(ii) If $\psi \in L_1(G) \cap P(G) \subseteq L_2(G) \cap P(G)$, then by [10, 13.8.6] there exists $\varphi \in L_2(G) \cap P(G)$ such that $\psi = \varphi \star \varphi = \varphi \star \varphi^*$. But $\psi \in L_1(G)$ implies that the operator L_ψ equal to convolution by ψ is bounded on $L_2(G)$, i. e., ψ is *modere* in [10, p. 268] terminology, or said otherwise, ψ is a bounded element of the Hilbert algebra of G. Moreover the proof in [10, p. 269-270] shows that ψ modere allows φ to be chosen modere. Also $L_\psi = L_\varphi L_\varphi$. Therefore if we denote by δ the canonical trace on the Hilbert algebra of G and use the definition of δ_G, we obtain

$$\delta_G(\psi) = \delta(L_\varphi L_\varphi) = (\varphi, \varphi) = \int |\varphi|^2 = \psi(e).$$

Q. E. D.

Several times in this paper we shall be confronted with a function $\psi \in C_0(G)$ such that $\pi(\psi)$ is positive for all $\pi \in \mathrm{Rep}(G)$. It follows from this lemma that $\psi \in C^*(G)^+$ and $\delta_G(\psi) = \psi(e)$. We are ready now for the main theorem of this section.

THEOREM 4.4. — *Let* N *be unimodular and type* I, *and let* G *be a compact extension of* N, *i. e.,* N *is a closed normal subgroup of* G *and* G/N *is compact. Then* G *is unimodular and type* I, *and for all* $\varphi \in L_1(G) \cap L_2(G)$:

$$\int_G |\varphi(g)|^2\, dg = \int_{\hat{N}/G} \sum_{\sigma \in (\widehat{G_\gamma/N})^{\overline{\omega}_\gamma}} \| \pi_{\gamma,\sigma}(\varphi) \|_2^2 \dim \sigma \, d\bar{\mu}_N(\bar{\gamma}). \tag{4.2}$$

Proof. — It is enough to prove (4.2) when $\varphi \in C_0(G)$. Let $\psi = \varphi \star_G \varphi^*$ and $\theta = \psi|_N$. Then $\theta \in C_0(N) \cap P(N)$, and we have the following computation :

$$\begin{aligned}
\int_G |\varphi(g)|^2\, dg &= \psi(e) = \theta(e) = \delta_N(\theta) && \text{(Lemma 4.3)}\\
&= \int_{\hat{N}} \mathrm{Tr}\, \gamma(\theta)\, d\mu_N(\gamma) && \text{[formula (4.1)]}\\
&= \int_{\hat{N}/G} \int_{G/N} \mathrm{Tr}\,(\gamma \cdot g)(\theta)\, d\bar{g}\, d\bar{\mu}_N(\bar{\gamma}) && \text{(Theorem 2.2)}\\
&= \int_{\hat{N}/G} \mathrm{Tr}\,(\mathrm{Ind}_N^G \gamma)(\psi)\, d\bar{\mu}_N(\bar{\gamma}) && \text{(Theorem 3.2)}\\
&= \int_{\hat{N}/G} \mathrm{Tr}\,(\oplus_{\tau \in \check{G}_\gamma} n_\gamma(\tau)\, \pi_\tau)(\psi)\, d\bar{\mu}_N(\bar{\gamma}) && \text{(Lemma 4.2)}\\
&= \int_{\hat{N}/G} \Sigma_{\tau \in \check{G}_\gamma} \mathrm{Tr}\, \pi_\tau(\psi)\, n_\gamma(\tau)\, d\bar{\mu}_N(\bar{\gamma})\\
&= \int_{\hat{N}/G} \sum_{\sigma \in (\widehat{G_\gamma/N})^{\overline{\omega}_\gamma}} \mathrm{Tr}\,[\pi_{\gamma,\sigma}(\varphi)\, \pi_{\gamma,\sigma}(\varphi)^*] \dim \sigma\, d\bar{\mu}(\bar{\gamma}).
\end{aligned}$$

Examples. — 1. The group of rigid motions of Euclidean space : Let $n \geqq 2$ be an integer. Denote $K = SO(n)$, $N = \mathbf{R}^n$, and $G = N.K$, where K acts on N *via* rotations. We identify N with its dual. The orbits in $\hat{N}$ are then spheres centered at the origin, and $\hat{N}/G \cong$ a ray emanating from the origin. For $\gamma \in \hat{N}$, $r = |\gamma|$, we have $G_\gamma = G$ if $r = 0$ and $G_\gamma \cong N.SO(n-1)$ if $r > 0$. Moreover $d\bar{\mu}_N = c_n r^{n-1}\, dr$, where c_n is the surface area of S^{n-1}. Thus $\hat{G}$ is a " fibre space " with base $\cong$ the non-negative reals (carrying the measure $c_n r^{n-1}\, dr$) and fibre $\cong SO(n-1)^\wedge$ [with discrete mass $\dim \sigma$ corresponding to each $\sigma \in SO(n-1)^\wedge$]. Note that the exceptional fibre over $r = 0$ [namely $SO(n)^\wedge$] has Plancherel measure zero.

2. Central groups : In case $N = \text{Cent } G$ and G/N is compact, formula (4.2) specializes to the Plancherel formula for central groups recently published in [16].

3. Moore groups : G is called a Moore group if $\pi \in \text{Irr}(G) \Rightarrow \dim \pi < \infty$. By [26, Theorem 3] every such group is a projective limit of finite extensions of central groups. Combining example 2, Theorem 4.4 and [21, Theorem 5.4], one can in principle write down the Plancherel measure of an arbitrary Moore group.

4. $GL(n, F)$: Let F be a locally compact, non-discrete field, with $\text{char}(F) = 0$. Then $[F^* : F^{*n}] < \infty$. If we set

$$G_n = \{ g \in GL(n, F) : \det g \in F^{*n} \},$$

then G_n is a closed normal subgroup of finite index in $GL(n, F)$. But the map

$$(g_{ij}) \times a \to (a g_{ij}),$$
$$SL(n, F) \times F^* \to G_n$$

is a continuous and open homomorphism onto G_n with finite kernel. Hence Theorem 4.4 provides the means of computing the Plancherel measure of $GL(n, F)$, given that of $SL(n, F)$. We have worked out the cases for which the latter is known ($F = \mathbf{C}$; $F = \mathbf{R}$; F a p-adic field with character of the residue class field $\neq 2$ and $n = 2$). The precise formulas can be computed fairly easily. We omit the various details here.

5. A LARGE ORBIT. — In this section we replace the assumption of compactness of the extension by " near transitivity ". The theorem we prove is not the most general possible, but it includes the important special case we have in mind. Therefore to avoid unenlightening technical difficulties, we omit further generalizations at this time.

THEOREM 5.1. — *Let* N *be a locally compact abelian group and* H *a locally compact unimodular group of automorphisms of* N. *Let* $G = N.H$ *and assume* G *is unimodular* (*i. e., the modulus of the* H-*action on* N *is* 1) *and type I. Assume that* N *is regularly embedded in* G. *In addition, assume that there is an orbit* $\mathcal{O}$ *in* $\hat{N}$ *whose complement has Haar measure zero. Let* $\gamma_1 \in \mathcal{O}$ *and* $V = G_{\gamma_1} \cap H$. *Assume finally that* V *is unimodular. Then we have the following Plancherel formula : for all* $\varphi \in L_1(G) \cap L_2(G)$:

$$\int_G |\varphi(g)|^2 \, dg = \int_{\hat{V}} \| \pi_{\gamma_1, \sigma}(\varphi) \|_2^2 \, d\mu_V(\sigma).$$

Proof. — We first note that since

$$\int_N f(n.h) \, dn = \int_N f(n) \, dn, \qquad f \in L_1(N),$$

it follows from a simple argument using the Plancherel theorem on N that

$$\int_{\hat{N}} f(\gamma.h) \, d\gamma = \int_{\hat{N}} f(\gamma) \, d\gamma.$$

That is, H also leaves Haar measure on $\hat{N}$ invariant. If follows easily that H leaves $d\gamma|_{\mathcal{O}}$ invariant. Since N is regularly embedded, H/V is Borel isomorphic to $\mathcal{O}$. But H/V carries a unique (up to a constant) H-invariant Borel measure $d\bar{h}$. Therefore with a suitable normalization of $d\bar{h}$, we must have

$$\int_{\mathcal{O}} f(\gamma) \, d\gamma = \int_{H/V} f(\gamma_1 . h) \, d\bar{h}, \qquad f \in L_1(\hat{N}). \tag{5.1}$$

Now proceed in a manner reminiscent of the proof of Theorem 4.4. We may assume $\varphi \in C_0(G)$. Set $\psi = \varphi \star_G \varphi^*$ and $\theta = \psi|_N$. Then

$$\begin{aligned} \int_G |\varphi(g)|^2 \, dg &= \psi(e) = \theta(e) \\ &= \int_{\hat{N}} \hat{\theta}(\gamma) \, d\gamma && \text{[Lemma 4.3 and formula (4.1)]} \\ &= \int_{\mathcal{O}} \hat{\theta}(\gamma) \, d\gamma && \text{(Hypothesis of theorem)} \\ &= \int_{H/V} \hat{\theta}(\gamma_1 . h) \, d\bar{h} && \text{[formula (5.1)]} \\ &= \int_{\hat{V}} \operatorname{Tr} \pi_{\hat{\gamma}_1, \sigma}(\psi) \, d\mu_V(\sigma) && \text{(to prove).} \quad (5.2) \end{aligned}$$

Note — the fact that V is type I is a consequence of the hypotheses of the theorem. It remains to establish equation (5.2).

We compute $\operatorname{Tr} \pi_{\gamma_1, \sigma} = \operatorname{Tr} \operatorname{Ind}_{N.V}^{N.H} \gamma_1 \sigma$ *via* Theorem 3.2. Indeed that result gives

$$\operatorname{Tr} \pi_{\gamma_1, \sigma}(\psi) = \int_{H/V} \operatorname{Tr}\left[\int_{NV} \psi(h^{-1} nvh)\, \gamma_1(n)\, \sigma(v)\, dn\, dv\right] d\bar{h}.$$

Now define $\Omega_h(v) = \int_N \psi(h^{-1} nvh)\, \gamma_1(n)\, dn$. Then

$$\operatorname{Tr} \pi_{\gamma_1, \sigma}(\psi) = \int_{H/V} \operatorname{Tr}\left[\int_V \Omega_h(v)\, \sigma(v)\, dv\right] d\bar{h} = \int_{H/V} \operatorname{Tr} \hat{\Omega}_h(\sigma)\, d\bar{h}.$$

Therefore

$$\int_{\hat{V}} \operatorname{Tr} \pi_{\gamma_1, \sigma}(\psi)\, d\mu_V(\sigma) = \int_{\hat{V}} \int_{H/V} \operatorname{Tr} \hat{\Omega}_h(\sigma)\, d\bar{h}\, d\mu_V(\sigma)$$

$$(5.3) \qquad = \int_{H/V} \int_{\hat{V}} \operatorname{Tr} \hat{\Omega}_h(v)\, d\mu_V(\sigma)\, d\bar{h}$$

$$(5.4) \qquad = \int_{H/V} \Omega_h(e)\, d\bar{h}$$

$$= \int_{H/V} \int_N \psi(h^{-1} nh)\, \gamma_1(n)\, dn\, d\bar{h}$$

$$= \int_{H/V} \hat{\theta}(\gamma_1 . h)\, d\bar{h}.$$

We still need to justify (5.3) and (5.4). Let σ be *any* unitary representation of V. Then the operator

$$\int_{NV} \psi(h^{-1}nvh)\, \gamma_1(n)\, \sigma(v)\, dn\, dv$$

is positive. In fact, it is nothing more than $\Psi(h)$ in the notation of Theorem 3.2. That is, $\hat{\Omega}_h(\sigma)$ is a positive operator for every $\sigma \in \operatorname{Rep}(V)$. The switch of integration in (5.3) is valid then by Tonelli's theorem and equation (5.4) follows from Lemma 4.3. That completes the proof.

Examples. — 1. Let F be any locally compact non-discrete field and consider the group $G = F^2 . SL(2, F)$. Identify F^2 with its dual and let $SL(2, F)$ act on it *via*

$$(u, v) . \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ua + vc,\ ub + vd).$$

There are two orbits, namely $\{0\}$ and $F^2 - \{0\}$. The stability group of $\gamma_1 = (1, 0)$ is

$$V = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} : c \in F \right\}.$$

And so, all the hypothesis of Theorem 5.1 are satisfied. There are two families of irreducible representations of G : those trivial on F^2, i. e., the representations of SL (2, F); and the one parameter family

$$\pi_{\gamma_1, \sigma} = \mathrm{Ind}_{F^2.V}^{G} \gamma_1 \sigma, \qquad \sigma \in \hat{V}.$$

By Theorem 5.1, the former has Plancherel measure zero, while the Plancherel measure on the latter is the image of Haar measure on F under the map $\sigma \to \pi_{\gamma_1, \sigma}$.

2. $G_n = F^n.SL(n, F)$. Once again, we identify F^n with its dual and let SL (n, F) act by matrix multiplication on the right. As before, there are two orbits $\{0\}$ and $F^n - \{0\}$. It is easy to check that the stability group of the latter is $\cong F^{n-1}.SL(n-1, F)$. The irreducible representations of G_n are : those trivial on F^n, i. e., the representations of SL (n, F); and the family

$$\pi_{\gamma_1, \sigma}, \quad \sigma \in [F^{n-1}.SL(n-1, F)]^\wedge.$$

The Plancherel measure on $\hat{G}_n$ " agrees " with the Plancherel measure on $\hat{G}_{n-1}$; and so it follows easily by an induction argument that

$$\hat{G}_n \cong SL(n, F)^\wedge \cup SL(n-1, F)^\wedge \cup \ldots \cup SL(2, F)^\wedge \cup \hat{F};$$

each of these sets has Plancherel measure zero with the exception of the latter, on which the Plancherel measure is an image of Haar measure on F.

Remark. — So far as we know, these are the first examples of a specific computation of the Plancherel measure for a *class* of type I, non-CCR, unimodular groups. That they are type I follows from [11, Proposition 2.3] (³), that they are not CCR can be shown in a fashion similar to [1, p. 172]. For example, if $G = F^2.SL(2, F)$; consider $\gamma_\varphi = (\varphi, 0) \in F^2 = \hat{F}^2$, $\varphi \in F^*$. Then $G_{\gamma_\varphi} = F^2.V$ in the notation of example 1. Write γ'_φ for the character $\gamma_\varphi \otimes 1$ on $F^2.V$. It is easily seen that $\gamma'_\varphi \to 1$ as $\varphi \to 0$ in the dual topology of $F^2.V$ (the latter is the " Heisenberg group " over F). By continuity of induction [14, Theorem 4.1] $\mathrm{Ind}_{F^2.V}^{G} \gamma'_\varphi \to \mathrm{Ind}_{F^2.V}^{G} 1$.

(³) At least for $F = \mathbf{R}$ or $\mathbf{C}$. For other local fields it appears reasonable to expect these to be type I, although (as Professor Dixmier has reminded us) this remains unproven.

But the representations $\mathrm{Ind}_{F^2.V}^{G}\ \gamma'_{\varphi}$ are all equivalent to $\pi_{\gamma_1,1}$; and the representation $\mathrm{Ind}_{F^2.V}^{G}\ 1$ is reducible. It follows that every irreducible representation in the support of $\mathrm{Ind}_{F^2.V}^{G}\ 1$ is contained in the closure of the point $\pi_{\gamma_1,1}$. Therefore $\pi_{\gamma_1,1}$ is not a CCR representation.

6. The Plancherel theorem for non-unimodular groups. — We begin with a generalized theory of Hilbert algebras (due to Dixmier [7]) which allows us to drop the unimodularity of the groups whose Plancherel formula we wish to consider.

Definition. — A quasi-Hilbert algebra $\mathfrak{A}$ is an associative algebra over $\mathbf{C}$ provided with an inner product $(.,.)$, an automorphism $x \to \check{x}$, and an involution $x \to x^*$ which satisfy :

(i) $(x, \check{x}) \geqq 0$, $x \in \mathfrak{A}$;

(ii) $(x, x) = (x^*, x^*)$, $x \in \mathfrak{A}$;

(iii) $(xy, z) = (y, (x^*)^{\vee} z)$, $x, y, z \in \mathfrak{A}$;

(iv) the map $y \to xy$ is continuous, $x \in \mathfrak{A}$;

(v) the set $\{ xy + (xy)^{\vee} : x, y \in \mathfrak{A} \}$ is dense in $\mathfrak{A}$.

The topology referred to in (iv) and (v) is that deduced from the norm defined by the inner product. Let $\mathcal{H}_{\mathfrak{A}}$ be the Hilbert space completion of $\mathfrak{A}$. Then the map $x \to x^*$ may be extended uniquely to a continuous conjugate linear map J of $\mathcal{H}_{\mathfrak{A}}$ onto itself which satisfies

$$J^2 = 1, \qquad (J\, a, J\, b) = (b, a), \qquad a, b \in \mathcal{H}_{\mathfrak{A}}.$$

The symmetric operator $x \to \check{x}$ admits a closure Λ which is invertible and satisfies

$$\Lambda^{-1} = J\, \Lambda\, J, \qquad \Lambda = J\, \Lambda^{-1} J.$$

Now for each $x \in \mathfrak{A}$, the operators $y \to xy$ and $y \to yx$ may be extended to bounded linear operators U_x and V_x on $\mathcal{H}_{\mathfrak{A}}$. The map $x \to U_x$ (respectively $x \to V_x$) is an algebra isomorphism (respectively anti-isomorphism). Furthermore

$$U_x^* = U_{(x^*)^{\vee}}, \qquad V_x^* = V_{(x^{\vee})^*}, \tag{6.1}$$

$$JU_x\, J = V_{x^*}, \qquad JV_x\, J = U_{x^*}. \tag{6.2}$$

Let $\mathcal{U}(\mathfrak{A})$ [respectively $\mathcal{V}(\mathfrak{A})$] be the von Neumann algebra generated by the collection $\{ U_x : x \in \mathfrak{A} \}$ (respectively $\{ V_x : x \in \mathfrak{A} \}$). Then $\mathcal{U}(\mathfrak{A})$

and $\mathcal{V}(\mathfrak{A})$, called the left and right rings of $\mathfrak{A}$, are commutants of each other [7, Theorem 1].

An element $x \in \mathcal{H}_{\mathfrak{A}}$ is called left bounded if there exists a bounded operator U_x such that for all $a \in \mathfrak{A}$, $U_x a = V_a x$. Similarly, x is right bounded if there exists a bounded operator V_x such that $V_x a = U_a x$, $a \in \mathfrak{A}$. If x is left (respectively right) bounded, then $U_x \in \mathcal{U}(\mathfrak{A})$ [respectively $V_x \in \mathcal{V}(\mathfrak{A})$]. Of course, the elements of $\mathfrak{A}$ are left and right bounded, and the notation is consistent.

LEMMA 6.1. — *If $\Lambda^n x$ exists and is left bounded for all $n \in \mathbf{Z}$, then $\Lambda J x$ is left bounded and $U_x^* = U_{\Lambda J x}$.*

Proof. — Let $\mathfrak{A}_1$ be the set of all x for which $\Lambda^n x$ exists and is left bounded for all $n \in \mathbf{Z}$. With the inner product inherited from $\mathcal{H}_{\mathfrak{A}}$ and the operations $\Lambda|_{\mathfrak{A}_1}$, $J|_{\mathfrak{A}_1}$, $xy = U_x y$, $\mathfrak{A}_1$ is a quasi-Hilbert algebra which contains $\mathfrak{A}$ [7, proof of Proposition 6]. The lemma then follows from the first of equations (6.1).

Next suppose that $\mathcal{U}(\mathfrak{A})$ is a semi-finite von Neumann algebra. Then there exist positive invertible self-adjoint operators M and M′ (in general, unbounded), affiliated (4) with $\mathcal{V}(\mathfrak{A})$ and $\mathcal{U}(\mathfrak{A})$ respectively, such that

$$M' = JMJ, \qquad \Lambda = \text{closure of } M'M^{-1}$$

[28, Theorem 1]. Moreover M and Λ commute [28, proof of Theorem 1]. The operators M and M′ are not uniquely determined; but if M_1, M'_1 have the same properties, there exists a positive invertible self-adjoint operator C, affiliated with $\mathcal{U}(\mathfrak{A}) \cap \mathcal{V}(\mathfrak{A})$, such that M_1 and M'_1 are the closures of CM and CM′ [7, Proposition 3]. Now if we let $\mathfrak{m}_1$ denote the set of operators in $\mathcal{U}(\mathfrak{A})$ of the form $\Sigma_{i=1}^n U_{a_i} U_{b_i}^*$, a_i, b_i left bounded and in the domain $\mathcal{D}_M$ of M, then the map

$$\Sigma\, U_{a_i} U_{b_i}^* \to \Sigma\, (M\, a_i, M\, b_i)$$

may be extended to a faithful normal semi-finite trace t_M on $\mathcal{U}(\mathfrak{A})$. We shall refer to $t = t_M$ as the trace defined by M, and shall call M a tracing operator.

M and t_M determine each other uniquely. In fact, if t is a faithful normal semifinite trace on $\mathcal{U}(\mathfrak{A})$, let $\mathfrak{m}_t$ be its ideal of definition and put

$$\mathfrak{n} = \mathfrak{n}_t = \mathfrak{m}_t^{1/2} = \{ T \in \mathcal{U}(\mathfrak{A}) : t(T^* T) < \infty \}.$$

(4) An unbounded operator T on $\mathcal{H}$ is affiliated with a von Neumann algebra $\mathcal{U}$ on $\mathcal{H}$ if $ST \subseteq TS$ for all $S \in \mathcal{U}'$.

$\mathfrak{n}$ is a Hilbert algebra with inner product $(S, T) = t(ST^*)$ [8, Chapter I, § 6, Theorem 1]. Let $\mathcal{H}_{\mathfrak{n}}$ be the completion of $\mathfrak{n}$, and let $\mathcal{H}_{\mathfrak{A}}(t)$ be the set of all left bounded elements x in $\mathcal{H}_{\mathfrak{A}}$ for which $U_x \in \mathfrak{n}$. Then there exists a tracing operator M, which is the closure of its restriction to $\mathcal{H}_{\mathfrak{A}}(t)$, such that $t(U_a U_b^*) = (M a, M b)$, $a, b \in \mathcal{H}_{\mathfrak{A}}(t)$. In addition, the map $Mx \to U_x : \mathcal{H}_{\mathfrak{A}}(t) \to \mathfrak{n}$, may be extended to a unitary map Υ_M of $\mathcal{H}_{\mathfrak{A}}$ onto $\mathcal{H}_{\mathfrak{n}}$ which carries $\mathcal{U}(\mathfrak{A})$ into $\mathcal{U}(\mathfrak{n})$ and $\mathcal{V}(\mathfrak{A})$ into $\mathcal{V}(\mathfrak{n})$ (*see* [7, proof of Theorem 3] and [28, Lemma 2]). We call $\Upsilon = \Upsilon_M$ the isomorphism defined by M. Because $\mathcal{U}(\mathfrak{A})' = \mathcal{V}(\mathfrak{A})$ and $\mathcal{U}(\mathfrak{n})' = \mathcal{V}(\mathfrak{n})'$, Υ must carry $\mathcal{U}(\mathfrak{A})$ onto $\mathcal{U}(\mathfrak{n})$ and $\mathcal{V}(\mathfrak{A})$ onto $\mathcal{V}(\mathfrak{n})$. If $\mathcal{D}$ is a dense subset of $\mathcal{H}_{\mathfrak{A}}(t)$, then since M is a self-adjoint invertible operator, the set $\{M x : x \in \mathcal{D}\}$ is dense in $\mathcal{H}_{\mathfrak{A}}$.

For $T \in \mathfrak{n}$, we write $U(T)$ and $V(T)$ for U_T and V_T; i. e., $U(T) S = TS$, $V(T) S = ST$. Let $j : T \to T^*$ be the canonical involution on $\mathfrak{n}$.

LEMMA 6.2. — *If* $a \in \mathcal{H}_{\mathfrak{A}}(t)$ *we have :*

(i) $\Upsilon U_a \Upsilon^{-1} = U(U_a)$;
(ii) $\Upsilon V_{a^*} \Upsilon^{-1} = V(U_a^*)$;
(iii) $\Upsilon J = j \Upsilon$.

Proof. — Let $a, b \in \mathcal{H}_{\mathfrak{A}}(t)$. Then

$$\Upsilon U_a M b = \Upsilon M U_a b = U_{U_a b} = U_a U_b = U(U_a) \Upsilon M b,$$

where the first equality results from the fact that M is affiliated with $\mathcal{V}(\mathfrak{A}) = \mathcal{U}(\mathfrak{A})'$, and the second from [7, Lemma 1]. But we have remarked that $M \mathcal{H}_{\mathfrak{A}}(t)$ is dense in $\mathcal{H}_{\mathfrak{A}}$. Thus it follows that $\Upsilon U_a = U(U_a) \Upsilon$ which proves (i).

Assume for the moment that (iii) is proven. Then, using (6.2) :

$$\Upsilon V_{a^*} \Upsilon^{-1} = \Upsilon J U_a J \Upsilon^{-1} = j \Upsilon U_a \Upsilon^{-1} j = j U(U_a) j = V(U_a^*).$$

This proves (ii) and so it remains to prove (iii).

Let $\mathfrak{A}_1$ be the set of $b \in \mathcal{H}_{\mathfrak{A}}$ for which $\Lambda^n b$ exists and is left bounded for all $n \in \mathbf{Z}$. If $b \in \mathcal{D}_M \cap \mathfrak{A}_1$, then by Lemma 6.1 and [7, Lemma 10] :

$$\Upsilon J M b = \Upsilon M J \Lambda^{-1} b = U_{J\Lambda^{-1} b} = U_{\Lambda J b} = U_b^* = j U_b = j \Upsilon M b.$$

By the continuity of j and J, it suffices to show $\mathcal{D}_M \cap \mathfrak{A}_1$ is dense in $\mathcal{H}_{\mathfrak{A}}$. For this we argue as in the proof of [7, Théorème 2]. Let $M = \int_0^\infty \alpha \, dE_\alpha$

be the spectral decomposition of M. Because M and Λ commute, Λ^n and E_α commute, $n \in \mathbf{Z}$, $\alpha \geqq 0$. If $x \in \mathfrak{A}$, then $E_\alpha x$ is left bounded [7, Lemma 8] and $\Lambda^n E_\alpha x = E_\alpha \Lambda^n x$ exists for all $n \in \mathbf{Z}$. Furthermore $E_\alpha x \in \mathcal{D}_M$. Since the set $\{ E_\alpha x : x \in \mathfrak{A}, \alpha \geqq 0 \}$ is dense in $\mathcal{H}_{\mathfrak{A}}$, we conclude that $\mathcal{D}_M \cap \mathfrak{A}_1$ is dense in $\mathcal{H}_{\mathfrak{A}}$.

With this preliminary material, we now consider a separable quasi-Hilbert algebra $\mathfrak{A}$ for which $\mathcal{U}(\mathfrak{A})$ is type I. Let t be a faithful normal semifinite trace on $\mathcal{U}(\mathfrak{A})$ and M the corresponding tracing operator. According to well-known results on the decomposition of Hilbert algebras (e. g., [10, 6.7.7 and the arguments in 8.8.5]), there exists a standard Borel space Z, a positive measure μ on Z, and a measurable field $\zeta \to \mathcal{K}_\zeta \otimes \tilde{\mathcal{K}}_\zeta$ of Hilbert spaces such that : if $\mathfrak{n} = \mathfrak{n}_t$, then

$$\mathcal{H}_{\mathfrak{n}} = \int_Z^{\oplus} \mathcal{K}_\zeta \otimes \tilde{\mathcal{K}}_\zeta \, d\mu(\zeta),$$

$$\mathcal{U}(\mathfrak{n}) = \int_Z^{\oplus} \mathcal{L}(\mathcal{K}_\zeta) \otimes \mathbf{C}_\zeta \, d\mu(\zeta),$$

$$\mathcal{V}(\mathfrak{n}) = \int_Z^{\oplus} \mathbf{C}_\zeta \otimes \mathcal{L}(\tilde{\mathcal{K}}_\zeta) \, d\mu(\zeta),$$

and $\mathcal{U}(\mathfrak{n}) \cap \mathcal{V}(\mathfrak{n})$ is the algebra of diagonalizable operators. [If $\mathcal{H}$ is a Hilbert space, $\tilde{\mathcal{H}}$ denotes its conjugate space realized as follows : $\tilde{\mathcal{H}} = \{ \xi \in \mathcal{H} : \alpha . \xi = \bar{\alpha}\xi, \alpha \in \mathbf{C}, (\xi, \eta)^{\sim} = (\eta, \xi) \}$; if T is an operator or a representation, then $\tilde{T}$ denotes the same object considered as acting on $\tilde{\mathcal{H}}$]. More precisely, let us write $U(T) : S \to TS$ and $V(T) : S \to ST$, $T \in \mathfrak{n}$, $S \in \mathcal{U}(\mathfrak{A})$, rather than U_T, V_T. If $T = \int_Z^{\oplus} T(\zeta)\, d\mu(\zeta)$, then

$$U(T) = \int_Z^{\oplus} T(\zeta) \otimes 1_\zeta \, d\mu(\zeta),$$

$$V(T) = \int_Z^{\oplus} 1_\zeta \otimes T(\zeta)^{*\sim} \, d\mu(\zeta).$$

If j is the canonical involution on $\mathfrak{n}$ and $j(\zeta)$ is the canonical involution $\xi \otimes \eta \to \eta \otimes \xi$ on $\mathcal{K}_\zeta \otimes \tilde{\mathcal{K}}_\zeta$, then

$$j = \int_Z^{\oplus} j(\zeta)\, d\mu(\zeta).$$

The measure μ is of course uniquely defined up to equivalence.

Now the map $\Upsilon = \Upsilon_t$ is an isomorphism of $\mathcal{H}_{\mathfrak{A}}$ onto $\int_Z^{\oplus} \mathcal{K}_\zeta \otimes \tilde{\mathcal{K}}_\zeta \, d\mu(\zeta)$ which carries $\mathcal{U}(\mathfrak{A})$ to $\int^{\oplus} \mathcal{L}(\mathcal{K}_\zeta) \otimes \mathbf{C}_\zeta \, d\mu(\xi)$, $\mathcal{V}(\mathfrak{A})$ to

$$\int^{\oplus} \mathbf{C}_\zeta \otimes \mathcal{L}(\tilde{\mathcal{K}}_\zeta) \, d\mu(\zeta) \qquad \text{and} \qquad \mathcal{U}(\mathfrak{A}) \cap \mathcal{V}(\mathfrak{A})$$

to the algebra of diagonalizable operators. If $a \in \mathcal{H}_{\mathfrak{A}}(t)$ and

$$U_a = \int^{\oplus} U_a(\zeta) \, d\mu(\zeta) \in \int^{\oplus} \mathcal{K}_\zeta \otimes \tilde{\mathcal{K}}_\zeta \, d\mu(\zeta);$$

that is, if

$$\Upsilon M a = \int^{\oplus} U_a(\zeta) \, d\mu(\zeta),$$

then by Lemma 6.2 :

$$\Upsilon U_a \Upsilon^{-1} = \int^{\oplus} U_a(\zeta) \otimes 1_\zeta \, d\mu(\zeta),$$

$$\Upsilon V_{a*} \Upsilon^{-1} = \int^{\oplus} 1_\zeta \otimes U_a(\zeta)^{\sim} \, d\mu(\zeta).$$

Moreover, since Υ is an isometry, we have for all $a, b \in \mathcal{D}_M^1$,

$$(M a, M b) = \int \operatorname{Tr}(U_a(\zeta) U_b(\zeta)^*) \, d\mu(\zeta).$$

Note we have written $\mathcal{D}_M^1$ for the set of left bounded elements in $\mathcal{D}_M$ [previously called $\mathcal{H}_{\mathfrak{A}}(t)$]. By [7, p. 293-294], $\mathcal{D}_M^1$ is a dense subset of $\mathcal{H}_{\mathfrak{A}}$ and $M|_{\mathcal{D}_M^1}$ is a positive invertible symmetric operator whose closure is M.

We wish to apply these results to the quasi-Hilbert algebra associated to a locally compact group. Let G be locally compact with right Haar measure dg and modular function Δ,

$$\Delta(y) \int_G f(yx) \, dx = \int_G f(x) \, dx, \qquad y \in G, \quad f \in C_0(G),$$

$$\int_G f(x^{-1}) \Delta(x^{-1}) \, dx = \int_G f(x) \, dx, \qquad f \in C_0(G).$$

If we impose the operations

$$(f \star h)(x) = \int_G f(xy^{-1}) h(y) \, dy,$$

$$f^*(x) = \bar{f}(x^{-1}) \Delta(x)^{-1/2},$$
$$f^{\vee}(x) = f(x) \Delta(x)^{+1/2}$$

and the inner product

$$(f, h) = \int_G f(x)\,\bar{h}(x)\,dx,$$

then $C_0(G)$ becomes a quasi-Hilbert algebra whose completion is $L_2(G)$. Note (in this section only) f^* differs from the usual adjoint (defined in Theorem 3.2 for example) by a factor of $\Delta^{-1/2}$. We can define the left and right regular representations of G as follows :

$$\lambda(x) f(y) = \Delta(x)^{-1/2} f(x^{-1}y), \qquad x, y \in G, \quad f \in L_2(G);$$
$$\rho(x) f(y) = f(yx), \qquad x, y \in G, \quad f \in L_2(G).$$

It is well-known that $\lambda(G)'' = \mathfrak{U}(C_0(G))$, $\rho(G)'' = \mathfrak{V}(C_0(G))$, facts which are easily seen from the equations

$$(6.2\,a) \qquad (f \star h)(x) = \int \Delta(y)^{-1} f(y)\, h(y^{-1}x)\,dy = \lambda(\Delta^{-1/2} f)\, h(x),$$

$$(6.2\,b) \qquad (f \star h^*)(x) = \int f(xy)\,\bar{h}(y)\,\Delta(y)^{-1/2}\,dy = \rho\left(\Delta^{-1/2}\bar{h}\right) f(x).$$

Before stating the Plancherel theorem for non-unimodular groups, we need one more idea. Let $\hat{G}_t$ be the subset of $\hat{G}$ consisting of traceable representations (*see* [10, 6.6] for this notion). Then $\pi \in \hat{G}_t$ if and only if $\pi(C^*(G))$ contains the compact operators [10, 6.7.5].

Lemma 6.3. — *$\hat{G}_t$ is a standard Borel subset of $\hat{G}$ on which the Mackey, Davies, and topological Borel structures coincide. Every Borel measure on $\hat{G}_t$ is canonical (i. e., arises from a central decomposition of a unitary representation), and $\hat{G}_t$ is a T_0 space which is invariant under every automorphism of* G.

Proof. — That $\hat{G}_t$ is a standard subset of $\hat{G}$ on which the Mackey and topological Borel structures coincide was shown by Guichardet [17, p. 22]. Since the Davies Borel structure lies between these in general [6, Theorem 4.1], it must also be the same. Therefore, by the main theorem of [13], every Borel measure on $\hat{G}_t$ is canonical. The natural map $\hat{G} \to \mathrm{Prim}(G)$ sending $\pi \to$ kernel π [the kernel formed in $C^*(G)$] is continuous and open [10, 3.1.5]. The image of $\hat{G}_t$ is precisely the set of all $J \in \mathrm{Prim}(G)$ which are kernels of precisely one irreducible representation. It follows easily that the restriction of $\pi \to$ kernel π to $\hat{G}_t$ is a homeomorphism. Since Prim (G) is a T_0 space, so therefore is $\hat{G}_t$. Finally, if α is an automorphism of G, then

$$\alpha\pi\,(C^*(G)) = \pi\,(\alpha^{-1}(C^*(G)) = \pi\,(C^*(G)).$$

Hence $\pi\,(C^*(G))$ contains the compact operators if and only if $\alpha\pi\,(C^*(G))$ does likewise.

Now comes the Plancherel theorem for non-unimodular groups.

THEOREM 6.4. — *Let* G *have a type I regular representation. Then there exists a positive standard Borel measure* μ *on* $\hat{G}$ *concentrated on* $\hat{G}$, *a measurable field* $\zeta \to \mathcal{H}_\zeta \otimes \tilde{\mathcal{H}}_\zeta$ *on* $\hat{G}$, *a measurable field of representations* $\zeta \to \pi_\zeta$ *acting on* $\mathcal{H}_\zeta$ *such that* $\pi_\zeta \in \zeta$ *for* μ*-almost all* ζ, *and an isomorphism* Υ *of* $L_2(G)$ *onto* $\int_{\hat{G}}^{\oplus} \mathcal{H}_\zeta \otimes \tilde{\mathcal{H}}_\zeta \, d\mu(\zeta)$ *which carries* λ *to* $\int^{\oplus} \pi_\zeta \otimes 1_\zeta \, d\mu(\zeta)$ *and* ρ *to* $\int^{\oplus} 1_\zeta \otimes \tilde{\pi}_\zeta \, d\mu(\zeta)$. *There exists a positive invertible self-adjoint operator* M *on* $L_2(G)$ *such that : whenever* f, $h \in \mathcal{D}_M$, $\Delta^{-1/2} f$, $\Delta^{-1/2} h \in L_1(G)$,

$$\Upsilon(Mf)(\zeta) = \pi_\zeta(\Delta^{-1/2} f), \tag{6.3}$$

$$\int_G Mf(x)\,\overline{Mh}(x)\,dx = \int_{\hat{G}} \operatorname{Tr}(\pi_\zeta(\Delta^{-1/2} f)\,\pi_\zeta(\Delta^{-1/2} h)^*)\,d\mu(\zeta). \tag{6.4}$$

The measure μ *is uniquely determined to within equivalence;* M *is uniquely determined to within multiplication by a positive invertible self-adjoint operator affiliated with* $\mathcal{U}(C_0(G)) \cap \mathcal{V}(C_0(G))$; *and* μ *and* M *determine each other uniquely. Finally, if the set* $\mathcal{D}_M^2 = \{ f \in \mathcal{D}_M : \Delta^{-1/2} f \in L_1(G) \}$ *is dense in* $L_2(G)$, *then* μ *is concentrated in* $\hat{G}_t$.

Proof. — We apply the preceding results on decomposition of quasi-Hilbert algebras to $\mathfrak{A} = C_0(G)$. There exists a standard Borel space Z, a positive measure μ on Z, a measurable field $\zeta \to \tilde{\mathcal{H}}_\zeta \otimes \mathcal{H}_\zeta$, and an isomorphism Υ of $L_2(G)$ onto $\int_Z^{\oplus} \mathcal{H}_\zeta \otimes \tilde{\mathcal{H}}_\zeta \, d\mu(\zeta)$ which carries

$$\mathcal{U}(C_0(G)) \quad \text{to} \quad \int^{\oplus} \mathcal{L}(\mathcal{H}_\zeta) \otimes \mathbf{C}_\zeta \, d\mu(\zeta), \tag{6.5}$$

$$\mathcal{V}(C_0(G)) \quad \text{to} \quad \int^{\oplus} \mathbf{C}_\zeta \otimes \mathcal{L}(\tilde{\mathcal{H}}_\zeta)\, d\mu(\zeta) \tag{6.6}$$

and $\mathcal{U}(C_0(G)) \cap \mathcal{V}(C_0(G))$ to the algebra of diagonalizable operators. Let M be a tracing operator defining Υ and t the corresponding trace. If $f \in \mathcal{D}_M^1$, then $U_f \in \mathfrak{n}_t$ and we have $U_f = \int^{\oplus} U_f(\zeta)\, d\mu(\zeta)$. By Lemma 6.2

then

$$\Upsilon \mathrm{U}_f \Upsilon^{-1} = \int^{\oplus} \mathrm{U}_f(\zeta) \otimes 1_\zeta \, d\mu(\zeta),$$

$$\Upsilon \mathrm{V}_{f^*} \Upsilon^{-1} = \int^{\oplus} l_\zeta \otimes \mathrm{U}_f(\zeta)^\sim \, d\mu(\zeta).$$

Because $\Upsilon \lambda(x) \Upsilon^{-1}$ is a decomposable operator for all $x \in \mathrm{G}$, $x \to \Upsilon \lambda(x) \Upsilon^{-1}$ is a representation, and by (6.5) we may choose an operator field $\zeta \to \pi_\zeta(x)$ such that

$$\Upsilon \lambda(x) \Upsilon^{-1} = \int^{\oplus} \pi_\zeta(x) \otimes 1_\zeta \, d\mu(\zeta) \tag{6.7}$$

and $x \to \pi_\zeta(x)$ is an irreducible representation for almost all ζ. We may replace the π_ζ which are not representations by the one-dimensional identity representation. To obtain the decomposition of ρ, note that $\rho(x) = \mathrm{J} \lambda(x) \mathrm{J}$. Then by Lemma 6.2 and (6.7) :

$$\begin{aligned} \Upsilon \rho(x) \Upsilon^{-1} &= \Upsilon \mathrm{J} \lambda(x) \mathrm{J} \Upsilon^{-1} \\ &= j \Upsilon \lambda(x) \Upsilon^{-1} j \\ &= \int^{\oplus} 1_\zeta \otimes \tilde{\pi}_\zeta(x) \, d\mu(\zeta). \end{aligned}$$

Next, using the fact that

$$\Upsilon \mathrm{U}_f \Upsilon^{-1} = \Upsilon \lambda(\Delta^{-1/2} f) \Upsilon^{-1} = \int \Delta^{-1/2}(x) f(x) \Upsilon \lambda(x) \Upsilon^{-1} \, dx,$$

and using (6.7) we can compute : for g, $h \in \mathcal{D}^1_{\mathrm{M}}$ and $\Delta^{-1/2} f \in \mathrm{L}_1(\mathrm{G})$,

$$\begin{aligned} &\int_{\mathrm{Z}} \mathrm{Tr}\,((\pi_\zeta(\Delta^{-1/2} f) \otimes 1_\zeta) \mathrm{U}_g(\zeta) \mathrm{U}_h(\zeta)^*) \, d\mu(\zeta) \\ &= \int_{\mathrm{Z}} \mathrm{Tr}\,(\pi_\zeta(\Delta^{-1/2} f) \mathrm{U}_g(\zeta) \mathrm{U}_h(\zeta)^*) \, d\mu(\zeta) \\ &= \int_{\mathrm{Z}} \int_{\mathrm{G}} \Delta^{-1/2}(x) f(x) \, \mathrm{Tr}\,(\pi_\zeta(x) \mathrm{U}_g(\zeta) \mathrm{U}_h(\zeta)^*) \, dx \, d\mu(\zeta) \\ &= \int_{\mathrm{G}} \Delta^{-1/2}(x) f(x) \int_{\mathrm{Z}} \mathrm{Tr}\,(\pi_\zeta(x) \mathrm{U}_g(\zeta) \mathrm{U}_h(\zeta)^*) \, d\mu(\zeta) \, dx \\ &= \int_{\mathrm{G}} \Delta^{-1/2}(x) f(x) \int_{\mathrm{Z}} \mathrm{Tr}\,((\pi_\zeta(x) \otimes 1_\zeta) \mathrm{U}_g(\zeta) \mathrm{U}_h(\zeta)^*) \, d\mu(\zeta) \, dx \\ &= \int_{\mathrm{G}} \Delta^{-1/2}(x) f(x) (\lambda(x) \mathrm{M}g, \mathrm{M}h) \, dx \\ &= (\lambda(\Delta^{-1/2} f) \mathrm{M}\, g, \mathrm{M}\, h) \\ &= (\mathrm{U}_f \mathrm{M}\, g, \mathrm{M}\, h). \end{aligned}$$

Note the use of Fubini's theorem is valid since $\Delta^{-1/2} f \in L_1(G)$ and $\zeta \to \operatorname{Tr}(U_g(\zeta)\, U_h(\zeta)^*)$ is μ-integrable. We conclude that

$$\Upsilon U_f \Upsilon^{-1} = \int_Z^{\oplus} \pi_\zeta(\Delta^{-1/2} f) \otimes 1_\zeta \, d\mu(\zeta)$$

and thus that

$$U_f = \int_Z^{\oplus} \pi_\zeta(\Delta^{-1/2} f)\, d\mu(\zeta).$$

If also $f \in \mathcal{O}_M^1$, then $U_f = \Upsilon M f$ and we have

$$\Upsilon M f(\zeta) = \pi_\zeta(\Delta^{-1/2} f) \quad \text{for almost all } \zeta.$$

Moreover, by (6.2 *a*) whenever $\Delta^{-1/2} f \in L_1(G)$, then U_f is left bounded. Hence $\mathcal{O}_M^2 \subseteq \mathcal{O}_M^1$. This proves (6.3). Since Υ is an isometry, (6.4) follows immediately.

Next if M' is another tracing operator, $M' = CM$ where C is a positive invertible self-adjoint operator affiliated with $\mathcal{U}(C_0(G)) \cap \mathcal{V}(C_0(G))$. Then C is diagonalizable, $C = \int_Z^{\oplus} c(\zeta)\, d\mu(\zeta)$ where $c(\zeta)$ is a positive Borel function on Z. In that case, (6.4) is preserved with M replaced by M' and μ by $\mu' = c\,\mu$. It follows easily that M and μ determine each other uniquely.

Finally, we need only transfer the measure μ to $\hat{G}$ and choose the representations π_ζ so that $\pi_\zeta \in \zeta \in \hat{G}$ — all of which we can do by arguing as in the proof of [10, 8.4.2]. That μ is concentrated in $\hat{G}$ follows from [10, 8.4.8] (*see* also [10, 7.3.6 (ii)]. To show that when $\mathcal{O}_M^2$ is dense in $L_2(G)$ μ is actually concentrated in $\hat{G}_t$, we reason as follows. Let $E = \hat{G} - \hat{G}_t$ and $f \in \mathcal{O}_M^2$. Then for any $\pi \in E$ either $\pi(\Delta^{-1/2} f) = 0$ or $\operatorname{Tr}(\pi(\Delta^{-1/2} f)\, \pi(\Delta^{-1/2} f)^*) = \infty$ [10, 4.1.10]. Therefore $\pi(\Delta^{-1/2} f) = 0$ for μ-almost all $\pi \in E$. But then $\nu = \mu.$(characteristic function of $\hat{G}_t$) is a standard Borel measure on $\hat{G}$ which satisfies all the conditions of the theorem. Therefore ν is equivalent to μ which implies $\mu(E) = 0$, that is μ is concentrated in $\hat{G}_t$.

Remarks 1. — If G is unimodular, then $\Delta = 1$, $\Lambda = 1$, and we may choose $M = 1$. The corresponding Plancherel measure $\mu = \mu_G$ is uniquely determined up to a constant (which depends only on the normalization of Haar measure on G). We thus obtain the usual Plancherel theorem [10, 18.8.1, 18.8.2] as a special case of our Theorem 6.4.

2. Moore [27] and Tatsuuma [31] have each independently proven a slightly different version of the Plancherel formula in the non-unimodular case. We indicate briefly here how their and our results are essentially the same. Replacing M by $M\Delta^{1/2}$ our Plancherel formula becomes

$$\int |Mf|^2 = \int \operatorname{Tr}(\pi_\zeta(f)\,\pi_\zeta(f)^*)\,d\mu(\zeta).$$

Let $M^{-1} = \int^{\oplus} D(\zeta)\,d\mu(\zeta)$ be a decomposition of M^{-1}; the $D(\zeta)$ are positive invertible self-adjoint operators. Letting $f_1 = Mf$, we obtain

$$\int |f_1|^2 = \int \operatorname{Tr}(D(\zeta)\,\pi_\zeta(f_1)\,\pi_\zeta(f_1)^*\,D(\zeta))\,d\mu(\zeta).$$

Setting $h(g) = \int f_1(gg_1)\,\bar{f}_1(g_1)\,dg_1$, we get Moore's formula [27] :

$$h(e) = \int \operatorname{Tr}(D(\zeta)\,\pi_\zeta(h)\,D(\zeta))\,d\mu(\zeta).$$

Note that although M has a degree of arbitrariness as general as a positive invertible self-adjoint operator, the operators $D(\zeta)$ are uniquely determined up to a constant. Moore has made some progress in computing these $D(\zeta)$ for certain kinds of Lie groups.

The technique of Moore and Tatsuuma is more constructive [they get the $D(\zeta)$ rather than M], but they need to make the additional assumption that $G_0 = \{g \in G : \Delta_G(g) = 1\}$ is type I and regularly embedded. Our techniques do not seem to require that assumption. However, it appears that the presence of some « reasonable » normal subgroup may be necessary in order to show $\mathcal{D}^2_M$ is dense. For example, L. Pukanszky has proven the density for connected solvable Lie groups (*in* [29, p. 592]) by using the unipotent radical. E. Carlton has done likewise for certain p-adic solvable groups. Unfortunately, we do not know to handle the general case at this time.

7. The projective Plancherel theorem.

— In order to deal with the Plancherel formula for group extensions we must be able to handle the Plancherel theorem, not only for non-unimodular groups, but also for projective representations. We develop that material in this section.

Let G be locally compact as usual, and suppose ω is a normalized [$\omega(x, x^{-1}) = 1$, $x \in G$] multiplier on G. Consider the left regular ω-representation λ_ω of G and also the right regular $\bar{\omega}$-representation ρ_ω of G.

These act on $L_2(G)$ as follows : for $x \in G$, $f \in L_2(G)$,

$$\lambda_\omega(x) f(y) = \Delta(x)^{-1/2} \omega(y^{-1}, x) f(x^{-1} y),$$
$$\rho_\omega(x) f(y) = \omega(x^{-1}, y^{-1}) f(yx).$$

We can associate these projective representations of G with ordinary representations of a group extension in the usual fashion. Let $G(\overline{\omega}) = \mathbf{T} \times G$ with the multiplication

$$(t, x)(s, y) = (ts\, \overline{\omega}(x, y), xy);$$

and the topology obtained by providing $\mathbf{T} \times G$ with the product of Haar measures and applying Weil's theorem [23, Theorem 7.1]. $G(\overline{\omega})$ is a locally compact group having $\mathbf{T}$ as a compact normal subgroup and $G(\overline{\omega})/\mathbf{T} \cong G$.

Consider the projection P on $L_2(G(\overline{\omega}))$ defined by

$$\mathrm{P} f(t, x) = t \int_{\mathbf{T}} s^{-1} f(s, x)\, ds, \qquad f \in L^2(G(\overline{\omega})).$$

Then $\operatorname{im} P$ is the subspace of $L_2(G(\overline{\omega}))$ consisting of all $f \in L_2(G(\overline{\omega}))$ satisfying

$$f(t, x) = t f(1, x), \quad \text{almost all } x \in G.$$

The map $f \to f(1, .)$ is an isometric isomorphism of $\operatorname{im} P$ with $L_2(G)$, and we use this map to identifiy $L_2(G)$ with $\operatorname{im} P$. Suppose λ and ρ denote the left and right regular representations of $G(\overline{\omega})$. P commutes with λ and ρ, and one checks readily that

$$\mathrm{P}\, \lambda(t, x) = t^{-1} \lambda_\omega(x)\, \mathrm{P},$$
$$\mathrm{P}\, \rho(t, x) = t\, \rho_\omega(x)\, \mathrm{P}.$$

Now assume that $G(\overline{\omega})$ has a type I regular representation. Then we can apply Theorem 6.4 to $G(\overline{\omega})$:

$$L_2(G(\overline{\omega})) \cong \int^{\oplus} \mathcal{K}_\zeta \otimes \tilde{\mathcal{K}}_\zeta\, d\mu_{G(\overline{\omega})}(\zeta), \qquad \lambda(t, x) = \int^{\oplus} \pi_\zeta(t, x) \otimes 1_\zeta\, d\mu_{G(\overline{\omega})}(\zeta),$$
$$\rho(t, x) = \int^{\oplus} 1_\zeta \otimes \pi_\zeta(t, x)^\sim\, d\mu_{G(\overline{\omega})}(\zeta)$$

and

$$\int_{G(\overline{\omega})} \mathrm{M} f(t, x)\, \overline{\mathrm{M}\, h}(t, x)\, dt\, dx = \int_{G(\overline{\omega})^\wedge} \operatorname{Tr}(\pi_\zeta(\Delta^{-1/2} f)\, \pi_\zeta(\Delta^{-1/2} h)^*)\, d\mu_{G(\overline{\omega})}(\zeta).$$

We have written Δ for the modular function $\Delta_{G(\overline{\omega})}(t, x) = \Delta_G(x)$. But P commutes with λ and ρ; so P is diagonalizable, $P = \int^{\oplus} P(\zeta)\, d\mu(\zeta)$. Because P is a projection $P(\zeta) = 0$ or 1 almost everywhere. Set

$$E = \{ \zeta \in G(\overline{\omega})^{\wedge} : P(\zeta) = I \}.$$

E is well-defined up to a set of measure zero. Now $P\lambda(t, x) = t^{-1} P\lambda(1, x)$; hence

$$\int_E^{\oplus} \pi_\zeta(t, x) \otimes 1_\zeta \, d\mu_{G(\overline{\omega})}(\zeta) = t^{-1} \int_E^{\oplus} \pi_\zeta(1, x) \otimes 1_\zeta \, d\mu_{G(\overline{\omega})}(\zeta).$$

Therefore $E = \{ \pi \in G(\overline{\omega})^{\wedge} : \pi(t, x) = t^{-1} \pi(1, x) \}$. It is straightforward to check that corresponding to any ω-representation σ of G, the representation $\pi(t, x) = t^{-1} \sigma(x)$ is an ordinary representation of $G(\overline{\omega})$. In fact the correspondence $\sigma \to \pi$ is a one-to-one correspondence between all ω-representations of G and ordinary representations of $G(\overline{\omega})$ having the property $\pi(t, x) = t^{-1} \pi(1, x)$. Note that unitary equivalence and irre-irreducibility are preserved by this correspondence. It follows that we may identify

$$E = \hat{G}^{\omega} = \text{equivalence classes of irreducible } \omega\text{-representations of } G.$$

The Borel structure of $\hat{G}^{\omega}$ that is carried over from E agrees with the Mackey Borel structure defined directly.

THEOREM 7.1. — *Let ω be a normalized multiplier on* G *and suppose* $G(\overline{\omega})$ *has a type I regular representation. Then there exists a positive standard Borel measure $\mu = \mu_{G,\omega}$ on $\widehat{G(\overline{\omega})}$ concentrated on $\hat{G}^{\omega}$, a measurable field $\zeta \to \mathcal{H}_\zeta \otimes \tilde{\mathcal{H}}_\zeta$ on $\widehat{G(\overline{\omega})}$, a measurable field of representations $\zeta \to \pi_\zeta$ acting on $\mathcal{H}_\zeta$ such that $\pi_\zeta \in \zeta$ for μ-almost all ζ, and an isomorphism Υ of* $L_2(G)$ *onto $\int_{\widehat{G(\overline{\omega})}}^{\oplus} \mathcal{H}_\zeta \otimes \tilde{\mathcal{H}}_\zeta \, d\mu(\zeta)$ which carries λ_ω to $\int^{\oplus} \pi_\zeta \otimes 1_\zeta \, d\mu(\zeta)$ and ρ_ω to $\int^{\oplus} 1_\zeta \otimes \tilde{\pi}_\zeta \, d\mu(\zeta)$. There exists a positive invertible self-adjoint operator* M *on* $L_2(G)$ *such that : whenever $f, h \in \mathcal{D}_M$, $\Delta^{-1/2} f$, $\Delta^{-1/2} h \in L_1(G)$:*

$$\Upsilon M f(\zeta) = \pi_\zeta(\Delta^{-1/2} f), \tag{7.1}$$

$$\int_G M f(x) \overline{M h}(x)\, dx = \int_{\hat{G}^{\omega}} \operatorname{Tr}(\pi_\zeta(\Delta^{-1/2} f)\, \pi_\zeta(\Delta^{-1/2} h)^*)\, d\mu(\zeta). \tag{7.2}$$

The measure μ is uniquely determined to within equivalence; M *is uniquely determined to within multiplication by a positive invertible self-adjoint operator affiliated with $\mathcal{U}(C_0(G(\overline{\omega}))) \cap \mathcal{V}(C_0(G(\overline{\omega})))$; and μ and* M *determine each other uniquely.*

Proof. — Set $\mu = \mu_{G(\overline{\omega})} \cdot \chi_E$, where χ_E denotes the characteristic function of E. Then by what we have already said, if $x \in G$,

$$\lambda_\omega(x) = \int_{\hat{G}^\omega} \pi_\zeta(x) \otimes 1_\zeta \, d\mu(\zeta),$$

$$\rho_\omega(x) = \int_{\hat{G}^\omega} 1_\zeta \otimes \pi_\zeta(x)^{\sim} \, d\mu(\zeta).$$

Note that if π_ζ is an ω-representation, then $\tilde{\pi}_\zeta$ is an $\overline{\omega}$-representation. More generally, the mapping $\pi \to \tilde{\pi}$ supplies a one-to-one Borel correspondence between ω and $\overline{\omega}$-representations of G.

Now let M_1 be the tracing operator that defines the isomorphism Υ_1 of $L_2(G(\overline{\omega}))$ onto $\int_{\widehat{G(\overline{\omega})}} \mathcal{H}_\zeta \otimes \tilde{\mathcal{H}}_\zeta \, d\mu_{G(\overline{\omega})}(\zeta)$. If $f \in \mathcal{D}_{M_1}$ then

$$\Upsilon_1(M_1 f)(\zeta) = \pi_\zeta(\Delta^{-1/2} f),$$

But M_1 and P commute [since M_1 is affiliated with $\mathcal{V}(C_0(G(\overline{\omega})))$ and P commutes ρ]. Therefore $f \in \mathcal{D}_{M_1} \Rightarrow P f \in \mathcal{D}_{M_1}$. Furthermore if $f \in \mathcal{D}^1_{M_1}$, then $U_f \in \mathfrak{n}$ and $U_{Pf} = PU_f \in \mathfrak{n}$ [7, p. 293-294]. Hence $P f \in \mathcal{D}^1_{M_1}$ as well. It follows that if we take

$$\Upsilon = \Upsilon_1 |_{L_2(G)}, \qquad M = M_1 |_{L_2(G)},$$

then equations (7.1) and (7.2) are fulfilled. We observe that for $\Delta^{-1/2} f \in L_1(G)$, $\sigma \in \hat{G}^\omega$, if we set $F(t, x) = tf(x)$, $\pi(t, x) = t^{-1}\sigma(x)$, then

$$\sigma(\Delta^{-1/2} f) = \pi(\Delta^{-1/2} F) = \int_{G(\overline{\omega})} \Delta^{-1/2}(t, x) \, F(t, x) \, \pi(t, x) \, dt \, dx$$

$$= \int_G \Delta^{-1/2}(x) \, f(x) \, \sigma(x) \, dx.$$

The correspondence between M and μ is easily derived as in the proof of Theorem 6.4. This completes the argument.

Remarks. — 1. It is an easy exercise to check that the results of Theorem 7.1 depend only on the class of ω. That is, if ω_1 is another normalized multiplier, similar to ω, then the objects $G(\overline{\omega})$, $\hat{G}^\omega$, $\mu_{G,\omega}$ transform « isomorphically » onto $G(\overline{\omega}_1)$, $\hat{G}^{\omega_1}$, μ_{G,ω_1}.

2. If G is unimodular, we may choose $M = 1$ and then $\mu_{G,\omega}$ is uniquely determined up to a constant.

3. With an appropriate density assumption, we can show that $\mu_{G,\omega}$ is concentrated in $\widehat{G(\omega)_t} \cap \hat{G}^{\omega}$, but we shall not need that in the following.

Example. — Let γ be a non-zero real number, $G = \mathbf{R}^2$, and consider the function on $G \times G$ defined by

$$\omega_\gamma((x, y), (\xi, \eta)) = e^{-i\gamma(\xi y - x\eta)/2}.$$

It is straightforward to check that ω is a normalized multiplier on G. It is well-known that there is only one irreducible ω_γ-representation σ_γ of G [24, § 9]. It can be realized on $L_2(\mathbf{R})$ as follows

$$\sigma_\gamma(x, y) f(u) = e^{i\gamma y(u+x/2)} f(u + x), \qquad f \in L_2(\mathbf{R}).$$

The group $\mathbf{R}^2(\overline{\omega}_\gamma)$ is easily seen to be isomorphic to a three-dimensional nilpotent Lie group, and so is type I. Therefore, by Theorem 7.1, there exists a constant c_γ such that

$$\int_G |\varphi(x, y)|^2 \, dx \, dy = c_\gamma \operatorname{Tr}[\sigma_\gamma(\varphi) \sigma_\gamma(\varphi)^*], \qquad \varphi \in L_1(G) \cap L_2(G).$$

We wish to compute c_γ.

Now

$$\sigma_\gamma(\varphi) = \iint \varphi(x, y) \sigma_\gamma(x, y) \, dx \, dy, \qquad \varphi \in L_1(G) \cap L_2(G).$$

Therefore

$$\begin{aligned} \sigma_\gamma(\varphi) f(u) &= \iint \varphi(x, y) e^{i\gamma y(u+x/2)} f(u + x) \, dx \, dy \\ &= \iint e^{i\gamma y(u+x)/2} \varphi(x - u, y) f(x) \, dx \, dy \\ &= \int \Phi_\varphi(u, x) f(x) \, dx, \end{aligned}$$

where

$$\Phi_\varphi(u, x) = \int e^{i\gamma y(u+x)/2} \varphi(x - u, y) \, dy.$$

It follows from Lemma 3.1 that

$$\operatorname{Tr}[\sigma_\gamma(\varphi) \sigma_\gamma(\varphi)^*] = \int \Psi(x) \, dx,$$

where

$$\Psi(x) = \int |\Phi_\varphi(x, z)|^2 \, dz.$$

But an easy calculation shows that

$$\int |\Phi_\varphi(x, z)|^2 \, dz = \int e^{i\gamma y x} \varphi_1(y) \, dy,$$

where

$$\varphi_1(y) = \iint e^{i\gamma y z/2} \varphi(z, y + \eta) \bar{\varphi}(z, \eta) \, dz \, d\eta.$$

φ_1 is easily seen to be integrable and continuous positive-definite. Therefore

$$\begin{aligned}
\operatorname{Tr}[\sigma_\gamma(\varphi) \sigma_\gamma(\varphi)^*] &= \iint e^{i\gamma y x} \varphi_1(y) \, dy \, dx \\
&= \int \hat{\varphi}_1(\gamma x) \, dx \\
&= |\gamma|^{-1} \int \hat{\varphi}_1(x) \, dx \\
&= 2\pi |\gamma|^{-1} \varphi_1(0) \\
&= 2\pi |\gamma|^{-1} \iint |\varphi(z, \eta)|^2 \, dz \, d\eta.
\end{aligned}$$

Hence $c_\gamma = |\gamma| / 2\pi$.

8. Extensions of the center. — We give in this section another case in which we can compute explicity the Plancherel formula for a group extension. We maintain the notation of paragraphs 4 and 5 with the additional assumption $N = \operatorname{Cent}(G)$. The action of G on $\hat{N}$ is then the complete opposite of the transitive case (§ 5), namely every point is an orbit.

Theorem 8.1. — *Let* G *be unimodular and type I, and set* $N = \operatorname{Cent}(G)$. *Assume that for almost all* $\gamma \in \hat{N}$, *the group* $(G/N)(\bar{\omega}_\gamma)$ *is type I. Then for all* $\varphi \in L_1(G) \cap L_2(G)$:

$$\int_G |\varphi(g)|^2 \, dg = \int_{\hat{N}} \int_{(\widehat{G/N})^{\bar{\omega}_\gamma}} \| \pi_{\gamma, \sigma}(\varphi) \|_2^2 \, d\mu_{G/N, \bar{\omega}_\gamma}(\sigma) \, d\gamma.$$

Proof. — We first note that G/N is unimodular, so that $\mu_{G/N, \bar{\omega}_\gamma}$ is uniquely defined. As in paragraphs 4 and 5, we may assume $\varphi \in C_0(G)$. Then, as before, we set $\psi = \varphi \star_G \varphi^*$ and $\theta = \psi|_N$. The following computation

$$\begin{aligned}
\int_G |\varphi(g)|^2 \, dg &= \psi(e) = \theta(e) \\
&= \int_{\hat{N}} \hat{\theta}(\gamma) \, d\gamma \\
&= \int_{\hat{N}} \int_{(\widehat{G/N})^{\bar{\omega}_\gamma}} \operatorname{Tr} \pi_{\gamma, \hat{\sigma}}(\psi) \, d\mu_{G/N, \bar{\omega}_\gamma}(\sigma) \, d\gamma, \qquad (8.1)
\end{aligned}$$

constitutes a proof of the theorem, as soon as we justify equation (8.1).

Now

$$\pi_{\gamma,\sigma}(\psi) = (\gamma'\sigma'')(\psi) = \int_G (\gamma'\sigma'')(g)\,\psi(g)\,dg$$
$$= \int_{G/N}\int_N \gamma'(ng)\,\sigma''(ng)\,\psi(ng)\,dn\,d\bar{g}$$
$$= \int_{G/N}\left(\int_N \gamma(n)\,\gamma'(g)\,\psi(ng)\,dn\right)\sigma(\bar{g})\,d\bar{g}$$
$$= \int_{G/N} \psi_1(\bar{g})\,\sigma(\bar{g})\,d\bar{g},$$

where

$$\psi_1(\bar{g}) = \int_N \gamma(n)\,\gamma'(g)\,\psi(ng)\,dn.$$

In order to proceed we have to consider briefly the twisted group algebra corresponding to the normalized multiplier ω. Let $L_1(G/N, \omega)$ be the collection of integrable functions on G/N with twisted convolution

$$(f \star_\omega h)(\bar{x}) = \int_{G/N} \omega(\bar{x}, \bar{y}^{-1})\,f(xy^{-1})\,h(\bar{y})\,dy.$$

Lemma 8.2. — *The map* $T : \varphi \to T_\varphi(\bar{g}) = \int_N \gamma(n)\,\gamma'(g)\,\varphi(ng)\,dn$ *is a homomorphism of* $L_1(G)$ *into* $L_1(G/N, \omega)$.

Proof. — The only thing that is not obvious is that T is an algebra homomorphism. But in fact we can compute for $\varphi, \psi \in L_1(G)$:

$$T(\varphi \star \psi)(\bar{g}) = \int_N \gamma(n)\,\gamma'(g)\,(\varphi \star \psi)(ng)\,dn$$
$$= \int_N \gamma(n)\,\gamma'(g)\int_G \varphi(ngh^{-1})\,\psi(h)\,dh\,dn$$
$$= \int_N \gamma(n)\,\gamma'(g)\int_{G/N}\int_N \varphi(ngh^{-1}n_1^{-1})\,\psi(n_1 h)\,dn_1\,d\bar{h}\,dn$$

(8.2) $$= \int_{G/N}\int_N\int_N \gamma(n)\,\gamma'(g)\,\varphi(nn_1^{-1}gh^{-1})\,\psi(n_1 h)\,dn\,dn_1\,d\bar{h}$$
$$= \int_{G/N}\int_N\int_N \gamma(nn_1)\,\gamma'(g)\,\varphi(ngh^{-1})\,\psi(n_1 h)\,dn\,dn_1\,d\bar{h}$$
$$= \int_{G/N}\int_N\int_N \gamma(n)\,\gamma(n_1)\,\gamma'(gh^{-1}h)\,\varphi(ngh^{-1})\,\psi(n_1 h)\,dn\,dn_1\,d\bar{h}$$

(8.3) $$= \int_{G/N} \int_N \int_N \gamma(n)\, \gamma'(gh^{-1})\, \varphi(ngh^{-1})$$
$$\times\, \gamma(n_1)\, \gamma'(h)\, \psi(n_1 h)\, dn\, dn_1\, \overline{\omega}(gh^{-1}, \bar{h})\, d\bar{h}$$

(8.4) $$= \int_{G/N} (T\varphi)(\bar{g}\bar{h}^{-1})\,(T\psi)(\bar{h})\, \omega(g, h^{-1})\, d\bar{h}$$
$$= (T\varphi \star_\omega T\psi)(\bar{g}).$$

We have used the fact that N is central in (8.2), that γ' is an ω-representation in (8.3), and the fact that $\omega(x, y)\, \omega(xy, y^{-1}) = 1$ in 8.4 [a simple consequence of the co-cycle identity for ω and the normalization $\omega(z, z^{-1}) = 1$].

The twisted convolution provides another service for us. Given an $\overline{\omega}$-representation σ of G/N, one checks easily that $\sigma(f \star_\omega h^*) = \sigma(f)\, \sigma(h)^*$, $f, h \in L_1(G/N, \omega)$. Putting these ideas together with the results of Theorem 7.1, we have

$$\int_{(\widehat{G/N})^{\overline{\omega}_\gamma}} \operatorname{Tr} \pi_{\gamma,\sigma}(\psi)\, d\mu_{G/N, \overline{\omega}_\gamma}(\sigma) = \int_{(\widehat{G/N})^{\overline{\omega}_\gamma}} \operatorname{Tr} \sigma(\psi_1)\, d\mu_{G/N, \overline{\omega}_\gamma}(\sigma)$$
$$= \int_{(\widehat{G/N})^{\overline{\omega}_\gamma}} \operatorname{Tr} \sigma(T\psi)\, d\mu_{G/N, \overline{\omega}_\gamma}(\sigma)$$
$$= \int_{(\widehat{G/N})^{\overline{\omega}_\gamma}} \operatorname{Tr}[\sigma(T\varphi)\, \sigma(T\varphi)^*]\, d\mu_{G/N, \overline{\omega}_\gamma}(\sigma)$$
$$= \int_{G/N} |T\varphi(\bar{g})|^2\, d\bar{g}$$
$$= (T\varphi \star_\omega (T\varphi)^*)(e)$$
$$= T\psi(e)$$
$$= \int_N \gamma(n)\, \psi(n)\, dn$$
$$= \hat{\theta}(\gamma).$$

This proves (8.1), and so completes the proof of Theorem 8.1.

Example. — Let G be the Heisenberg group,

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R} \right\}.$$

More precisely, G is the group of all real triples $\{(x, y, z) : x, y, z \in \mathbf{R}\}$ with multiplication

$$(x, y, z)(\xi, \eta, \zeta) = (x + \xi, y + \eta, z + \zeta + x\eta).$$

We take Lebesgue measure $dx\,dy\,dz$ for Haar measure on G. Let $\mathrm{N} = \mathrm{Cent}\ \mathrm{G} = \{(0, 0, z) : z \in \mathbf{R}\}$ with Haar measure dz. Then

$$\hat{\mathrm{N}} = \{\gamma : (0, 0, z) \to e^{i\gamma z},\ \gamma \in \mathbf{R}\},$$

and the Plancherel measure on $\hat{\mathrm{N}}$ is normalized Lebesgue measure $d\gamma/2\pi$. Clearly $\mathrm{G}_\gamma = \mathrm{G}$ for any γ. When $\gamma = 0$, we obtain the representations $\pi_{0,\sigma}$, i. e., the one-dimensional characters σ of $\mathrm{G}/\mathrm{N} \cong \mathbf{R}^2$ lifted to G. When $\gamma \neq 0$, we obtain the following :

$$\gamma'(x, y, z) = e^{i\gamma(z - xy/2)},$$
$$\omega_\gamma[(x, y), (\xi, \eta)] = e^{i\gamma(\xi y - x\eta)/2}.$$

Note that this multiplier is precisely $\overline{\omega}_\gamma = \omega_{-\gamma}$ in the notation of the example in paragraph 7. Taking σ_γ as in that section, we obtain a one-parameter family of infinite-dimensional representations

$$\pi_{\gamma,\sigma_\gamma} = \gamma' \otimes \sigma''_\gamma,$$
$$\pi_{\gamma,\sigma_\gamma}(x, y, z) f(u) = e^{i\gamma(z + yu)} f(u + x), \qquad f \in \mathrm{L}^2(\mathbf{R}).$$

According to Theorem 8.1 and the example in paragraph 7, Plancherel measure on $\hat{\mathrm{G}}$ is exactly the image of $(2\pi)^{-2}\,|\gamma|\,d\gamma$ under the map $\gamma \to \pi_{\gamma,\sigma_\gamma}$ (compare e. g., [9, § 4] where the normalizations of Haar measure are less precise).

9. The selection lemma. — In this section we prove the crucial measure-theoretic result that we need for the proof of our general Plancherel formula. The selection lemma is involved with the choice of measurable sections and, as one might expect, is quite technical. Another matter with which we will be concerned here is to generalize the results of [24, § 8] by dropping the type I assumption on N in favor of the " type I subset " $\hat{\mathrm{N}}_t$ of $\hat{\mathrm{N}}$.

For each cardinal n, $1 \leq n \leq \infty$, let $\mathcal{H}_n$ be a fixed n-dimensional Hilbert space. We let $\mathrm{Rep}_n(\mathrm{G})$ be the set of all (continuous unitary) representations of G on $\mathcal{H}_n$, with the usual topology [the weakest for which all the maps $\pi \to (\pi(x)\xi, \eta)$, $x \in \mathrm{G}$, $\xi, \eta \in \mathcal{H}_n$, are continuous]. If we set $\mathrm{Rep}(\mathrm{G}) = \cup_n \mathrm{Rep}_n(\mathrm{G})$ with the sum topology, then $\mathrm{Rep}(\mathrm{G})$ is a polonaise space [10, 3.7.1]. Let $\mathrm{Irr}_n(\mathrm{G})$ denote the subset of $\mathrm{Rep}_n(\mathrm{G})$ consisting of irreducible representations, and set $\mathrm{Irr}(\mathrm{G}) = \cup_n \mathrm{Irr}_n(\mathrm{G})$. Since $\mathrm{Irr}_n(\mathrm{G})$ is a G_δ set in $\mathrm{Rep}_n(\mathrm{G})$ [10, 3.7.4], $\mathrm{Irr}(\mathrm{G})$ is polonaise.

Let M (Irr (G)) be the space of all finite Borel measures on Irr (G). Now the topology of Irr_n (G) arises from embedding the space in a countable product of discs (as in [10, 3.7.1]). Thus Irr_n (G) is a G_δ subset of a compact metric space S_n, and Irr (G) is a G_δ subset of a locally compact metric space $S = \bigcup_n S_n$ (where S has the sum topology). Let $C_b(S)$ be the space of all bounded continuous real-valued functions on S with the uniform norm. Then $C_b(S')$ may be identified with the space of all finite Borel measures on S, and M (Irr (G)) is the subspace of $C_b(S)'$ consisting of measures concentrated in Irr (G).

We provide $C_b(S)'$ with the weak $\star$-topology. This induces a topology on M (Irr (G)) which may be described as the weakest for which all the functions $\nu \to \langle \nu, f \rangle = \int d\nu$, $f \in C_u(\mathrm{Irr}(G))$, are continuous — where $C_u(\mathrm{Irr}(G))$ is the space of all bounded real functions on Irr (G) whose restriction to each Irr_n (G) is uniformly continuous, i. e.,

$$C_u(\mathrm{Irr}(G)) = \{ f|_{\mathrm{Irr}(G)} : f \in C_b(S) \}.$$

Let $M_1^+(\mathrm{Irr}(G))$ be the subset of all positive measures of norm ≤ 1.

Lemma 9.1. — $M_1^+(\mathrm{Irr}(G))$ *is a Borel subset of the unit ball of* $C_b(S)'$. *In particular*, $M_1^+(\mathrm{Irr}(G))$ *is a standard Borel space.*

Proof. — The space of positive measures of norm ≤ 1 is weak $\star$-compact in $C_b(S)'$. Thus to show $M_1^+(\mathrm{Irr}(G))$ is Borel, it is enough to show M (Irr (G)) is Borel. But $\mathrm{Irr}(G) = \bigcap_n T_n$, where each T_n is open in S. Furthermore, it is easily seen that $M(\mathrm{Irr}(G)) = \bigcap_n M(T_n)$, where $M(T_n)$ is the space of measures concentrated in T_n. Thus it is enough to show that for any open subset T of S, M (T) is Borel.

Let f_0 be the characteristic function of T, and let $\{ f_n \}$ be a sequence of continuous bounded functions on S such that $f_n \to f_0$ pointwise. Then

$$\nu \in M(T) \iff (1 - f_0)\nu = \lim (1 - f_n)\nu = 0.$$

Let $\{ h_j \}_{j \geq 1}$ be a countable dense subset of the unit ball in $C_b(S)$. Then

$$\lim (1 - f_n)\nu = 0 \iff \lim \langle (1 - f_n)\nu, h_j \rangle = 0 \qquad (j = 1, 2, \ldots).$$

It is clear that the last condition defines a Borel subset of $C_b(S)'$; hence M (T) is Borel.

Next let $m : \mathrm{Irr}(G) \to \{ 1, 2, \ldots, \infty \}$ be a Borel function. We have in mind a certain function which we will describe explicitly later. If $\pi \in \mathrm{Rep}(G)$, we shall denote by $m(\pi)\pi$ the representation $1_{m(\pi)} \otimes \pi$ on the space $\mathcal{H}_{m(\pi)} \otimes \mathcal{H}_\pi$, where $\mathcal{H}_\pi = \mathcal{H}_n$ if $\pi \in \mathrm{Rep}_n(\mathrm{G})$. The representation $m(\pi)\,\pi$ is of course equivalent to $m(\pi)$ copies of π. For each $\nu \in \mathrm{M}_1^+(\mathrm{Irr}(\mathrm{G}))$, we put

$$\sigma_\nu = \int_{\mathrm{Irr}(\mathrm{G})}^{\oplus} m(\pi)\,\pi\, d\nu(\pi).$$

σ_ν is a representation on

$$\int^{\oplus} m(\pi)\,\mathcal{H}_\pi\, d\nu(\pi) = \int^{\oplus} \mathcal{H}_{m(\pi)} \otimes \mathcal{H}_\pi\, d\nu(\pi) = \mathcal{H}_{\sigma_\nu}.$$

We set $d(\nu) = \dim \sigma_\nu$. If $\nu = 0$, we let σ_ν be the one-dimensional identity representation.

LEMMA 9.2. — *For each* $\nu \in \mathrm{M}_1^+(\mathrm{Irr}(\mathrm{G}))$ *it is possible to choose a unitary operator* A_ν *of* $\mathcal{H}_{\sigma_\nu}$ *onto* $\mathcal{H}_{d(\nu)}$ *such that the map*

$$\nu \to \mathrm{A}_\nu\,\sigma_\nu\,\mathrm{A}_\nu^{-1},$$
$$\mathrm{M}_1^+(\mathrm{Irr}(\mathrm{G})) \to \mathrm{Rep}(\mathrm{G})$$

is Borel.

The proof of this lemma is extremely technical and we postpone it.

Now let H be a closed subgroup of a locally compact group G. Write $\bar{x} = \mathrm{H}x$, $x \in \mathrm{G}$, as usual, and let $s : \mathrm{G}/\mathrm{H} \to \mathrm{G}$ be a Borel cross-section such that $s(\bar{e}) = e$. Assume $\gamma \in \mathrm{Rep}(\mathrm{H})$ and set $\pi = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\,\gamma$. In paragraph 3 we have shown how to realize π on the space $\mathrm{L}_2(\mathrm{G}/\mathrm{H}; \mathcal{H}_\gamma)$ with respect to a quasi-invariant measure $d\bar{g}$ on G/H. In this form the group action becomes

$$\pi(y) f(\bar{x}) = \gamma(\beta(\bar{x}, y))\, f(\overline{xy})\, [q(xy)^{1/2}/q(x)]^{1/2}$$

where $\beta(\bar{x}, y) : \mathrm{G}/\mathrm{H} \times \mathrm{G} \to \mathrm{H}$ is the Borel function

$$\beta(\bar{x}, y) = s(\bar{x})\, y s(\overline{xy})^{-1},$$

and q is an equivariant function defining the quasi-invariant measure. If $\gamma \in \mathrm{Rep}_n(\mathrm{H})$, then

$$\mathrm{L}_2(\mathrm{G}/\mathrm{H}; \mathcal{H}_\gamma) = \mathrm{L}_2(\mathrm{G}/\mathrm{H}) \otimes \mathcal{H}_n$$

and we may suppose, by abuse of notation, that this is one of our fixed Hilbert spaces; that is, we may suppose $\pi = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\,\gamma \in \mathrm{Rep}(\mathrm{G})$.

LEMMA 9.3. — *The map* $\gamma \to \pi = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\,\gamma$ *is a Borel map from* Rep (H) *to* Rep (G).

Proof. — It suffices to show $(\gamma \to \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} \gamma)|_{\mathrm{Rep}_n(\mathrm{H})}$ is Borel for each n. Therefore, by the definition of the topology of Rep (G), what we must show is that for each $f, f' \in \mathrm{L}_2(\mathrm{G}/\mathrm{H}; \mathcal{H}_n)$, $y \in \mathrm{G}$, the map

$$\gamma \to (\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} \gamma(y) f, f') = \int_{\mathrm{G}/\mathrm{H}} (\gamma(\beta(\bar{x}, y)) f(\overline{xy}), f'(\bar{x})) [q(xy)/q(x)]^{1/2} d\bar{x}$$

is Borel. We may write f, f' as countable sums

$$f = \Sigma_i f_i \xi_i, \qquad f_i \in \mathrm{L}_2(\mathrm{G}/\mathrm{H}), \quad \xi_i \in \mathcal{H}_n,$$
$$f' = \Sigma_j f'_j \eta_j, \qquad f'_j \in \mathrm{L}_2(\mathrm{G}/\mathrm{H}), \quad \eta_j \in \mathcal{H}_n;$$

and so

$$(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} \gamma(y) f, f') = \Sigma_{i,j} \int_{\mathrm{G}/\mathrm{H}} (\gamma(\beta(\bar{x}, y)) \xi_i, \eta_j) f_i(\overline{xy}) \bar{f}'_j(\bar{x}) [q(xy)/q(x)]^{1/2} d\bar{x}.$$

Replacing $f(\overline{xy})$ by $f(\overline{xy})[q(xy)/q(x)]^{1/2}$, we see it is enough to show that

$$\gamma \to \int_{\mathrm{G}/\mathrm{H}} (\gamma(\beta(\bar{x}, y)) \xi, \eta) f(\bar{x}) \bar{f}'(\bar{x}) d\bar{x}$$

is Borel for fixed $\xi, \eta \in \mathcal{H}_n$, $y \in \mathrm{G}$, $f, f' \in \mathrm{L}_2(\mathrm{G}/\mathrm{H})$.

Now there exists an ordinal Γ such that for each y, $\bar{x} \to \beta(\bar{x}, y)$ is a Borel function of class Γ (in fact Γ depends only on the Borel class of s). Thus $(\gamma, \bar{x}) \to (\gamma(\beta(\bar{x}, y)) \xi, \eta)$ is of Borel class Γ in $\bar{x}$, and continuous in γ. It follows from [19, § 27, V] that $(\gamma(\beta(\bar{x}, y)) \xi, \eta)$ is a jointly Borel function of $(\gamma, \bar{x})$. We may assume that f, f' are Borel functions, and then

$$(\gamma, \bar{x}) \to (\gamma(\beta(\bar{x}, y)) \xi, \eta) f(\bar{x}) \bar{f}'(\bar{x})$$

is Borel. By the Borel version of the Fubini theorem

$$\gamma \to \int_{\mathrm{G}/\mathrm{H}} (\gamma(\beta(\bar{x}, y)) \xi, \eta) f(\bar{x}) \bar{f}'(\bar{x}) d\bar{x}$$

is also Borel. That completes the proof.

We return once again to the general group extension situation as described in paragraph 1. Let G be locally compact, N a closed normal subgroup, $\gamma \in \mathrm{Irr}(\mathrm{N})$, and suppose the stability group G_γ is closed; let γ' be an extension of γ to G_γ with normalized multiplier ω_γ. Denote by $\mathrm{Rep}_n(\mathrm{G}_\gamma/\mathrm{N}, \omega_\gamma)$ the set of ω_γ-representations of $\mathrm{G}_\gamma/\mathrm{N}$ on $\mathcal{H}_n$ and

$$\mathrm{Rep}(\mathrm{G}_\gamma/\mathrm{N}, \omega_\gamma) = \bigcup_n \mathrm{Rep}_n(\mathrm{G}_\gamma/\mathrm{N}, \omega_\gamma),$$

We identify $\operatorname{Rep}(G_\gamma/N, \omega_\gamma)$ with a subset of $\operatorname{Rep}((G_\gamma/N)(\overline{\omega}_\gamma))$ as in paragraph 7, and we use this identification to topologize $\operatorname{Rep}(G_\gamma/N, \omega_\gamma)$. If $\sigma \in \operatorname{Rep}(G_\gamma/N, \omega_\gamma)$, then $\tilde{\sigma} \in \operatorname{Rep}(G_\gamma/N, \overline{\omega}_\gamma)$, and as usual we write $\tilde{\sigma}''$ for the lift of $\tilde{\sigma}$ to G_γ. By an abuse of notation we write $\gamma' \otimes \tilde{\sigma}'' \in \operatorname{Rep}(G_\gamma)$.

Lemma 9.4. — *The map* $\sigma \to \operatorname{Ind}_{G_\gamma}^G \gamma' \otimes \tilde{\sigma}''$ *is a Borel map from* $\operatorname{Rep}(G_\gamma/N, \omega_\gamma)$ *to* $\operatorname{Rep}(G)$.

Proof. — The map $\sigma \to \tilde{\sigma}$ is a homeomorphism of $\operatorname{Rep}(G_\gamma/N, \omega_\gamma)$ onto $\operatorname{Rep}(G_\gamma/N, \overline{\omega}_\gamma)$. Furthermore the map $\tilde{\sigma} \to \tilde{\sigma}''$ is a homeomorphism of $\operatorname{Rep}(G_\gamma/N, \overline{\omega}_\gamma)$ into $\operatorname{Rep}(G_\gamma, \overline{\omega}_\gamma)$. Next, for each $\xi,\ \eta \in \mathcal{H}_\gamma \otimes \tilde{\mathcal{H}}_\sigma$, $x \in G_\gamma$, the function $\tilde{\sigma}'' \to (\gamma' \otimes \tilde{\sigma}''(x)\xi, \eta)$ may be expressed as a countable sum of Borel functions of the form $(\gamma'(x)\xi_1, \xi_2)(\tilde{\sigma}''(x)\eta_1, \eta_2)$; thus, it is Borel. But the Borel structure generated by the weak topology agrees with the Borel structure generated by its matrix coefficients. Therefore $\tilde{\sigma}'' \to \gamma' \otimes \tilde{\sigma}''$ is a Borel map from $\operatorname{Rep}(G_\gamma, \overline{\omega}_\gamma)$ to $\operatorname{Rep}(G_\gamma)$. Applying Lemma 9.3 finally, we conclude that $\sigma \to \operatorname{Ind}_{G_\gamma}^G \gamma' \otimes \tilde{\sigma}''$ is Borel.

Now we define a function m on $\operatorname{Irr}(G)$ as follows. For $\pi \in \operatorname{Irr}(G)$, we set $m(\pi) = p$ if $\pi|_N$ is of uniform multiplicity p, $1 \leq p \leq \infty$, and $m(\pi) = 1$ otherwise. This is the function alluded to prior to Lemma 9.2.

Lemma 9.5. — *m is a Borel function.*

Proof. — The map $\pi \to \pi|_N$ is a continuous map of $\operatorname{Rep}(G)$ into $\operatorname{Rep}(N)$. Thus all we need to show is that for each p, the subset

$$M_p(N) = \{\gamma \in \operatorname{Rep}(N) : \gamma \text{ is of uniform mutiplicity } p\}$$

is a Borel set. For each j, let $M_{p,j} = M_p(N) \cap \operatorname{Rep}_j(N)$. Then of course it suffices to show $M_{p,j}$ is Borel for any j and any p. We choose a unitary operator $A_{p,j}$ of $p\,\mathcal{H}_j$ ($= \mathcal{H}_j \oplus \mathcal{H}_j \oplus \dots$, p times) onto $\mathcal{H}_{jp}$. Then $\gamma \to A_{p,j} . p\,\gamma . A_{p,j}^{-1}$ is a Borel isomorphism of $M_{1,j}$ onto $M_{p,j}$. Since $\operatorname{Rep}_j(N)$ is standard, it will follow that $M_{p,j}$ is Borel once we can show that $M_{1,j}$ is Borel [23, Theorem 3.2 and Corollary 1].

Let $\mathfrak{A}(\mathcal{H}_j)$ be the Borel space of von Neumann algebras on $\mathcal{H}_j$ introduced by Effros [12]. This is a standard Borel space and the maps $(A, B) \to A \cap B$, $A \to A'$, are Borel [12, Theorem 3, Corollaries 1, 2]. It follows from this that the set

$$\begin{aligned} \mathfrak{M}_{1,j} &= \{A \in \mathfrak{A}(\mathcal{H}_j) : A' \text{ abelian}\} \\ &= \{A \in \mathfrak{A}(\mathcal{H}_j) : A \cap A' = A'\} \end{aligned}$$

is a Borel set. But the map $\gamma \to \gamma(G)''$ is a Borel map from $\mathrm{Rep}_j(N)$ into $\mathfrak{A}(\mathcal{H}_j)$ [12, Theorem 4, Corollary 1]. Since $\gamma \in M_{1,j}$ if and only if $\gamma(G)'' \in \mathfrak{M}_{1,j}$, it follows immediately that $M_{1,j}$ is a Borel set.

Our next lemma is an almost obvious result, so we omit the proof.

LEMMA 9.6. — *Let* X *be a standard Borel space,* Y *a separable metric space and* f, $h : X \to Y$ *two Borel functions. Then* $\{ x \in X : f(x) = h(x) \}$ *is a Borel set.*

In order to proceed we have to weaken the assumption « N is type I » in [24, Theorem 8.1]. We accomplish this by restricting our attention to $\hat{N}_t$.

LEMMA 9.7. — *Let* $\gamma \in \hat{N}_t$. *Then* G_γ *is closed. For any* $\sigma \in \left(\widehat{G_\gamma/N}\right)^{\overline{\omega}_\gamma}$, *the representation*

$$\pi_{\gamma,\sigma} = \mathrm{Ind}_{G_\gamma}^{G} \gamma' \otimes \sigma''$$

is irreducible. Also if σ_1, $\sigma_2 \in \left(\widehat{G_\gamma/N}\right)^{\omega_\gamma}$, $\sigma_1 \neq \sigma_2$, *or if* γ_1 *and* γ_2 *are not in the same* G-*orbit, then* $\pi_{\gamma_1,\sigma_1} \neq \pi_{\gamma_2,\sigma_2}$. *Finally* $\pi_{\gamma,\sigma}|_N$ *is of uniform multiplicity* $\dim \sigma$.

Proof. — It follows from Lemma 6.3 that $\hat{N}_t$ is G-invariant. Since $\hat{N}_t$ is standard, the stability groups G_γ, $\gamma \in \hat{N}_t$, are necessarily closed [1, Proposition 3.7]. Next we remark that the arguments used by Mackey in [24, Theorem 8.1] to show the $\pi_{\gamma,\sigma}$ are irreducible and pairwise inequivalent will be valid if we know that $\pi_{\gamma,\sigma}|_N$ is of uniform multiplicity. In Mackey's work this comes about because N is type I [24, p. 293]. We give an alternate proof here.

We have already seen several times how to realize $\pi_{\gamma,\sigma}$ on $L_2(G/G_\gamma; \mathcal{H}_\gamma \otimes \mathcal{H}_\sigma)$. If we restrict the representation to N, we obtain the following formulas

$$\pi_{\gamma,\sigma}(n) F(\bar{g}) = \gamma(gng^{-1}) \otimes 1_\sigma F(\bar{g}), \qquad F \in L_2(G/G_\gamma; \mathcal{H}_\gamma \otimes \mathcal{H}_\sigma).$$

That is

$$\pi_{\gamma,\sigma}|_N = \int_{G/G_\gamma}^{\oplus} \gamma \cdot g \, d\bar{g} \otimes 1_\sigma;$$

and so it suffices to show that $\int_{G/G_\gamma}^{\oplus} \gamma \cdot g \, d\bar{g}$ is multiplicity-free. By [23, Theorem 10.5] it is enough to show that the representation π_1 is type I, where

$$\pi_1 = \int_{G/G_\gamma}^{\oplus} \gamma \cdot g \, d\bar{g}. \tag{9.1}$$

For that is it enough to establish that (9.1) is the central decomposition of π_1. But G/G_γ is Borel isomorphic to the orbit $\gamma . G \subseteq \hat{N}_t$, and so this is an immediate consequence of Lemma 6.3.

Now let Z be a Borel subset of Irr (N) such that its image in $\hat{N}$ is contained in $\hat{N}_t$ and for each $\gamma \in Z$ the right regular representation of (G_γ/N) $(\overline{\omega}_\gamma)$ is type I. We already know that

$$\mathrm{Ind}_N^G \gamma = \mathrm{Ind}_{G_\gamma}^G \mathrm{Ind}_N^{G_\gamma} \gamma = \mathrm{Ind}_{G_\gamma}^G (\gamma' \otimes \rho''_\gamma)$$

where ρ_γ is the right regular $\overline{\omega}_\gamma$-representation; and by Theorem 7.1, we have

$$\rho_\gamma = \int^{\oplus}_{(\widehat{G_\gamma/N})^{\omega_\gamma}} 1_\sigma \otimes \tilde{\sigma} \, d\mu_{G_\gamma/N, \omega_\gamma}(\sigma).$$

Writing μ_γ for $\mu_{G_\gamma/N, \omega_\gamma}$, we see that

$$\mathrm{Ind}_N^G \gamma = \int^{\oplus}_{(\widehat{G_\gamma/N})^{\omega_\gamma}} 1_\sigma \otimes \mathrm{Ind}_{G_\gamma}^G (\gamma' \otimes \tilde{\sigma}'') \, d\mu_\gamma(\sigma). \tag{9.2}$$

We write $\pi = \mathrm{Ind}_{G_\gamma}^G \gamma' \otimes \tilde{\sigma}''$ and since, π is irreducible (Lemma 9.7), we may suppose $\pi \in \mathrm{Irr}(G)$. Once again, letting $m(\pi)$ be the multiplicity of $\pi|_N$, we see from Lemma 9.7 that $m(\pi) = \dim \sigma$. Therefore we have

$$\mathrm{Ind}_N^G \gamma = \int^{\oplus} m(\pi)\, \pi \, d\nu'_\gamma(\pi),$$

where ν'_γ is the image of μ_γ under the map $\sigma \to \pi = \mathrm{Ind}_{G_\gamma}^G \gamma' \otimes \tilde{\sigma}''$, which is Borel according to Lemma 9.4.

Lemma 9.8. — *For each* $\gamma \in Z$ *it is possible to choose a measure* $\nu_\gamma \in M_1^+(\mathrm{Irr}(G))$ *such that :*

(i) $\mathrm{Ind}_N^G \gamma \cong \int^{\oplus} m(\pi)\, \pi \, d\nu_\gamma(\pi)$;

(ii) *The map* $\gamma \to \nu_\gamma$ *from* Z *to* $M_1^+(\mathrm{Irr}(G))$ *is universally measurable.*

Proof. — We know from Lemma 9.2 that for each $\nu \in M_1^+(\mathrm{Irr}(G))$, there exists a unitary operator A_ν such that

$$\nu \to A_\nu \sigma_\nu A_\nu^{-1}, \quad M_1^+(\mathrm{Irr}(G)) \to \mathrm{Rep}(G)$$

is Borel. By Lemma 9.3 the map

$$\gamma \to \mathrm{Ind}_N^G \gamma, \quad \mathrm{Rep}(N) \to \mathrm{Rep}(G)$$

is also Borel. Let $X = M_1^+(\mathrm{Irr}(G)) \times Z$, $Y = \mathrm{Rep}(G)$ and set

$$f: \quad X \to Y, \qquad f(\nu, \gamma) = A_\nu \sigma_\nu A_\nu^{-1},$$
$$h: \quad X \to Y, \qquad h(\nu, \gamma) = \mathrm{Ind}_N^G \gamma.$$

Clearly f and h are Borel; therefore, by Lemma 9.6 the set

$$Q = \{(\nu, \gamma) \in X : A_\nu \sigma_\nu A_\nu^{-1} = \mathrm{Ind}_N^G \gamma\}$$

is Borel. According to the previous calculations, for each $\gamma \in Z$ there exists some $\nu'_\gamma \in M_1^+(\mathrm{Irr}(G))$ such that $(\nu'_\gamma, \gamma) \in Q$. Applying the cross-section Lemma [8, Appendice 5, § 5] there exists a universally measurable function $\gamma \to \nu_\gamma$ such that $(\nu_\gamma, \gamma) \in Q$, i. e., such that

$$\mathrm{Ind}_N^G \gamma \cong \int^\oplus m(\pi)\, \pi \, d\nu_\gamma(\pi).$$

That completes the proof.

If $\zeta \in \hat{N}$ and $\gamma \in \zeta$, then the unitary equivalence class of $\mathrm{Ind}_N^G \gamma$ depends only on ζ and may be denoted $\mathrm{Ind}_N^G \zeta$. At times we write $\zeta = [\gamma]$. We are now ready for the main result of this section.

Lemma 9.9 (*Selection Lemma*). — *Let μ_0 be a Borel measure on $\hat{N}$ concentrated in the standard Borel subspace $\hat{N}_0 \subseteq \hat{N}_t$. Suppose that for all $\zeta \in \hat{N}_0$ there exists $\nu \in M_1^+(\mathrm{Irr}(G))$ such that $\mathrm{Ind}_N^G \zeta = [\sigma_\nu]$. Then for μ_0-almost all $\zeta \in \hat{N}_0$ it is possible to choose a measure $\nu_\zeta \in M_1^+(\mathrm{Irr}(G))$ such that*

(i) $\mathrm{Ind}_N^G \zeta = [\sigma_{\nu_\zeta}]$;

(ii) $\zeta \to \nu_\zeta$ is *μ_0-measurable.*

Proof. — Let $Z_0 = \{\gamma \in \mathrm{Irr}(N) : [\gamma] \in \hat{N}_0\}$. By deleting a μ_0-null Borel, set in $\hat{N}_0$ if necessary, we may suppose there exists a Borel section $s : \zeta \to \gamma_\zeta$, $\hat{N}_0 \to Z_0$ [1, Proposition 2.15]. Set $Z = s(\hat{N}_0)$. Since $\hat{N}_0$ is standard Z is a Borel set in $\mathrm{Irr}(N)$ [23, Theorem 3.2]. Let $t : \gamma \to \nu_\gamma$, $Z \to M_1^+(\mathrm{Irr}(G))$ be the universally measurable map whose existence is guaranteed by Lemma 9.8. Finally set $w = t \circ s : \hat{N}_0 \to M_1^+(\mathrm{Irr}(G))$. Then $\zeta \to \nu_\zeta = w(\zeta)$ is μ_0-measurable and

$$\mathrm{Ind}_N^G \zeta = [\mathrm{Ind}_N^G \gamma_\zeta] = \left[\int^\oplus m(\pi)\, \pi \, d\nu_{\gamma_\zeta}(\pi)\right] = [\sigma_{\nu_\zeta}],$$

since by definition $\nu_{\gamma_\zeta} = \nu_\zeta$.

We devote the remainder of this section to the proof of Lemma 9.2. First we introduce the following objects :

$\mathcal{H}_0 =$ complex vector space of countable dimension;
$\mathrm{T} =$ metrizable space;
$(.,\ .)_t =$ positive semi-definite inner product on $\mathcal{H}_0$ such that for all $\xi, \eta \in \mathcal{H}_0$, $t \to (\xi, \eta)_t$, $\mathrm{T} \to \mathbf{C}$ is Borel;
$\mathcal{H}_t =$ separated completion of $\mathcal{H}_0$ with respect to $(.,\ .)_t$;
$\mathrm{T}'_k = \{ t \in \mathrm{T} : \dim \mathcal{H}_t \leqq k \},\ 1 \leqq k < \infty$;
$\mathrm{T}_k = \mathrm{T}'_k - \mathrm{T}_{k-1},\ 1 \leqq k < \infty,\ \mathrm{T}_0 = \emptyset$;
$\mathrm{T}_\infty = \mathrm{T} - \bigcup_1^\infty \mathrm{T}_k$.

Our first task is to prove that each T_k is a Borel set, or equivalently that each T'_k is a Borel set. Let U be the vector space over $\mathbf{Q}(i)$ of complex rational linear combinations of a basis of $\mathcal{H}_0$. For each k, $1 \leqq k < \infty$, set

$$\mathrm{U}_k = \{ (u_1, \ldots, u_k) : u_j \in \mathrm{U} \},$$
$$\mathrm{R}_k = \{ (r_1, \ldots, r_k) : r_j \in \mathbf{Q}(i),\ 1/2 \leqq \Sigma \mid r_j \mid^2 \leqq 1 \}.$$

Then

$$t_0 \in \mathrm{T}'_{k-1} \iff (\forall\, u \in \mathrm{U}_k)\ (\forall\, m,\ 1 \leqq m < \infty)\ (\exists\, r \in \mathrm{R}_k) \ni \| r_1 u_1 + \ldots + r_k u_k \|_{t_0} < 1/m$$
$$\iff t_0 \in \bigcap_{u \in \mathrm{U}_k} \bigcap_{m=1}^{\infty} \bigcup_{r \in \mathrm{R}_k} \{ t : \| r_1 u_1 + \ldots + r_k u_k \|_t < 1/m \},$$

where $\|.\|_t$ is the norm on $\mathcal{H}_t$ deduced from $(.,\ .)_t$. Since

$$\{ t \in \mathrm{T} : \| r_1 u_1 + \ldots + r_k u_k \|_t < 1/m \}$$

is a Borel set in T, it follows that T'_{k-1} is also Borel.

Now fix n, $1 \leqq n \leqq \infty$, and let $\mathcal{H}_n$ be as in the beginning of the section. We shall show that for each $t \in \mathrm{T}_n$, there exists an operator $\mathrm{A}_t : \mathcal{H}_0 \to \mathcal{H}_n$ such that for $\xi \in \mathcal{H}_0$, $t \to \mathrm{A}_t \xi$ is Borel and A_t is unitary for $(.,\ .)_t$. It will follow that A_t extends uniquely to a unitary operator $\tilde{\mathrm{A}}_t$ of $\mathcal{H}_t$ onto $\mathcal{H}_n$. For this, let $\{ e_1, e_2, \ldots \}$ denote a basis for $\mathcal{H}_0$. For each t, let $\{ e_1(t), e_2(t), \ldots \}$ be the result of applying the Gram-Schmidt procedure to $\{ e_1, e_2, \ldots \}$ with respect to $(.,\ .)_t$. Then

$$e_1(t) = \begin{cases} 0, & \text{if } \| e_1 \|_t = 0, \\ e_1/\| e_1 \|_t, & \text{otherwise}; \end{cases}$$
$$e_2(t) = \begin{cases} 0, & \text{if } \| e_2 - (e_2, e_1(t))_t\, e_1(t) \|_t = 0, \\ \dfrac{e_2 - (e_2, e_1(t))_t\, e_1(t)}{\| e_2 - (e_2, e_1(t))_t\, e_1(t) \|_t}, & \text{otherwise}; \end{cases}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The non-zero vectors among this set form an orthonormal set for $(.,\ .)_t$, and their images in $\mathcal{H}_t$ form an orthornormal basis in that space. For each t, let $\{ j_1 (t), j_2 (t), \ldots \}$ be the indices of the non-zero vectors among $\{ e_1 (t), e_2 (t), \ldots \}$. Let $\{ f_1, f_2, \ldots \}$ be an orthonormal basis of $\mathcal{H}_n$. We define A_t by setting

$$A_t e_j (t) = \begin{cases} f_k, & \text{if } j = j_k (t) \text{ for some } k, \\ 0, & \text{otherwise} \end{cases}$$

and then extending to all of $\mathcal{H}_0$ in the obvious fashion. If we denote by $\tilde{\xi}$ or $\tilde{\xi}_t$ the image in $\mathcal{H}_t$ of a vector $\xi \in \mathcal{H}_0$, then we can define $\tilde{A}_t$ by setting $\tilde{A}_t \tilde{e}_j (t) = A_t e_j (t)$ and extending to $\mathcal{H}_t$ by continuity. It is a triviality to check that $\tilde{A}_t$ is a unitary map of $\mathcal{H}_t$ onto $\mathcal{H}_n$.

To show the Borel property, since each $\xi \in \mathcal{H}_0$ is a finite linear combination of $\{ e_1, e_2, \ldots \}$, we need only prove that $t \to A_t e_j$ is Borel on T_n. But in fact, once we carry out the (tedious) solution of the Gram-Schmidt equations for the e_j as functions of the $e_j (t)$, the fact that $t \to A_t e_j$ is Borel is immediately clear.

Since T_n is a Borel set, we can conclude : given $t \in T$ there is an operator $A_t : \mathcal{H}_0 \to \mathcal{H}_n$, $n = \dim \mathcal{H}_t$, such that for any $\xi \in \mathcal{H}_0$, $t \to A_t \xi$, $T \to \bigcup_n \mathcal{H}_n$ is Borel, and A_t extends uniquely to a unitary operator of $\mathcal{H}_t$ onto $\mathcal{H}_n$. $\bigcup_n \mathcal{H}_n$ is of course given the sum topology.

We are now going to apply these results to the situation described in Lemma 9.2. Let $T = M_1^+ (\mathrm{Irr}(G))$ provided with the weak $\star$-topology defined by $C_u (\mathrm{Irr}(G))$. For each $\pi \in \mathrm{Irr}(G)$, let $\mathcal{K}_\pi$ be a fixed separable Hilbert space. Denote by $\mathcal{B} = \{ (\pi, \xi) : \xi \in \mathcal{K}_\pi \}$ the Hilbert bundle with base $\mathrm{Irr}(G)$, fibres $\mathcal{K}_\pi$ and Borel structure defined by a sequence of sections $e_1, e_2, \ldots$ such that

(1) $E \subseteq \mathrm{Irr}(G)$ is Borel if and only if $p^{-1}(E)$ is Borel, where $p : \mathcal{B} \to \mathrm{Irr}(G)$ is the natural projection;

(2) $(\pi, \xi) \to (e_j (\pi), \xi)$ is Borel, $j = 1, 2, \ldots$;

(3) $\pi \to (e_j (\pi), e_k (\pi))$ is Borel, $j, k = 1, 2, \ldots$;

(4) the functions $(\pi, \xi) \to (e_j (\pi), \xi)$ separate points;

see, e. g., [25, § 9]. Multiplying each e_j by a suitable scalar-valued function if necessary, we may suppose $\pi \to \| e_j (\pi) \|$ is a bounded function, $j = 1, 2, \ldots$. Let $\mathcal{H}_0$ be the vector space consisting of complex linear combinations of the sections $e_1, e_2, \ldots$. For each $\nu \in T$, we define an inner product on $\mathcal{H}_0$ by

$$(f, h)_\nu = \int (f(\pi), h(\pi)) \, d\nu (\pi).$$

The separated completion $\mathcal{H}_\nu$ of $\mathcal{H}_0$ is precisely $\int^{\oplus} \mathcal{K}_\pi \, d\nu (\pi)$. We need to show $\nu \to (f, h)_\nu$ is Borel.

Now whenever S is a locally compact separable space, $M(S) = C_b(S)'$ with the weak $\star$-topology, the mapping $\nu \to \langle \alpha, \nu \rangle = \int \alpha(s)\, d\nu(s)$ is Borel, for any bounded Borel function α on S. Indeed the bounded Borel functions for which this is true contains $C_b(S)$ and is closed under the formation of bounded pointwise limits. Now since Irr (G) is a Borel subset of a separable locally compact space S, and M (Irr (G)) is a Borel subset of M (S), it follows that for any bounded Borel function α on Irr (G), the map $\nu \to \int \alpha(\pi)\, d\nu(\pi)$ is Borel.

Returning to our set-up, we conclude that for all $j, k \geqq 1$,

$$\nu \to \int (e_j(\pi), e_k(\pi))\, d\nu(\pi)$$

is Borel. Hence for every $f, h \in \mathcal{H}_0$, the map

$$\nu \to \int (f(\pi), h(\pi))\, d\nu(\pi) = (f, h)_\nu$$

is also Borel. Therefore we may appeal to our earlier results to conclude that for each ν, there exists a unitary operator A_ν of $\mathcal{H}_\nu$ onto $\mathcal{H}_{d(\nu)}$, where $d(\nu) = \dim \mathcal{H}_\nu$ and $\mathcal{H}_{d(\nu)}$ is one of our fixed Hilbert spaces, such that $\nu \to A_\nu f$ is Borel, $f \in \mathcal{H}_0$.

Let $m : \mathrm{Irr}(G) \to \{1, 2, \ldots, \infty\}$ be a Borel function, and take for $\mathcal{K}_\pi$ the Hilbert space $\mathcal{K}_\pi = \mathcal{H}_{m(\pi)} \otimes \mathcal{H}_\pi$. Again we put

$$\sigma_\nu = \int^{\oplus} m(\pi)\, \pi\, d\nu(\pi), \qquad \nu \in M_1^+(\mathrm{Irr}(G)).$$

Let $T_n = \{\nu : d(\nu) = n\}$, a Borel subset of T. What we must show to finish the proof of Lemma 9.2 is that : for all $y \in G$, $\xi, \eta \in \mathcal{H}_n$, the map $\nu \to (A_\nu \sigma_\nu(y) A_\nu^{-1} \xi, \eta)$ is Borel on T_n.

For each ν, let $\{e_1(\nu), e_2(\nu), \ldots\}$ be the result of applying the Gram-Schmidt procedure with respect to $(.,.)_\nu$ to the set $\{e_1, e_2, \ldots\}$. It is obvious from the Gram-Schmidt formulas that there are certain complex-valued Borel functions $c_{jk}(\nu)$, $j \geqq 1$, $1 \leqq k \leqq j$ such that

$$e_j(\nu) = \Sigma_{k=1}^{j} c_{jk}(\nu)\, e_k.$$

But then

$$\begin{aligned}(A_\nu \sigma_\nu(y) A_\nu^{-1} \xi, \eta) &= (\sigma_\nu(y) A_\nu^{-1} \xi, A_\nu^{-1} \eta)_\nu \\ &= \Sigma_{j=1}^{\infty} (\sigma_\nu(y) A_\nu^{-1} \xi, e_j(\nu))_\nu (e_j(\nu), A_\nu^{-1} \eta)_\nu.\end{aligned}$$

For each j, we have

$$(e_j(\nu), A_\nu^{-1} \eta)_\nu = \Sigma_{k=1}^{j} c_{jk}(\nu)\, (A_\nu e_k, \eta)$$

and by our previous remarks this is a Borel function of ν.

It only remains to show that

$$\nu \to (\sigma_\nu(y) A_\nu^{-1} \xi, e_j(\nu))_\nu = (A_\nu^{-1} \xi, \sigma_\nu(y^{-1}) e_j(\nu))_\nu$$

is Borel. Hence it suffices to prove $\nu \to (A_\nu^{-1} \xi, \sigma_\nu(y^{-1}) e_j)_\nu$ is Borel. But then

$$(A_\nu^{-1} \xi, \sigma_\nu(y^{-1}) e_j)_\nu = \Sigma (A_\nu^{-1} \xi, e_k(\nu))_\nu (e_k(\nu), \sigma_\nu(y^{-1}) e_j)_\nu.$$

The function $\nu \to (A_\nu^{-1} \xi, e_k(\nu))_\nu = (\xi, A_\nu e_k(\nu))$ is Borel and

$$(e_k(\nu), \sigma_\nu(y^{-1}) e_j)_\nu = \Sigma_l c_{kl}(\nu) (e_l, \sigma_\nu(y^{-1}) e_j)_\nu.$$

Hence it only remains to consider $\nu \to (\sigma_\nu(y) e_i, e_j)_\nu$. But finally we have

$$(\sigma_\nu(y) e_i, e_j)_\nu = \int (m(\pi) \pi(y) e_i(\pi), e_j(\pi)) \, d\nu(\pi);$$

and since $\pi \to (m(\pi) \pi(y) e_i(\pi), e_j(\pi))$ is a bounded Borel function, our proof is at last completed.

10. THE PLANCHEREL FORMULA FOR GROUP EXTENSIONS. — In this section we prove our most general formula. Under fairly broad hypotheses, we will compute the equivalence class of the Plancherel measure of a group G in terms of the Plancherel measures of a closed normal subgroup N and the little groups G_γ/N.

We begin with some material on automorphisms and equivalent representations. Let G be a locally compact group and α a (bicontinuous) automorphism of G. Let H be a closed subgroup of G such that $\alpha|_H$ is an automorphism of H. If γ (respectively π) is a unitary representation of H (respectively G), we write $\alpha\gamma$ (respectively $\alpha\pi$) for the representation $\alpha\gamma(h) = \gamma(\alpha^{-1} h)$ [respectively $\alpha\pi(g) = \pi(\alpha^{-1} g)$]. Our next result is considerably more general than we need, but we present it partially for its own sake.

LEMMA 10.1. — *With the above notation, we have* $\alpha \operatorname{Ind}_H^G \gamma = \operatorname{Ind}_H^G \alpha\gamma$.

Proof. — For functions f on G, we put $\alpha f(x) = f(\alpha^{-1} x)$, $x \in G$. Denote by $\Delta(\alpha) = \Delta_G(\alpha)$ the modulus of α, i. e.,

$$\Delta(\alpha) \int_G (\alpha f)(x) \, dx = \int_G f(x) \, dx, \qquad f \in C_0(G).$$

If $i_x(y) = x^{-1}yx$, then $\Delta(i_x) = \Delta(x) = \Delta_G(x)$, the usual modular function (*see* § 6). Now for $y \in G$, we compute

$$\begin{aligned}\int f(x)\,dx &= \Delta(\alpha\, y)\int f((\alpha\, y)\, x)\,dx \\ &= \Delta(\alpha\, y)\int f(\alpha\,(y\,\alpha^{-1}\,x))\,dx \\ &= \Delta(\alpha\, y)\int (\alpha^{-1} f)\,(y\,\alpha^{-1}\,x)\,dx \\ &= \Delta(\alpha\, y)\,\Delta(\alpha)^{-1}\int (\alpha^{-1} f)\,(yx)\,dx \\ &= \Delta(\alpha\, y)\,\Delta(\alpha)^{-1}\,\Delta(y)^{-1}\int (\alpha^{-1} f)\,(x)\,dx \\ &= \Delta(\alpha\, y)\,\Delta(y)^{-1}\int f(x)\,dx.\end{aligned}$$

Therefore

$$\Delta(\alpha\, y) = \Delta(y), \qquad y \in G. \tag{10.1}$$

We denote by $\bar{\alpha}$ the homeomorphism of G/H obtained by passage to the quotient. Fix a choice of right Haar measures dg, dh, and let q be an equivariant function defining a quasi-invariant measure on G/H (*see* § 3). We use the notation of that section : for $f \in C_0(G)$, $f'(\bar{x}) = \int_H f(hx)\,dh$, and

$$\int_{G/H} f'(\bar{x})\,d\mu(\bar{x}) = \int_G f(x)\,q(x)\,dx.$$

Now if we set $\Delta_H(\alpha) = \Delta_H(\alpha\,|_H)$, then we have

$$\begin{aligned}(\alpha f)'(\bar{x}) &= \int (\alpha f)\,(hx)\,dh = \int f(\alpha^{-1}\,h\,\alpha^{-1}\,x)\,dh \\ &= \Delta_H(\alpha)^{-1}\int f(h\,\alpha^{-1}\,x)\,dh = \Delta_H(\alpha)^{-1} f'\,(\overline{\alpha^{-1}\,x}) \\ &= \Delta_H(\alpha)^{-1}\,(\bar{\alpha} f')\,(\bar{x}).\end{aligned}$$

Let $\bar{\alpha}\mu$ be the measure

$$\langle f', \bar{\alpha}\mu \rangle = \langle \bar{\alpha} f', \mu \rangle.$$

We wish to compute the Radon-Nikodym derivative of $\bar{\alpha}\mu$ with respect to μ. By the preceding

$$\int_{G/H} \bar{\alpha} f'\,d\mu = \Delta_H(\alpha)\int_{G/H} (\alpha f)'\,d\mu.$$

Then making use of the definition of μ,

$$\begin{aligned}\Delta_{\mathrm{H}}(\alpha)\int_{\mathrm{G/H}}(\alpha f)'\,d\mu &= \Delta_{\mathrm{H}}(\alpha)\int_{\mathrm{G}}(\alpha f)(x)\,q(x)\,dx \\ &= \Delta_{\mathrm{H}}(\alpha)\,\Delta_{\mathrm{G}}(\alpha)^{-1}\int_{\mathrm{G}} f(x)\,(\alpha^{-1}q)(x)\,dx \\ &= \Delta_{\mathrm{H}}(\alpha)\,\Delta_{\mathrm{G}}(\alpha)^{-1}\int f(x)\,q(\alpha x)\,dx \\ &= \Delta_{\mathrm{H}}(\alpha)\,\Delta_{\mathrm{G}}(\alpha)^{-1}\int f(x)\frac{q(\alpha x)}{q(x)}\,q(x)\,dx.\end{aligned}$$

But by (10.1) and (3.2), for $h \in \mathrm{H}$, $x \in \mathrm{G}$,

$$\frac{q(\alpha(hx))}{q(hx)} = \frac{q(\alpha h \alpha x)}{q(hx)} = \frac{\Delta_{\mathrm{H}}(\alpha h)\,\Delta_{\mathrm{G}}(\alpha h)^{-1}\,q(\alpha x)}{\Delta_{\mathrm{H}}(h)\,\Delta_{\mathrm{G}}(h)^{-1}\,q(x)} = \frac{q(\alpha x)}{q(x)}.$$

Thus $q(\alpha x)/q(x)$ depends only on $\bar{x}$. Therefore

$$\int_{\mathrm{G/H}} \bar{\alpha} f'\,d\mu = \Delta_{\mathrm{H}}(\alpha)\,\Delta_{\mathrm{G}}(\alpha)^{-1}\int_{\mathrm{G/H}} f'(\bar{x})\frac{q(\alpha x)}{q(x)}\,d\mu(\bar{x}).$$

From this it follows that

$$\frac{d\bar{\alpha}\mu}{d\mu}(\bar{x}) = \frac{\Delta_{\mathrm{H}}(\alpha)}{\Delta_{\mathrm{G}}(\alpha)}\,\frac{q(\alpha x)}{q(x)}.$$

We rewrite this slightly

$$(10.2)\qquad \frac{\Delta_{\mathrm{G}}(\alpha)}{\Delta_{\mathrm{H}}(\alpha)}\int \bar{\alpha} f(\bar{x})\frac{q(\alpha^{-1}x)}{q(x)}\,d\mu(\bar{x}) = \int f(\bar{x})\,d\mu(\bar{x}), \qquad f \in \mathrm{C}_0(\mathrm{G/H}),$$

and go on to see how it figures in the proof of the lemma.

Let γ be a unitary representation of H and set $\pi_\gamma = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\,\gamma$. For $f \in \mathcal{H}(\pi_\gamma) = \mathcal{H}(\alpha\pi_\gamma)$, we put

$$\mathrm{T} f(x) = (\alpha f)(x)\,[q(\alpha^{-1}x)/q(x)]^{1/2}\,[\Delta_{\mathrm{G}}(\alpha)/\Delta_{\mathrm{H}}(\alpha)]^{1/2}.$$

It follows immediately from (10.2) that T is a unitary mapping of $\mathcal{H}(\alpha\pi_\gamma)$ onto $\mathcal{H}(\pi_\gamma)$. Moreover T intertwines the representations $\alpha\pi_\gamma$ and $\pi_{\alpha\gamma}$, because

$$\begin{aligned}\mathrm{T}\,\alpha\pi_\gamma(y)\,f(x) &= \mathrm{T}\,\pi_\gamma(\alpha^{-1}y)\,f(x) \\ &= \alpha(\pi_\gamma(\alpha^{-1}y)\,f)(x)\,[q(\alpha^{-1}x)/q(x)]^{1/2}\,[\Delta_{\mathrm{G}}(\alpha)/\Delta_{\mathrm{H}}(\alpha)]^{1/2} \\ &= f(\alpha^{-1}x\,\alpha^{-1}y)\,[q(\alpha^{-1}x\,\alpha^{-1}y)/q(\alpha^{-1}x)]^{1/2}[q(\alpha^{-1}x)/q(x)]^{1/2}[\Delta_{\mathrm{G}}(\alpha)/\Delta_{\mathrm{H}}(\iota)]^{1/2}\end{aligned}$$

while

$$\begin{aligned}\pi_{\alpha\gamma}(y)\,\mathrm{T} f(x) &= \mathrm{T} f(xy)\,[q(xy)/q(x)]^{1/2} \\ &= f(\alpha^{-1}(xy))\,[q(xy)/q(x)]^{1/2}\,[q(\alpha^{-1}(xy))/q(xy)]^{1/2}\,[\Delta_{\mathrm{G}}(\alpha)/\Delta_{\mathrm{H}}(\alpha)]^{1/2}.\end{aligned}$$

The two are the same and the proof is complete.

Given an automorphism of G, it is clear how to obtain an action on $\hat{G}$ and thus on the measures on $\hat{G}$. With this in mind we have a

COROLLARY. — *Let* ρ_G *be the right regular representation of* G, *and let* $\rho_G = \int_{\hat{G}} \pi \, d\mu(\pi)$ *be its central decomposition. If* α *is an automorphism of* G, *then* μ *and* $\alpha\mu$ *are equivalent.*

Proof. — Let $H = \{e\}$ and let γ_0 be the one-dimensional identity representation of H. Then $\rho_G = \mathrm{Ind}_H^G \gamma_0$. But by Lemma 10.1,

$$\alpha\rho_G = \mathrm{Ind}_H^G \alpha\gamma_0 = \mathrm{Ind}_H^G \gamma_0 = \rho_G.$$

Then

$$\rho_G = \int_{\hat{G}} \alpha\pi \, d\mu(\pi) = \int_{\hat{G}} \pi \, d(\alpha\mu)(\pi).$$

It follows from the essential uniqueness of the central decomposition that μ and $\alpha\mu$ are equivalent.

We are finally ready for our main result on extensions. Let G be locally compact, N a closed normal subgroup. Our first assumption is :

(I) The right regular representation ρ_N of N is type I and μ_N is concentrated in $\hat{N}_t$.

Then, according to Theorem 6.4, we may write

$$\rho_N = \int_{\hat{N}}^{\oplus} 1_\gamma \otimes \tilde{\gamma} \, d\mu_N(\gamma),$$

μ_N is a representative of the Plancherel measure class on $\hat{N}$, which we may (and do) take to be finite. μ_N is concentrated in the G-invariant, standard Borel subet $\hat{N}_t$; and by the corollary to Lemma 10.1, μ_N is also quasi-invariant under the action of G. Our next assumption is that up to a μ_N-null set, N is regularly embedded (*see* [15, Theorem 1]) :

(II) The measure $\bar{\mu}_N$ is countably separated.

But then we may apply Theorem 2.1 to write μ_N as an integral of quasi-invariant measures each concentrated on an orbit. Explicitly

$$\rho_N = \int_{\hat{N}_t/G}^{\oplus} \int_{G/G_\gamma}^{\oplus} (1_\gamma \otimes \tilde{\gamma}).g \, d_\gamma(\bar{g}) \, d\bar{\mu}_N(\bar{\gamma}),$$

where $d_\gamma(\bar{g})$ denotes a quasi-invariant measure on G/G_γ. Since

$$(1_\gamma \otimes \tilde{\gamma}).g = 1_\gamma \otimes \tilde{\gamma}.g,$$

we have

$$\rho_N = \int_{\hat{N}_t/G}^{\oplus} \int_{G/G_\gamma}^{\oplus} 1_\gamma \otimes \tilde{\gamma}.g \, d_\gamma(\bar{g}) \, d\bar{\mu}_N(\bar{\gamma}).$$

Next let ρ_G denote the right regular representation of G. Then

$$\rho_G = \mathrm{Ind}_N^G \rho_N = \int_{\hat{N}_t/G}^{\oplus} \int_{G/G_\gamma}^{\oplus} \mathrm{Ind}_N^G (1_\gamma \otimes \tilde{\gamma}.g) \, d_\gamma(\bar{g}) \, d\bar{\mu}_N(\bar{\gamma}).$$

But

$$\mathrm{Ind}_N^G (1_\gamma \otimes \tilde{\gamma}.g) = 1_\gamma \otimes \mathrm{Ind}_N^G \tilde{\gamma}.g = 1_\gamma \otimes (\mathrm{Ind}_N^G \tilde{\gamma}).g = 1_\gamma \otimes \mathrm{Ind}_N^G \tilde{\gamma} = 1_\gamma \otimes (\mathrm{Ind}_N^G \gamma)^{\sim}.$$

Therefore

$$\int_{G/G_\gamma}^{\oplus} \mathrm{Ind}_N^G (1_\gamma \otimes \tilde{\gamma}.g) \, d_\gamma(\bar{g}) = i(\gamma)\,(1_\gamma \otimes (\mathrm{Ind}_N^G \gamma)^{\sim}),$$

where

$$i(\gamma) = \min([G : G_\gamma], \infty).$$

Thus

$$\rho_G = \int_{\hat{N}_t/G}^{\oplus} i(\gamma)\,(1_\gamma \otimes (\mathrm{Ind}_N^G \gamma)^{\sim}) \, d\bar{\mu}_N(\bar{\gamma}). \tag{10.3}$$

We wish next to appeal to a formula from paragraph 9, and for that we make our third assumption :

(III) For μ_N-almost all γ, the group $(G_\gamma/N)(\bar{\omega}_\gamma)$ has a type I regular representation.

Then by formula (9.2), we have

$$\mathrm{Ind}_N^G \gamma = \int_{(\widehat{G_\gamma/N})^{\omega_\gamma}}^{\oplus} 1_\sigma \otimes \mathrm{Ind}_{G_\gamma}^G (\gamma' \otimes \tilde{\sigma}'') \, d\mu_\gamma(\sigma), \tag{10.4}$$

where μ_γ is a Plancherel measure on $\left(\widehat{G_\gamma/N}\right)^{\omega_\gamma}$.

Now by assumption II, $\bar{\mu}_N$ is concentrated in a standard Borel set S, for which we may assume by [1, Propositions 2.15 and 2.5], that there exists a standard Borel subset S_1 of $\hat{N}$ which meets each G orbit over S in exactly one point. As in the proof of the Selection Lemma [*see* the paragraph containing (9.2)], we use Lemma 9.4 to identify the measures μ_γ with their images on Irr (G). Then we use the Selection Lemma 9.9 (applied to S_1) to choose a μ_N-measurable family $\gamma \to \mu_\gamma, \gamma \in S_1$. Putting $\mu_{\bar{\gamma}} = \mu_\gamma$,

$\gamma \in S$, we obtain a $\bar{\mu}_N$-measurable family. Combining (10.3) and (10.4) then, our formula for ρ_G becomes

$$\rho_G = \int_{\hat{N}_t/G}^{\oplus} \int_{(\widehat{G_\gamma/N})^{\bar{\omega}_\gamma}}^{\oplus} i(\gamma)\,(1_\gamma \otimes 1_\sigma \otimes (\mathrm{Ind}_{G_\gamma}^G \gamma' \otimes \sigma'')^{\sim})\, d\mu_{\bar{\gamma}}(\sigma)\, d\bar{\mu}_N(\bar{\gamma}).$$

Note we have employed the Borel isomorphism $\sigma \to \tilde{\sigma}$ of $(\widehat{G_\gamma/N})^{\omega_\gamma}$ onto $(\widehat{G_\gamma/N})^{\bar{\omega}_\gamma}$ to adjust the notation slightly.

Now since $\gamma \to \mu_{\bar{\gamma}}$ is $\bar{\mu}_N$-measurable, it follows that $d\mu_{\bar{\gamma}}(\sigma)\, d\bar{\mu}_N(\bar{\gamma})$ is in fact a measure on $\mathrm{Irr}(G)$. We transfer it to $\hat{G}$ *via* the Borel map $\mathrm{Irr}(G) \to \hat{G}$. (Note we have written $d\mu_{\bar{\gamma}}(\sigma)\, d\bar{\mu}_N(\bar{\gamma})$ to stand for the measure $\int \mu_{\bar{\gamma}}\, d\bar{\mu}_N(\bar{\gamma})$ in the sense of [3].) We note also that

$$i(\gamma) \dim \gamma \dim \sigma = \dim(\mathrm{Ind}_{G_\gamma}^G \gamma' \otimes \sigma'').$$

As usual we write $\pi_{\gamma,\sigma} = \mathrm{Ind}_{G_\gamma}^G \gamma' \otimes \sigma''$. Finally we set $1_{\gamma,\sigma} = 1_{\pi_{\gamma,\sigma}}$. Then we have proven most of the following

THEOREM 10.2 (Plancherel Formula for Group Extensions). — *Let* G *and* N *satisfy the assumptions I, II, III. Then we have the direct integral decomposition*

$$\rho_G = \int_{\hat{G}}^{\oplus} 1_{\gamma,\sigma} \otimes \tilde{\pi}_{\gamma,\sigma}\, d\mu_{\bar{\gamma}}(\sigma)\, d\bar{\mu}_N(\bar{\gamma}). \tag{10.5}$$

If in addition we have

(IV) ρ_G *is type* I.

Then the measure $d\mu_{\bar{\gamma}}(\sigma)\, d\bar{\mu}_N(\bar{\gamma})$ *is indeed the Plancherel measure* μ_G.

Proof. — There is very little left to show. By the preceding arguments, formula (10.5) is valid whenever assumptions I, II, III are satisfied. If IV also holds, then by Theorem 6.4 there is a unique measure (class) μ_G on $\hat{G}$ such that $\rho_G = \int_{\hat{G}}^{\oplus} 1_\pi \otimes \tilde{\pi}\, d\mu_G(\pi)$. Since the $\pi_{\gamma,\sigma}$ are irreducible and the multiplicities are " correct " in formula (10.5), the measure $d\mu_{\bar{G}}(\sigma)\, d\bar{\mu}_N(\bar{\gamma})$ must be in the Plancherel measure class.

Examples. — 1. We give a brief computation of the Plancherel formula for the group of affine transformations of the line. Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbf{R},\ a \neq 0 \right\}$$

with the usual topology and matrix multiplication. The modular function of G is $\Delta_G(a, b) = |a|$. Take for the closed normal subgroup

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbf{R} \right\}.$$

Then

$$\hat{N} = \left\{ \gamma : \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \to e^{i\gamma b}, \gamma \in \mathbf{R} \right\}$$

and one checks easily that

$$G_\gamma = \begin{cases} G & \text{if } \gamma = 0, \\ N & \text{if } \gamma \neq 0. \end{cases}$$

There are two orbits in $\hat{N}$: $\mathcal{O}_1 = \{0\}$, $\mathcal{O}_2 = \hat{N} - \{0\}$. Let f be a strictly positive Borel function on $\hat{N}$ with total integral 1. Then we may take $f(\gamma)\,d\gamma$ for μ_N. The quotient measure on $\hat{N}/G$ is $\bar{\mu}_N$ which assigns 0 to $\mathcal{O}_1$ and 1 to $\mathcal{O}_2$. The irreducible representations of G are the one-parameter family of characters σ'', $\sigma \in (G/N)^\wedge$, and the single infinite-dimensional representation

$$\pi_1 = \mathrm{Ind}_N^G \gamma_1, \qquad \gamma_1 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = e^{ib}.$$

According to Theorem 10.2, the Plancherel formula for G takes the form

$$\rho_G = 1 \otimes \tilde{\pi}_1 = 1 \otimes \pi_1,$$

that is, the regular representation is a multiple of the irreducible representation π_1. We understand that Calvin Moore has shown that the operator $D(\pi_1)$ is $\pi_1(\partial/\partial b)$ [27].

2. The inhomogeneous Lorentz groups. For $n \geqq 2$, let H_n denote the neutral component of the group of transformations of $\mathbf{R}^{n+1}$ leaving invariant the quadratic form $-x_0^2 + x_1^2 + \ldots + x_n^2$. The H_n are the so-called (homogeneous) Lorentz groups. We shall use the notation of [20, § 2] in describing the structure of H_n. The groups $G_n = R^{n+1} . H_n$ are called the inhomogeneous Lorentz groups. As usual, we identify $\mathbf{R}^{n+1}$ with its dual and consider the orbit space $\mathbf{R}^{n+1}/H_n$. It is well-known that there are four families of orbits : $\{0\}$,

$$\begin{aligned}
\mathcal{C}^+ &= \{(x_0, x_1, \ldots, x_n) : x_0^2 = x_1^2 + \ldots + x_n^2, x_0 > 0\}, \\
\mathcal{C}^- &= \{(x_0, x_1, \ldots, x_n) : x_0^2 = x_1^2 + \ldots + x_n^2, x_0 < 0\}; \\
\mathcal{J}_r^+ &= \{(x_0, x_1, \ldots, x_n) : -x_0^2 + x_1^2 + \ldots + x_n^2 = -r^2, x_0 \geqq r\}, \qquad r > 0, \\
\mathcal{J}_r^- &= \{(x_0, x_1, \ldots, x_n) : -x_0^2 + x_1^2 + \ldots + x_n^2 = -r^2, x_0 \leqq -r\}, \qquad r > 0;
\end{aligned}$$

and

$$\mathcal{O}_r = \{(x_0, x_1, \ldots, x_n) : -x_0^2 + x_1^2 + \ldots + x_n^2 = r^2\}, \qquad r > 0.$$

$\mathcal{C}^+$ and $\mathcal{C}^-$ are the light cones, $\mathcal{J}_r^+ \cup \mathcal{J}_r^-$ constitutes a two-sheeted hyperboloid, and $\mathcal{O}_r$ is a hyperboloid of revolution. Using the notation of [20, p. 941-942] one computes readily that the stability groups are as follows : for $\gamma = 0$, $H_\gamma = H_n$; for $\gamma = (1, 1, 0, \ldots, 0) \in \mathcal{C}^+$, $H_\gamma = MN$ and for $\gamma = (-1, 1, 0, \ldots, 0) \in \mathcal{C}^-$, $H_\gamma = MV$; for $\gamma = (\pm r, 0, \ldots, 0) \in \mathcal{J}_r^\pm$, $H_\gamma = K$; and finally for $\gamma = (0, 0, \ldots, r)$, $H_\gamma \simeq H_{n-1}$. Now the base $\mathbf{R}^{n+1}/H_n$ of our " fibre space " has (up to a set of measure zero) three rays $t_r^\pm$, o_r, meeting at a point. The orbit $\mathcal{O}_r$ (the others are similar) can be parametrized

$$x_0 = r \operatorname{sh} u,$$
$$(x_1, \ldots, x_n) = r \operatorname{ch} u \, (\theta_1, \ldots, \theta_{n-1}),$$

$(\theta_1, \ldots, \theta_{n-1})$ = polar coordinates on S^{n-1}. The Jacobian of this transformation has the form $r^n \, J(u, \theta_1, \ldots, \theta_{n-1})$. Therefore (at least the equivalence class of) the measure on the base is $r^n \, dr$ on each of the three rays. Finally the fiber over the points of $t_r^\pm$ carry the Plancherel measure of $K^\wedge = SO(n)^\wedge$, a discrete fibre; and the fibre over the points of o_r carry the Plancherel measure of $\hat{H}_{n-1}$ — a well-known split-rank one semisimple Lie group. We note that the case $n = 3$ has been studied in [30].

Remarks. — 1. The above groups are type I, but not CCR (*see*, e. g., [1, p. 172]). We tried, without success, to give an example of a Plancherel formula for a non-unimodular CCR group. In fact, we do not know *any* example of a non-unimodular CCR group.

2. A reasonable alternative to assumption III would be

(III′) For μ_N almost all γ, the right regular $\overline{\omega}_\gamma$-representation of G_γ/N is type I.

However, we have not yet been able to replace III by III′ in the proof of Theorem 10.2.

3. We suspect, but have not been able to prove, that assumptions I, II, III imply the truth of assumption IV. Using the results of Effros [13] as we did in paragraph 9, one sees that what is needed is a proof of : γ, σ traceable $\Rightarrow \pi_{\gamma,\sigma}$ is traceable.

4. Another matter which we have touched skimpily here — and which we feel is quite important — is to give a complete description of the Borel structure of $\hat{G}$ in terms of those for $\hat{N}$ and the $\left(\widehat{G_\gamma/N}\right)^{\overline{\omega}_\gamma}$.

5. If G is unimodular, Theorem 10.2 does not specify which is *the* Plancherel measure in the measure class of $d\mu_{\overline{\gamma}}(\sigma) \, d\overline{\mu}_N(\overline{\gamma})$. Even if all

the groups G_γ/N are unimodular (a fact which is not necessarily true), we have only determined the measure class. The explicit determination of the precise measure in the class has of course been carried out under various general assumptions (§ 4, 5 and 8). It is in these cases that the "appropriate" choice of the pseudo-image of μ_N is evident. What is needed in general is a method for determining the correct choice of $\overline{\mu}_N$.

We hope to return to some of these matters at a later time [5].

11. Discrete series. — We close this paper with some remarks on the relation of discrete series representations of G to those of N and the little groups G_γ/N. We begin with a general

Definition. — Let G be locally compact with a normalized multiplier ω. We call the discrete series (respectively ω-discrete series) of G the collection of $\pi \in \hat{G}$ (respectively $\pi \in \hat{G}^{\omega}$) such that π is equivalent to a subrepresentation of the left regular representation (respectively left regular ω-representation) of G.

If the left regular representation of G [respectively $G(\overline{\omega})$] is type I, then by Theorem 6.4 (respectively Theorem 7.1) and [10, 8.6.8], a representation $\pi \in \hat{G}$ (respectively $\pi \in \hat{G}^{\omega}$) is in the discrete series (respectively ω-discrete series) if and only if $\mu_G(\{\pi\}) > 0$ [respectively $\mu_{G,\omega}(\{\pi\}) > 0$]. If furthermore G is unimodular, then $\pi \in \hat{G}$ (respectively $\pi \in \hat{G}^{\omega}$) is in the discrete series (respectively ω-discrete series) if and only if its matrix coefficients are square integrable functions.

Having these notions, we obtain immediately the following consequence of Theorem 10.2.

Corollary 11.1. — *With assumptions* I-IV *of Theorem* 10.2, *the representation* $\pi_{\gamma,\sigma} = \mathrm{Ind}_{G_\gamma}^{G}(\gamma' \otimes \sigma'')$ *is in the discrete series of* G *if and only if both* (i) σ *is in the* $\overline{\omega}_\gamma$*-discrete series of* G_γ/N, *and* (ii) *the orbit containing* γ *has positive measure, i. e.,* $\mu_N(\gamma . G) > 0$.

Proof. — This follows from (10.5) as soon as we observe that the positivity of $\mu_G(\{\pi_{\gamma,\sigma}\})$ is independent of the choice of representative of μ_G, ditto for $\mu_{G(\overline{\omega})}(\{\sigma\})$, and that the positivity of $\mu_N(\gamma . G)$ also does not depend on which quasi-invariant measure in the class of μ_N is chosen.

Note that if γ is in the discrete series of N, then $\mu_N(\gamma . G) > 0$; but the converse is false. This is illustrated by the first of the following examples.

[5] *Added during proof.* The authors have solved the problems posed in 3 and 5. *See* their article in the Proc. 1972 AMS Summer Inst. E. Carlton has settled 2,

Examples. — 1. Let $N = \mathbf{Q}_p =$ the field of p-adic numbers, $U =$ the (multiplicative) subgroup of units in $\mathbf{Q}_p$. Set $G = N.U$ where U acts on N by multiplication. Identifying $N = \hat{N}$, we see that $\hat{N}/G$ consists of $\{0\} \cup \bigcup_{n=-\infty}^{\infty} \mathcal{O}_n$, where $\mathcal{O}_n = \{x \in N : |x| = p^{-n}\}$ is a compact-open orbit. Moreover, an easy calculation shows that $\mu_N(\mathcal{O}_n) = p^{-n}[1-(1/p)]$. Also, the stability groups are

$$G_\gamma = \begin{cases} G & \text{if } \gamma = 0, \\ N & \text{if } \gamma \neq 0. \end{cases}$$

Thus aside from the 0-orbit (to which corresponds the characters of G trivial on $N \simeq \hat{U}$), all the other irreducible representations $\pi_\gamma = \mathrm{Ind}_N^G \gamma$, $|\gamma| = p^{-m}$, are in the discrete series of G-this in spite of the fact that N has no discrete series. Note also that $\inf\{\mu_G(\{\pi\}) : \pi$ in the discrete series of $G\} = 0$. This is contrary to a difficult and significant conjecture of Harish-Chandra's for reductive algebraic groups to the effect that $\inf\{\mu_G(\{\pi\}) : \pi$ in the discrete series of $G\} > 0$. Of course, the group $G = N.U$ is solvable.

2. Instead of the Heisenberg group G (*see* the example of paragraph 8), let us consider $G_1 = G/D$, where D is the discrete central subgroup

$$D = \left\{ \begin{pmatrix} 1 & 0 & 2\pi n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\}.$$

Take for N the image of the center of G under the canonical projection $G \to G_1$. N is a compact central subgroup of G_1. $\hat{N}$ is free abelian on one generator and $G_\gamma = G$, all $\gamma \in \hat{N}$. The representations of G fall into two classes : $\pi_{0,\sigma} =$ the one-dimensional characters σ of $G/N = R^2$ lifted to G, and $\pi_n =$ the unique irreducible (infinite-dimensional) representation of G, whose restriction to the center acts via $z \to z^n$, $n \neq 0$. Without bothering about the precise normalization of Haar measures, we see easily that the Plancherel formula in this case is

$$\int_{G_1} |\varphi(g_1)|^2 \, dg_1 = \int_{\mathbf{R}^2} |\pi_{0,\sigma}(\varphi)|^2 \, d\sigma + \Sigma_{n \neq 0} |n| \, \mathrm{Tr}(\pi_n(\varphi)\pi_n(\varphi)^*), \qquad \varphi \in L_1(G_1) \cap L_2(G_1).$$

This illustrates the fact (which is apparently not widely-known) that unimodular groups other than reductive groups can have discrete series.

3. We observe finally that in the example of paragraph 10 (the "$ax+b$" group), the sole infinite-dimensional irreducible representation constitutes the discrete series of that group.

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