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Annales scientifiques de l’É.N.S. 4e série, tome 5, n° 2 (1972), p. 261-264

<http://www.numdam.org/item?id=ASENS_1972_4_5_2_261_0>
WEIL-CHÂTELET GROUPS
OVER LOCAL FIELDS : ADDENDUM

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By using the structure theorems for the Néron minimal model of an abelian variety with semi-stable reduction, as presented in [2], it is possible to complete the proof of the following theorem. (Notations are as in [3].)

**Theorem.** — *Let $A$ be an abelian variety over a local field $K$ (with finite residue field) and let $\hat{A}$ be the dual abelian variety. Then the pairings*

$$H^r(K, A) \times H^{1-r}(K, \hat{A}) \to H^2(K, G_m) \cong \mathbb{Q}/\mathbb{Z},$$

*as defined by Tate [4], are non-degenerate for all $r$.*

After [3], we need only consider the case where $K$ has characteristic $p \neq 0$. Also we have only to prove that the map

$$\theta_K(A)_p : H^1(K, A)_p \to (\hat{A}(K)^0)^*$$

is injective, and it suffices to do this after making a finite separable field extension. Thus we may assume that $A$ and $\hat{A}$ have semi-stable reduction ([2], § 3.6) and that

$$A_p(K) = A_p(\overline{K}), \quad \hat{A}_p(K) = \hat{A}_p(\overline{K}).$$
Let $\mathfrak{C}$ be the Néron minimal model of $A$ over $R$. The Raynaud group $\mathfrak{C}^\circ$ of $\mathfrak{C}$ over $R$ is a smooth group scheme over $R$ such that: (a) there are canonical isomorphisms $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}^\circ$ and $\mathfrak{C}^\circ \cong \mathfrak{C}^\circ_0$ (where $\mathfrak{C}$ denotes the formal completion of a scheme $\mathfrak{C}$ of $\mathfrak{C}$ over $R$) and (b) there is an exact sequence $0 \to \mathfrak{C} \to \mathfrak{C}^\circ \to \mathfrak{C}^\circ_0 \to 0$ in which $\mathfrak{C}$ is an abelian scheme and $\mathfrak{C}$ is a torus ([2], § 7.2). $\mathfrak{C} = (\mathfrak{C}^\circ)^p$ is identified through the isomorphism in (a) with the maximal finite flat subgroup scheme of the quasi-finite group scheme $\mathfrak{C}^\circ$. If we write $B = \mathfrak{C} \otimes_R K$, $N = \mathfrak{C} \otimes_R K$, ... , then we get a filtration $A_p = \mathfrak{C}_p \otimes_R K \supset N \supset T_p \supset 0$ of $A_p$ in which $N/T_p \cong B_p$.

Let $\mathfrak{C}', \mathfrak{C}^\prime, \mathfrak{C}^\prime', ...$ be the schemes corresponding, as above, to $\mathfrak{A}$. The canonical non-degenerate pairing $A_p \times \mathfrak{A}_p \to G_m$ respects the filtrations on $A_p$ and $\mathfrak{A}_p$, i.e. $N$ and $T_p$ are the exact annihilators of $T_p'$ and $N'$ respectively. Indeed, the induced pairing $N \times N' \to G_m$ has a canonical extension to a pairing $\mathfrak{C}_p \times \mathfrak{C}_p' \to G_{m,p}$ ([2], § 1.4). This pairing is trivial on $\mathfrak{C}_p$ and $\mathfrak{C}_p'$ and the quotient pairing $\mathfrak{C}_p \times \mathfrak{C}_p' \to G_{m,p}$ is the non-degenerate pairing defined by a Poincaré divisorial correspondence on $(\mathfrak{C}, \mathfrak{C}^\prime)$ ([2], § 7.4, 7.5). This shows that $T_p$ (resp. $T_p'$) is the left (resp. right) kernel in the pairing $N \times N' \to G_m$. The pairing $A_p/T_p \times N' \to G_m$ is right non-degenerate. But $A_p/T_p$ has rank $p^{2n-\mu}$ where $n = \dim (A)$ and $\mu = \dim (\mathfrak{C})$ and $N'$ has rank $p^{\mu+2z}$, where $z = \dim (\mathfrak{C})$ (cf. [2], § 2.2.7). This shows that the pairing is also left non-degenerate (because $n = \mu + z$), which completes the proof of our assertion.

Consider the commutative diagram:

$$
\begin{array}{ccc}
\mathfrak{C}^\circ (R)^{(p)} & \longrightarrow & H^1 (R, \mathfrak{C}^\circ) \\
\downarrow & & \downarrow \\
A (K)^{(p)} & \longrightarrow & H^1 (K, A_p)
\end{array}
$$

in which the horizontal maps are boundary maps in the cohomology sequences for multiplication by $p$ on $A$ and $\mathfrak{C}$. $H^1 (R, \mathfrak{C}^\circ_p) \cong H^1 (R, \mathfrak{C})$ because $\mathfrak{C}^\circ_p/\mathfrak{C}$ is smooth over $R$ with zero special fibre and so has zero cohomology groups ([1], § 11.7) (including in dimension 0). The top arrow is an isomorphism because $H^1 (R, \mathfrak{C}^\circ_p) = 0$ (loc. cit). The cokernel of the left vertical arrow is $\Phi_\circ (k)^{(p)}$, where $\Phi_\circ$ is the group of connected components of $\mathfrak{C} \otimes_R k$ (cf. [2], § 11.1). Using all of this, one can extract from the top diagram on p. 275 of [3] (with $m = p$) an exact commutative
diagram:

\[
\begin{array}{c}
0 \longrightarrow \Phi (k)^{\rho} \longrightarrow H^1 (K, \Lambda_p) / H^1 (R, \Lambda) \longrightarrow H^1 (K, \Lambda_p) \longrightarrow 0 \\
\downarrow \psi_1 \downarrow \psi_1 \downarrow \psi_1 \\
0 \longrightarrow H^1 (R, \Lambda') / \Lambda (R)^{\rho} \longrightarrow \Lambda (R)^{\rho} \longrightarrow 0
\end{array}
\]

It is easy to see that \( \psi_1 \) is an isomorphism if and only if

\[ [\ker \psi_1] = [\Lambda (K)^{\rho} / \Lambda (R)^{\rho}] \],

i.e. \( [\ker \psi_1] = [\Phi (k)^{\rho}] \).

We shall show that

\[ [\ker \psi_1] = p^{\alpha}, \quad [\Phi (k)^{\rho}] = p^{\alpha} = [\Phi (k)^{\rho}], \]

and as \( [\ker \psi_2] [\Phi (k)^{\rho}] = [\ker \psi_1] \), this completes the proof.

Consider first the situation: \( \Lambda \) is a finite group scheme over \( K \) and \( \Lambda \) and \( \Lambda' \) are finite flat group schemes over \( R \) with given embeddings \( N \to M, \Lambda' \to \Lambda \). If \( \Lambda = \mathcal{O}_p \) for some abelian scheme \( \mathcal{O} \) over \( R \) and \( M = \Lambda', \Lambda' = \hat{\Lambda}, \) then

\[ \psi : H^1 (K, M) / H^1 (R, \Lambda) \to H^1 (R, \Lambda')^*, \]

the map defined by the cup-product pairing

\[ H^1 (K, M) \times H^1 (K, \hat{\Lambda}) \to H^1 (K, \hat{\Lambda}), \]

is an isomorphism \([3]\). If \( \Lambda = \mu_p, M = N, \) and \( \Lambda' = 0 \), then \( [\ker \psi] = p \) because \([3]\)

\[ H^1 (K, \mu_p) / H^1 (R, \mu_p) \cong H^1 (R, \mathbb{Z}/p \mathbb{Z})^* \cong H^1 (k, \mathbb{Z}/p \mathbb{Z})^*. \]

If \( M = \mathbb{Z}/p \mathbb{Z}, \Lambda = 0, \) and \( \Lambda' = \mu_p \), then \( [\ker \psi] = p \) because \([3]\)

\[ \ker \psi = H^1 (R, \mathbb{Z}/p \mathbb{Z}). \]

It follows from this, and the above discussion of the structures of \( \Lambda_p \) and \( \hat{\Lambda}_p \), that \( [\ker \psi_1] = p^{\alpha \rho}. \)

Finally, let \( \Phi = \Lambda_\rho / \Lambda_\rho^0. \) It is a finite étale group scheme over \( R \) such that \( \Phi \otimes_k k = \Phi_v \), and there is an exact sequence

\[ 0 \to \Lambda \to \Lambda_\rho \to \Phi \to 0. \]

\( \Lambda_\rho^0 (R) \cong \Lambda_\rho (R) \), because \( \Lambda_\rho^0 \) and \( \Lambda_\rho \) differ only by a scheme with empty special fibre, and \( \Lambda_\rho (R) \cong \Lambda_\rho (K). \) It follows that \( \Phi (K) = \Lambda_\rho (K) / N (K) \) has \( \alpha \rho \) elements. But

\[ \Phi (K) \cong \Phi (R) \cong \Phi_v (k) \quad \text{and so} \quad [\Phi (k)^{\rho}] = [\Phi_v (k)] = p^{\rho}. \]
REFERENCES


(Manuscrit reçu le 2 novembre 1971.)

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