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ON THE INTERSECTION MATRIX
OF A HYPERSURFACE

By Nicholas M. KATZ.

Introduction. — Let (R, m) be a complete discrete valuation ring whose residue field has q elements, and whose fraction field K has characteristic zero. Let X be a projective and smooth hypersurface of dimension n and degree d over R, in " general position " (*see* section 1 for a precise definition). Dwork [3] has constructed a finite-dimensional K-vector spece W^S, and an endomorphism α of W^S, which determines the zeta function [10] of the special fibre $\overline{X}$ as follows :

$$(0.1)\qquad Z_{\overline{X}}(t) = \frac{\det\left(1 - t\frac{\alpha}{q}\,\middle|\, W^S\right)^{(-1)^{n+1}}}{\prod_{i=0}^{n}(1-q^i t)}.$$

The verity of the then-conjectural functional equation [10] for $Z_{\overline{X}}$,

$$(0.2)\qquad Z_{\overline{X}}\left(\frac{1}{q^n t}\right) = \pm\, t^{\chi}\, q^{\frac{n\chi}{2}} Z_{\overline{X}}(t)$$

[here χ is minus the degree of $Z_{\overline{X}}(t)$ as rational function] is equivalent, by (0.1), to the fact that the eigenvalues of $\frac{\alpha}{q}$ are mapped bijectively to themselves by $x \mapsto q^n/x$. In order to verify this, Dwork constructed a non-degenerate bilinear pairing

$$W^S \times W^S \to K$$

with respect to which $q^{-\frac{n}{2}-1}\,\alpha$ was an isometry ([4], p. 266 and th. 9.2).

The space W^S, together with its pairing, is of a purely algebraic nature, and may be constructed for every hypersurface X, projective, smooth and in general position over an arbitrary field K of characteristic zero. There is an isomorphism, Θ [7], between W^S and $\mathrm{Prim}^n(X/K)$, the subspace of $H^n_{DR}(X/K)$ consisting of primitive (in the sense of Hodge theory) classes. W. Cassellman conjectured that Dwork's pairing should correspond via Θ to the cup-product pairing.

Oun purpose is to prove that this is essentially the case, i. e., that there exists a non-zero rational number, c, depending only on the dimension, n, and the degree, d, of the hypersurface X, such that the following diagram is commutative :

$$\begin{array}{ccc} W^S \times W^S & \xrightarrow{c\times(\text{Dwork's pairing})} & K \\ \downarrow{\scriptstyle \Theta\times\Theta} & & \wr\downarrow \\ \mathrm{Prim}^n(X/K) \times \mathrm{Prim}^n(X/K) & \xrightarrow{\text{cup-product}} & H^{2n}_{DR}(X/K) \end{array}$$

(The commutativity of this diagram is the only known property of the constant c.)

The basic idea is this : the two constructions, W^S and Prim^n, globalize to give locally free sheaves on T, the classifying space for nonsingular hypersurfaces of dimension n and degree d in general position. These sheaves are both endowed with integrable connections. The isomorphism Θ globalizes to an isomorphism of these sheaves, which is compatible with the connections. The desired theorem is then the commutativity of the following diagram of sheaves on T :

$$\begin{array}{ccc} W^S \times W^S & \xrightarrow{c\times(\text{Dwork's pairing})} & \mathcal{O}_T \\ \downarrow{\scriptstyle \Theta\times\Theta} & & \wr\downarrow \\ \mathrm{Prim}^n(X/T) \times \mathrm{Prim}^n(X/T) & \xrightarrow{\text{cup-product}} & H^{2n}_{DR}(X/T) \end{array}$$

Two reductions are now possible. First, it suffices to prove that the above diagram commutes after the base-change which replaces T by the *formal* neighborhood of a point $s \in T$ (i. e. the completion of the local ring at s). Second, both pairings respect the connection (i. e. the " product formula " holds) in the formal neighborhood of the point in T which corresponds to the diagonal hypersurface $\sum_{i=1}^{n+2} X_i^d = 0$; because the connection is *integrable*, both pairings are then determined by their *value* at this point.

As explicit formulas for Dwork's pairing at this point are available, the problem is to compute the intersection matrix of the hypersurface $\sum_{i=1}^{n+2} X_i^d = 0$, with respect to a *given* basis of its De Rham cohomology. The shape of this matrix is dictated by the presence of a large group of automorphisms. We succeed in the *precise* determination only by recourse to a deformation-theoretic method, which consists in differentiating (with respect to the parameters) certain relations in the cohomology ring of the general hypersurface.

1. Some De Rham Cohomology. — Fix positive integers n and d, and an ordered set of homogeneous coordinates $X_1, \ldots, X_{n+2}$ for the projective space $\mathbf{P} = \mathbf{P}^{n+1}$. Let S be an arbitrary scheme of characteristic zero (i. e. over $\mathbf{Q}$). Let X be a nonsingular hypersurface of dimension n and degree d, over S, in general position with respect to the coordinates axes. This means X is defined by the vanishing of a homogeneous form $F(X_1, \ldots, X_{n+2})$ of degree d, and that, denoting by $\mathcal{O}_S[X]$ the sheaf of graded algebras on S given by $\mathcal{O}_S[X_1, \ldots, X_{n+1}] = \bigoplus_{k\in\mathbf{Z}} \mathrm{pr}_{2*}(\mathcal{O}_{\mathbf{P}\times S}(k))$, the Koszul complex defined by the $X_i \dfrac{\partial F}{\partial X_i}$

$$\text{(1.0)} \qquad 0 \leftarrow \mathcal{O}_S[X] \leftarrow \bigoplus_{1\leq i\leq n+2} \mathcal{O}_S[X] \leftarrow \bigoplus_{1\leq i\leq j\leq n+2} \mathcal{O}_S[X] \leftarrow$$

is exact.

Let us explicitly write the defining equation $\sum_{w} A_w X^w = 0$, the sum taken over the $\binom{n+d+1}{d}$ monomials X^w of degree d in $X_1, \ldots, X_{n+2}$. The coefficients A_w " are " a morphism from S to the projective space over $\mathbf{Q}$ of $\binom{n+d+1}{d} - 1$ dimensions. This projective space is thus the classifying space for (flat) families of hypersurfaces of degree d in the " coordinatized " projective space $\mathbf{P}^{n+1}$ over $\mathbf{Q}$. It is easily seen (theory of the resultant) that the classifying space for those families which are nonsingular and in general position is a non-empty Zariski open-subset T of $\mathbf{P}^{\star}_{\mathbf{Q}}$, $\star = \binom{n+d+1}{d} - 1$.

Over T we dispose of the universal non-singular hypersurface in general position, $X \to T$, its complement $\mathbf{P}_T - X \to T$, and the constant projective space $\mathbf{P}_T \to T$.

Recall that for any smooth morphism $f : A \to B$, the relative De Rham cohomology sheaves on B, written simply $H^q_{DR}(A/B)$, are, by definition, the $\mathbf{R}^q f_*(\Omega^{\cdot}_{A/B})$, and that [6], when B is (the spectrum of) the complex numbers $\mathbf{C}$, $H^{\cdot}_{DR}(A/B) \simeq H^{\cdot}(A^{an}, \mathbf{C})$, the second member being the " ordinary " cohomology groups of the assosciated complex manifold A^{an}. Recall also that the sheaf $H^q_{DR}(A/B)$ is said to commute with base change if, for any morphism $g : S \to B$, the homomorphism

$$g^* H^q_{DR}(A/B) \to H^q_{DR}(A \times_B S/S)$$

is an isomorphism.

PROPOSITION 1. — *The sheaves* $H^i_{DR}(X/T)$ *and* $H^i_{DR}(\mathbf{P}_T - X/T)$ *are locally free, and commute with base change.*

Proof. — We introduce the *coherent* complex $\Omega^{\bullet}_{\mathbf{P}/T}(\log X)$ on $\mathbf{P}_T$, whose local sections are the differentials of the form $\omega + \tau \wedge \frac{df}{f}$, where ω and τ are local sections of $\Omega^{\bullet}_{\mathbf{P}/T}$ and f is a local equation for X (we write $\mathbf{P}/T$ for $\mathbf{P}_T/T$). Equivalently, $\Omega^{\bullet}_{\mathbf{P}/T}(\log X)$ may be described as the tensor product

$$l(X)^{-1} \otimes_{\mathcal{O}_{\mathbf{P}}} (\text{kernel of } \Omega^{\bullet}_{\mathbf{P}/T} \to \Omega^{\bullet}_{X/T}),$$

where $l(X)$ is the sheaf of ideals defining X on $\mathbf{P}_T$. Because the inclusion $\Omega^{\bullet}_{\mathbf{P}/T}(\log X) \to \Omega^{\bullet}_{\mathbf{P}-X/T}$ is a quasi-isomorphism (i. e., induces an isomorphism of cohomology sheaves), the induced map of sheaves on T, $\mathbf{R}^i f_*(\Omega^{\bullet}_{\mathbf{P}/T}(\log X)) \to \mathbf{R}^i f_*(\Omega^{\bullet}_{\mathbf{P}-X/T})$ is an isomorphism. Thus from the exact sequence of complexes on $\mathbf{P}_T$,

$$(1.1) \qquad 0 \to \Omega^{\bullet}_{\mathbf{P}/T} \to \Omega^{\bullet}_{\mathbf{P}/T}(\log X) \to \Omega^{\bullet -1}_{X/T} \to 0$$

$\Big($the last map, the residue, is defined locally by extracting the coefficient of $\frac{df}{f}$ and restricting to X$\Big)$ we obtain the long cohomology sequence

$$(1.2) \qquad \to H^{i-2}_{DR}(X/T) \to H^i_{DR}(\mathbf{P}/T) \to H^i_{DR}(\mathbf{P} - X/T) \to H^{i-1}_{DR}(X/T) \to$$

We will apply the base-changing theorems of ([2], § 7), this being permissible because we are dealing with T-flat coherent complexes, T-linear differentials, and projective morphisms. Let $s \in T$ be any point, $K(s)$ its residue field, $\mathbf{P}_s$ and X_s the corresponding fibres, which are varieties over $K(s)$. We have a corresponding exact sequence of $K(s)$-vector space (by repeating the above construction over $K(s)$)

$$\to H^{i-2}_{DR}(X_s) \to H^i_{DR}(\mathbf{P}_s) \to H^i_{DR}(\mathbf{P}_s - X_s) \to H^{i-1}_{DR}(X_s) \to$$

and by the Lefschetz theorems [8], $H^i_{DR}(\mathbf{P}_s - X_s)$ is zero unless $i = 0$ or $i = n+1$. Thus ([2], 7.8.3 and 7.8.4) the sheaves $H^i_{DR}(\mathbf{P} - X/T)$ are locally free and commute with base change, because these sheaves vanish save for $i = 0$ and $i = n+1$. Next we remark that each

$$H^i_{DR}(\mathbf{P}/T) = H^i_{DR}(\mathbf{P}_{\mathbf{Q}} \times_{\mathbf{Q}} T) \simeq H^i_{DR}(\mathbf{P}_{\mathbf{Q}}/\mathbf{Q}) \otimes_{\mathbf{Q}} \mathcal{O}_T$$

is certainly locally free and commutes with base change, and thus the map $H^i_{DR}(\mathbf{P}/T) \to H^i_{DR}(\mathbf{P} - X/T)$ is zero for $i \geq 1$, this being so at each $K(s)$.

Referring to the exact sequence (1.2), we then see that

$$H^i_{DR}(X/T) \xrightarrow{\sim} H^{i+2}_{DR}(\mathbf{P}/T) \qquad \text{for} \quad i \neq n.$$

Hence, by ([2], 7.8.3), $H^n_{DR}(X/T)$ is also locally free, and commutes with base change.

Q. E. D.

Recall now that a local section of $H^n_{DR}(X/T)$ is called *primitive* [8] if its cup-product with the cohomology class of a hyperplane section is zero.

Corollary. — *The residue mapping* $H^{n+1}_{DR}(\mathbf{P}-X/T) \to H^n_{DR}(X/T)$ *establishes an isomorphism between* $H^{n+1}_{DR}(\mathbf{P}-X/T)$ *and* $\mathrm{Prim}^n(X/T)$, *the sheaf of primitive cohomology classes in* $H^n_{DR}(X/T)$. *In particular,* $\mathrm{Prim}^n(X/T)$ *is locally free and commutes with base change.*

Proof. — From the exact sequence, we have

$$0 \to H^{n+1}_{DR}(\mathbf{P}-X/T) \to H^n_{DR}(X/T) \to H^{n+2}_{DR}(\mathbf{P}/T) \to 0.$$

Since the restriction mapping $H^{n+2}_{DR}(\mathbf{P}/T) \to H^{n+2}_{DR}(X/T)$ is an isomorphism, we may rewrite the above sequence

$$0 \to H^{n+1}_{DR}(\mathbf{P}-X/T) \to H^n_{DR}(X/T) \to H^{n+2}_{DR}(X/T) \to 0,$$

where the last map is given by cup-product with the " class of a hyperplane section ", whence the corollary, by definition of primitive.

Q. E. D.

We conclude this section by recalling that all the De Rham sheaves are equipped with an integrable connection, that of Gauss-Manin [9], with which all the maps in (1.2) are compatible. The compatibility is most easily seen as follows. For any smooth $g: Y \to T$, the Gauss-Manin connection is obtained as the differential d_1 in the spectral sequence of the filtered object $\Omega^{\cdot}_{Y/\mathbf{Q}}$ with respect to the functor $\mathbf{R}^{\cdot}g_*$, the filtration being defined by

$$F^i(\Omega^{\cdot}_{Y/\mathbf{Q}}) = \mathrm{image}(\Omega^{\cdot-i}_{Y/\mathbf{Q}} \otimes g^*(\Omega^i_{T/\mathbf{Q}}) \to \Omega^{\cdot}_{Y/\mathbf{Q}}).$$

These considerations apply *mutatis mutantis* to the complex $\Omega^{\cdot}_{\mathbf{P}\times T/\mathbf{Q}}(\log X)$, and provide the $\mathbf{R}^q f_*(\Omega^{\cdot}_{\mathbf{P}/T}(\log X))$ with an integrable connection. The only point to be checked is that, under the filtration

$$F^i(\Omega^{\cdot}_{\mathbf{P}\times T/\mathbf{Q}}(\log X)) = \mathrm{image}\,\Omega^{\cdot-i}_{\mathbf{P}\times T/\mathbf{Q}}(\log X) \otimes g^*(\Omega^i_{T/\mathbf{Q}}) \to \Omega^{\cdot}_{\mathbf{P}\times T/\mathbf{Q}}(\log X)$$

the associated graded objects are given by

$$\mathrm{gr}^i(\Omega_{\mathbf{P}\times T/\mathbf{Q}}(\log X)) = F^i/F^{i+1} = \Omega^{\cdot-i}_{\mathbf{P}/T}(\log X) \otimes g^*(\Omega^i_{T/\mathbf{Q}}).$$

As the quasi-isomorphism $\Omega^{\cdot}_{\mathbf{P}/T}(\log X) \to \Omega^{\cdot}_{\mathbf{P}-X/T}$ is the gr^0 of the map $\Omega^{\cdot}_{\mathbf{P}\times T/\mathbf{Q}}(\log X) \to \Omega^{\cdot}_{\mathbf{P}\times T-X/\mathbf{Q}}$, the connection on $\mathbf{R}^q f_*(\Omega^{\cdot}_{\mathbf{P}/T}(\log X))$ corresponds, via the isomorphism $\mathbf{R}^q f_*(\Omega^{\cdot}_{\mathbf{P}/T}(\log X)) \to H^q_{DR}(\mathbf{P}-X/T)$, to the

Gauss-Manin connection on $H^a_{DR}(\mathbf{P}-X/T)$. Finally, the exact sequence (1.1) is the gr^0 of the exact sequence of filtered (as above) complexes

$$0 \to \Omega^{\cdot}_{\mathbf{P}\times T/\mathbf{Q}} \to \Omega^{\cdot}_{\mathbf{P}\times T/\mathbf{Q}}(\log X) \to \Omega^{\cdot -1}_{X/\mathbf{Q}} \to 0,$$

whence the compatibility of the maps in (1.2) with the connections.

2. THE CONSTRUCTIONS OF DWORK. — We begin by introducing certain modules over T.

$\mathcal{L}$ the free module with base those monomials $X_1^{w_1}\dots X_{n+2}^{w_{n+2}}$, or simply X^w, with all $w_i \geq 0$, and $\sum_{i=1}^{n+2} w_i$ divisible by d. One defines $w_0 = \frac{1}{d}\sum_{i=1}^{n+2} w_i$.

$\mathcal{L}^S$, the submodule with base those of the above monomials which in addition have all $w_i \geq 1$.

$W^S = \mathcal{L}^S / \mathcal{L}^S \cap \left(\sum_{i=1}^{n+2} D_i \mathcal{L}\right)$, where D_i is the T-linear mapping

$$D_i : \quad \mathcal{L} \to \mathcal{L}, \quad D_i(X^w) = w_i X^w + X_i \frac{\partial F}{\partial X_i} X^w.$$

PROPOSITION 2. — *The module* W^S *is locally free, and commutes with base change.*

Proof. — By the exactness of the Koszul complex for X *and* its intersections with coordinate hyperplanes, we have

$$\mathcal{L}^S \cap \sum D_i \mathcal{L} = \sum D_i \mathcal{L}^{S-i},$$

where $\mathcal{L}^{S-i}$ is the span of those X^w having $w_j > 0$, $\forall j \neq i$. Thus W^S as the cokernel of a map between free modules, commutes with base change. By ([3]. lemma 3.19), for each point $s \in T$, the corresponding vectorspace $W^S_{K(s)}$ over the residue field $K(s)$ of s has constant dimension $d^{-1}[(d-1)^{n+2}+(-1)^n(d-1)]$, whence W^S is locally free.

Q. E. D.

We recall that W^S is provided with an integrable connection [9], deduced by passage to quotients from an integrable connection on $\mathcal{L}$. It is defined by having $D \in \mathrm{Der}_{\mathbf{Q}}(\mathcal{O}_T, \mathcal{O}_T)$ act through $\sigma_D : \mathcal{L} \to \mathcal{L}$ by

$$\sigma_D(tX^w) = D(t)X^w + F^D X^w,$$

where $t \in \mathcal{O}_T$, and F^D denotes the result of applying D to the coefficients of F.

3. THE COMPARISON THEOREM. — There is a (T-linear) map

$$\mathcal{R} : \quad \mathcal{L}^S \to H^{n+1}_{DR}(\mathbf{P}-X/T)$$

given by

$$\mathcal{R} : \quad X^{w} \to (-1)^{w_0-1}(w_0-1)!\,\frac{X^{w}}{F^{w_0}}\,\frac{d(X_1/X_{n+2})}{X_1/X_{n+2}} \wedge \ldots \wedge \frac{d(X_{n+1}/X_{n+2})}{X_{n+1}/X_{n+2}}.$$

It is quickly verified that, because all $w_i \geq 1$, the above differential form is regular on all of $\mathbf{P} - X$, and thus defines a section of $H^{n+1}_{DR}(\mathbf{P} - X/T)$.

The map $\Theta : \mathcal{L}^S \to \mathrm{Prim}^n(X/T)$ is defined to be the composition $\mathcal{L}^S \xrightarrow{\mathcal{R}} H^{n+1}_{DR}(\mathbf{P} - X/T) \xrightarrow{\sim} \mathrm{Prim}^n(X/T)$.

THEOREM 3. — *The maps $\mathcal{R}$ and Θ induce, by passage to quotients, isomorphisms*

$$\mathcal{R} : \quad W^S \xrightarrow{\sim} H^{n+1}_{DR}(\mathbf{P} - X/T),$$
$$\Theta : \quad W^S \xrightarrow{\sim} \mathrm{Prim}^n(X/T),$$

which are compatible with the connections.

Proof. — We first show that $\mathcal{L}^S \cap \sum D_i \mathcal{L}$ lies in the kernel of $\Theta : \mathcal{L}^S \to \mathrm{Prim}^n(X/T)$. As $\mathrm{Prim}^n(X/T)$ is locally free and commutes with base change, it suffices to show that for every $s \in T$, the corresponding map $\Theta_s : \mathcal{L}^S \to \mathrm{Prim}^n(X_s)$ of $K(s)$ vector spaces annihilates $\mathcal{L}^S \cap \sum D_i \mathcal{L}$. By ([7], th. 1.17), the composition

$$\mathcal{L}^S \xrightarrow{\Theta_s} \mathrm{Prim}^n(X_s) \xrightarrow{\text{restriction}} H^n_{DR}(X_s^{\varnothing})$$

(here $X_s^{\varnothing}$ is the open subset of X_s where all coordinates X_i are invertible) annihilates $\mathcal{L}^S \cap \sum D_i \mathcal{L}$. By ([7], cor. 1.11), temporarily writing $X_s^{(1)}$ for the open subset of X_s where the first coordinate X_1 is invertible, the restriction map

$$H^n(X_s^{(1)}) \to H^n(X_n^{\varnothing})$$

is injective. And

$$H^n(X_s) = \mathrm{Prim}^n(X_s) \oplus \text{Kernel of } (H^n(X_s) \to H^n(X_s^{(1)})),$$

whence $\mathrm{Prim}^n(X_s) \to H^n_{DR}(X_s^{\varnothing})$ is injective.

To show that $\Theta : W^S \to \mathrm{Prim}^n(X/T)$ is an isomorphism, it suffices, since both W^S and $\mathrm{Prim}^n(X/T)$ are locally free, to show it over the residue field of each $s \in T$. This is so in virtue of ([7], th. 1.17). That $\mathcal{R}$ (and hence Θ) transforms σ into the Gauss-Manin connection is seen directly

$$\mathcal{R}(\sigma_D(X^{w})) = \mathcal{R}(F^D X^{w}) = (-1)^{w_0} w_0!\,\frac{F^D X^{w}}{F^{1+w_0}}\,\frac{d(X_1/X_{n+2})}{X_1/X_{n+2}} \wedge \ldots \wedge \frac{d(X_{n+1}/X_{n+2})}{X_{n+1}/X_{n+2}}$$

while

$$D(\mathcal{R}(X^{w})) = D\left\{(-1)^{w_0-1}(w_0-1)!\,\frac{X^{w}}{F^{w_0}}\,\frac{d(X_1/X_{n+2})}{X_1/X_{n+2}} \wedge \ldots \wedge \frac{d(X_{n+1}/X_{n+2})}{X_{n+1}/X_{n+2}}\right\}$$

and, as D annihilates all X_i/X_{n+2}, $i=1, \ldots, n+1$, this *is*

$$(-1)^{w_0} w_0! \frac{F^D X^w}{F^{w_0+1}} \frac{d(X_1/X_{n+1})}{X_1/X_{n+1}} \wedge \ldots \wedge \frac{d(X_{n+1}/X_{n+2})}{X_{n+1}/X_{n+2}}.$$

Q. E. D.

4. The Intersection Matrix. — Any T-bilinear pairing

$$\mathrm{Prim}^n(X/T) \times \mathrm{Prim}^n(X/T) \to H^{2n}_{DR}(X/T) \simeq \mathcal{O}_T$$

may be thought of as a global section of the locally free sheaf

$$\mathit{Hom}_{\mathcal{O}_T}(\mathrm{Prim}^n(X/T) \otimes_{\mathcal{O}_T} \mathrm{Prim}^n(X/T), \mathcal{O}_T).$$

Because T is (reduced and) irreducible, for any non-void open subsets $U \subset V$ of T, and any locally free sheaf F on T, the restriction mapping on sections $\Gamma(V, F) \to \Gamma(U, F)$ is injective. Fixing a point $s \in T$, we have $\Gamma(V, F) \subset \varinjlim_{U \ni s} \Gamma(U, F) = F_s$, the stalk at s. Since the local ring $\mathcal{O}_{T,s}$ at s is noetherian, it follows by Krull's Intersection Theorem that $\mathcal{O}_{T,s}$ is a subring of its completion $\hat{\mathcal{O}}_{T,s}$; F_s being free over $\mathcal{O}_{T,s}$, we have $F_s \subset \hat{F}_s = F_s \otimes_{\mathcal{O}_{T,s}} \hat{\mathcal{O}}_{T,s}$. Denote by $\hat{X}_s$ the fibre product $X \times_T \hat{\mathcal{O}}_{T,s}$, which may be thought of as a tubular neighborhood of the special fibre $X_s = X \times_T K(s)$.

Proposition 4. — *Let $s \in T$ be a closed point which is* Q-*rational. Then the $\hat{\mathcal{O}}_{T,s}$-module* $\mathrm{Prim}^n(\hat{X}_s/\hat{\mathcal{O}}_{T,s})$ *admits a basis of horizontal (for the Gauss-Manin connection) elements.*

Proof. — The hypersurface $\hat{X}_s$ is given by an equation

$$X_1^d + \sum_w (a_w + A_w) X^w = 0,$$

where the sum is over all monomials X^w of degree d other than X_a^d, and the A_w are independent variables, with $\hat{\mathcal{O}}_{T,s} = \mathbf{Q}[[A_w]]$. The hypersurface X_s is given by $X_1^d + \sum a_w X^w = 0$.

[We have used the fact that all the monomials X_i^d necessarily occur in the equation of an X in general position.] Let $\{\eta_i\}$ be a basis of $\mathrm{Prim}^n(\hat{X}_s/\hat{\mathcal{O}}_{T,s})$. Each $\frac{\partial}{\partial A_w}$ acts; say $\frac{\partial \eta_i}{\partial A_w} = \sum_j B_{i,j}(w) \eta_j$. If it possible to find a basis of horizontal sections τ_i, which agree with the η_i over the residue field of s, they will necessarily be linear combinations of the η_i,

$$\tau_i = \sum_j u_{ij} \eta_j, \qquad u_{ij} \in \mathbf{Q}[[A_w]],$$

where the matrix $U = (u_{ij})$ satisfies the equation

$$\begin{cases} \dfrac{\partial U}{\partial A_w} = -UB(w) \quad \text{for each } A_w; \qquad \text{here} \quad B(w) = (B_{i,j}(w)), \\ \qquad\qquad U(0) = I. \end{cases}$$

That this system has a unique solution results from the formal version of the Frobenius theorem on " complete integrability ", applicable precisely because the Gauss-Manin connection is integrable ([1], p. 304).

Q. E. D.

Remark. — The inverse P of the matrix U, by means of which the η_i may be expressed in terms of the horizontal basis, is nothing other than the classical period matrix. Let us explain why this is so. Consider a proper and smooth (holomorphic) map $f: Y \to \Delta$, where Δ is the unit disc $|t| < 1$. Replacing Δ if necessary by a smaller disc around 0, we may suppose that $f: Y \to \Delta$ is a differentiably trivial fibre bundle. Choose a C^∞ trivialization $\gamma : Y \xrightarrow{\sim} Y_0 \times \Delta$ [here $Y_0 = f^{-1}(0)$]. Let $\{\eta_i(t)\}$ be a basis of $H^{\text{anal}}_{\text{DR}}(Y/\Delta)$. The horizontal basis $\{\tau_i(t)\}$ is obtained as follows; for each $t \in \Delta$, γ induces $\gamma_t : Y_t \xrightarrow{\sim} Y_0$, and we take $\tau_i(t) = \gamma_t^*(\eta_i(0))$. Let $\alpha_i(0)$ be the basis of $H.(Y_0, \mathbf{C})$ dual to the $\eta_i(0)$, and define a basis $\alpha_i(t)$ of $H.(Y_t, \mathbf{C})$ by $\alpha_i(t) = \gamma_t^{-1}(\alpha_i(0))$. Clearly we have

$$\int_{\alpha_j(t)} \tau_i(t) = \int_{\alpha_j(t)} \gamma_t^*(\eta_i(0)) = \int_{\gamma_t(\alpha_j(t))} \eta_i(0) = \int_{\alpha_i(0)} \eta_i(0) = \delta_{i,j}.$$

Hence $\eta_i(t) = \sum_j \left(\int_{\alpha_j(t)} \eta_i(t)\right) \tau_j(t)$ in $H.(Y_t, \mathbf{C})$, as both sides have the same integral over each cycle $\gamma_j(t)$. Thus the " variable " sections $\eta_i(t)$ are expressed in terms of the horizontal sections $\tau_j(t)$ by means of the matrix of periods $\int_{\alpha_j(t)} \eta_i(t)$.

Proposition 5. — *Assumptions as in Proposition 4, under the cup-product pairing*

$$\text{Prim}^n(\hat{X}_s/\hat{\mathcal{O}}_{T,s}) \times \text{Prim}^n(\hat{X}_s/\hat{\mathcal{O}}_{T,s}) \to H^{2n}_{\text{DR}}(\hat{X}_s/\hat{\mathcal{O}}_{T,s}) \simeq \hat{\mathcal{O}}_{T,s}$$

the cup-product of two horizontal classes, $\tau_i \wedge \tau_j$, is a rational number.

Proof. — Each $\dfrac{\partial}{\partial A_w}$ acts, through the Gauss-Manin connection, as a *derivation* of the cohomology algebra [9], and the connection on $H^{2n}_{\text{SR}}(\hat{X}_s/\hat{\mathcal{O}}_{T,s}) \simeq \hat{\mathcal{O}}_{T,s}$ is just ordinary differentiation. Thus $\tau_i \wedge \tau_j$ " is " an element τ_{ij} of $\mathbf{Q}[[A_w]]$, and

$$\frac{\partial}{\partial A_w}(\tau_{ij}) = \frac{\partial}{\partial A_w}(\tau_i \wedge \tau_j) = \frac{\partial \tau_i}{\partial A_w} \wedge \tau_j + \tau_i \wedge \frac{\partial \tau_j}{\partial A_w} = 0 + 0.$$

Q. E. D.

COROLLARY. — *Let* $\{\eta_i\}$ *be a basis of* $\mathrm{Prim}^n(\hat{X}_s/\hat{\mathcal{O}}_{T,s})$, *and* $\{\tau_i\}$ *the basis of horizontal sections such that for each* i, $\tau_i\text{-}\eta_i$ *induces* o *in* $\mathrm{Prim}^n(X_s/K(s))$. *Let* $P=(P_{ij})$ *be the period matrix*

$$\eta_i=\sum P_{ij}\tau_j.$$

The intersection matrix $M=(M_{ij})$, $M_{ij}=\eta_i\wedge\eta_j$, *is given by*

$$M=PJP', \tag{4.1}$$

where J *is the value of* M *at* s, *i. e. the matrix of rational numbers* $\tau_{ij}=\eta_i(s)\wedge\eta_j(s)$ $(\eta_i(s)\in\mathrm{Prim}^n(X_s/\mathbf{Q})$ *being the class induced by* $\eta_i)$.

5. THE PAIRING OF DWORK. — The pairing was constructed in a bootstrap way (historically). Near the point $s\in T$ which corresponding to the diagonal hypersurface of equation $\sum_{i=1}^{n+2} X_i^d=0$, a basis of W^S is given by the monomials

$$X^w \text{ satisfying } \begin{cases} dw_0=\sum_{i=1}^{n+2} w_i, & w_0\in\mathbf{Z},\\ 1\leq w_i\leq d-1 & \text{for } i=1,\ldots,n+2.\end{cases}$$

Such an exponent system $w=(w_1,\ldots,w_{n+2})$ is called admissible. For each admissible exponent system $w=(w_1,\ldots,w_{n+2})$, define the *dual* system by $\bar{w}=(d-w_1,\ldots,d-w_{n+2})$. Clearly $w\mapsto\bar{w}$ is an involution of admissible exponent systems.

The pairing ([4], p. 248-251) on the **Q**-vector space $W^S(s)$ is defined by

$$(X^u,X^v)_s=\begin{cases}(-1)^{u_0} & \text{if } u=\bar{v},\\ 0 & \text{if not.}\end{cases}$$

It is extended " locally " ([4], p. 266-267), i. e. to $W^S\otimes\hat{\mathcal{O}}_{T,s}$ by requiring $\hat{\mathcal{O}}_{T,s}$ bilinearity and that, for any two *horizontal* elements τ_1 and τ_2, the product (τ_1,τ_2) is the rational number defined above by $(\tau_1(s),\tau_2(s))$. This does define a pairing, since, just as in Proposition 4, $W^S\otimes_T\hat{\mathcal{O}}_{T,s}$ admits an $\hat{\mathcal{O}}_{T,s}$ base of horizontal elements. Let τ_w be the horizontal section whose specialization at s is X^w.

By transport of structure by Θ, the expression of the elements X^w in terms of the horizontal elements τ_w is given by a matrix $P=(P_{w,u})$:

$$X^w=\sum_u P_{w,u}\tau_u \quad \text{in } W^S\otimes\hat{\mathcal{O}}_{T,s}.$$

P is the *period* matrix, expressing

$$\Theta(X^w) = \sum_u P_{w,u}\,\Theta(\tau_u)$$

in $\mathrm{Prim}^n(\hat{X}_s/\hat{\mathcal{O}}_{T,s})$.

Thus Dwork's local intersection matrix $M' = M'_{u,v} = (X^u, X^v)$ is given by

$$M' = PJ'P^t, \tag{5.1}$$

where

$$J'_{u,v} = \begin{cases} (-1)^{u_0} & \text{if } u = v, \\ 0 & \text{if not.} \end{cases}$$

Dwork then constructs an $\mathcal{O}_T$-bilinear pairing

$$W^S \times W^S \to \mathcal{O}_T$$

which extends the one described above ([4], p. 267-282, esp. th. 6.2). As explained in section 4, such an extension is necessarily unique. Since we will make use only of local properties, we will not describe the extension.

6. The Intersection Matrix of a Diagonal Hypersurface. — This section is devoted to proving

Theorem 6. — *Dwork's pairing corresponds to (a constant multiple of) the cup-product, i. e. the following diagram is commutative for some non-zero constant $c \in \mathbf{Q}$, which depends only on the dimension, n, and the degree, d,*

$$\begin{tikzcd} W^S \times W^S \arrow[r, "c\times(\text{Dwork's pairing})"] \arrow[d, "\Theta\times\Theta"] & \mathcal{O}_T \arrow[d, "\wr"] \\ \mathrm{Prim}^n(X/T) \times \mathrm{Prim}^n(X/T) \arrow[r, "\text{cup-product}"] & H^{2n}_{DR}(X/T) \end{tikzcd}$$

Proof. — As explained in section 4, it suffices to show this is so after the base extension $\mathrm{Spec}(\hat{\mathcal{O}}_{T,s}) \to T$, i. e., that it is so in the formal neighborhood of the point $s \in T$ corresponding to the hypersurface $X(n, d)$ of equation $\sum_{i=1}^{n+2} X_i^d = 0$. Comparing (4.1) and (5.1), the pairings are given by PJP^t and $PJ'P^t$ respectively. Thus we must show that there is a constant c such that $J' = cJ$, i. e., that under the cup-product pairing

$$\mathrm{Prim}^n(X(n, d)) \times \mathrm{Prim}^n(X(n, d)) \to \mathbf{Q}$$

we have

$$\Theta(X^u) \wedge \Theta(X^v) = c(-1)^{u_0}\delta_{u,\bar{v}},$$

for each pair u, v of admissible exponent systems.

LEMMA 7. — *Let u and v be admissible, and $u \neq \bar{v}$. Then*

$$\Theta(X^u) \wedge \Theta(X^v) = 0.$$

Proof. — Extending the ground field from $\mathbf{Q}$ to $K = \mathbf{Q}(\exp(2\pi i/d))$, we introduce the action of the group $G = (\mu_d)^{n+2}$, which acts on $\mathbf{P}^{n+1}$, $X(n, d)$, $\mathbf{P}^{n+1} - X(n, d)$, etc., via

$$g(X_1, \ldots, X_{n+2}) = (\zeta_1 X_1, \ldots, \zeta_{n+2} X_{n+2}).$$

Here $g = (\zeta_1, \ldots, \zeta_{n+2}) \in (\mu_d)^{n+2}$, and $(X_1, \ldots, X_{n+2})$ are the homogeneous coordinates.

To each admissible exponent system w we assosciate the character χ_w of G defined by

$$\chi_w(g) = \zeta_1^{w_1} \ldots \zeta_{n+2}^{w_{n+2}} \quad \text{for} \quad g = (\zeta_1, \ldots, \zeta_{n+2}).$$

For each admissible system w, we claim

$$g(\Theta(X^w)) = \chi_w(g)\, \Theta(X^w).$$

This is so because :

1° $\Theta(X^w) = \text{residue}(\mathcal{R}(X^w))$;

2° $\mathcal{R}(X^w) = (-1)^{w_0 - 1}(w_0 - 1)!\, \dfrac{X^w}{\left(\sum X_i^d\right)^{w_0}} \dfrac{d(X_1/X_{n+2})}{(X_1/X_{n+2})} \wedge \ldots \wedge \dfrac{d(X_{n+1}/X_{n+2})}{(X_{n+1}/X_{n+2})}$,

a differential which obviously transforms under G by χ_w, and 3° the residue map $H^{\bullet}(P^{n+1} - X(n, d)) \otimes K \to H^{\bullet-1}(X(n, d)) \otimes K$ commutes with the action of G.

The product $\Theta(X^u) \wedge \Theta(X^v)$ is thus an element of $H_{DR}^{2n}(X(n, d))$ which transforms under G by $\chi_u \chi_v$. But G acts *trivially* on $H_{DR}^{2n}(X(n, d)) \otimes K$, and thus either

$$\Theta(X^u) \wedge \Theta(X^v) = 0, \quad \text{or} \quad \chi_u = \chi_v^{-1},$$

in which case $u = \bar{v}$.

Q. E. D.

Thus it remains to show that for each admissible exponent system u,

$$\Theta(X^u) \wedge \Theta(X^{\bar{u}}) = c(-1)^{u_0},$$

where c is *independent* of u. We first do this in some special cases :

I. $d = 1$, any n; then $\text{Prim}^n(X(n, 1)) = 0$;

II. $d = 2$, any n; $\text{Prim}^n(X(n, 2))$ is zero if n is odd, and is one-dimensional, spanned by $\Theta\left(\prod_{i=1}^{n+2} X_i\right)$, is n if even;

III. $d = 3$, $n = 2$. The six admissible exponent systems

$$w = (w_1, w_2, w_3, w_4)$$

are

$$(1, 1, 2, 2), \quad (2, 2, 1, 1), \quad (1, 2, 1, 2), \quad (2, 1, 2, 1), \quad (1, 2, 2, 1), \quad (2, 1, 1, 2);$$

with dual pairs written together.

Let the symmetric group $\mathfrak{S}_4$ act on $\mathcal{L}^S$, $\mathbf{P}^3$, $X(2, 3)$, ... by permuting the homogeneous coordinates :

$$\sigma(X_1, X_2, X_3, X_4) = (X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}, X_{\sigma(4)}).$$

By functoriality, the mapping residue : $H^3(P^3 - X(2, 3)) \to \mathrm{Prim}^2(X(2, 3))$ commutes with the action of σ, and by direct computation

$$\mathcal{R} : \quad W^S \to H^3(\mathbf{P}^3 - X(2, 3))$$

commutes " up to the signature ", i. e.

$$\mathcal{R}(\sigma(X^w)) = \mathrm{sgn}(\sigma)\, \mathcal{R}(X^w).$$

Thus $\Theta(\sigma(X^w)) = \mathrm{sgn}(\sigma)\, \Theta(X^w)$, and, as the involution $w \to \bar{w}$ commutes with the action of $\mathfrak{S}_4$, we have

$$\Theta(\sigma(X^u)) \wedge \Theta(\sigma(X^{\bar{u}})) = \Theta(X^u) \wedge \Theta(X^{\bar{u}}).$$

As $\mathfrak{S}_4$ acts transitively on the six admissible systems for $X(2, 3)$, this completes case III.

The proof of Theorem 6 will be completed by the next proposition, in which we return to the *universal* $X \to T$.

PROPOSITION 8. — *Suppose* $n \geq 1$, $d \geq 3$, *and either* $n \neq 2$ *or* $d \neq 3$. *For each admissible exponent system* w, *the element* $X^w \in \mathcal{L}^S$ *determines a global section,* $\Theta(X^w)$, *of* $\mathrm{Prim}^n(X/T)$. *For any two admissible systems* u *and* v, *one has*

$$\Theta(X^u) \wedge \Theta(X^{\bar{u}}) = (-1)^{u_0 - v_0}\, \Theta(X^v) \wedge \Theta(X^{\bar{v}}) \quad \textit{in } \mathcal{O}_T.$$

Proof. — It suffices to demonstrate the equality over the local ring of the generic point of T, i. e. to consider the hypersurface X of equation $X_1^d + \sum_w A_w X^w$ (where the sum is over all monomials X^w of degree d other than X_1^d, and the A_w are independent variables) over the field $K = \mathbf{Q}(A_w)$.

The spectral sequence assosciated to the filtration of $\Omega^{\cdot}_{X/K}$ by the subcomplexes

$$F^i(\Omega^{\cdot}_{X/K}) = \underbrace{0 \to 0 \to \ldots \to 0}_{i \text{ zéroes}} \to \Omega^i_{X/K} \to \Omega^{i+1}_{X/K} \to \ldots,$$

has $E_1^{p,q} = H^q(X, \Omega^p_{X/K}) \Rightarrow H^{p+q}_{DR}(X/K)$.

By Hodge theory, this spectral sequence degenerates at E_1 for *any* projective nonsingular X over a field K of characteristic zero. This means that the assosciated graded objects $\mathrm{gr}^p\,H^{p+q}_{\mathrm{DR}}(X/K)$ are given by $H^q(X, \Omega^p_{X/K})$. In particular, because $n = \dim X/K$, $\mathrm{gr}^p\,H^{2n}_{\mathrm{DR}}(X/K)$ vanishes for $p \neq n$. Thus $F^p\,H^{2n}_{\mathrm{DR}}(X/K) = 0$ for $p > n$.

Because the filtration F^i of $\Omega^{\cdot}_{X/K}$ is " multiplicative ",

$$F^i\Omega^{\cdot}_{X/K} \wedge F^j\Omega^{\cdot}_{X/K} \subset F^{i+j}\Omega^{\cdot}_{X/K},$$

it follows that

$$F^i H^{\cdot}_{\mathrm{DR}}(X/K) \wedge F^j H^{\cdot}_{\mathrm{DR}}(X/K) \subset F^{i+j} H^{\cdot}_{\mathrm{DR}}(X/K).$$

Thus if $\xi \in F^i\,H^n_{\mathrm{DR}}(X/K)$, $\eta \in F^j\,H^n_{\mathrm{DR}}(X/K)$ and $i + j > n$, then $\xi \wedge \eta = 0$.

Now consider any monomial $X^w \in \mathcal{L}^S$. We have $\Theta(X^w) = \mathrm{residue}(\mathcal{R}(X^w))$. The class $\mathcal{R}(X^w)$ is represented by a form holomorphic on $\mathbf{P}^{n+1} - X$, which has a pole of order (at most) w_0 along X. It follows ([5], § 8] that its residue lies in $F^{n+1-w_0}\,H^n_{\mathrm{DR}}(X/K)$.

Thus if u and v are admissible exponent systems with $u_0 + v_0 \leqq n + 1$, *one has* $\Theta(X^u) \wedge \Theta(X^v) = 0$.

Let us say that two admissible exponent systems u and v are *connected* if either $u \geqq v$ (meaning $u_i \geqq v_i$ for all i) or $v \geqq u$.

As a first step towards Proposition 8, we prove

LEMMA 9. — *When u and v are admissible exponent systems which are connected, the conclusion of Proposition 8 is valid.*

Proof. — Say $u \geqq v$; since we may clearly interpolate a sequence of admissible systems

$$u = u^{(0)} \geqq u^{(1)} \geqq \ldots \geqq u^{(r)} = v$$

with $u_0^{(i)} = 1 + u_0^{(i+1)}$, it is enough to check the case $u_0 = 1 + v_0$. Let us write $u = v + w$; then $\bar{v} = \bar{u} + w$ (because $v + \bar{v} = u + \bar{u}$).

Recall the connection σ on W^S, induced by $\sigma_D = D + F^D$. In particular, taking the derivation $\dfrac{\partial}{\partial A_w}$, defined by $\dfrac{\partial}{\partial A_w}(A_y) = \delta_{w,y}$, and writing σ_w for σ_{A_w}, we have

$$\sigma_w = \frac{\partial}{\partial A_w} + X^w$$

whence

$$\begin{cases} \sigma_w(X^v) = X^u, \\ \sigma_w(X^{\bar{u}}) = X^{\bar{v}} \end{cases}$$

and so

$$\begin{cases} \dfrac{\partial}{\partial A_w}\Theta(X^v) = \Theta(X^u), \\ \dfrac{\partial}{\partial A_w}\Theta(X^{\bar{u}}) = \Theta(X^{\bar{v}}). \end{cases}$$

Because $v_0 + \bar{u}_0 = u_0 - 1 + \bar{u}_0 = n + 1$, one has

$$\Theta(X^v) \wedge \Theta(X^{\bar{u}}) = 0.$$

Differentiating with respect to A_w,

$$0 = \frac{\partial}{\partial A_w}(\Theta(X^v)) \wedge \Theta(X^{\bar{u}}) + \Theta(X^v) \wedge \frac{\partial}{\partial A_w}(\Theta(X^{\bar{u}}))$$

whence

$$0 = \Theta(X^u) \wedge \Theta(X^{\bar{u}}) + \Theta(X^v) \wedge \Theta(X^{\bar{v}}).$$

Q. E. D.

LEMMA 10. — *Let a be the least integer with $ad \geqq n + 2$. Every admissible exponent system u is connected to an admissible v having $v_0 = a$.*

Proof. — It follows from the definition of admissible that $u_0 \geqq a$. Consider *all* $n + 2$-tuples $t = (t_1, \ldots, t_{n+2})$ of integers satisfying

$$0 \leqq t_i \leqq u_i - 1 \quad \text{for} \quad i = 1, \ldots, n + 2.$$

As t varies over such systems, the function $\sum_{i=1}^{n+2} t_i$ assumes all integral values from 0 to $\sum_{i=1}^{n+2}(u_i - 1)$. Because

$$\sum_{i=1}^{n+2}(u_i - 1) = du_0 - (n + 2) \geqq du_0 - ad,$$

there exists a t as above having $\sum_{i=1}^{n+2} t_i = d(u_0 - a)$. The desired v is $u - t$.

Q. E. D.

LEMMA 11. — *If $n \geqq 1$ and $d \geqq 3$, then either $n + 1 \geqq 2a$ or $n = 2$ and $d = 3$.*

Proof. — Suppose $n + 1 < 2a$. As $n \geqq 1$, we have $a \geqq 2$. As $n + 2 > (a - 1)d$ by definition of a, we have $2a \geqq n + 2 > (a - 1)d$, so $2a/a - 1 > d \geqq 3$, or $a/a - 1 > 3/2$, possible only if $a = 2$. Then $d < 2a/a - 1$ gives $d = 3$. Finally, $2a \geqq n + 2 > (a - 1)d$ gives $n = 2$.

Q. E. D.

LEMMA 12. — *Suppose $n + 1 \geqq 2a$. Then any two admissible exponent systems u and v having $u_0 = v_0 = a$ are connected to a common admissible exponent system w.*

Proof. — For $1 \leqq i \leqq n + 2$, put $r_i = \max(u_i, v_i)$. We have $1 \leqq r_i \leqq d - 1$. If $\sum_{i=1}^{n+2} r_i$ is divisible by d, take $w = (r_1, \ldots, r_{n+2})$; if not, there are positive integers e and f with

$$\sum_1^{n+2} r_i = de - f, \qquad 1 \leqq f \leqq d - 1.$$

We can take

$$w = (r_1 + t_1,\ r_2 + t_2,\ \ldots,\ r_{n+2} + t_{n+2}),$$

if we can find $(t_1, \ldots, t_{n+2})$ satisfying $0 \leqq t_i \leqq d - 1 - r_i$, and $\sum_{i=1}^{n+} t_i = f$.

As $(t_1, \ldots, t_{n+2})$ varies over all $n+2$-tuples satisfying only $0 \leqq t_i \leqq d - 1 - r_i$, the function $\sum_{i=1}^{n+2} t_i$ assumes all integral values between 0 and

$$\sum_{i=1}^{n+2} (d - 1 - r_i) = (n+2)\,(d-1) - \sum_{i=1}^{n+2} r_i.$$

Because $u_i \geqq 1$ and $v_i \geqq 1$, $r_i = \max(u_i, v_i) \leqq u_i + v_i - 1$.

Thus $\sum_{i=1}^{n+2} r_i \leqq du_0 + dv_0 - (n+2) = 2ad - (n+2)$, and so

$$\begin{aligned}(n+2)\,(d-1) - \sum_{i=1}^{n+2} r_i &\geqq (n+2)\,(d-1) - 2ad + (n+2) \\ &= (n+2)\,d - 2ad = (n+1-2a)\,d + d \geqq d.\end{aligned}$$

Hence there exists $(t_1, \ldots, t_{n+2})$ with

$$0 \leqq t_i \leqq d - 1 - r_i \qquad \text{and} \qquad \sum_{i=1}^{n+2} t_i = f.$$

Q. E. D.

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