MARIUSZ LEMANCZYK

The centralizer of Morse shifts

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Abstract We examine the centralizer of Morse shifts.

1/ Let $x = b^0 x b^1 \ldots$ be a regular Morse sequence and $|b_i| = r$. Then

a/ $C(T) = \{ T^{i,j} : i \in \mathbb{Z}, j = 0,1 \}$ where $T$ is the shift and $\sigma$ is the mirror map.

b/ There are no roots of $T$.

2/ There are Morse shifts with uncountable centralizer.

Let $\gamma^{n_i}$ be the class of all ergodic automorphisms $\gamma$ with $\exp(2 \pi i/n_i)$ in the point spectrum of $\gamma$. We introduce some number $d^{n_i}(\gamma)$ for $\gamma \in \gamma^{n_i}$ and prove that if $d^{n_i}(\gamma) < \infty$ then $\gamma$ is coalescent.

Introduction Let $(X, B, \mu)$ be a Lebesgue space and $T$ an invertible transformation of $(X, B, \mu)$. By $C(T)$ we mean the centralizer of $T$ i.e. the group of all automorphisms $S$ of $(X, \mu)$ with $TS = ST$. The centralizer is an important invariant in ergodic theory. It can state some ergodic properties of $T$. In particular knowing $C(T)$ we can usually answer whether $T$ has roots or $T$ is embeddable in measurable flows. Moreover, if $P$ is a finite generator of $T$ then $SP, S \in C(T)$ are the only generators with the same finite distributions as $P$.

In the present paper we investigate centralizers of Morse shifts. These shifts play an important role in ergodic theory in providing concrete examples of dynamical systems with required properties /[7], [12], [15]/.
There are some direct reasons to compute the centralizers of Morse shifts. As we shall see in Section 5 the property to have an uncountable centralizer is a typical one in the class of all automorphisms acting in a fixed Lebesgue space. On the other hand examples of automorphisms with the trivial centralizer i.e. \( C(T) = \{ T^i, \ i \in \mathbb{Z} \} \) are well-known/mixing rank one, minimal self-joining automorphisms [8][20]. Our main theorem / Theorem 1 provides a large class of automorphisms with countable but not trivial centralizer.

Consider Morse dynamical systems as examples in topological dynamic / [16] /. We see that their topological properties are usually common for all Morse sequences / [3],[16] /; In particular the group of all homeomorphisms of \( \mathcal{C}_\text{x} \) commuting with the shift \( C^{\text{top}}(x) \) is equal to \( \{ T^{i \sigma j} : i \in \mathbb{Z}, j=0,1 \} \). It is interesting to know whether \( C^{\text{top}}(x)=C(x) \) or not. Surprisingly it turns out that in our class the answer can be negative as well as positive.

2. Notations Now, we introduce a bit of terminology: Each element \( B=(b_0,\ldots,b_{k-1}) \in \{0,1\}^k \) will be called a block, \( k \) is called the length of \( B \) and we denote it by \( |B| \). Denote \( B[i,j] = (b_i, b_{i+1}, \ldots, b_j) \), \( B[i,i]=B[i] \). The block \( \overline{B}=(\overline{b}_0,\ldots,\overline{b}_{k-1}) \) is defined by setting \( \overline{b}_i=0 \) if \( b_i=1 \) and \( \overline{b}_i=1 \) if \( b_i=0 \). Let \( \overline{C}=(c_0,\ldots,\overline{c}_{m-1}) \) be another block. Then the product \( B \times C \) is defined by \( B \times C = B^0 C^0 \times B^{c_1} C^{c_1} \times \cdots \times B^{c_{m-1}} C^{c_{m-1}} \) where \( B^0=B \), \( B^\delta=\overline{B} \). Let \( |B|=|C|=k \). Then \( d(B,C) = \frac{1}{k} \text{card} \{ i: 0 \leq i \leq k-1 \ B[i] \neq C[i] \} \). If \( |B| \leq |C| \) then \( fr(B,C)=\text{card} \{ i: 0 \leq i \leq |C|-|B| \ C[i,i+|B|-1]=B \} \). We will say \( B \) appears in \( C \) at \( i \) within \( \delta \) if \( d(B,C[i,i+|B|-1]) < \delta \). If \( d(B,C[i,i+|B|-1]) = 0 \) we say simply \( B \) appears in \( C \) at \( i \).
Now, let $b^0, b^1, b^2, \ldots$ be finite blocks of lengths at least two beginning with zero and put

\[ x = b^0 \cdot b^1 \cdot b^2 \cdots. \]

We set $\lambda_i = |b^i|$, $r_i = \min \left\{ \frac{1}{\lambda_i} \text{fr}(0, b^i), \frac{1}{\lambda_i} \text{fr}(1, b^i) \right\}$, $i = 0, 1, \ldots$.

The sequence $x$ defined in (1) is said to be a Morse sequence if

\( i \) infinitely many of the $b^i$'s are different from $0 \cdots 0$,

\( ii \) infinitely many of the $b^i$'s are different from $01 \cdots 010$ and

\( iii \) $\sum \limits_{i=0}^{\infty} r_i = \infty$

Obviously $i \Rightarrow$ follows from $iii$.

If $x$ is a Morse sequence then one can find an almost periodic point $w \in X = \{0, 1\}^Z$ such that $w[k] = x[k]$, $k \geq 0$:

Put $\mathcal{O}_x = \{ T^i w : i \in Z \}$ where $T$ is the shift on $X$.

It is known that $(\mathcal{O}_x, T)$ is strictly ergodic [12]. The unique $T$-invariant measure / ergodic / we shall denote by $\mu_x$ and the system $(\mathcal{O}_x, T, \mu_x)$ will said to be a Morse dynamical system / Morse shift /.

Denote by $\sigma$ the mirror map on $\mathcal{O}_x$, i.e. $\sigma(y) = \overline{y}$, $\overline{y}[i] = \overline{y}[i]$, $i \in Z$.

Then $T \sigma = \sigma T$ and by strictly ergodicity of $\mathcal{O}_x$ $\sigma$ preserves $\mu_x$.

Kwiatkowski in [13] has found out the structure of $\mathcal{O}_x$ / the set $X(x)$ described there is contained in $\mathcal{O}_x$ and the set $\mathcal{O}_x \setminus X(x)$ is countable / Namely, let

\[ D_{t}(j) = \{ y \in \mathcal{O}_x : y[-i+kn_t, -i+(k+1)n_t-1] = c^i, k = 0, \pm 1, \pm 2, \ldots \} \]

\[ i = 0, \ldots, n_t - 1, t \geq 0, j = 0, 1, a_t = b^x \cdot b^y \cdots b^t \text{ and put } D_{t}^n = D_{t}(0) \cup D_{t}^n(1) \]

Then $D^x = (D_{0}^x, \ldots, D_{n_t-1}^x)$ is a partition of $\mathcal{O}_x$ into open and closed subsets. Moreover, for every $y \in X(x)$ and every $t \in N$ there is only one $i, 0 \leq i \leq n_t - 1$ such that $y[-i+kn_t, -i+(k+1)n_t-1] = c^i$ or $\overline{c}^i$.

Denote $n_t = \lambda^t_1 \cdots \lambda^t_{\lambda^t_m}$, $c_t = b^x \cdots b^t m\geq0$, $c^0 = c_m$, $n^0 = n_m$, $x_t = b^x \cdot b^{x+1} \cdots \mu_t = \mu_t \cdot a_t$ or $\overline{c}^t$ will be called $t$-symbols.

In what follows we will say about properties of $x$ instead of $T$.
on $\mathcal{O}_x$ and for example if no confusion becomes we shall write $G(x)$ instead of $G(T)$.

3. Coalescence Let $(X,\mathcal{B},\mu)$ be a Lebesgue space. We say an automorphism $\tau : X^2$ is coalescent if every endomorphism of $(X,\mu)$ commuting with $\tau$ is necessarily invertible. Consider the class of all ergodic automorphisms $\tau$ of $(X,\mathcal{B},\mu)$ for which $\text{Sp}(\tau) \supset G\{\lambda_t : t \geq 0\}$ where $\text{Sp}(\tau)$ is the group of all eigenvalues of unitary operator $U_\tau$ defined in the following way $U_\tau (f) = f\tau$. Here $\lambda_t = \lambda_1 \cdots \lambda_t$, $t \geq 0$ and $\mathcal{G}(\lambda_t : t \geq 0)$ denotes the group generated by $\{\exp 2\pi i/\lambda_t\}$.

Let us notice that $\exp(2\pi i/\lambda_t) \in \text{Sp}(\tau)$ iff there is a $\lambda_t$-stack for $\tau$ i.e. a partition $(A, \tau A, \ldots, \tau^n A)$ of $X/\langle z \rangle$. Moreover it is not difficult to verify that ergodicity of $\tau$ implies that there is only one / reordering if necessary elements of another $\lambda_t$-stack / $\lambda_t$-stack for $\tau$, so we denote it by $D^\lambda_t = (D_0^\lambda, \ldots, D_n^\lambda)$. In addition if $n_t | n_{t+1}$ then $D^n_t \subseteq D^{n+1}_t$.

If $\tau \in \mathcal{G}(\bar{\lambda}_t)$ then we get a sequence of $\tau$-invariant partitions $D^\tau_t \subseteq D^\tau_{t+1}$. Let $D=(D_t)_{t \in \mathbb{I}}$ be the limit partition. We assert $D_t$, is a constant number for all $i \in \mathbb{I}$ i.e. either $D_i = \infty$ or $D_i = m$ for some natural $m$ / a.e. Indeed $D$ is $\tau$-invariant and measurable partition, so our claim easily follows from [1].

Put $d^{\lambda_t}(\tau) \equiv \text{card } D_t$, $i \in \mathbb{I}$. Let us observe that $d^{\lambda_t}(\tau)$ is an invariant of isomorphy.

**Proposition 1** If $d^{\lambda_t}(\tau)$ is finite then $\tau$ is coalescent.

**Proof** Let $\tau : (X,\mathcal{B},\mu)^2$ and $\xi$ be any $\tau$-invariant and measurable partition of $X$ and let $f : X \to X/\xi$ be canonical map. It is sufficient to show $/\langle \xi \rangle$ that if $(\tau, X, \mathcal{B}, \mu)$ and $(\tau/\xi, X/\xi, \mathcal{B}/\xi, \mu/\xi)$ are
isomorphic then \( \mathcal{Z} \) is equal to the partition into points.

So, let us suppose it. Thus there is the sequence \( \{ \mathcal{D}^n \} \to \mathcal{D} \)
of \( n \)-\( \mathcal{A} \)-stacks and \( d^{n_3}(\mathcal{A}) = d^{n_3}(\mathcal{A}/\mathcal{Z}) \).
Let \( \mathcal{D}_i \) be any "typical" atom from \( \mathcal{D} \). Therefore \( \mathcal{D}_i = \bigcap_{t>0} \mathcal{D}^n_t \) so \( f^{-1}(\mathcal{D}_i) = \bigcap_{t>0} f^{-1}(\mathcal{D}^n_t) = \bigcap_{t>0} \mathcal{D}^n_t \cap \mathcal{D}_j \cap \mathcal{D} \)
because the preimage carries \( n \)-\( \mathcal{A} \)-stacks into \( n \)-\( \mathcal{A} \)-stacks. Hence \( f \) cannot stick together points as soon as they belong to the same atom \( \mathcal{D}_j \).

Finally, \( f^{-1}(\mathcal{A}/\mathcal{Z}) \) contains \( \mathcal{A} \)-algebra generated by \( n \)-\( \mathcal{A} \)-stacks \( t \geq 0 \), so \( \mathcal{A} \geq \mathcal{D} \). Therefore \( \mathcal{Z} \) must be equal to the partition into points.

**Remark 1** For every Morse sequence \( x \), \( d^{n_3}(x) = 2 \).

**Remark 2** If \( d^{n_3}(\mathcal{A}) = \infty \) then \( \mathcal{A} \) need not be coalescent. For instance if \( \mathcal{A} \) is a Morse shift and \( \mathcal{A} \) any Bernoulli automorphism then \( \mathcal{A} \times \mathcal{A} \) cannot be coalescent /[3],[18]/.

4. Centralizer and simple spectrum In this section we formulate and prove some characterization of automorphism having simple spectra that we need in the following.

**Proposition 2** Let \( \mathcal{T} : (X, \mu) \to \mathcal{L}^2 \) be an automorphism of a Lebesgue space. Then \( U_\mathcal{T} \) has a simple spectrum iff the unitary centralizer of \( \mathcal{T} \), \( \mathcal{C}^{un}(\mathcal{T}) = \{ V : L^2(X, \mu) \to \mathcal{L}^2 \} \), \( V \) is unitary, \( \mathcal{V} = \mathcal{U}_\mathcal{C} \mathcal{V} \mathcal{J} \) is abelian.

**Proof** If \( U \) has a simple spectrum then every unitary operator \( \mathcal{V} \), \( \mathcal{V} = \mathcal{U}_\mathcal{C} \mathcal{V} \mathcal{J} \) is a function of \( \mathcal{T} \) i.e. there exists a bounded function \( f \) such that \( \mathcal{V} = f(\mathcal{T}) = \int_{\mathcal{T}} f d\mathcal{E} \), where \( \mathcal{E} \) is the spectral measure of \( U_\mathcal{T} \). Let \( V' \in \mathcal{C}^{un}(\mathcal{T}) \) then \( \mathcal{V}' = f'(\mathcal{T}) \). Hence \( \mathcal{V} = f \mathcal{E} \mathcal{E} = f f \mathcal{E} \mathcal{E} = f f \mathcal{E} \mathcal{E} = \mathcal{V} \mathcal{V} \mathcal{J} \).

Now, suppose \( \mathcal{T} \) does not have simple spectrum. Then there are \( f_1, f_2 \in \mathcal{L}^1(X, \mu) \) such that \( \mathcal{L}^2(X, \mu) = \mathcal{B}_1 \mathcal{B}_2 \mathcal{C} \), where \( \mathcal{B}_1 \) is the
cyclic space generated by $f_i$, i.e. $B_i = \text{span} \left( U_{i,j} f_i, \ j \in \mathbb{Z} \right)$, $i=1,2$, $C$ is $U, \omega$-invariant and there exists $U_1 : B_1 \rightarrow B_2$ which is unitary and $U_1^* U_1 = \mathbb{1}_{B_2} \circ U_1$. We define two unitary operators $V, V'$ on $L^2(X, \mu)$ setting

\[
V(b_1) = U_1(b_1), \quad V'(b_1) = U_2(b_1), \quad b_1 \in B_1,
\]

\[
V(b_2) = U_1^{-1}(b_2), \quad V'(b_2) = b_2, \quad b_2 \in B_2,
\]

\[
V(c) = 0, \quad V'(c) = c, \quad c \in C.
\]

It is easy to see that $V, V' \in C^{\text{unit}}(\tau)$ but $VV' \neq VV_1$. Indeed, if $VV' \neq VV_1$, $\mathbb{1}_{B_2}$ and $\mathbb{1}_{B_2}$ are identity and a contradiction to ergodicity of $\tau$.

It is known that every Morse sequence $\tau$ has a simple spectrum. Combining this with Proposition 2 we have obtained

**Corollary** 1. For every Morse sequence $\tau$, $C(\tau)$ is abelian.

**5. A class of Morse sequences with uncountable centralizer**

In this section we give a class of Morse sequences with uncountable centralizer. We also provide some arguments that the property to have an uncountable centralizer is a typical one.

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue space and $\tau$ be an ergodic automorphism of $(X, \mu)$. Let us consider the group $S$ of all automorphisms $S : (X, \mu)^2$ with the weak topology $\mathcal{W}$ defined in the following way

\[
S \xrightarrow{\text{w}} S \iff \lim_{n \to \infty} S_n \Delta S_n \delta_n = 0 \quad \text{for every } \delta_n \in \mathcal{B}.
\]

Now, we recall some known results on the weak topology:

1. $(S, \mathcal{W}, \circ)$ is a topological group.
2. $(S, \mathcal{W})$ is completely metrizable.
3. $S_n \xrightarrow{\text{w}} S$ iff $U_{S_n} \Rightarrow U_S$ i.e. $\|U_{S_n} f - U_S f\|_n \to 0$.
4. $C(\tau)$ is a close set in $\mathcal{W}$.
5. If $\tau_i \xrightarrow{i} S$, $i \xrightarrow{\infty}$ then $S \in C(\tau)$, $\tau_i \xrightarrow{i} \text{id}$ and $C(\tau)$ is a perfect set, so from the Baire's property $C(\tau)$ is uncountable.
We let $S^4$ denote the class of all $S \in S$ with $S^t \to \text{id}$ for some sequence $\{i_t\} \to \infty$. Then $S^4$ contains a dense $G_\delta$ set of automorphisms of $(X, \mu)$. Indeed, if $S$ admits a cyclic approximation with speed $o(1/n)$ then $U^t \to \text{id}$ for some sequence $\{i_t\}$ and moreover the class of all automorphisms admitting a cyclic approximation with a fixed speed contains a dense $G_\delta$ set $/[m]/$.

So, we have proved the property to have an uncountable centralizer is a typical one in $\sqrt{\cdot}$. Let us observe that the class $S^4$ is closed under taking factors, so if $S \in S^4$ then $S$ does not have mixing factors, in particular $h(S) = 0$. But a stronger fact is true: If $S \in S^4$ then $S$ is disjoint from all mixing transformations $/[m]/$.

Now, we are able to show there are Morse shifts with uncountable centralizer.

Given a Morse sequence $x = b_0^x b_1^x \ldots$ we denote
\[ p_t = \mu_t(00) + \mu_t(11), \quad q_t = \mu_t(01) + \mu_t(10) \]

Proposition 3 Let $x = b_0^x b_1^x \ldots$ be a Morse sequence.

If $\lim_{t \to \infty} p_t = 0$ then $C(x)$ is uncountable.

Proof We will prove that $x$ admits a cyclic approximation with speed $o(1/n)$.

We have $3_t = \sum D_i^t(j)$: $i=0, \ldots, n_t-1$, $j=0,1$, $t \geq 0$ / see Section 2 /.

From it follows that $3_t \uparrow \xi$

We define a cyclic approximation $S_t$ putting
\[ S_t D_i^t(j) = D_{i+1}^t(j) \quad i=0, \ldots, n_t-2, j=0,1 \]

Now, we wish to estimate $A_t = \sum_{i=0}^{n_t-1} \sum_{j=0}^{n_t-1} \mu_t(TD_i^t(j) \Delta S_t D_{i+1}^t(j))$

We then get
\[ A_t = 2 \mu(TD_{n_t-1}^t(0) \Delta S_t D_{n_t-1}^t(0)) \leq 2 \left[ \frac{1}{n_{t+1}} \mu(00, b_t^{+4}) + \mu(11, b_t^{+4}) \right] \]

or
\[ = 2 \frac{\mu(00, b_t^{+4}) + \mu(11, b_t^{+4})}{n_t} \leq 2 p_{t+1} \frac{1}{n_t} \]
Therefore $x$ admits desired cyclic approximation.

6. The measure-theoretic centralizer of regular Morse sequences. This section is devoted to prove the main result of the paper.

Theorem 1. Let $x = b^0 x b^1 x \ldots$ be a regular Morse sequence satisfying (11) and let $S \in C(x)$. Then $S = T^i \delta^j$ for some $i \in \mathbb{Z}$, $j = 0, 1$.

We start with presenting our main techniques (Proposition 4, 5) needed in proving of Theorem 1.

Let $x = b^0 x b^1 x \ldots$ be a Morse sequence.

A measurable function $\varphi : X = \{0, 1\}^\mathbb{N}$ is said to be a code of length $k$ if

/i/ $\varphi T = T \varphi$,

/ii/ $\varphi(y) [0]$ depends only on $y[-k, k]$, i.e., if $y[-k, k] = y'[0, k]$ then $\varphi(y) [0] = \varphi(y') [0]$,

/iii/ $k$ is the smallest natural number satisfying /ii/ and we denote it by $|\varphi|$.

The following Proposition establishes a list of properties of finite codes that we will need.

Proposition 4. /a/ Let $\varphi$ be finite code. Then for a.e. $y, y' \in \mathcal{O}_x$ if $y[-|\varphi| + t, t + |\varphi|] = y'[-|\varphi| + u, u + |\varphi|]$ then $\varphi(y)[t] = \varphi(y')[u]$.

/b/ Let $S \in C(x)$ and $\delta > 0$. There is a finite code $\varphi$ such that

/8/ $d(Sy, \varphi y) < \delta$,

/9/ $d(\varphi y, \varphi \bar{y}) > 1 - 2 \delta$ for a.e. $y \in \mathcal{O}_x$.

Proof. The proof is straightforward and we use only ergodic theorem.
Following [13] we say $x$ is a regular Morse sequence if there is $\gamma > 0$ such that

\[ q < p_t < 1 - q \quad \text{and} \quad q < q_t < 1 - q, \quad t \geq 0. \]

In addition we assume

\[ \sup_{t \in \mathbb{N}} \gamma_t = \lambda < \infty. \]

The following characterization of regular Morse sequences satisfying /11/ can be found in [14].

**Proposition 5** Let $x = b^0, b^1, \ldots$ be a regular Morse sequence and let /11/ holds. Then

\[ (\exists d > 0)(\exists L > 0)(\forall \gamma \text{-block}) (\forall t \in \mathbb{N}) \left[ \text{if } \gamma = o_t - B, \ |B| = L \text{ appears in } x \text{ at } i \text{ within } d \text{ then } n_t \mid i \text{ and } \eta \text{ appears in } x \text{ at } i \right] \]

In the sequel we will need some facts of combinatorial nature.

Let $x = b^0, b^1, \ldots$ be a regular Morse sequence satisfying /11/ and let $c > 0$, $L > 0$ be determined by Proposition 5.

Let us take $\varepsilon > 0$ and assume $\varphi : X^\omega$ is a code of length $k$ so that

\[ d(\varphi_y, S_y) < \varepsilon \quad \text{for a.e. } y \in \mathcal{O}_x \]

Fix $y \in \mathcal{O}_x$ for which /12/ holds.

Next, we find $t \in \mathbb{N}$ so large that

\[ k / n_t < \varepsilon / 2 \]

\[ d(\hat{e}_t, \hat{e}_t) > 1 - 3 \varepsilon \quad \text{where } e_t / \hat{e}_t / \text{ is the code of } o_t / \sigma_t / \]

via $\varphi$ i.e. $|e_t| = n_t - 2k$, $e_t[j] = \varphi(o_t[k+j, 2k+j-1]), j = 0, \ldots, n_t < 2k - 1$

\[ (\forall m \geq n_t) \quad d(\varphi_y[-m, m], S_y[-m, m]) < \varepsilon. \]

Assume in addition

\[ y \in D_n^t \]

Now we shall define some map $H : [0, 1]^2$ in the following way

\[ d(e_t^r[i_o, i_1], \sigma_t^h[i_o, i_1]) = \min \left\{ d(e_t^r[i_o, i_4], o_t[i_o, i_4]), d(e_t^r[i_o, i_1], \sigma_t[i_o, i_1]) \right\} \]

where $e_t^r = e_t$ if $r = 0$ or $\hat{e}_t$ otherwise and $|i_o - i_4| = |i_o - i_1| \geq \frac{1}{2} |e_t|$.

/ see Picture 1/.
Let us observe that

\(17/\) \(d(e^r_t[i_0, i_1], c^{H(r)}_t[j_0, j_1]) < 20\varepsilon, \ r=0,1\)

Indeed, otherwise we would have \(d(e^r_t[i_0, i_1], c^{H(r)}_t[j_0, j_1]) \geq 20\varepsilon, \ s=0,1\)

Choose a sector of \(y\), say \(y[-m, m]\), \(m > n_t\) such that

\(y[-m, m]\) consists of \(p\) \(t\)-symbols and this sector contains

\(18/\) at least \((1/2-\varepsilon)p\) of \(c^r_t\), \(r=0,1\) calculated only in the
places of the form \(-u+vn_t\), \(v=0, \pm 1, \pm 2, \ldots\)

To see \(18/\) it is sufficient to use ergodic theorem and the fact that \(\mu_t(r) = \frac{1}{t}\) for every \(r=0,1\), \(t=0,1, \ldots\). Hence

\[\varepsilon > d(\varphi y [-m,m], Sy [-m,m]) \geq (1/2-\varepsilon)p \frac{20\varepsilon}{\varepsilon} |e_t^r| (2m+1) \geq (1/2-\varepsilon)p 10 \varepsilon |e_t^r| / p n_t = 5\varepsilon (1-2\varepsilon) (1-2k/p) \geq 5\varepsilon (1-2\varepsilon) (1-\varepsilon) \geq \varepsilon\]

a contradiction.

Now, we show \(H: \{0, 1\}^y \) is one-to-one. Indeed let us suppose

\(H(0) = H(1)\). Then

\[d(e^r_t[i_0, i_1], e^r_t[i_0, i_1]) \leq d(e^r_t[i_0, i_1], c^{\mu(r)}_t[j_0, j_1]) + d(e^r_t[i_0, i_1], c^{H(r)}_t[j_0, j_1]) < 40\varepsilon\]

But from \(14/\)

\[d(e^r_t[i_0, i_1], e^r_t[i_0, i_1]) \geq (1-3\varepsilon)\frac{1}{2} |e_t^r| / |e_t^r| \geq \frac{1}{2}-2\varepsilon, \ a \ contradiction.\]

At present, we estimate \(d(e^r_t[i_0, i_1], c^{H(r)}_t[j_0, j_1])\). We have

\[d(e^r_t[i_0, i_1], e^{\mu(r)}_t[i_0, i_1]) \leq d(e^r_t[i_0, i_1], c^{H(r)}_t[j_0, j_1]) + d(c^{H(r)}_t[j_0, j_1], e^{\mu(r)}_t[i_0, i_1])\]

Hence

\(19/\) \(d(e^r_t[i_0, i_1], c^{H(r)}_t[j_0, j_1]) \geq \frac{1}{2}-2\varepsilon\)

Let us consider again the sector \(y[-m,m]\) satisfying \(18/\)

and we match by arrow \(e^r_t\) with \(c^{H(r)}_t\) / Picture 2/.
We wish to estimate the number \( R \) of \( e^\gamma_t \), \( r=0,1 \) without arrow. We have
\[
d(Q_y [-m,m], S_y [-m,m]) \geq R (1 - 22\epsilon) \frac{1}{n_t} |e^\gamma_t| / p_n_t, \text{ therefore}
\]

\[
/20/ \quad R < 8 \epsilon p
\]

Proof of Theorem 1 From the invertibility of \( H \) we have
\[
/21/ \quad c_t^{H(\gamma)} = o_t^{H(\gamma)+1} \quad \text{for } r=0,1
\]

Take now \( T^5 y^{H(\gamma)} \) where \( T^5 y [-u+in_t, -u+(i+1)n_t - 1] \) is always \( t \)-symbol. Then
\[
d(T^5 y^{H(\gamma)}[-m+s,m-s], S_y[-m+s,m-s] < \frac{R}{p} < 8 \epsilon
\]

Find the greatest \( t_o \) such that \( y[-m,m] \) contains \( t_o \)-symbols.

Using the condition of boundness of \( \{ \lambda_t \} \) we get \( t_o \rightarrow \infty \) whenever \( p \rightarrow \infty \). So choosing \( \epsilon \) as small as we need and applying Proposition 5 we obtain
\[
T^5 y^{H(\gamma)}[v, v+Ln_t - 1] = S_y [v, v+Ln_t - 1]
\]
for some \( v \in \mathbb{Z} \). Letting \( p \rightarrow \infty \) we get at once \( T^5 c^{H(\gamma)} y = S_y \).

Let us set \( A_h = \{ y \in \mathcal{Q} : S_y = T^5 c^{H(\gamma)} y \} \), \( h=0,1 \). So either \( \mu_x(A_0) > 0 \) or \( \mu_x(A_1) > 0 \). But \( A_h \) is \( T \)-invariant and ergodicity of \( T \) forces \( A_h \) with positive measure / to have full measure. Finally \( S = T^5 c^h \).

Corollary 2 For every regular Morse sequence with /11/ there are no roots of the shift induced by \( x \).

Corollary 3 For every regular Morse sequence with /11/ \( C(T) \neq \{ T^i : i \in \mathbb{Z} \} \)

Proof In the case of the equality \( C(T) \) is uncountable.

Final remarks Let us now consider the class of all nonperiodic substitutions on two symbols of constant length / for definition and properties see [4] /
There are two kinds of them:

/i/ discrete substitutions: if \( \Theta \) defined in /22/ has the property \( b_i = c_i \) for some \( i, 0 \leq i \leq \lambda - 1 \)

/ii/ continuous substitutions: otherwise.

Their topological centralizer was calculated in [3]. It is equal to \( \{ T^k : k \in \mathbb{Z} \} \) for /i/ and \( \{ T^k \sigma^j : k \in \mathbb{Z}, j=0,1 \} \) for /ii/

Here \( \sigma \) is again the mirror map.

Now, we are able to show measure-theoretic centralizer for such a \( \Theta \). Let \( \Theta \) be discrete substitution. Then \( \Theta \) may be considered from the measure-theoretic point of view as a discrete, ergodic dynamical system with \( \text{Sp}(\Theta) = \mathbb{G}\{ h^t : t \geq 0 \} \).

From [4] it follows that \( \text{C}(\Theta) = \text{End}(\mathbb{G}\{ h^t : t \geq 0 \}) \). It is easy to see that the last group is equal to the \( \lambda \)-adic integers.

Let \( \Theta \) be a continuous substitution. Then the dynamical system arising from \( \Theta \) is equal to \( (\mathbb{G}^x, \mathcal{T}, \mu^x) \) where \( x = B \cdot B^\gamma \cdots \) is a Morse sequence. If \( B \) does not start with zero we replace \( B \) by \( B \cdot B^\gamma, /B\gamma/ \). So from Theorem 1 \( \text{C}^t(\Theta) = \text{C}(\Theta) = \{ T^t \sigma^j : i \in \mathbb{Z}, j=0,1 \} \).

Consider the class of Morse sequences over a fixed finite Abelian group \( G \) / see [5], [7] / . Let \( x = b^0_1 \cdot b^1_1 \cdots \) be such a one. Let us call it regular if \( \sup_{t \in \mathbb{N}} s_t = s < \infty \) where

\[
s_t \sup_{g, \xi} \{ |B| : B = 0 \xi_1(0) \cdots \xi_{(\gamma)}(0) \cdots 0 \xi_2(0) \cdots 0 \xi_\lambda(0) \cdot 0 \xi_\lambda(0) \cdots \xi_{\gamma_k}(0) \} \text{and} \ B \text{appears in} \ x_t \}
\]

\( t \geq 0, |g| \) denotes the order of \( g \) and \( \xi_i(1) = i + g, i, g \in G \). If \( \{ \lambda_t \} \) is bounded then Proposition 5 holds for these regular Morse sequences over \( G \). The concept of finite code \( \Psi : \mathbb{G}^\lambda \) and Proposition 4 go as in Section 6. Let us assume \( S \in C(x) \) and in addition \( S \xi_g = \xi_g S \) for every \( g \in G \). Repeating considerations of Section 6 we see that the only formula which is not quite clear is the following \( H(g) = H(0) + g, g \in G \). To prove it we take
p as in /18/. There must exist an \( i \epsilon \mathbb{Z} \) such that
\[
\varphi_i(y) [-u + \iota \eta_i - u + (i+1) \eta_i - 1] = \varphi_i(y) \quad \text{with an arrow for every } g \epsilon G
\]
it is a simple consequence of /18/,/20/ and \( \varphi_i \varphi_j = \varphi_j \varphi_i \). This proves that if \( S \epsilon G(x) \), \( \varphi_j = \varphi_j \varphi_i \), \( g \epsilon G \) then \( S = T \varphi_j \varphi_i \) for some \( i \epsilon \mathbb{Z} \), \( g \epsilon G \). To get \( \varphi_j = \varphi_j \varphi_i \) it is sufficient to know that
\( (\varphi_j, T_j \mu) \) has a simple spectrum. In general it is still unknown whether they have simple spectra or not. Recently Kwiatkowski have communicated me that he knows examples of Morse sequences /over any cyclic group/of the form \( x=B^*B^* \ldots \) having simple spectrum.

References


M. LEMANCYK
Nicholas Copernicus University
Institute of Mathematics
TORUN
Poland

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