F. UTZET

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ON A NON SYMMETRIC OPERATION FOR
TWO-PARAMETER MARTINGALES

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ABSTRACT: In this paper we define a martingale \( M^*N \) such that by symmetrization provides the martingale \( \widetilde{MN} \) which takes part in the multi-dimensional Itô formula for continuous two-parameter martingales.

0. INTRODUCTION

In the compact version of Itô formula (see [3], [8]) for a continuous two-parameter \( L^4 \)-martingale \( M \), a new martingale \( \tilde{M} \) is involved. By polarization, we can define \( \tilde{MN} \), and this martingale takes part in the multi-dimensional Itô formula (see [8]). In this work, we define a martingale \( M^*N \) (\( M \) and \( N \) continuous \( L^p \)-martingales, \( p \geq 2 \)). Roughly speaking, \( M^*N \) is the limit of sums like \( \sum_{i,j} M(\Delta_{ij}^1)N(\Delta_{ij}^2) \). Then, \[
\tilde{MN} = \frac{1}{2} M^*N + \frac{1}{2} N^*M.
\]

We prove a convergence in \( H^{p/2} \) for \( M^*N \) which is very useful to compute \( M^*N \) and \( \tilde{M} \) in several cases. We also compute the quadratic variation of \( M^*N \).

We should point out that the martingale \( \tilde{M} \), written \( J_M \), for a continuous, strong \( L^4 \)-martingale, was defined by Cairoli-Walsh [2]. The martingale \( M^*N \), written \( J_{MN} \), for two
continuous $L^4$-martingales appeared in Guyon-Prum [4], where it was defined by a double stochastic integral of a corner function.

1. NOTATIONS AND DEFINITIONS

We consider on $\mathbb{R}_+^2$ the usual partial ordering $(s,t) \leq (s',t')$ if $s \leq s'$ and $t \leq t'$; we will write $(s,t) < (s',t')$ if $s < s'$ and $t < t'$. For $z, z' \in \mathbb{R}_+^2$, $z < z'$, $[z,z']$ will be the set

$$\{z \in \mathbb{R}_+^2 : z < z'\},$$

and similarly we define $[z,z']$. Put

$$R_z = ]0,z].$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let

$$\{\mathcal{F}_z, z \in \mathbb{R}_+^2\}$$

be an increasing, complete, right-continuous family of sub-$\sigma$-fields of $\mathcal{F}$; we also assume that $\{\mathcal{F}_z, z \in \mathbb{R}_+^2\}$ satisfies the condition (F4) of Cairoli-Walsh [2]: If we define

$$\mathcal{F}_s = \bigvee_{v \geq 0} \mathcal{F}_{sv} \quad \text{and} \quad \mathcal{F}_\infty = \bigvee_{u \geq 0} \mathcal{F}_{ut},$$

then $\mathcal{F}_s$ and $\mathcal{F}_\infty$ are conditionally independent given $\mathcal{F}_st$.

A stochastic process $M = \{M_z, z \in \mathbb{R}_+^2\}$ adapted to

$$\{\mathcal{F}_z, z \in \mathbb{R}_+^2\}$$

and integrable is said to be a martingale if for each $z \leq z'$,

$$E[M_{z'}, |\mathcal{F}_z] = M_z.$$

We will denote for $M^p_{\infty}$ (p ≥ 1) the set of all sample continuous $L^p$-martingales

$$M = \{M_z, z \in \mathbb{R}_+^2\}$$

(that is, $E[|M_z|^p] < \infty$, for all $z \in \mathbb{R}_+^2$); and for $M^p_{\infty}(z_0)$ the set of all sample continuous martingales

$$M = \{M_z, z \in [0,z_0]\}$$

with $E[|M_{z_0}|^p] < \infty$.

For a process $X = \{X_z, z \in \mathbb{R}_+^2\}$ the increment over $[z,z']$

$(z=(s,t), z'=(s',t'))$ is $X([z,z']) = X_{z'} - X_s - X_{s't} + X_z$.

A process $A = \{A_z, z \in \mathbb{R}_+^2\}$ is said to be increasing
if it is right-continuous, $A_z = 0$ on the axes, and $A([z, z']) \geq 0$ for all rectangle $[z, z']$. Given a martingale $M$ of $\mathbb{M}^2_C$, we will denote by $\langle M \rangle = \{\langle M \rangle_z, z \in \mathbb{R}^2_+ \}$ a continuous version of the quadratic variation of $M$ (see [7]).

This process $\langle M \rangle$ is increasing.

Let $z = (s, t)$ be a point of $\mathbb{R}^2_+$. A grid over $[0, z]$ will be a finite sub-set $\Gamma = \Gamma^1 \times \Gamma^2$ of $[0, z], \Gamma^1 = \{s_1, \ldots, s_p\}$, $0 = s_1 < s_2 < \ldots < s_p < s$, $\Gamma^2 = \{t_1, \ldots, t_q\}$, $0 = t_1 < t_2 < \ldots < t_q < t$. For $z' \in [0, z]$, $\Gamma_{z'}$ will be the set $\{z'' \in \Gamma : z'' < z'\}$. If $u = (s_i, t_j)$ is a point of the grid $\Gamma$, then, we will write

$\Delta_u = [s_i, s_{i+1}] \times [t_j, t_{j+1}]$, $\Delta^i_u = [s_i, s_{i+1}] \times ]0, t_j]$ and $\Delta^2_u = ]0, s_i] \times [t_j, t_{j+1}]$, with the convention $s_{p+1} = s$ and $t_{q+1} = t$. The norm of the grid is the number

$$|\Gamma| = \max_{i=1, \ldots, p} \max_{j=1, \ldots, q} (|s_{i+1} - s_i| + |t_{j+1} - t_j|).$$

Let $\{\Gamma^n, n \geq 1\}$ be a sequence of grids over $[0, z]$. $\{\Gamma^n, n \geq 1\}$ is said to be a standard one if $\Gamma^{n+1}$ is a refinement of $\Gamma^n$ and $\lim_{n \to \infty} |\Gamma^n| = 0$.

If $M$ is a martingale of $\mathbb{M}^p_C$ ($p \geq 2$), then there exists a martingale $\tilde{M}$ of $\mathbb{M}^{p/2}_C$ (see [7]) such that for all $z_0$ and for all standard sequence $\{\Gamma^n, n \geq 1\}$ of grids over $[0, z_0]$,

$$\lim_{n \to \infty} \sup_{z \in [0, z_0]} E_0 \left[ \sum_{u \in \Gamma^n_z} M(\Delta^1_u) M(\Delta^2_u) - \tilde{M}_z |^{p/2} \right] = 0.$$ 

The next result about one-parameter martingales will be needed (cf. lemma 2.1 of Nualart [7]).

**Lemma 1.1 (Nualart):** Let $M = \{M_t, t \in \mathbb{R}^+_+\}$ be a square integrable continuous martingale with respect to an increasing
family of σ-fields \( \{ \mathcal{F}_t, t \in \mathbb{R}_+ \} \) satisfying the usual conditions. Suppose \( M_0 = 0 \). Fix \( t_0 \) and denote by \( \Lambda = \{ s_1, \ldots, s_n \} \), \( 0 = s_1 \leq s_2 \leq \cdots \leq s_n < t_0 \) a finite set of points of \( [0, t_0] \). Consider another finite set \( \Lambda' \supset \Lambda \), whose points can always be written as \( \sigma_{k_i}^i \), \( i = 1, \ldots, N; k = 1, \ldots, r_i \), in such a way that \( s_i = \sigma_{k_i}^i < \cdots < \sigma_{k_i}^i < s_{i+1} \) for all \( i \). Set \( |\Lambda| = \max \{|s_{i+1} - s_i|\}, i = 1, \ldots, n \) where \( s_{n+1} = t_0 \). Then,

\[
\lim_{|\Lambda| \to 0} \sup_{\Lambda' \supset \Lambda} E\left[ \sup_{k=1}^{r_i} (M(\sigma_{k+1}^i) - M(\sigma_k^i))^2 \right] = 0,
\]

where by convention, we put \( \sigma_{r_i+1}^i = s_{i+1} \).

2. THE MARTINGALE \( M*N \)

**THEOREM 2.1:** Let \( M \) and \( N \) be martingales of \( \mathbb{M}_{c}(\mathbb{R}) \), \( p > 2 \). Then there exists a continuous martingale \( M*N \) of \( \mathbb{M}_{c}(\mathbb{R}) \) such that for every standard sequence of grids \( \{ \Gamma^n, n \geq 1 \} \) over \( [0, \infty) \), if we define the martingales \( S^n \) as

\[
S^n_z = \sum_{u \in \Gamma^n \cap z} M(\Delta^1_u)N(\Delta^2_u), \quad z \in [0, \infty),
\]

then

(i) \( \lim_{n \to \infty} \sup_{z \in [0, \infty)} E\left[ |S^n_z - M*N_z|^{p/2} \right] = 0 \quad (2.1) \)

(ii) For any \( n \), the martingales \( S^n \) and \( S^n_{s_0} = \{ S^n_{s_0}, t \in \Gamma^n \} \) are in \( \mathbb{H}^{p/2} \), and \( \lim_{n \to \infty} S^n_{s_0} = M*N_{s_0} \) in the convergence of \( \mathbb{H}^{p/2} \), that is,

\[
\lim_{n \to \infty} E\left[ \sup_{s \in \Gamma^n} |S^n_{s_0} - M*N_{s_0}|^{p/2} \right] = 0 \quad (2.2)
\]
and
\[ \lim_{n \to \infty} \mathbb{E}\left[ \sup_{t \in [0,1]} \left| S^n_{s,t} - M^*N^*_{s,t}\right|^{p/2} \right] = 0. \]

PROOF

Without loss of generality, we can suppose $M$ and $N$ are zero on the axes.

The part (i) of the theorem will be proved adapting the proof of lemma 3.2 of Nualart [7]. The part (ii) for $p > 2$ is an obvious consequence of maximal Doob's inequality; for $p = 2$ it is proved using a modification of that lemma. We detail the main steps of this proof.

For simplicity, $z_0 = (1,1)$ is supposed.

a) Let $p > 2$. We consider a grid $\Gamma^n$ over $[0,1]^2$ and we denote its points by $u = (s_i, t_j)$, $i=1, \ldots, p_n$, $j=1, \ldots, q_n$, $0 = s_1 < s_2 < \ldots < s_{p_n} < 1$, $0 = t_1 < t_2 < \ldots < t_{q_n} < 1$ (we put $(s_{p_n+1}, t_{q_n+1}) = (1,1)$).

By $\Gamma^n$ we will indicate a grid over $[0,1]^2$ such that has the same projection on the "t" axes as $\Gamma^n$. The points of $\Gamma^n$ will be denoted by $u' = (\sigma_i, \tau_j)$, $i'=1, \ldots, p'_n$, $j'=1, \ldots, q'_n$.

Let $I_i$ the set $\{ i' : \sigma_i, \tau_j \in [s_i, s_{i+1}] \}$. We define
\[ S^n_{z} = \sum_{u' \in I^n} M(\Delta_{u'}^1)N(\Delta_{u'}^2), \quad z \leq (1,1). \]

Then
\[ \lim_{n \to \infty} \sup_{\Gamma^n} \mathbb{E}\left[ \left| S^n_{1,1} - S^n_{1,1}\right|^{p/2} \right] = 0. \quad (2.3) \]

In fact, this convergence would follow the same argument used in the proof of (3.6) in [7].

Similarly, we denote by $\Gamma^{2n}$ a grid over $[0,1]^2$ which contains $\Gamma^n$ and has the same projection on the "s"
axes. The points of $2_T^n$ will be written by $u'=(s_i, \tau_j')$, \[i=1, \ldots, p_n; \quad j'=1, \ldots, q_n',\] and let $J_j$ be the set \[\{j' : \tau_j' \in [t_j, t_{j+1}]\}.\] We define \[
\frac{\tilde{S}^n_z}{S^n_{z}} = \bigcup_{u' \in 2_T^n} M(\Delta^1_u, )N(\Delta^2_{u'}).\]

By symmetry we obtain
\[
\lim_{n \to \infty} \sup_{2_T^n} E[|\tilde{S}^n_{1,1} - S^n_{1,1}|^{p/2}] = 0. \quad (2.4)
\]

By the conjunction of (2.3) and (2.4) we obtain (i). By Cairoli-Doob's inequality, there exists a continuous version of $M^*N$. The part (ii) is an immediate consequence of the maximal Doob's inequality.

b) Let $p=2$. With the same notation as above,
\[
\lim_{n \to \infty} \sup_{2_T^n} E[\sup_{s \leq 1} |\tilde{S}^n_{s,1} - S^n_{s,1}|] = 0. \quad (2.5)
\]

This can be shown as (3.8) of [7].

By symmetry,
\[
\lim_{n \to \infty} \sup_{2_T^n} E[\sup_{t \leq 1} |\tilde{S}^n_{1,t} - S^n_{1,t}|] = 0. \quad (2.6)
\]

(2.5) and (2.6) imply (i). The continuity of $M^*N$ is proved like [7]. It remains to show (ii) for $p=2$. We claim that
\[
\lim_{n \to \infty} \sup_{2_T^n} E[\sup_{s \leq 1} |\tilde{S}^n_{s,1} - S^n_{s,1}|] = 0. \quad (2.7)
\]

In fact,
\[
E[\sup_{s \leq 1} |\tilde{S}^n_{s,1} - S^n_{s,1}|]
\]
where $\Delta^2_u = \{0, s_1 \} \times \{ \tau_j, \tau_{j+1} \}$, and $N(\Delta^2_u)$ does not depend on $s$. For all $i$,

$$J - V_j$$

is a martingale in $s$ with respect to $s_{s_1}$. If $i \neq i'$, these martingales are orthogonal. Indeed, let $\{\xi_1, \ldots, \xi_k\}$,

$0 = \xi_1 < \xi_2 < \cdots < \xi_k < 1$, be a partition of $[0,1]$ which is a refinement of $0 = s_{s_1} < s_2 < \cdots < s_p < 1$. Then,

$$\langle M(\sum_{k+1}^{\xi_k} \xi_k \tau_j), (s_{i+1} \wedge \xi_k \tau_j) \rangle - \langle M(\sum_{k+1}^{\xi_k} \xi_k \tau_j), (s_{i+1} \wedge \xi_k \tau_{j'} - \xi_k \tau_j) \rangle = 0,$$

for all $i = i'$, for all $\tau_j$, $\tau_j'$, $\tau_j$, $\tau_j'$, because one of the two factors is always zero. Then,

$$\langle M(\sum_{k+1}^{\xi_k} \xi_k \tau_j), (s_{i+1} \wedge \xi_k \tau_j) \rangle - \langle M(\sum_{k+1}^{\xi_k} \xi_k \tau_j), (s_{i+1} \wedge \xi_k \tau_{j'} - \xi_k \tau_j) \rangle = 0.$$

By Davis inequality,

$$E[\sup_{s_1 \leq s} | S_n^{s_1} - S_n^{s_1} |] \leq C E[\sum_{i \leq i'} \sum_{j' \neq j} N(\Delta^2_u) M(\sum_{1}^{\xi_i} \xi_i \tau_j - \sum_{1}^{\xi_i} \xi_i \tau_{j'} - \xi_i \tau_j) \xi_i^{1/2}]$$

For each $i$, let $\Lambda_i = \{s_{i1}, s_{i2}, \ldots, s_{i1}\}$ be a finite partition of $[s_{i1}, s_{i1}+1]$. By Fatou's inequality,

$$E[\sup_{s_1 \leq s} | S_n^{s_1} - S_n^{s_1} |] \leq$$
\[ \begin{align*}
\sum \mathbb{E}[\sum \lim_{i \to 0} \sum_{i,k,j'} M_i^*(s_i, t_j, s_k, t_{j'}^j)] - M_i^*(s_i, t_j, s_k, t_{j'})^2 \cdot |1/2| &= \\
= \mathbb{E}[\sum \lim_{i \to 0} \sum_{i,k,j'} M_i^*(s_i, t_j, s_k, t_{j'})^2 \cdot |1/2|] \leq \\
\leq \mathbb{E}[\sup_{i,k} F_{i,k}(t) \cdot dt] = \\
= \mathbb{E}[\sup_{i,k} \int_0^1 F_{i,k}(t) \cdot dt] \leq \\
\leq \mathbb{E}[\sup_{i,k} \sup_{j} N_i^*(s_i, t_j) \cdot M_i^*(s_i, t_j, s_k, t_{j'})^2 \cdot |1/2|] \cdot \mathbb{E}[\sup_{i,k} M_i^*(s_i, t_j, s_k, t_{j'})^2]^{1/2}. \\
\end{align*} \]

To bound the first factor of (2.8), let \( \{f_j, j' \in J_j\} \) be a family of Rademacher functions over \([0,1]\). By Khintchine inequality,

\[ \mathbb{E}[\sup_{i,j} N_i^*(s_i, t_j) \cdot M_i^*(s_i, t_j, s_k, t_{j'})^2] \leq \]

\[ \mathbb{E}[\sup_{i,j} \sup_{j' \in J_j} \int_0^1 N_i^*(s_i, t_j) \cdot f_j(t) \cdot dt^2] \leq \]
By maximal Doob's inequality, we can bound the second factor of (2.8):

\[
\mathbb{E} \left[ \sum_{i,j,k} \sup_j M(\Lambda_{ij}^k) \right] = \\
= \sum_{i,j,k} \mathbb{E} \left[ \sup_j M(\Lambda_{ij}^k) \right] \leq C \sum_{i,j,k} \mathbb{E} \left[ M(\Lambda_{ij}^k) \right] = C \mathbb{E} \left[ M_{1,l,1}^2 \right],
\]

where \( \Lambda_{ij}^k = \left[ s_{k,l}^i, s_{k+1,l}^i \right] \times \left[ t_{j,l}, t_{j+1,l} \right] \).

By (2.5) and (2.8) we have

\[
\lim_{n,m \to \infty} \mathbb{E} \left[ \sup_{s \leq l} \left| S_{s,1}^n - S_{s,1}^m \right| \right] = 0. \tag{2.9}
\]

In fact, for \( n,m \), let \( \Gamma_{nm} \) be a grid over \([0,1]\) which has the same projection on the "t" axes as \( \Gamma_n \), and on the "s" axes as \( \Gamma_m \). We define

\[
S_{z}^{nm} = \sum_{u \in \Gamma_{nm}} M(\Lambda_{u}^1) N(\Delta_{u}^2).
\]

Then,

\[
\mathbb{E} \left[ \sup_{s \leq l} \left| S_{s,1}^n - S_{s,1}^m \right| \right] \leq \sup_{m \leq n} \mathbb{E} \left[ \sup_{s \leq l} \left| S_{s,1}^n - S_{s,1}^{nm} \right| \right] + \sup_{m \leq n} \mathbb{E} \left[ \sup_{s \leq l} \left| S_{s,1}^m - S_{s,1}^{nm} \right| \right],
\]

and taken \( n,m \to \infty \), (2.9) holds.

Finally, the convergence in (2.9) is the convergence in \( H_{1}^1 \), and since the space \( H_{1}^1 \) is complete, there exists a martingale \( S_{s,1} \) of \( H_{1}^1 \) such that \( \lim_{n \to \infty} \mathbb{E} \left[ \sup_{s \leq l} \left| S_{s,1}^n - S_{s,1} \right| \right] = 0 \).

By (2.1) \( \lim \sup_{n \to \infty} \mathbb{E} \left[ \left| S_{s,1}^n - M S_{s,1}^n \right| \right] = 0 \). It follows \( S_{s,1} = M S_{s,1} \).
REMARKS

1) The operation \( * \) is not commutative, but it is distributive with respect to the sum either from the right or from the left.

2) \( \tilde{M} = M \ast M \) and \( \tilde{M} \tilde{N} = \frac{1}{2} M \ast N + \frac{1}{2} N \ast M \).

3) \( \tilde{M} + \tilde{N} = \tilde{M} + \tilde{N} + M \ast N + N \ast M \).

This last remark allows to compute \( \tilde{M} \) when \( M \) is a sum of factors. Specifically,

**COROLLARY 2.2:** Let \( M_1, \ldots, M_n \) be martingales of \( \mathbb{M}^p(z_0) \), \( p \geq 2 \). Then

\[
\tilde{\sum_{i=1}^n M_i} = \tilde{\sum_{i=1}^n M_i} + 2 \sum_{i \neq j} \tilde{M_i \tilde{M_j}}.
\]

3. AN EXAMPLE: THE FILTRATION PRODUCT OF FILTRATIONS

GENERATED BY MULTI-DIMENSIONAL BROWNIAN MOTIONS

On the complete probability space \((\Omega, \mathcal{F}, P)\) we consider two independent multi-dimensional brownian motions

\( W = \{(W^1_s, \ldots, W^n_s) : s \in \mathbb{R}_+\} \) and \( \hat{W} = \{\hat{W}^1_t, \ldots, \hat{W}^m_t) : t \in \mathbb{R}_+\} \).

We will denote by \( \{\mathbb{F}^{1}_S, s \in \mathbb{R}_+\} \) and \( \{\mathbb{F}^{2}_t, t \in \mathbb{R}_+\} \) the completed filtrations generated by \( W \) and \( \hat{W} \) respectively.

Set \( \mathbb{V} \mathbb{F}^{1}_S = \mathbb{F}^{1}_\infty \) and \( \mathbb{V} \mathbb{F}^{2}_t = \mathbb{F}^{2}_\infty \). (We might suppose \( \mathbb{F} = \mathbb{F}^{1}_\infty \mathbb{V} \mathbb{F}^{2}_\infty \)). We define the product filtration \( \{\mathbb{F}^{1}_S, z \in \mathbb{R}_+^2\} \) by \( \mathbb{F}^{1}_S \mathbb{V} \mathbb{F}^{2}_t = \mathbb{F}^{1}_S \mathbb{V} \mathbb{F}^{2}_t \). It is known that this filtration is right-continuous and satisfies (F4).

We define the bi-brownian process \( W^{ij} = \{W^{ij}_z, z \in \mathbb{R}_+^2\} \) by

\[
W^{ij}_S(\omega) = W^i_s(\omega) \hat{W}^j_t(\omega).
\]
Let \( L^2_1(\mathbb{R}_+ \times \Omega) \) be the set of equivalence classes of measurable processes \( g: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R} \) adapted to \( \{ F^1_s, \ s \in \mathbb{R}_+ \} \) and such that \( E \int_0^S g^2(x)dx < \infty \), for all \( S \). Similarly, we define \( L^2_2(\mathbb{R}_+ \times \Omega) \). We will denote by \( L^2(\mathbb{R}_+^{2} \times \Omega) \) the set of equivalence classes of measurable processes \( f: \mathbb{R}_+^{2} \times \Omega \rightarrow \mathbb{R} \) adapted to \( \{ F^2_z, \ z \in \mathbb{R}_+^{2} \} \) and such that \( E \int_0^Z f^2(\zeta)d\zeta < \infty \), for all \( Z \).

The results of Brossard-Chevalier ([1]) are extended without difficulty to the multi-dimensional case and we obtain

**Proposition 3.1:** Let \( M = \{ M_z, F^2_z, \ z \in \mathbb{R}_+^{2} \} \) be a square-integrable martingale. Then there exists unique functions \( g_1, \ldots, g_n \in L^2_1(\mathbb{R}_+ \times \Omega), h_1, \ldots, h_m \in L^2_2(\mathbb{R}_+^{2} \times \Omega), f_{11}, \ldots, f_{nm} \in L^2(\mathbb{R}_+^{2} \times \Omega) \) such that

\[
M_{st} = M_{00} + \sum_{i=1}^{n} \int_0^s g_i(x)dW^i(x) + \sum_{j=1}^{m} \int_0^t h_j(y)dW^j(y) + \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{R_{st}} f_{ij}(x,y)dW^{ij}(x,y).
\]

If \( M'_st = \int_{R_{st}} f(z)dW^{ij}(z) \) and \( M''_{st} = \int_{R_{st}} \bar{f}(z)dW^{jk}(z) \), by means of the Itô formula we can compute \( \tilde{M}' \) (see [3]) and by means of the multi-dimensional version, we can compute \( \tilde{M}'' \). Exactly,

\[
\tilde{M}'_{st} = \int_{R_{st}} (\int_0^t f(s',y)dW^j(y)) (\int_0^s f(x,t')dW^i(x))dW^{ij}(s',t')
\]

and

\[
\tilde{M}''_{st} = \frac{1}{2} \int_{R_{st}} (\int_0^t f(s',y)dW^j(y)) (\int_0^s \bar{f}(x,t')dW^d(x))dW^{ik}(s',t') + \frac{1}{2} \int_{R_{st}} (\int_0^t \bar{f}(s',y)dW^k(y)) (\int_0^s f(x,t')dW^d(x))dW^{dj}(s',t')
\]

and

\[
\tilde{M}''_{st} = \frac{1}{2} \int_{R_{st}} (\int_0^t \bar{f}(s',y)dW^k(y)) (\int_0^s \bar{f}(x,t')dW^d(x))dW^{ik}(s',t') + \frac{1}{2} \int_{R_{st}} (\int_0^t f(s',y)dW^j(y)) (\int_0^s \bar{f}(x,t')dW^d(x))dW^{dj}(s',t')
\]

and

\[
\tilde{M}''_{st} = \frac{1}{2} \int_{R_{st}} (\int_0^t \bar{f}(s',y)dW^k(y)) (\int_0^s \bar{f}(x,t')dW^d(x))dW^{ik}(s',t') + \frac{1}{2} \int_{R_{st}} (\int_0^t f(s',y)dW^j(y)) (\int_0^s \bar{f}(x,t')dW^d(x))dW^{dj}(s',t')
\]

and

\[
\tilde{M}''_{st} = \frac{1}{2} \int_{R_{st}} (\int_0^t \bar{f}(s',y)dW^k(y)) (\int_0^s \bar{f}(x,t')dW^d(x))dW^{ik}(s',t') + \frac{1}{2} \int_{R_{st}} (\int_0^t f(s',y)dW^j(y)) (\int_0^s \bar{f}(x,t')dW^d(x))dW^{dj}(s',t')
\]

and

\[
\tilde{M}''_{st} = \frac{1}{2} \int_{R_{st}} (\int_0^t \bar{f}(s',y)dW^k(y)) (\int_0^s \bar{f}(x,t')dW^d(x))dW^{ik}(s',t') + \frac{1}{2} \int_{R_{st}} (\int_0^t f(s',y)dW^j(y)) (\int_0^s \bar{f}(x,t')dW^d(x))dW^{dj}(s',t')
\]
REMARK: The computation of $M^1*M^2$ cannot be reached by means of Itô formula, because this martingale does not appear in this formula. The expression of $M^1*M^2$ can be deduced calculating the limit of (2.1). In order to obtain the explicit formula for $M^1*M^2$ we need the convergence in $H^1$ given by (2.2). The result is

$$M^{1*}M^2_{st} = \int_{R_{st}} \left( \int_0^{t'} f(s', y) d\hat{W}^j(y) \right) \left( \int_0^{s'} f(x, t') dW^d(x) \right) dW^i(s', t').$$

We introduce some notation. We restrict our study to the martingales vanishing on the axes: If $M$ is a $L^2$-martingale, zero on the axes, with representation

$$M_{st} = \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{R_{st}} f_{ij}(x, y) dW^i(x, y),$$

we define

$$Y^j(s, t) = \sum_{i=1}^{n} \int_0^{s} f_{ij}(x, t) dW^i(x), \quad j = 1, \ldots, m;$$

$$\hat{Y}^i(s, t) = \sum_{j=1}^{m} \int_0^{t} f_{ij}(s, y) d\hat{W}^j(y), \quad i = 1, \ldots, n.$$

A Fubini theorem for bi-brownian stochastic integrals allows us to write

$$M_{st} = \sum_{j=1}^{m} \int_0^{t} Y^j(s, y) d\hat{W}^j(y) = \sum_{i=1}^{n} \int_0^{s} \hat{Y}^i(x, t) dW^i(x).$$

The expression of $\hat{N}$ and $\hat{M}^2$ and the corollary 2.2 give:

PROPOSITION 3.2: Let $M$ be a $L^2$-martingale, zero on the axes. With the preceding notations,

$$\hat{M}_{st} = \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{R_{st}} \hat{Y}^i(x, y) Y^j(x, y) dW^i(x, y).$$
4. THE QUADRATIC VARIATION OF M*N

Some definitions are required: A stochastic process \( \{M^z, z \in \mathbb{R}^2_+\} \) adapted to \( \{\mathcal{F}^z, z \in \mathbb{R}^2_+\} \) and integrable is said to be a 1-martingale if for any fixed \( t \), the process \( \{M^t_s, \mathcal{F}^t_s, s \in \mathbb{R}_+\} \) is a martingale. Similarly, we define 2-martingales. Because of (F4), we have that \( M \) is a martingale if and only if it is a 1 and 2-martingale. For a \( L^2 \) 1-martingale, we denote by \( \langle M^t \rangle_s \) the process \( \langle M^t_s \rangle \), that is, the quadratic variation of the one parameter martingale \( \{M^t_s, s \in \mathbb{R}_+\} \).

Let \( M \) be a 1-martingale. \( M \) is said to have 1-orthogonal increments if for any couple of disjoints rectangles \( \]z_1,z_1^i]\) and \( \]z_2,z_2^i]\) we have

\[
\mathbb{E}[\mathbb{E}([M(z_1^1,z_1^1)]M(z_2^i,z_2^i))|\mathcal{F}_{s_1^i,s_2^i,\infty}] = 0,
\]

where \( z_i = (s_i,t_i), z_i^i = (s_i^i,t_i^i), i = 1,2 \).

Similarly we define 2-martingales with 2-orthogonals increments. A martingale is said to have orthogonal increments if it has 1 and 2-orthogonal increments.

If \( M \) is a \( i \)-martingale with \( i \)-orthogonal increments, then the process \( \langle M^i \rangle_z \) is increasing, \( i = 1,2 \) (see [6]).

**Lemma 4.1**: Let \( M \) and \( N \) be two martingales of \( \mathbb{M}^\infty_4(z_0) \) zero on the axes, and let \( \Gamma \) a grid over \( [0,z_0] \). We consider the martingale

\[
S_z = \sum_{u \in \Gamma_z} M(u^1)N(u^2), \quad z \leq z_0.
\]

Then

\[
\langle S \rangle_z = \sum_{u \in \Gamma_z} \langle M^u \rangle \langle N^u \rangle < 0.
\]
PROOF

We consider two points of \([0, z_0] : z = (s, t) \) and \(z' = (s', t')\), \(z < z'\). Let \(\Delta^1_{z, z'} = ]s, s'] \times ]0, t']\) and \(\Delta^2_{z, z'} = ]0, s'] \times ]t, t']\). Then

\[
S_z = \sum_{u \in \Gamma} M(\Delta^1_u \cap R_z) N(\Delta^2_u \cap R_z),
\]

and

\[
S([z, z']) = \sum_{u \in \Gamma} (M(\Delta^1_u \cap R_z) N(\Delta^2_u \cap R_z))([z, z']).
\]

By considering different cases it can be proved that

\[
S([z, z']) = \sum_{u \in \Gamma \setminus z'} M(\Delta^1_u \cap \Delta^1_{z, z'}) N(\Delta^2_u \cap \Delta^2_{z, z'}). \]

We denote by \(A_Z\) the process \(\sum_{u \in \Gamma \setminus z'} M^j(\Delta^1_u) <N^\Delta(\Delta^2_u)\), which has the following properties:

a) \(A\) is increasing: Just as before,

\[
A([z, z']) = \sum_{u \in \Gamma \setminus z'} <M^j(\Delta^1_u \cap \Delta^1_{z, z'}) <N^\Delta(\Delta^2_u \cap \Delta^2_{z, z'}) \geq 0.
\]

b) \(A\) is continuous and adapted. So, it is predictable.

c) \(S^2 - A\) is a weak martingale. That means, we have to show that

\[
E[S^2([z, z']) | F_z] = E[A([z, z']) | F_z].
\]

In fact,

\[
E[S^2([z, z']) | F_z] = E[(S([z, z']))^2 | F_z] =
\]

\[
= \sum_{u \in \Gamma \setminus z'} E[M(\Delta^1_u \cap \Delta^1_{z, z'})^2 N(\Delta^2_u \cap \Delta^2_{z, z'})^2 | F_z] +
\]

\[
+ 2 \sum_{u, u' \in \Gamma \setminus z', u \neq u'} E[M(\Delta^1_u \cap \Delta^1_{z, z'}) N(\Delta^2_u \cap \Delta^2_{z, z'}) M(\Delta^1_{u'} \cap \Delta^1_{z, z'}) N(\Delta^2_{u'} \cap \Delta^2_{z, z'}) | F_z].
\]

The second term is zero: If \(u, u' \in [z, z']\), \(u = (u_1, u_2)\),
We similarly calculate the other possibilities for $u$ and $u'$. For the first term, we suppose $u \in [z, z']$ (the other cases are equally computed). Using the conditional independence we obtain

$$E[M(\Delta_u^1)N(\Delta_u^2)M(\Delta_{u'}^1)N(\Delta_{u'}^2) | \mathbb{F}_Z] =$$

$$= E[M(\Delta_u^1)N(\Delta_u^2)M(\Delta_{u'}^1)E[N(\Delta_{u'}^2) | \mathbb{F}_u^2] | \mathbb{F}_Z] = 0.$$  

By the unicity of the quadratic variation of $S$, we obtain $<S> = A.$

**LEMMA 4.2:** Let $\{M^n, n \geq 1\}$ be a sequence of martingales of $M^2(z_0)$ such that

$$\lim_{n \to \infty} E[|M^n_{z_0} - M^z_{z_0}|^2] = 0.$$ 

Then

$$\lim_{n \to \infty} E[|<M^n>_{z_0} - <M>_z |] = 0.$$ 

**PROOF**

By the Kunita-Watanabe inequality we have

$$<M>^{1/2} - <N>^{1/2} \leq <M-N>^{1/2}.$$ 

Then, by the Burkholder inequalities for the continuous two-
parameter martingales (see [9]) and by Cairoli-Doob inequality,

$$E[|M^n_{z_0} - M_{z_0}|] =$$

$$= E[|M^n_{z_0} - M_{z_0}|] \leq$$

$$\leq \{2E[|\{M^n_{z_0} + |M_{z_0}|^2\}] \cdot E[|\{M^n_{z_0} - |M_{z_0}|^2\}]\}^{1/2} \leq$$

$$\leq \{2E[|\{M^n_{z_0} + |M_{z_0}|^2\}] \cdot E[|\{M^n_{z_0} - |M_{z_0}|^2\}]\}^{1/2} \leq$$

$$\leq C\{E[|M^n_{z_0} - |M_{z_0}|^2\}]^{1/2}. \blacksquare$$

THEOREM 4.3: Let $M$ and $N$ be two martingales of $M^4_{C(z_0)}$, zero on the axes. For any standard sequence of grids $\{n^n, n \geq 1\}$ over $[0, z_0]$ we have

$$<M^*N>_{z} = \lim_{n \to \infty} \sum_{u \in [n^n]_{Z}} <M^*(\Delta^1_u)N^*(\Delta^2_u) >_{z} \in L^1,$$

and if $M$ has 1-orthogonal increments and $N$ has 2-orthogonal increments, then

$$<M^*N>_{z} = \int_{R^2_z \times R^2_z} \psi(\zeta, \zeta') \, d<M^*(\zeta)\,d<N^*(\zeta'),$$

where $\psi: R^2_+ \times R^2_+ \to R$ is the deterministic corner function

$$\psi(z, z') = \begin{cases} 1 & \text{if } s > s' \text{ and } t < t' \\ 0 & \text{otherwise.} \end{cases}$$

(z=(s,t) and z'=(s',t')).
PROOF

The first part is a consequence of the preceding lemmas.

The second part holds because the functions

$$
\Psi_n = \sum_{\mu \in \Gamma} n^{1/2} \Delta_u^1 \Delta_u^2
$$

converge pointwise to $\Psi$. And then, by the dominated convergence theorem, they converge to $\Psi$ in the norm

$$
\|\varphi\| = \left( \iint_{R \times R} \varphi^2(z, z') \, d\langle M^\varphi(z) \rangle d\langle N^\varphi(z') \rangle \right)^{1/2}.
$$

REMARK: The martingale $M^*N$ can always be written as

$$
M^*_N = \iint_{R \times R} \Psi(\zeta, \zeta') \, dM_\zeta \, dN_{\zeta'},
$$

where this integral must be understood as a double stochastic integral of a corner function (see [4], [10]). On the contrary, $<M^*N>$ cannot, generally, be expressed as an integral with respect to $<M>_Z^1$ and $<N>_Z^2$, because these processes are not increasing in general.

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F. UTZET
Trav. de les Corts, 272, A, 11-1
08014 BARCELONE
Espagne

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