R. Zaharopol

A zero-two theorem for a certain class of positive contractions in finite dimensional $L^p$-spaces ($1 \leq p < +\infty$)

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A Zero-Two Theorem for a certain class of positive Contractions in Finite Dimensional $L^p$-spaces ($1 \leq p < +\infty$)

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Summary

Our goal here is to extend Theorem 1.1 from [2] (which is sometimes called the zero-two law for positive contractions in $L^1$-spaces) to a class of positive contractions in finite dimensional $L^p$-spaces ($1 \leq p < +\infty$).

1. A General Lemma

Let $(X, \Sigma, \mu)$ be a measure space and $L^p(X, \Sigma, \mu)$ ($1 \leq p < +\infty$) the usual Banach spaces. By a positive contraction $T: L^p(X, \Sigma, \mu) \rightarrow L^p(X, \Sigma, \mu)$ we mean that $T$ is a linear bounded operator which transforms non-negative functions into non-negative functions and its norm is not more than one.

Lemma 1. Let $1 \leq p < +\infty$ and let $T: L^p(X, \Sigma, \mu) \rightarrow L^p(X, \Sigma, \mu)$ be a positive contraction. Suppose that there exist $\delta > 0$ and $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\|T^{n_0+1} - T^{n_0}\|_p^p < 2(1 - \delta).$$

Let $f \in L^p(X, \Sigma, \mu)$ be such that for every $n \in \mathbb{N} \cup \{0\}$ $T^nf \cdot T^{n+1}f = 0$. Then

$$\lim_{n \rightarrow +\infty} \|T^nf\|_p = 0.$$

Proof. Clearly it is enough to prove the lemma for $\|f\|_p = 1$.

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If \( f \) is as above then

\[
\| T^{n_0+1}f - T^{n_0}f \|_p^p = \| T^{n_0+1}f \|_p^p + \| T^{n_0}f \|_p^p < 2(1 - \varepsilon) .
\]

Using the fact that \( \left( \| T^n f \|_p \right)_n \) is a decreasing sequence we obtain that

\[
\| T^{n_0+1}f \|_p < (1 - \varepsilon)^{1/p} .
\]

It follows that if we note \( \rho = (1 - \varepsilon)^{1/p} \) then

\[
\| (T^{n_0+1} - T^{n_0})T^{n_0+1}f \|_p^p = \| T^{n_0+1}T^{n_0+1}f \|_p^p + \| T^{n_0}T^{n_0+1}f \|_p^p < 2(1 - \varepsilon)\rho^p
\]

and we obtain that

\[
\| T^{2(n_0+1)}f \|_p < \rho^2 .
\]

By induction it follows that for every \( h \in \mathbb{N} \) \( \| T^hf \|_p^p < \rho^h \) and using the fact that \( \left( \| T^n f \|_p \right)_n \) is a decreasing sequence it follows that \( \lim_{n \to \infty} \| T^n f \|_p = 0 \).

Remark. In Lemma 1 we may drop the assumption of \( T \) being positive. However we will need this assumption later on.

2. The Finite Dimensional \( L^p \)-spaces and the Theorem

We will now consider the following case:

Let \( k \in \mathbb{N} , \ k \geq 2 \) be and we note \( X = \{1, 2, \ldots, k\} \), \( \Sigma = \mathcal{P}(X) \). Let \( m_1, \ldots, m_k \) be \( k \) non-zero positive real numbers. We will denote by \( m \) the measure generated by \( m_1, \ldots, m_k \) (that is \( m(\{i\}) = m_i \), \( i = 1, \ldots, k \)). We will call the space \( L^p(X, \Sigma, m) \) a finite dimensional \( L^p \)-space and we will note
\[ T_p(k,m) = L^p(X,\Sigma,\mu). \] A positive contraction \( T: T_p(k,m) \to T_p(k,m) \) is generated by a matrix \( (a_{ij})_{i,j=1,2,\ldots,k} \) and the resulting positive contraction \( T^n(n \in \mathbb{N}) \)

is generated by \( (a^{(n)}_{ij})_{i,j=1,\ldots,k} \).

If \( x \in T_p(k,m), x = (x_1,\ldots,x_k) \) then \( Tx = (\sum_{i=1}^{k} x_i a_{ij})_{j=1,\ldots,k} \) and \( T^n = (\sum_{i=1}^{k} x_i a^{(n)}_{ij})_{j=1,\ldots,k} \).

If \( T \) is a positive contraction on \( T_p(k,m) \) we will note \( \Omega = \{i \in X \mid \text{for every } n \in \mathbb{N} \cup \{0\} \text{ and } \mathbf{1} = 1,\ldots,k \sum_{j=1}^{k} a_{ij} m_j = m_i \text{ and } a^{(n)}_{ij} a^{(n+1)}_{ij} = 0 \} \).

**Lemma 2.** Let \( T: T_p(k,m) \to T_p(k,m) \) be a positive contraction. Then the following are equivalent:

a) for every \( n \in \mathbb{N} \cup \{0\}, \|T^{n+1} - T^n\|_1 = 2 \)

b) \( \Omega \neq \emptyset \).

**Proof.** a) \( \Rightarrow \) b) Suppose \( \Omega = \emptyset \). It follows that for every \( i \in \{1,2,\ldots,k\} \) there exists \( n_i \in \mathbb{N} \) such that \( \sum_{j=1}^{k} a_{ij} m_j < m_i \) or there exists \( j_0 \) such that \( a^{(n)}_{ij_0} a^{(n+1)}_{ij_0} \neq 0 \). In other words for every \( i \in \{1,2,\ldots,k\} \) there exists \( n_i \) such that \( \|T_i^{n+1} - T_i^n\|_{1 \{i\}} < 2m_i = 2\|1_{\{i\}}\|_1 \).

If we note \( n_0 = \max\{n_1,\ldots,n_k\} \) it follows that for every \( n \geq n_0 \) and for every \( i \in \{1,2,\ldots,k\} \) \( \|T_i^{n+1} - T_i^n\|_{1 \{i\}} < 2m_i \) (as for every \( i \in \{1,2,\ldots,k\} \))
the sequence \((\|T_{n+1} - T_n\|_1)_{n=1}^\infty\) is a decreasing one).

It follows that for \(n \geq n_0\), \(\|T_{n+1} - T_n\|_1 < 2\).

b) \(\Rightarrow\) a) If \(\emptyset \neq \Phi\) then for every \(n \in N \cup \{0\}\) and \(i \in \Phi\), \(\|T_{n+1}^{i} - T_n^{i}\|_1 = 2\|1\{i\}\|_1\) and as \(T\) is a positive contraction it follows that for every \(n \in N \cup \{0\}\), \(\|T_{n+1} - T_n\|_1 = 2\).

Now we are able to prove the desired result:

**Theorem 3.** Let \(p\) be such that \(1 \leq p \leq +\infty\) and let \(T\) be simultaneously a positive contraction of \(L^p(k,m)\) and \(L^p(k,m)\). If there exists \(n_0 \in N \cup \{0\}\) such that \(\|T_{n+1}^{n+1} - T_{n_0}^{n_0}\|_p < 2^{1/p}\) then \(\lim_{n \to \infty} \|T_{n+1}^{n+1} - T_n^{n_0}\|_p = 0\).

**Proof.** If \(\lim_{n \to \infty} \|T_{n+1}^{n+1} - T_n^{n_0}\|_p \neq 0\) then \(\lim_{n \to \infty} \|T_{n+1}^{n+1} - T_n^{n_0}\|_1 \neq 0\) (as every two norms in \(R^{k^2}\) are equivalent). Using the zero-two law for positive contractions in \(L^1\)-spaces (Theorem 1.1 from [2]) it follows that for every \(n \in N \cup \{0\}\), \(\|T_{n+1}^{n+1} - T_n\|_1 = 2\) and by Lemma 2 it follows that \(\emptyset \neq \Phi\). If \(i \in \Phi\) then the characteristic function \(1\{i\}\) satisfies the conditions of Lemma 1 and it follows that \(\lim_{n \to \infty} \|T_{n+1}^{n+1} 1\{i\} - T_n^{n_0} 1\{i\}\|_p = 0\). We obtain that \(\lim_{n \to \infty} \|T_{n+1}^{n+1} 1\{i\} - T_n^{n_0} 1\{i\}\|_1 = 0\) which contradicts the fact that \(i \in \Phi\).
References


R. ZAHAROPOL
Département de Mathématiques
Université Hébraïque
GIVAT-RAM
Jérusalem
Israël

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