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RATE OF CONVERGENCE TOWARDS A FRECHET TYPE LIMIT DISTRIBUTION

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Abstract : Let $\{X_i\}$ $i \in N$ be a sequence of i.i.d. random variables, and $M_n = \max(X_1,\dots,X_n)$. It is well-known that $F_{M_n}(x) = F^n_{X_1}(x)$, and that if there are attraction coefficients $\{a_n\}$ $n \in N$ $(a_n > 0)$ and $\{b_n\}$ $n \in N$ $(b_n \in R)$ such that $F^n(a_n x + b_n) \to G(x)$, $x \in C_G$, then G is one of the three extreme value stable types :

$\Lambda(x) = \exp(-e^{-x})$ if $-\infty < x < +\infty$ (Gumbel type)

$\Phi_\alpha(x) = \exp(-x^{-\alpha})$ if $0 < x < +\infty$ (Fréchet type)

$\Psi_\alpha(x) = \exp(-(-x^{\alpha}))$ if $-\infty < x < 0$ (Weibull type)

There are no definite results on the rate of convergence of F^n towards the limiting form but in the case F is of normal type. Under mild conditions on the taiweight of F, we study the rate of convergence in the case of a Fréchet type limit distribution.

1. Intoduction

Let $M_n = \max(X_1,\dots,X_n)$, where $\{X_i\}$ $i \in N$ is a sequence of independent identically distributed (i.i.d.) random variables (r.v.) with distribution function (d.f.) F. It is known that, when the weak limit of M_n, suitably normalized, exists, it has distribution function which belongs to one of the three stable types - Gumbel, Fréchet or Weibull (see, Gnedenko,1943) - and we say that, the d.f. F belongs to the domain of attractionof the limit distribution function. In particular, we say that, F belongs to the domain of attraction of the Fréchet d.f. $\Phi_\alpha (\alpha>0)$ - $F \in \mathcal{D}(\Phi_\alpha)$ - that is, there are constants $\{a_n\}$ $n \in N$, $\{b_n\}$ $n \in N$, $a_n>0$, $b_n \in R$ such that

(1.1) $F^n(a_n x + b_n) \to \Phi_\alpha(x) = \exp(-x^{-\alpha})$, $x>0$, $\alpha>0$

if and only if

(1.2.a) $\sup\{x: F(x)<1\} = +\infty$ and

(1.2.b) $1-F(x) = x^{-\alpha} L(x)$, $L(x)$ is a slowly varying function (in Karamara's, 1933 sense)

A general description of limit laws and domains of attraction concerning maximums of i.i.d. r.v. may be found in Galambos (1978). The normalizing constants a_n are usually defined in terms of levelcrossings; in the special case of attraction towards the Fréchet distribution, it is easy to show that we can take,

(1.3) $a_n = \inf\{x: 1-F(x)<1/n\}$ and $b_n = 0$ (cf. Galambos 1978, p.51)

Observe that

(1.4) $a_n = n^{1/\alpha}\psi(n)$, where $\psi(x)$ is a slowly varying function (cf. Iglésias, 1982.a)

The case $\psi(x) = A>0$ is particularly interesting. This happens if and only if

(1.5) $1-F(x) = c\, x^{-\alpha} + o(x^{-\alpha})$, $c>0$, $x \to \infty$

i.e., if and only if F has a Paretian tail.

The rate of convergence of $F^n(x)$ towards the limit distribution may be extremely slow (see, Fisher and Tippett, 1928; Gomes, 1978, 1982). In the present paper we study this problem under certain mild conditions on the tail behavior (assumed to be paretian) of F(.).

2. An asymptotic result

Let us suppose that for $0<\alpha<\beta$ we have

$$(2.1) \quad 1-F(x) = c_1\, x^{-\alpha} + c_2\, x^{-\beta} + r(x) \quad , \; x \to +\infty$$

where $c_1>0$, $c_2 \in R$ and $r(x) = o(x^{-\beta})$

Theorem 2.1 : Let $F(x)$ be a d.f. which belongs to the domain of attraction of Φ_α, with normalizing constants $a_n = A\, n^{1/\alpha}$, $A>0$, $b_n = 0$ and satisfying condition (2.1); then for $x>0$ and $n \to +\infty$ we have :

a) if $\beta<2\alpha$

$$(2.2) \quad F^n((c_1 n)^{1/\alpha}\, x) = \Phi_\alpha(x) - \frac{c_1^{-\beta/\alpha}\, c_2\, x^{-\beta}}{n^{\beta/\alpha - 1}}\, e^{-x^{-\alpha}} + o\left(\frac{1}{n^{\beta/\alpha-1}}\right)$$

b) if $\beta = 2\alpha$

$$(2.3) \quad F^n((c_1 n)^{1/\alpha}\, x) = \Phi_\alpha(x) - e^{-x^{-\alpha}} \frac{(1/2 + c_1^{-2}\, c_2)\, x^{-2\alpha}}{n} + o\left(\frac{1}{n}\right)$$

c) if $\beta >2\alpha$

$$(2.4) \quad F^n((c_1 n)^{1/\alpha} x) = \Phi_\alpha(x) - e^{-x^{-\alpha}} \Big\{ \sum_{i=1}^{(\beta/\alpha)-1} \frac{x^{-(i+1)\alpha}}{(i+1)\, n^i} - \sum_{2\leq i+k\leq(\beta/\alpha)-1} \frac{x^{-(i+k+2)\alpha}}{(i+1)\,(k+1)\, n^{i+k}} + \frac{c_1^{-\beta/\alpha}\, c_2\, x^{-\beta}}{n^{\beta/\alpha - 1}} \Big\} +$$

$$+ o\left(\frac{1}{n^{\beta/\alpha-1}}\right)$$

Proof : In expression (2.1), replacing x by $(c_1 n)^{1/\alpha}\, x$ we have

$$(2.5) \quad 1-F((c_1 n)^{1/\alpha}\, x) = \frac{x^{-\alpha}}{n} + \frac{c_1^{-\beta/\alpha}\, c_2\, x^{-\beta}}{n^{\beta/\alpha}} + r((c_1 n)^{1/\alpha}\, x)$$

for $x>0$ and $n \to +\infty$.
To simplify the notation we shall put $d_n = (c_1 n)^{1/\alpha}$. Expanding $\log F(d_n x)$ we obtain

$$(2.6) \quad n.\log F(d_n x) = -n\, \{(1-F(d_n x)) + o\,((1-F(d_n x))\}$$

$$= -x^{-\alpha} - \frac{c_1^{-\beta/\alpha}\, c_2\, x^{-\beta}}{n^{\beta/\alpha - 1}} - n\, r(d_n x) - n.o\,(1-F(d_n x))$$

and hence

$$(2.7)\quad F^n(d_n x) = e^{-x^{-\alpha}} . \exp\{ - \frac{c_1^{-\beta/\alpha}\ c_2\ x^{-\beta}}{n^{\beta/\alpha - 1}} - n\ r(d_n x) - n.o\ (1-F(d_n x))\}$$

Expanding now the factor exp {... } we obtain

$$(2.8)\quad F^n(d_n x) = e^{-x^{-\alpha}} \{1 - \frac{c_1^{-\beta/\alpha}\ c_2\ x^{-\beta}}{n^{\beta/\alpha - 1}} - n\ r(d_n x) - n.o\ (1-F(d_n x)) +$$

$$+ (\ n.o\ (1-F(d_n x))^2 \} + o\ (\frac{1}{n^{\beta/\alpha-1}})$$

We now study the asymptotic behavior of the 3^{rd}, 4^{th} and 5^{th} summands in the right hand side of (2.8). According to (2.1) :

$$(2.9)\quad n\ r(d_n x) = \frac{n^{\beta/\alpha}\ c_1^{\beta/\alpha}\ x^{\beta}\ r(d_n x)}{n^{\beta/\alpha-1}\ c_1^{\beta/\alpha} x^{\beta}} = o\ (\frac{1}{n^{\beta/\alpha-1}})$$

The 4^{th} summand is equal to :

$$(2.10)\quad n.o\ (1-F(d_n x)) = n.\{(1-F(d_n x))^2/2 + (1-F(d_n x))^3/3 + \ldots\}$$

and by (2.5)

$$(2.11)\quad n.o\ (1-F(d_n x)) = n.\ \sum_{j=2}^{\infty} \frac{\{\frac{x^{-\alpha}}{n} + \frac{c_1^{-\beta/\alpha} c_2\ x^{-\beta}}{n^{\beta/\alpha}} + r(d_n x)\}^j}{j}$$

or else,

$$(2.12)\quad n.o\ (1-F(d_n x)) = \sum_{i=1}^{(\beta/\alpha)-1} \frac{x^{-(i+1)\alpha}}{(i+1)\ n^i} + o\ (\frac{1}{n^{\beta/\alpha-1}})$$

And hence the 5^{th} summand may be written

$$(2.13)\quad \{n.o\ (1-F(d_n x))\}^2 = \sum_{2\leq i+k\leq (\beta/\alpha)-1} \frac{x^{-(i+k+2)\alpha}}{(i+1)\ (k+1)\ n^{i+k}} + o\ (\frac{1}{n^{\beta/\alpha-1}})$$

And finally from (2.8), (2.9), (2.12) and (2.13) we have

$$(2.14)\quad F^n(d_n x) = \Phi_\alpha(x) - e^{-x^{-\alpha}} \{ \sum_{i=1}^{(\beta/\alpha)-1} \frac{x^{-(i+1)\alpha}}{(i+1)\ n^i} -$$

$$- \sum_{2\leq i+k\leq (\beta/\alpha)-1} \frac{x^{-(i+k+2)\alpha}}{(i+1)\ (k+1)\ n^{i+k}} + \frac{c_1^{-\beta/\alpha} c_2\ x^{-\beta}}{n^{\beta/\alpha-1}} \} + o\ (\frac{1}{n^{\beta/\alpha-1}})$$

The result follows considering the magnitude relation between β and α .

Example : Let X be a standard normal r.v. with d.f. Φ_X and define the r.v. $Y = 1/X^2$ whose d.f. is given by

$$(2.15) \quad F_Y(y) = 2(1-\Phi(1/\sqrt{y})) \quad , \quad y>0$$

It is known from the theory of the addition of independent random variables that, if we have a sequence $\{Y_i\}$ $i \in N$ of i.i.d. r.v. with d.f. F_Y, then there are constants $a_n > 0$ and $b_n \in R$ such that

$$(2.16) \quad \frac{\sum_{i=1}^{n} Y_i - b_n}{a_n} \overset{d}{=} Y \qquad \forall n \in N$$

it is easy to verify that the norming constants must have the form $a_n = A\, n^{1/\alpha}$, $A>0$, $\alpha \in]0,2]$ and the definition of the centering constants b_n is unessential unless $\alpha = 1$ (cf. Feller,1966). As (2.16) may be rewritten :

$$(2.17) \quad \frac{\sum_{i=1}^{n} Y_i}{A\, n^2} \overset{d}{=} Y$$

we have that the r.v. Y is stable with characteristic exponent $\alpha = 1/2$. In particular we can say that F belongs to its own domain of normal attraction, and according to Gnedenko and Kolmogorov (1968) we have :

$$(2.18) \quad 1-F_Y(y) = c\, y^{-1/2} + o(y^{-1/2}) \quad , c>0 \quad , y \to +\infty$$

On the other hand, as $1-F_Y(y)$ satisfies conditions (1.2.a), (1.2.b) and (1.5) we may conclude that $F_Y \in \mathfrak{L}(\Phi_{1/2})$, where $\Phi_{1/2}$ is the Fréchet distribution with parameter $\alpha = 1/2$, and attraction coefficients $a_n = A\, n^2$, $A>0$ and $b_n = 0$. Finally expanding $\Phi_X(.)$ on power series (Abramowitz, 1972, p.932) we have :

$$(2.19) \quad 1-F_Y(y) = 2\Phi_X(1/\sqrt{y}) - 1 = \frac{2}{\sqrt{2\pi}} x^{-1/2} - \frac{x^{-3/2}}{3\sqrt{2\pi}} + \frac{2}{\sqrt{2\pi}} \sum_{n=2}^{\infty} \frac{(-1)^n x^{-(n+1/2)}}{n!\, 2^n (2n+1)}$$

taking $c_1 = \frac{2}{\sqrt{2\pi}}$, $c_2 = -\frac{1}{3\sqrt{2\pi}}$, $\alpha = 1/2$, $\beta = 3/2$ and

$r(x) = \frac{2}{\sqrt{2\pi}} \sum_{n=2}^{\infty} \frac{(-1)^n x^{-(n+1/2)}}{n!\, 2^n (2n+1)}$ we see that the d.f. F_Y satisfies the conditions of theorem 2.1 and so :

$$(2.20) \quad F^n\left(\left(\frac{2}{\sqrt{2\pi}} n\right)^2 x\right) = e^{-x^{-1/2}} - e^{-x^{-1/2}} \left\{ \frac{x^{-1}}{n} + \frac{(1-\pi/4)\, x^{-3/2}}{3\, n^2} - \frac{x^{-2}}{4\, n^2} \right\} +$$

$$+ o\left(\frac{1}{n^2}\right)$$

for $x > 0$, $n \to +\infty$.

For the particular choice $x = 1/64$ we obtain

n	$G^n((\frac{2}{\sqrt{2\pi}} n)^2 . 1/64)$	$e^{-8}\{1 - \frac{64}{2n} - \frac{(1-\pi/4)\, 64^{3/2}}{3 n^2} + \frac{64^2}{4 n^2}\}$
10	0,000010379	0,0025742551
40	0,0001513444	0,0002741096
100	0,0002498394	0,0002612373
500	0,0003144483	0,0003153179

3. Optimal choice of attraction coefficients

In theorem 2.1 we used a particular form for the attraction coefficients. In the next theorem we will show that this particular choice is essentially optimal.

Let us consider arbitrary constants $a_n' > 0$ and $b_n' \in R$ for which it remains true that

$$(3.1) \qquad F^n(a_n' x + b_n') \to \Phi_\alpha(x) \quad , \quad x \in C_{\Phi_\alpha}, \quad n \to +\infty$$

Theorem 3.1 : Let F be a distribution function satisfying conditions (2.1) and (3.1). Let

$$(3.2) \qquad A_n = \frac{a_n'}{a_n} \quad , \quad B_n = \frac{(b_n' - b_n)}{a_n}$$

where a_n and b_n are the same constants of theorem 2.1. Then,

a) if $\beta < 2\alpha$

$$(3.3) \qquad F^n(a_n' x + b_n') = \Phi_\alpha(x) + (x(A_n - 1) + B_n)\Phi_\alpha'(x) -$$

$$- \Phi_\alpha(x + x(A_n - 1) + B_n) \frac{c_1^{-\beta/\alpha}\, c_2\, (x + x(A_n - 1) + B_n)^{-\beta}}{n^{\beta/\alpha - 1}} +$$

$$+ o\left(\frac{1}{n^{\beta/\alpha - 1}}\right) + o\,(x(A_n - 1) + B_n)$$

b) if $\beta = 2\alpha$

$$(3.4)\quad F^n(a_n' x + b_n') = \Phi_\alpha(x) + (x(A_n - 1) + B_n)\,\Phi_\alpha'(x) - \Phi_\alpha(x + x(A_n - 1) + B_n).\frac{(1/2 + c_1^{-2} c_2)(x + x(A_n - 1) + B_n)^{-2\alpha}}{n} + o\left(\frac{1}{n}\right) + o\,(x(A_n - 1) + B_n)$$

c) if $\beta > 2\alpha$

$$(3.5)\quad F^n(a_n' x + b_n') = \Phi_\alpha(x) + (x(A_n - 1) + B_n)\Phi_\alpha'(x) - \Phi_\alpha(x + x(A_n - 1) + B_n).\Big\{\sum_{i=1}^{(\beta/\alpha)-1} \frac{(x + x(A_n - 1) + B_n)^{-(i+1)\alpha}}{(i+1)\,n^i} + \sum_{2\le i+k\le(\beta/\alpha)-1} \frac{(x + x(A_n - 1) + B_n)^{-(i+k+2)\alpha}}{(i+1)(k+1)\,n^{i+k}} + \frac{c_1^{-\beta/\alpha} c_2 (x + x(A_n - 1) + B_n)^{-\beta}}{n^{\beta/\alpha - 1}}\Big\} + o\left(\frac{1}{n^{\beta/\alpha - 1}}\right) + o\,(x(A_n - 1) + B_n)$$

Proof : By hypothesis we have

$$(3.6)\quad F^n(a_n' x + b_n') = F^n(a_n A_n x + a_n B_n) = F^n((c_1 n)^{1/\alpha}(A_n x + B_n)) \to \Phi_\alpha(x)$$

for $x\ (>0) \in C_{\Phi_\alpha}$ and $n \to +\infty$, and where $A_n \to 1$ and $B_n \to 0$.
On the other hand by condition (2.1) and putting, as before, $d_n = (c_1 n)^{1/\alpha}$

$$(3.7)\quad 1 - F(d_n(A_n x + B_n)) = \frac{(A_n x + B_n)^{-\alpha}}{n} + \frac{c_1^{-\beta/\alpha} c_2 (A_n x + B_n)^{-\beta}}{n^{\beta/\alpha}} + r(d_n(A_n x + B_n))$$

for $x > 0$ and $n \to +\infty$.
Let us take $x_n = A_n x + B_n$ and apply the expansion of the logarithm on power series to get :

$$(3.8)\quad n.\log F(d_n x_n) = -n.\{(1 - F(d_n x_n)) + o\,(1 - F(d_n x_n))\}$$

$$= - x_n^{-\alpha} - \frac{c_1^{-\beta/\alpha} \; c_2 \; x_n^{-\beta}}{n^{\beta/\alpha \; - \; 1}} - n \; r(d_n \; x_n) - n.o \; (1-F(d_n \; x_n))$$

i.e.

$$(3.9) \quad F^n(d_n \; x_n) = e^{-x_n^{-\alpha}} \; . \; \exp \{ - \frac{c_1^{-\beta/\alpha} \; c_2 \; x_n^{-\beta}}{n^{\beta/\alpha \; - \; 1}} - n \; r(d_n \; x_n) - n.o \; (1-F(d_n \; x_n))\}$$

Expanding on power series the factor exp {...} we obtain :

$$(3.10) \quad F^n(d_n \; x_n) = e^{-x_n^{-\alpha}} \; \{1 - \frac{c_1^{-\beta/\alpha} \; c_2 \; x_n^{-\beta}}{n^{\beta/\alpha \; - \; 1}} - n \; r(d_n \; x_n) - n.o \; (1-F(d_n \; x_n)) +$$

$$+ \; (n.o \; (1-F(d_n \; x_n))^2 \; \} + o \; (\frac{1}{n^{\beta/\alpha-1}})$$

Observing that from (2.1) we get :

$$(3.11) \quad n \; r(d_n \; x_n) = \frac{n^{\beta/\alpha} \; c_1^{\beta/\alpha} x_n^{\beta} \; r((c_1 n)^{1/\alpha} \; x_n)}{n^{\beta/\alpha \; -1} c_1^{\beta/\alpha} \; x_n^{\beta}} = o \; (\frac{1}{n^{\beta/\alpha-1}})$$

Let us now study the behavior of the 4^{th} and 5^{th} summands of the expansion in (3.10)

$$(3.12) \quad n.o \; (1-F(d_n \; x_n)) = n. \; \{(1-F(d_n \; x_n))^2/2 + (1-F(d_n \; x_n))^3/3 + \ldots\}$$

and by (3.7)

$$(3.13) \quad n.o \; (1-F(d_n \; x_n)) = n.\{ \sum_{j=2}^{\infty} \frac{\{\frac{x_n^{-\alpha}}{n} + \frac{c_1^{-\beta/\alpha} \; c_2 \; x_n^{-\beta}}{n^{\beta/\alpha}} + r(d_n \; x_n)\}^j}{j} \}$$

or else, after some calculations,

$$(3.14) \quad n.o \; (1-F(d_n \; x_n)) = \sum_{i=1}^{(\beta/\alpha) \; -1} \frac{x_n^{-(i+1)\alpha}}{(i+1) \; n^i} + o \; (\frac{1}{n^{\beta/\alpha-1}})$$

Then the 5^{th} summand becomes

$$(3.15) \quad (n.o \; (1-F(d_n \; x_n))^2 = \sum_{2 \leq i+k \leq (\beta/\alpha)-1} \frac{x_n^{-(i+k+2)\alpha}}{(i+1) \; (k+1) \; n^{i+k}} + o(\frac{1}{n^{\beta/\alpha-1}})$$

From (3.11), (3.14) and (3.15) we can write

$$(3.16) \quad F^n(d_n \; x_n) = e^{-x_n^{-\alpha}} - e^{-x_n^{-\alpha}} \; \{ \frac{c_1^{-\beta/\alpha} \; c_2 \; x_n^{-\beta}}{n^{\beta/\alpha \; - \; 1}} + \sum_{i=1}^{(\beta/\alpha)-1} \frac{x_n^{-(i+1)\alpha}}{(i+1) \; n^i} +$$

$$+\sum_{2\leq i+k\leq(\beta/\alpha)-1} \frac{x_n^{-(i+k+2)\alpha}}{(i+1)\,(k+1)\,n^{i+k}}\Big\} \quad + o\,(\frac{1}{n^{\beta/\alpha-1}})$$

As $\phi_\alpha(x_n) = e^{-x_n^{-\alpha}}$ admits an expansion on Taylor series and considering that $x_n = A_n\, x + B_n = x + (A_n-1)x + B_n$ we have

$$(3.17) \qquad \phi_\alpha(x_n) = \phi_\alpha(x) + (x(A_n-1) + B_n)\phi_\alpha'(x) + o\,(x(A_n-1) + B_n)$$

Then,

$$(3.18) \qquad F^n(d_n\, x_n) = \phi_\alpha(x) + (x(A_n-1) + B_n)\,\phi_\alpha'(x) -$$

$$- \phi_\alpha(x + x(A_n-1) + B_n).\Big\{ \frac{c_1^{-\beta/\alpha}\; c_2\; (x + x(A_n-1) + B_n)^{-\beta}}{n^{\beta/\alpha\, -\, 1}} +$$

$$+ \sum_{i=1}^{(\beta/\alpha)-1} \frac{(x+ x(A_n-1) + B_n)^{-(i+1)\alpha}}{(i+1)\; n^i} +$$

$$+\sum_{2\leq i+k\leq(\beta/\alpha)-1} \frac{(x + x(A_n-1) + B_n)^{-(i+k+2)\alpha}}{(i+1)\,(k+1)\,n^{i+k}}\Big\} + o\,(\frac{1}{n^{\beta/\alpha-1}}) +$$

$$+ o\,(x(A_n-1) + B_n)$$

The result follows considering the possible ordering of β and $2\,\alpha$.

From the analysis of (3.18) we can see that the rate of convergence hasn't improved; in fact if $(x(A_n-1) + B_n)$ converges faster than $(n^{-\beta/\alpha\,+1})$ the overall convergence in (3.18) is still of the order of $(n^{-\beta/\alpha\,+1})$ and if $(x(A_n-1) + B_n)$ converges more slowly, then convergence in (3.18) is slower than $(n^{-\beta/\alpha\,+1})$. In this sense, we may say that the constants a_n and b_n in theorem 2.1 are optimal.

4. Pareto distributions

In this paragraph our aim is to study the rate of convergence of suitably normalized maximum of Pareto r.v.'s with d.f. of the form :

$$(4.1.a) \qquad F(x) = 1 - (a/x)^\alpha \qquad x\geq a \quad ,\ \alpha>0$$

where for simplicity we take $a = 1$

$$(4.1.b) \qquad F(x) = 1 - (1/x)^\alpha \qquad x\geq 1 \quad ,\ \alpha>0$$

The limiting distribution is, of course, of Fréchet type . More precisely, by (1.5) we know that $F \in \mathfrak{L}(\Phi_\alpha)$ with normalizing constants $a_n = A\, n^{1/\alpha}$ and $b_n = 0$. Hence,

(4.2) $$\lim_n F^n(n^{1/\alpha}x) = \lim_n (1 - \frac{x^{-\alpha}}{n})^n = e^{-x^{-\alpha}} \qquad x > 0$$

Using the methods developed in paragraph 2 we may estabilish the following result:

Theorem 4.1 : Let $F(x)$ be a Pareto d.f. defined in (4.1.b). Then for $x > 0$ and $n \to \infty$ we have :

(4.3) $$F^n(n^{1/\alpha}x) = \Phi_\alpha(x) - \frac{x^{-2\alpha}}{2n}\Phi_\alpha(x) + o(\frac{1}{n})$$

Proof : Expanding the logarithm in power series we obtain :

(4.4) $$-n.\log F(n^{1/\alpha}x) = n.\{(1-F(n^{1/\alpha}x)) + o'(1-F(n^{1/\alpha}x))\}$$

$$= n.\{\frac{x^{-\alpha}}{n} + \sum_{j=2}^{\infty}\frac{x^{-\alpha j}}{j\, n^j}\} = x^{-\alpha} + \sum_{j=2}^{\infty}\frac{x^{-\alpha j}}{j\, n^{j-1}}$$

(4.5) $$F^n(n^{1/\alpha}x) = e^{-x^{-\alpha}}.\exp\{-\sum_{j=2}^{\infty}\frac{x^{-\alpha j}}{j\, n^{j-1}}\}$$

$$= e^{-x^{-\alpha}}.\{1 - \sum_{j=2}^{\infty}\frac{x^{-\alpha j}}{j\, n^{j-1}} + o(\frac{1}{n})\}$$

$$= e^{-x^{-\alpha}} - \frac{x^{-2\alpha}}{2n}.e^{-x^{-\alpha}} + o(\frac{1}{n})$$

It is well-known that the above choice of the constaants a_n and b_n isn't the only possible one. According to (1.4) we may take $a'_n = n^{1/\alpha}\psi(n)$ where $\psi(.)$ is a slowly varying function, as long as,

(4.6) $$\lim \frac{a'_n}{a_n} = 1$$
$$\lim (b'_n - b_n) / a_n = 0$$

Let us take $a'_n = (1 - e^{-1/n})^{1/\alpha} = n^{1/\alpha}(1 - \frac{1}{2n} + \frac{1}{6\, n^2} + \dots)^{-1/\alpha}$

(4.7) $$a'_n = n^{1/\alpha}\psi(n) \qquad \text{and} \qquad \lim_n(a'_n / a_n) = 1$$

Further,

(4.8) $F^n(a_n' x) = (1-x^{-\alpha}(1 - e^{-1/n}))^n = (1 - \frac{x^{-\alpha}}{n} + o(\frac{1}{n}))^n \to e^{-x^{-\alpha}} \quad x>0$

After some algebra in the line of what had been done to prove theorem 4.1 we arrive to :

(4.9) $F^n(a_n' x) = \Phi_\alpha(x) + \frac{x^{-\alpha} - x^{-2\alpha}}{2n}\Phi_\alpha(x) + o(\frac{1}{n})$

Related results appear in Anderson (1971) .

Comparing expressions (4.3) and (4.9) we see that the overall rate of convergence is still of order (1/n). As in paragraph 2 we shall prove that the constants $a_n = n^{1/\alpha}$ and $b_n = 0$ are essentially ptimal. In fact tables 1 and 2 illustrate what has been said. Let,

$F(x) = 1 - 1/x \quad x \geq 1, \alpha = 1$

$a_n = n \;,\; b_n = 0 \;,\; a_n' = (1 - e^{-1/n})^{-1}$ and $b_n' = 0$

and take x = 0.5 (table 1) and x = 2 (table 2) .

TABLE 1

n	$F^n(n.(0.5)) = (1-2/n)^n$ $e^{-2}(1-2/n)$	$F^n(a_n'.(0.5)) = (1-2(1-e^{-1/n}))^n$ $e^{-2}(1-1/n)$
10	0.107374 0.108268	0.121059 0.121802
10^2	0.132619 0.132628	0.133975 0.133982
10^3	0.1350645 0.1350646	0.1351998 0.1351999

TABLE 2

n	$F^n(n.(2)) = (1-1/2n)^n$ $e^{-1/2}(1-1/8n)$	$F^n(2.a_n') = (1-1/2(1-e^{-1/n}))^n$ $e^{-1/2}(1+1/8n)$
10	0.598736 0.598949	0.614156 0.614112
10^2	0.605770 0.605772	0.607289 0.607288
10^3	0.60645482 0.60645484	0.60660648 0.60660647

Theorem 4.2 : Let $F(x)$ be a Pareto d.f. and $a'_n > 0$, $b'_n \in R$ normalizing constants such that,

$$(4.10)\qquad \begin{aligned} A_n &= a'_n / a_n \to 1 \\ B_n &= (b'_n - b_n) / a_n \to 0 \end{aligned}$$

where $a_n = n^{1/\alpha}$, $b_n = 0$. Then,

$$(4.11)\qquad F^n(a'_n x + b'_n) = \Phi_\alpha(x) + (x(A_n-1) + B_n)\Phi'_\alpha(x) -$$

$$- \Phi_\alpha(x + x(A_n-1) + B_n).(x + x(A_n-1) + B_n)^{-2\alpha} / 2n +$$

$$+ o\,(x(A_n-1) + B_n) + o\,(1/n)$$

Proof: According to (4.10)

$$(4.12)\qquad F^n(a'_n x + b'_n) = F^n(n^{1/\alpha}(A_n x + B_n)) \to \Phi_\alpha(x) \quad , \quad x \in C_{\Phi_\alpha}$$

for $x>0$ and $n \to +\infty$.

Besides,

$$(4.13)\qquad 1-F(n^{1/\alpha}(A_n x + B_n)) = (A_n x + B_n)^{-\alpha} / n$$

and expanding the logarithm on power series we have

$$(4.14)\qquad -n.\log F(n^{1/\alpha}(A_n x + B_n)) = n.\ \{(1-F(n^{1/\alpha}(A_n x + B_n))) +$$

$$+ o\,(1-F(n^{1/\alpha}(A_n x + B_n)))\}$$

$$(4.15)\qquad -n.\log F(n^{1/\alpha}(A_n x + B_n)) = (A_n x + B_n)^{-\alpha} + \sum_{j=2}^{\infty} (A_n x + B_n)^{-\alpha j} / (j\, n^{j-1})$$

$$= (A_n x + B_n)^{-\alpha} + (A_n x + B_n)^{-2\alpha} / 2n + o\,(1/n)$$

$$(4.16)\qquad F^n(n^{1/\alpha}(A_n x + B_n)) = e^{-(A_n x + B_n)^{-\alpha}} .\ \exp\{- \frac{(A_n x + B_n)^{-2\alpha}}{2n} + o\,(1/n)\}$$

$$= e^{-(A_n x + B_n)^{-\alpha}} \{ 1- \frac{(A_n x + B_n)^{-2\alpha}}{2n} + o\,(1/n)\}$$

Taking $A_n x + B_n = x + x(A_n - 1) + B_n$ and expanding the exponencial on power series at the neighborhood of point x we get

$$(4.17) \quad F^n(n^{1/\alpha}(A_n x + B_n)) = e^{-x^{-\alpha}} + (x(A_n-1) + B_n)\ \phi'_\alpha (x) -$$

$$- \phi_\alpha(x + x(A_n-1) + B_n).(x + x(A_n-1) + B_n)^{-2\alpha} / 2n +$$

$$+ o\ (x(A_n-1) + B_n) + o\ (1/n)$$

Finally,

$$(4.18) \quad F^n(a'_n x + b'_n) = \Phi_\alpha(x) + (x(A_n-1) + B_n)\ \Phi'_\alpha (x) -$$

$$- \Phi_\alpha(x + x(A_n-1) + B_n).(x + x(A_n-1) + B_n)^{-2\alpha} / 2n +$$

$$+ o\ (x(A_n-1) + B_n) + o\ (1/n)$$

As in paragraph 3 we conclude that the overall convergence is still of order (1/n) or slower than (1/n), according as $(x(A_n-1) + B_n)$ converges faster or slower than (1/n) .

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