

ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Mathématiques

DIMITRII E. PAL'CHUNOV

ALAIN TOURAILLE

On some connections between Boolean algebras and Heyting algebras

Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 98, série *Mathématiques*, n° 28 (1992), p. 5-16

http://www.numdam.org/item?id=ASCFM_1992__98_28_5_0

© Université de Clermont-Ferrand 2, 1992, tous droits réservés.

L'accès aux archives de la revue « Annales scientifiques de l'Université de Clermont-Ferrand 2 » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

On some connections between Boolean algebras and Heyting algebras.

par

Dimitrii. E. Pal'chunov and Alain Touraille

Abstract.

We present a finitely axiomatizable class of Heyting algebras (with identity $(-x) + (--x) = 1$) such that this class and the class of Boolean algebras with n distinguished ideals are mutually first order definable.

As a corollary some results on countably categoricity, finitely axiomatizability, decidability, prime and countably saturated models for Heyting algebras are obtained.

I - Constructions of Boolean algebras with particular subsets from Heyting algebras, and converse problems of representation.

For each element x of an Heyting algebra $H = (H, +, \cdot, \rightarrow, 0, 1)$, the element $x \rightarrow 0$ is denoted by $-x$; x is called dense if $-x = 0$, and regular if $--x = x$.

It is well-known (see for example [14]) that one obtains a Boolean algebra $A(H)$ by endowing the set of all regular elements of H with the constants 0 and 1, and with the operations $+^*$, \cdot , $-$, where $+^*$ is defined by $x +^* y = --(x + y)$. Similarly the set $\nabla(H)$ of all dense elements of H is an implicative lattice and a filter in H .

Notation.

Let H be an Heyting algebra. For any $a \in A(H)$ and $b \in \nabla(H)$ put :

$$\begin{aligned} \nabla_a &= \{c \in \nabla(H) : a \leq c\} \quad \text{and} \\ P_b &= \{c \in A(H) : c \leq b\}. \end{aligned}$$

It is shown in [7] that each ∇_a is a filter in H , that each P_b is decreasing (i.e. $x \leq y$ and $y \in P_b$ imply $x \in P_b$), and that H satisfies the identity $(-x) + (--x) = 1$ if and only if P_b is an ideal of $A(H)$ for each $b \in \nabla(H)$ (it is well - known that this identity is equivalent to the identity $(-x) + (-y) = --((-x) + (-y))$).

Now let A be a Boolean algebra and $\{\nabla_a\}_{a \in A}$ be a set of filters in an implicative lattice ∇ . Put $L = (\{(a,b) : a \in A, b \in \nabla\} / \sim ; \leq / \sim)$ where $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ and $b_1 \rightarrow b_2 \in \nabla_{a_1}$; $(a_1, b_1) \sim (a_2, b_2)$ if $(a_1, b_1) \leq (a_2, b_2)$ and $(a_2, b_2) \leq (a_1, b_1)$.

It is proved in [7] that L is a lower semilattice.

The following question may then be interesting : when L is an Heyting algebra ?

A necessary condition for A and $\{\nabla_a\}_{a \in A}$ to generate an Heyting algebra L was presented in the theorem 1 of [7].

However, S.I. Mardaev have given an example showing that this condition is not sufficient.

So, the theorem 1 of [7] saying that the presented condition is necessary and sufficient, and its corollary 2, turned out to be wrong.

Call P-algebra each Boolean algebra with distinguished decreasing subsets. As a consequence of the previous situation the following lemma, based on this corollary 2, has lost a proof :

Lemma [7].

Let us consider a finite implicative lattice $\nabla = (\{b_1, \dots, b_n\}, \leq)$ and a P-algebra $(A, P_{b_1}, \dots, P_{b_n})$. There exists an Heyting algebra H such that $\nabla = \nabla(H)$ and $(A, P_{b_1}, \dots, P_{b_n}) = (A(H), P_{b_1}, \dots, P_{b_n})$ if and only if :

$$(*) \quad P_c \cap P_d = P_{c \cdot d} \quad \text{and} \quad 1 \in P_e \Leftrightarrow e = 1 \quad \text{for any } c, d, e \in \nabla.$$

It is obvious that the condition (*) is necessary for existing such an Heyting algebra H. So it is natural to consider the following problems :

Problem 1. Do every implicative lattice ∇ and P-algebra $(A, P_b)_{b \in \nabla}$ verifying (*) can be represented as $\nabla = \nabla(H)$ and $(A, P_b)_{b \in \nabla} = (A(H), P_b)_{b \in \nabla}$ for some Heyting algebra H ?

A P-algebra $(A, P_j)_{j \in J}$ will be called I-algebra if every P_j is an ideal.

Problem 1_i is the problem 1 for I-algebra $(A, P_b)_{b \in \nabla}$; problem 1_f is the problem 1 for P-algebra $(A, P_b)_{b \in \nabla}$ with finite ∇ , and problem 1_{if} is the problem 1 for I-algebra $(A, P_b)_{b \in \nabla}$ with finite ∇ .

Problem 2. Does every I-algebra (A, I_1, \dots, I_n) can be represented by $A = A(H)$ and $I_j = P_{b_j}$ for some Heyting algebra H with finite set $\nabla(H)$ and $b_1, \dots, b_n \in \nabla(H)$?

Problem 3. Does every I-algebra (A, I_1, \dots, I_n) can be represented as $A = A(H)$ and $I_j = P_{b_j}$ for some Heyting algebra H and $b_1, \dots, b_n \in \nabla(H)$?

Problem 4. Does every I-algebra (A, I_1, \dots, I_n) can be represented as $A = A(H)$ and $I_j = P_{b_j}$ for some Heyting algebra H and some definable $b_1, \dots, b_n \in \nabla(H)$ (it means that there exist formulas $\phi_1(x), \dots, \phi_n(x)$ of the Heyting algebra first order language such that $\{c \in H : H \models \phi_i(c)\} = \{b_i\}$) ?

Problems 2', 3' and 4' are problems 2, 3 and 4 for Heyting algebra H satisfying the identity $(-x) + (--x) = 1$.

Interest to these problems is connected with the following result :

Theorem [7].

Let H and H' be Heyting algebras.

Then $H \cong H'$ if and only if there exists an isomorphism $\phi : \nabla(H) \rightarrow \nabla(H')$ such that $(A(H), P_b)_{b \in \nabla(H)} \cong (A(H'), P_{\phi(b)})_{b \in \nabla(H)}$

A special interest to problems 1, 1f and 2 is connected with the following facts :

Theorem [7].

Let H and H' be Heyting algebras and $\nabla(H)$ be finite. Then $H \equiv H'$ if and only if there exists an isomorphism

$$\phi: \nabla(H) \rightarrow \nabla(H')$$

such that $(A(H), P_b)_{b \in \nabla(H)} \equiv (A(H'), P_{\phi(b)})_{b \in \nabla(H)}$.

Corollary [7].

Let H be an Heyting algebra with $\nabla(H)$ finite.

The theory of H is countably categorical (finitely axiomatizable, decidable) if and only if the theory of $(A(H), P_b)_{b \in \nabla(H)}$ is the same.

Interest to problems 1i, 1fi, 2', 3' and 4' is also connected with results on different model theoretical properties of Boolean algebras with distinguished ideals (see all bibliographic references except [14]).

II - Some negative answers

Proposition 1.

Let ∇ be an implicative semilattice, $(A, P_b)_{b \in \nabla}$ be a P -algebra verifying the condition () and such that there exist b_0 with P_{b_0} maximal for inclusion in $\{P_b\}_{b \neq 1}$, and $a \in A$ with $a, -a \notin P_{b_0}$. Then there does not exist an Heyting algebra H with $\nabla = \nabla(H)$ and $(A, P_b)_{b \in \nabla} \cong (A(H), P_{b(H)})_{b \in \nabla}$.*

Proof. Suppose that such an H exists.

Then $a \rightarrow b_0 \geq b_0 \in \nabla(H)$ gives $P_{b_0} \subseteq P_{a \rightarrow b_0}$ and $a \notin P_{b_0}$ gives $P_{a \rightarrow b_0} \neq P_1$; as moreover $-a = a \rightarrow 0 \leq a \rightarrow b_0$ implies $-a \in P_{a \rightarrow b_0}$, we obtain a contradiction with the maximality of P_{b_0} .

Corollary 1 .

Problems 1, 1i, 1f and 1if have a negative solution.

For each element a of a Boolean algebra, let (a) be the principal ideal generated by a .

Proposition 2 .

Let H be an Heyting algebra with $A(H)$ atomless. If there exists $d \in A(H)$ with $d \neq 0$ and $(d) \cap P_f = (0)$ for some $f \in \nabla(H)$, then $\nabla(H)$ is infinite.

Proof .

We can take an infinite sequence of regular elements $d_1 < d_2 < \dots < d$.

Suppose $f + d_i = f + d_{i+1}$ for one i : then

$d_{i+1} \cdot (-d_i) \leq (f + d_i) \cdot (-d_i) \leq f$ gives $d_{i+1} \cdot (-d_i) \in (d) \cap P_f = (0)$, so that $d_{i+1} \leq --d_i = d_i$ which is contradictory.

Thus the $f + d_i$'s form a strictly increasing sequence of elements of $\nabla(H)$.

Corollary 2 .

Problems 2 and 2' have a negative solution.

III - A finitely axiomatizable class of Heyting algebras.

If an Heyting algebra H contains a least dense element α_1 , $\nabla(H)$ is an Heyting algebra for the operations $+, \cdot, \rightarrow$, and the constants α_1 and 1 . We then put $\nabla^1(H) = \nabla(H)$, $\nabla^2(H) = \nabla(\nabla^1(H))$, and continue the process by putting $\nabla^{i+1}(H) = \nabla(\nabla^i(H))$ as long as $\nabla^i(H)$ contains a least dense element α_{i+1} .

Notice that $\alpha_{n+1} = 1$ for a number n if and only if $\nabla^n(H)$ is a Boolean algebra.

Definition .

For a number n different from 0 , let C_n be the class of all Heyting algebras verifying :

1) $(-x) + (--x) = 1$,

2) $\nabla^i(H)$ has a least element α_i for each $i \in \{1, \dots, n\}$, and $\nabla^n(H)$ is a Boolean algebra ,

- 3) For each $x \in [\alpha_i, \alpha_{i+1}]$ there exists $y \in A(H)$ satisfying $x = y \cdot \alpha_{i+1} + \alpha_i$ (with $i \in \{0, \dots, n\}$, putting $\alpha_0 = 0$ and $\alpha_{n+1} = 1$).

Notice that if α_i exists in an Heyting algebra, the condition 3) is satisfied for $i = 0$ by taking $y = \neg\neg x$. It is easy to see that :

Remark .

C_n is finitely axiomatizable.

Definition .

Let $A = (A, I_1, \dots, I_n)$ be a Boolean algebra A with distinguished ideals $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n$.

We put (identifying $(A/I_i) / (I_{i+1}/I_i)$ with A/I_{i+1}) :

$H(A) = \{(a_0, a_1, \dots, a_n) \in A \times (A/I_1) \times \dots \times (A/I_n) : a_0/I_1 \geq a_1$ and $a_i/(I_{i+1}/I_i) \geq a_{i+1} \forall i \in \{1, \dots, n-1\}\}$,

and we endow this set with the pointwise order (i.e. $(a_0, \dots, a_n) \leq (b_0, \dots, b_n)$ if $a_i \leq b_i$ for $i \in \{0, \dots, n\}$).

Proposition 3.

$H(A)$ is an Heyting algebra of the class C_n , with

$A(H(A)) = \{(a_0, \dots, a_n) \in H(A) : a_i = a_0/I_i \forall i \in \{1, \dots, n\}\}$,

and for $i \in \{1, \dots, n\}$:

$\nabla^i(H(A)) = \{(a_0, \dots, a_n) \in H(A) : a_0 = 1, a_1 = 1/I_1, \dots, a_{i-1} = 1/I_{i-1}\}$

(so that $\alpha_i = (1, 1/I_1, \dots, 1/I_{i-1}, 0/I_i, \dots, 0/I_n)$).

Proof. Obviously $H(A)$ is a bounded lattice with $+$ and \cdot computed pointwise, such that for each elements (a_0, \dots, a_n) and (b_0, \dots, b_n) there exists $(c_0, \dots, c_n) = (a_0, \dots, a_n) \rightarrow (b_0, \dots, b_n)$, defined by

$$c_0 = b_0 + (-a_0) \text{ and}$$

$$c_i = ((b_0 + (-a_0))/I_i) \text{ ou au moins } ((b_1 + (-a_1))/(I_i/I_1)) \dots$$

$$(b_{i-1} + (a_{i-1}))/I_i \dots (b_i + (-a_i)) .$$

So $H(A)$ is an Heyting algebra, and for each $(a_0, \dots, a_n) \in H(A)$ we have

$$-(a_0, \dots, a_n) = \left(-a_0, -a_0/I_1, -a_0/I_2, \dots, -a_0/I_n \right), \text{ which shows that } H(A)$$

verifies the identity $(-x) + (--x) = 1$ and gives the expected description of $A(H(A))$ and $\nabla^1(H(A))$.

Supposing now that α_i exists for $i \geq 1$ and takes the value given in the proposition, we have

$$(a_0, \dots, a_n) \rightarrow \alpha_i = (1, 1/I_1, \dots, 1/I_{i-1}, -a_i, -a_i/(I_{i+1}/I_i), \dots, -a_i/(I_n/I_i)),$$

which gives the existence and value of α_{i+1} if $i < n$, and shows that

$\nabla^n(H(A))$ is a Boolean algebra isomorphic to A/I_n if $i = n$.

Finally each element x of $[\alpha_i, \alpha_{i+1}]$ has the form $x = (1, 1/I_1, \dots, 1/I_{i-1}, a_i, 0/I_{i+1}, \dots, 0/I_n)$; taking $a_0 \in A$ with $a_i = a_0/I_i$ and putting

$$y = (a_0, a_0/I_1, \dots, a_0/I_n) \text{ we obtain } y \in A(H(A)) \text{ with } x = y \cdot \alpha_{i+1} + \alpha_i.$$

Definition .

For each Heyting algebra H of the class C_n , put $I_i(H) = A(H) \cap (\alpha_i]$ for $i \in \{1, \dots, n\}$ (where $(\alpha_i] = \{x \in H : x \leq \alpha_i\}$), and put $A(H) = (A(H), I_1(H), \dots, I_n(H))$.

Theorem 1 .

- a) Let $A = (A, I_1, \dots, I_n)$ be a Boolean algebra A with distinguished ideals $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n$. Then $A \simeq A(H(A))$.
- b) Let H be an Heyting algebra of the class C_n . Then $H \simeq H(A(H))$.

Proof .

- a) From proposition 3 we see that we define an isomorphism $f : A \rightarrow A(H(A))$ by putting $f(a) = (a, a/I_1, \dots, a/I_n)$ for each $a \in A$. Moreover for each $i \in \{1, \dots, n\}$:
 $f(a) \in I_i(H(A)) \Leftrightarrow f(a) \leq (1, 1/I_1, \dots, 1/I_{i-1}, 0/I_i, \dots, 0/I_n)$
 $\Leftrightarrow a \in I_i$
- b) If $a \in H$ then for each $i \in \{0, \dots, n\}$ there exists $b_i \in A(H)$ with $a \cdot \alpha_{i+1} + \alpha_i = b_i \cdot \alpha_{i+1} + \alpha_i$. Notice that if b'_i is suitable too, then $b_i/I_i(H) = b'_i/I_i(H)$ (do not write $I_i(H)$ when $i = 0$) : indeed

$b_i \cdot a_{i+1} \leq b'_i \cdot a_{i+1} + a_i$ gives $b_i \cdot (-b'_i) \cdot a_{i+1} \leq a_i$, so that
 $b_i \cdot (-b'_i) \leq a_{i+1} \rightarrow a_i = a_i$ and thus $b_i \cdot (-b'_i) \in I_i(H)$; as
 $1 = b'_i + (-b'_i)$ we see that $b_i / I_i(H) = (b_i \cdot b'_i) / I_i(H)$, and obtain
 $b'_i / I_i(H) = (b_i \cdot b'_i) / I_i(H)$ by a similar argument. This allows us to define
 $\phi: H \rightarrow H(A(H))$ by putting $\phi(a) = (b_0, b_1 / I_1(H), \dots, b_n / I_n(H))$.

Conversely we define $\psi: H(A(H)) \rightarrow H$ by putting $\psi(x) = x_0 \cdot (x_1 + a_1) \dots$
 $(x_n + a_n)$ for each element $x = (x_0, x_1 / I_1(H), \dots, x_n / I_n(H))$ of $H(A(H))$.

Then by some simple verifications we see that ϕ and ψ preserve the orders and
that each of them is inverse of the other, which gives the result.

IV. Positive answers

First notice that from any ideals I_1, \dots, I_n of a Boolean algebra, an increasing
sequence (for inclusion) of $2^n - 1$ ideals can be computed by using Heyting algebras
operations, such that I_1, \dots, I_n are computable from them in a similar way. We can thus
apply the results of the previous section to any Boolean algebra with distinguished ideals
 I_1, \dots, I_n :

Corollary 3.

Problems 3, 3', 4 and 4' have a positive solution.

Corollary 4.

For any Heyting algebras H, H' of the class C_n :

- a) $H \cong H'$ if and only if $A(H) \cong A(H')$,
- b) $H \equiv H'$ if and only if $A(H) \equiv A(H')$,
- c) *The theory of H is countably categorical (finitely axiomatizable, decidable) if and only if the theory of $A(H)$ is the same,*
- d) *The theory of H has a prime (countably saturatel) model if and only if the theory of $A(H)$ has a same model,*

e) H is prime (countably saturated) if and only if $A(H)$ is the same.

The decidability of the theory of Boolean algebras with a sequence of distinguished ideals is due to Rabin [13]; a description of countably categorical Boolean algebras with a finite number of distinguished ideals is due to Macintyre and Rosenstein [2] (see also [6] and [17]); descriptions of finitely axiomatizable such algebras are given in [8] and [17]; numbers of theories of (A, I) 's according to fixed A 's, or decidability or undecidability of theories of all (A, I) 's for fixed A 's, are given in [1] and [4]. Descriptions of elementary equivalence of Boolean algebras with distinguished ideals are presented in [3], [6], [8], [15] and [16]; a complete description of decidability of elementary theories of Boolean algebras with distinguished ideals is obtained in [8]; a classification of complete theories using an axiomatization of structures of definable ideals is given in [17]; prime models were studied in [5], [9], [12] and [17], countably saturated in [5] and [12].

In particular these results imply :

Corollary 5.

- 1) *The theory of the class C_n is decidable.*
- 2) *For any non-superatomic Boolean algebra A there exists a continuum of Heyting algebras H with $A \cong A(H)$, satisfying $(-x) + (--x) = 1$, and having different theories which :*
 - a) *have not a prime model,*
 - b) *have a prime model but no countably saturated model,*
 - c) *have a countably saturated model.*

The results of this paper were obtained in a collaboration of the authors during July 1991, when the first author was invited by the University Blaise Pascal of Clermont-Ferrand. Grateful acknowledgements are due to Professors Guillaume, Maksimova, Mardoev and Tomasik for interesting discussions.

R E F E R E N C E S

- [1] P.F. JURIE and A. TOURAILLE :
Idéaux élémentairement équivalents dans une algèbre booléenne,
Comptes-Rendus des Séances de l'Académie des Sciences. Série I : Mathématique,
vol 299 (1984) pp. 415-418.
- [2] A. MACINTYRE and J.G. ROSENSTEIN,
 \aleph_0 -categoricity for rings without nilpotent elements and for Boolean structures ,
Journal of Algebra, vol. 43 (1976) pp. 129-154.
- [3] B. MOLZAN,
On the theory of Boolean algebras with Ramsey quantifiers,
Proceedings of the third Easter conference on model theory (Gros-Köris, 1985) ,
Seminarberichte N₀ 70,
Sektion Mathematik, Humboldt-Universität, Berlin, 1985, pp. 186-192.
- [4] D.E. PAL'CHUNOV,
On undecidability of theories of Boolean algebras with distinguished ideals,
Algebra and Logic, vol. 25 (1986) , n°3 , pp. 326-346.
- [5] D.E. PAL'CHUNOV,
On prime and countably saturated Boolean algebras with distinguished ideals,
Proceedings of the 8th Soviet Union Mathematical Logic Conference, Moskow, 1986,
p. 147.
- [6] D.E. PAL'CHUNOV,
Countably categorical Boolean algebras with distinguished ideals,
Studia Logica, vol. XLVI (1987) , n° 2, pp. 121-135.
- [7] D.E. PAL'CHUNOV,
On Heyting algebras with finite number of dense elements,
Computable invariants in the algebraic system theory, Novosibirsk, 1987, pp. 35-45.

- [8] D.E. PAL'CHUNOV,
Finitely axiomatizable Boolean algebras with distinguished ideals,
 Algebra and Logic, vol. 26 (1987) , n° 4 , pp. 435-455.
- [9] D.E. PAL'CHUNOV,
On the prime models of the theory of Boolean algebras with distinguished ideals,
 Proceedings of the Summer School and Conference "Kleene-90 ", Sofia, 1990 , p. 59.
- [10] D.E. PAL'CHUNOV,
Unlocal Boolean algebras with distinguished ideals,
 Proceedings of the Soviet and France Colloquium on Model Theory , Karaganda,
 1990, pp. 35-36.
- [11] D.E. PAL'CHUNOV,
Direct Summands of Boolean algebras with distinguished ideals,
 Proceedings of the Logic Colloquium' 90 , Helsinki, 1990, p. 60.
- [12] D.E. PAL'CHUNOV,
Prime and countably saturated Boolean algebras,
 Proceedings of the Institute of Mathematics, Nauka, Novosibirsk, to appear.
- [13] M.O. RABIN,
Decidability of second-order theories and automata on infinite trees,
 Transactions of the American Mathematical Society, vol. 141 (1969) , pp. 1 - 35.
- [14] H. RASIOWA and R. SIKORSKI,
The Mathematics of Metamathematics,
 Polska Akademia Nauk Monographi Matematyczne 41, Warszawa, 1963.
- [15] A. TOURAILLE,
*Elimination des quantificateurs dans la théorie élémentaire des algèbres de Boole
 munies d'une famille d'idéaux distingués,*
 Comptes-Rendus des Séances de l'Académie des Sciences , Série I : Mathématique,
 vol. 300 (1985) , pp. 125-128.

[16] A. TOURAILLE,

Théories d'algèbres de Boole munies d'idéaux distingués I,
Journal of Symbolic Logic, vol. 52 (1987) , pp. 1027-1043.

[17] A. TOURAILLE ,

Théories d'algèbres de Boole munies d'idéaux distingués II,
Journal of Symbolic Logic, vol. 55 (1990) , pp. 1192-1212.

PAL'CHUNOV Dimitrii.E.
Institute of Mathematics
Universitetsky pr.4
Novosibirsk 90 630090
U.S.S.R.

TOURAILLE Alain
Département de Mathématiques Pures
Université Blaise Pascal
Clermont-Ferrand
FRANCE