ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Mathématiques

MIRCEA SOFONEA

Quasistatic processes for elastic-viscoplastic materials with internal state variables

Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 94, série *Mathématiques*, n° 25 (1989), p. 47-60

<http://www.numdam.org/item?id=ASCFM_1989__94_25_47_0>

Ann. Sci. Blaise Pascal (Clermont II), Sér. Math., Fasc. 25, pp. 47-60

QUASISTATIC PROCESSES FOR ELASTIC-VISCOPLASTIC MATERIALS WITH INTERNAL STATE VARIABLES

MIRCEA SOFONEA
Department of Mathematics, INCREST, Bd. Pacii 220,
79622 Bucharest, ROMANIA

1. Introduction

An initial and boundary value problem for materials with a constitutive equation of the form

$$(1.1)\quad \dot{\varepsilon} = A\dot{\sigma} + G(\sigma,\varepsilon,\chi)$$

is considered, in which ε is the small strain tensor, σ is the stress tensor and χ is an internal state variable (in (1.1) and everywhere in this paper the dot above a quantity represents the derivative with respect to the time variable of that quantity). Such type of equations are used in order to model the behaviour of real bodies like rubbers, metals, rocks and so on, for which the plastic rate of deformation depends also on an internal state variable. There exists a large scope in the choice of the internal state variable, authors having individual preferences varying even from paper to paper (for references in the field see Cristescu and Suliciu [1] , ch. VI). Some of the internal state variables considered by many authors are the plastic strain, a number of tensor variables that take into account the spatial display of dislocations and the work-hardening of the material. A major and still remaining open problem in viscoplasticity concerns the way of establishing the evolution equations for the internal state variables. Here we suppose that χ is a vector-valued function which satisfies the equation

(1.2) $\dot{\chi} = \phi(\sigma, \varepsilon, \chi)$

in which ϕ is a given function.

In the case when G depends only on σ existence and uniqueness results for dynamic or quasistatic problems involving (1.1) for differents form of G were obtained by Duvaut and Lions [2] ch. 5, Suquet [3], [4], [5], Djaoua and Suquet [6].

In the case when G depends only on σ and ε, existence and uniqueness results for models of the form (1.1) were given by Ionescu and Sofonea [7] in the quasistatic case and by Ionescu [8] in the dynamic case.

Initial and boundary value problems for models (1.1), (1.2) for different forms of G and ϕ vere studied by many authors. So, existence and uniqueness results were given by Necas and Krotochivil [9], John [10], Laborde [11] (the case when G does not depends on ε) and by Sofonea [12], [13], Ionescu [8], [14] (the case when G depends also on ε). Energy estimates for one -dimensional problems in the study of models (1.1) in which χ is the work hardening parameter were obtained by Suliciu and Sabac [15].

The aim of this paper is to study a quasistatic problem governed by the constitutive equations (1.1), (1.2). Thus an existence and uniqueness result is proved (theorem 3.1) and the continuous dependence of the solution with respect the input data is also given (theorem 4.1). No monotony properties for the functions G and ϕ are required and no monotony arguments for evolution equations are used. The technique presented here is based on the equivalence between the studied problem and an ordinary differential equation in a product Hilbert space. This technique was used before by Necas and Krotochivil [9], Suquet [5], Ionescu and Sofonea [7]. The results presented here generalized some results of Sofonea [12]. Theorem 3.1 represent the quasistatic version of theorem 4.2 of Ionescu [8].

2. Problem statement and preliminaries

Let $\Omega \subset \mathbb{R}^N$ (N = 1, 2, 3) be a bounded domain with a smooth boundary $\partial\Omega = \Gamma$ and let Γ_1 be an open subset of Γ such that meas $\Gamma_1 > 0$; we denote by $\Gamma_2 = \Gamma \setminus \overline{\Gamma_1}$, ν the outward unit normal vector on Γ and by S_N the set of second order symmetric tensors on $\mathbb{R}^N$. Let T be a real positive constant. We consider the following mixed problem :

(2.1) $\quad \mathrm{Div}\,\sigma + b = 0$

(2.2) $\quad \dot{\varepsilon}(u) = A\dot{\sigma} + G(\sigma, \varepsilon(u), \chi)$ $\qquad$ in $\Omega \times (0,T)$

(2.3) $\quad \dot{\chi} = \phi(\sigma, \varepsilon(u), \chi)$

(2.4) $\quad u = f \quad$ on $\Gamma_1 \times (0,T)$

(2.5) $\quad \sigma\nu = g \quad$ on $\Gamma_2 \times (0,T)$

(2.6) $\quad u(0) = u_0, \quad \sigma(0) = \sigma_0, \ \chi(0) = \chi_0$ $\qquad$ in Ω

in which the unknows are the displacement function $u : \Omega \times [0,T] \to \mathbb{R}^N$, the stress function $\sigma : \Omega \times [0,T] \to S_N$ and the internal state variable $\chi : \Omega \times [0,T] \to \mathbb{R}^M$ $(M \in \mathbb{N})$. This problem represents a quasistatic problem for rate-type models of the form (2.2), (2.3) in which A is a forth order tensor, G and ϕ are given constitutive functions and $\varepsilon(u)$ defines the small strain tensor (i.e. $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u)$) ; in (2.1) Div σ represents the divergence of the vector-valued function σ and b is the given body force ; the functions f and g in (2.4), (2.5) are the given boundary data and finally the function u_0, σ_0, χ_0 in (2.6) ar the initial data.

In the sequel we denote by " . " the inner product on the spaces $\mathbb{R}^N$, $\mathbb{R}^M$, S_N and by $|\cdot|$ the Euclidean norms on these spaces. The following notations are also used :

$$H = [L^2(\Omega)]^N, \quad H_1 = [H^1(\Omega)]^N, \quad \mathcal{H} = [L^2(\Omega)]_s^{N \times N}, \quad \mathcal{H}_1 = \{\sigma \in \mathcal{H} \mid \mathrm{Div}\,\sigma \in H\},$$

$$Y = [L^2(\Omega)]^M .$$

The spaces H, H_1, $\mathcal{H}$, $\mathcal{H}_1$ and Y are real Hilbert spaces endowed with the cannonical inner products denoted by $< , >_H$, $< , >_{H_1}$, $< , >_{\mathcal{H}}$, $< , >_{\mathcal{H}_1}$ and $< , >_Y$ respectively.

Let $H_\Gamma = [H^{1/2}(\Gamma)]^N$ and $\gamma_1 : H_1 \to H_\Gamma$ be the trace map. We denote by V the closed subspace of H_1 defined by $V = \{u \in H_1 \mid \gamma_1 u = 0 \text{ on } \Gamma_1\}$ and let $V_\Gamma = \gamma_1(V)$. We also denote by H'_Γ and V'_Γ the duals of H_Γ and V_Γ. The operator $\varepsilon : H_1 \to \mathcal{H}$ given by $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ is linear and continuous and moreover, since meas $\Gamma_1 > 0$, Korn's inequality holds :

(2.7) $\|\varepsilon(u)\|_{\mathcal{H}} \geq C\|u\|_{H_1}$ for all $u \in V$

where C is a strictely positive constant (everywhere in this paper C will represents strictely positive generic constants that may depend on Ω, Γ_1, A, G, ϕ and T and do not depend on time or on unput data).

If $\sigma \in \mathcal{H}_1$ there exists $\gamma_2 \sigma \in H'_\Gamma$ such that

$$< \gamma_2 \sigma, \gamma_1 u >_{H'_\Gamma, H_\Gamma} = < \sigma, \varepsilon(u) >_{\mathcal{H}} + < \mathrm{Div}\,\sigma, u >_H$$

for all $u \in H_1$. By $\sigma\nu_{|\Gamma_2}$ we shall understand the restriction of $\gamma_2 \sigma$ on V_Γ and we denote by $\mathcal{V}$ the closed subspace of $\mathcal{H}_1$ defined by

$$\mathcal{V} = \{\sigma \in \mathcal{H}_1 \mid \mathrm{Div}\,\sigma = 0 \ , \ \sigma\nu_{|\Gamma_2} = 0\} .$$

Here we consider V and $\mathcal{V}$ as real Hilbert spaces endowed with the inner products of H_1 and $\mathcal{H}_1$ respectively ; it is well known that $\varepsilon(V)$ is the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}$, hence

(2.8) $< \sigma, \varepsilon(u) >_{\mathcal{H}} = 0$ for all $\sigma \in \mathcal{V}$ and $u \in V$.

Finally for every real Hilbert space X we denote by $\|\cdot\|_X$ the norm on X and by $C^j(0,T,X)$ $(j = 0,1)$ the spaces defined as follows :

$$C^0(0,T,X) = \{z : [0,T] \to X \mid z \text{ is continuous}\}$$
$$C^1(0,T,X) = \{z : [0,T] \to X \mid \text{there exists } \dot{z} \text{ the derivative of } z \text{ and } \dot{z} \in C^0(0,T,X)\} .$$

$C^j(0,T,X)$ are real Banach spaces, endowed with the norms

$$\|z\|_{0,T,X} = \max_{t \in [0,T]} \|z(t)\|_X \quad \text{and} \quad \|z\|_{1,T,X} = \|z\|_{0,T,X} + \|\dot{z}\|_{0,T,X}$$

respectively.

3. Existence and uniqueness of the solution

The following hypotheses are considered :

(3.1)
$A : \Omega \times S_N \to S_N$ is a symmetric and positively definite bounded tensor, i.e. :

(a) $A_{ijkh} \in L^\infty(\Omega)$ for all $i,j,k,h = \overline{1,N}$

(b) $A(x)\sigma.\tau = \sigma.A(x)\tau$ for all $\sigma,\tau \in S_N$, a.e. in Ω

(c) there exists $\alpha > 0$ such that $A(x)\,\sigma.\sigma \geq \alpha\,|\sigma|^2$ for all $\sigma \in S_N$, a.e. in Ω.

(3.2)
$G : \Omega \times S_N \times S_N \times \mathbb{R}^M \to S_N$ has the following properties :

(a) there exists $L > 0$ such that $|G(x,\sigma_1,\varepsilon_1,\chi_1) - G(x,\sigma_2,\varepsilon_2,\chi_2)| \leq$ $\leq L(|\sigma_1-\sigma_2| + |\varepsilon_1-\varepsilon_2| + |\chi_1-\chi_2|)$ for all $\sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2, \in S_N$, $\chi_1,\chi_2 \in \mathbb{R}^M$, a.e. in Ω.

(b) $x \to G(x,\sigma,\varepsilon,\chi)$ is a measurable function with respect to the Lebesgue measure on Ω, for all $\sigma,\varepsilon \in S_N$ and $\chi \in \mathbb{R}^M$

(c) $x \to G(x,0,0,0) \in H$.

(3.3)
$\phi : \Omega \times S_N \times S_N \times \mathbb{R}^M \to \mathbb{R}^M$ has the following properties :

(a) there exists $\tilde{L} > 0$ such that $|\phi(x,\sigma_1,\varepsilon_1,\chi_1) - \phi(x,\sigma_2,\varepsilon_2,\chi_2)| \leq$ $\leq \tilde{L}(|\sigma_1-\sigma_2| + |\varepsilon_1-\varepsilon_2| + |\chi_1-\chi_2|)$ for all $\sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2 \in S_N$, $\chi_1,\chi_2 \in \mathbb{R}^M$, a.e. in Ω.

(b) $x \to \phi(x,\sigma,\varepsilon,\chi)$ is a measurable function with respect to the Lebesgue measure on Ω, for all $\sigma,\varepsilon \in S_N$ and $\chi \in \mathbb{R}^M$

(c) $x \to \phi(x,0,0,0) \in Y$.

(3.4) $b \in C^1(0,T,H)$, $f \in C^1(0,T,H_\Gamma)$, $g \in C^1(0,T,V'_\Gamma)$

(3.5) $u_0 \in H_1$, $\sigma_0 \in \mathcal{H}_1$, $\chi_0 \in Y$

(3.6) $\mathrm{Div}\,\sigma_0 + b(0) = 0$ in Ω, $\gamma_1 u_0 = f(0)$ on Γ_1, $\sigma_0 \nu = g(0)$ on Γ_2.

The main result of this section is the following :

Theorem 3.1 :

Suppose that the hypotheses (3.1) - (3.6) *are fulfilled. Then there exists a unique solution* $u \in C^1(0,T,H_1)$, $\sigma \in C^1(0,T,\mathcal{H}_1)$, $\chi \in C^1(0,T,Y)$ of the problem (2.1) - (2.6).

<u>Remark 3.1</u> :

Let us observe that if the problem (2.1) - (2.6) has a solution (u, σ, χ) such that $u \in C^1(0,T,H_1)$, $\sigma \in C^1(0,T,\mathcal{H}_1)$, $\chi \in C^1(0,T,Y)$ then the hypotheses (3.4) - (3.6) are fulfilled.

<u>Remark 3.2</u> :

A relative simple example of constitutive functions G ans ϕ satisfying (3.2) and (3.3) may be obtained by taking $M = 1$ and

(3.7) $$G(\sigma, \varepsilon, \chi) = \frac{1}{2\mu}(\sigma - P_{K(\chi)}\sigma)$$

(3.8) $$\phi(\sigma, \varepsilon, \chi) = \varnothing(|\dot{\varepsilon}^I|)$$

(3.9) $$\dot{\varepsilon}^I = G(\sigma, \varepsilon, \chi)$$

for all $\sigma, \varepsilon \in \mathcal{S}_N$ and $\chi \in \mathbb{R}$ where $\mu > 0$ is a viscosity coefficient, $P_{K(\chi)}$ is the projection map on von Mises plasticity convex $K(\chi) = \{\sigma \in \mathcal{S}_N \mid |\sigma^D| \leq k(\chi)\}$ defined by the Lipschitz yield function $\chi \rightarrow k(\chi)$ and $\varnothing$ is a Lipschitz continuous function.

<u>Remark 3.3</u> :

Formula (3.9) defines the plastic strain rate $\dot{\varepsilon}^I$ for the model (1.1) . The internal state variable defined by (1.2) , (3.8) , (3.9) is called the strain hardening parameter. Concrete examples for viscoplastic models involving a strain harderning parameter were proposed by Cristescu [16] for rock-like materials ; in this paper χ is the irreversible equivalent strain i.e. $\phi(r) = \frac{2}{3} r$ in (3.8) .

In order to prove theorem 3.1 we need some preliminary results. Let $X = V \times \mathcal{V} \times Y$ be the product space endowed with the inner product $<,>_X$ defined by

(3.10) $$<x, y>_X = <A^{-1}\varepsilon(u), \varepsilon(v)>_{\mathcal{H}} + <A\sigma, \tau>_{\mathcal{H}} + <\chi, \eta>_Y$$

for all $x = (u, \sigma, \chi) \in X$ and $y = (v, \tau, \eta) \in X$. Using (3.1) and (2.7) we get that the norm $\|\cdot\|_X$ generated by (3.10) is equivalent with the natural norm on X.

Let $\tilde{u} \in C^1(0,T,H_1)$ and $\tilde{\sigma} \in C^1(0,T,\mathcal{H}_1)$ be the solution of the elastic problem

$$\left.\begin{array}{lll} (3.11) & \mathrm{Div}\,\tilde{\sigma} + b = 0 & \\ & & \text{in } \Omega \times (0,T) \\ (3.12) & \varepsilon(\tilde{u}) = A\tilde{\sigma} & \end{array}\right]$$

$$(3.13) \qquad \tilde{u} = f \qquad\qquad \text{on } \Gamma_1 \times (0,T)$$

$$(3.14) \qquad \tilde{\sigma}\nu = g \qquad\qquad \text{on } \Gamma_2 \times (0,T)$$

(the existence of $\tilde{u}$ and $\tilde{\sigma}$ can be easily proved using (3.4) and standard arguments of linear elasticity) and let $F : [0,T] \times X \to X$ be the operator defined by

$$(3.15) \quad \left[\begin{array}{l} < F(t,x), y >_X \; = \; < A^{-1} G(\sigma + \tilde{\sigma}(t), \varepsilon(u) + \varepsilon(\tilde{u}(t)), \chi), \; \varepsilon(v) >_{\mathcal{H}} \; - \\ \quad - \; < G(\sigma + \tilde{\sigma}(t), \varepsilon(u) + \varepsilon(\tilde{u}(t)), \chi), \tau >_{\mathcal{H}} \quad + \\ \quad + \; < \phi(\sigma + \tilde{\sigma}(t), \varepsilon(u) + \varepsilon(\tilde{u}(t)), \chi), \eta >_Y \end{array}\right.$$

for all $\quad x = (u, \sigma, \chi) \in X, \quad y = (\upsilon, \tau, \eta) \in X \quad$ and $\quad t \in [0,T]$.

Lemma 3.1

The operator F given by (3.15) is continuous and there exists $C > 0$ such that

$$(3.16) \qquad \| F(t,x_1) - F(t,x_2) \|_X \leq C \, \| x_1 - x_2 \|_X$$

for all $\quad x_1, x_2 \in X \quad$ and $\quad t \in [0,T]$.

Proof : Let $t_1, t_2 \in [0,T]$ and $x_1 = (u_1, \sigma_1, \chi_1)$, $\quad x_2 = (u_2, \sigma_2, \chi_2) \in X$.
For every $\quad y = (\upsilon, \tau, \eta) \in X$ we have

$$\begin{aligned} | < F(t_1, x_1) - F(t_2, x_2), y >_X | \; \leq & \; | < A^{-1} G(\sigma_1 + \tilde{\sigma}(t_1), \varepsilon(u_1) + \varepsilon(\tilde{u}(t_1)), \chi_1) \; - \\ & - A^{-1} G(\sigma_2 + \tilde{\sigma}(t_2), \varepsilon(u_2) + \varepsilon(\tilde{u}(t_2)), \chi_2), \; \varepsilon(\upsilon) >_{\mathcal{H}} | \quad + \\ + \; & | < G(\sigma_1 + \tilde{\sigma}(t_1), \varepsilon(u_1) + \varepsilon(\tilde{u}(t_1)), \chi_1) - G(\sigma_2 + \tilde{\sigma}(t_2), \varepsilon(u_2) + \varepsilon(\tilde{u}(t_2)), \chi_2), \tau >_{\mathcal{H}} | \; + \\ + \; & | < \phi(\sigma_1 + \tilde{\sigma}(t_1), \varepsilon(u_1) + \varepsilon(\tilde{u}(t_1)), \chi_1) - \phi(\sigma_2 + \tilde{\sigma}(t_2), \varepsilon(u_2) + \varepsilon(\tilde{u}(t_2)), \chi_2), \eta >_Y | \end{aligned}$$

and using (3.1) - (3.3) we get

$$\begin{aligned} | < F(t_1, x_1) - F(t_2, x_2), y >_X | \leq & \; C (\| x_1 - x_2 \|_X + \| \tilde{\sigma}(t_1) - \tilde{\sigma}(t_2) \|_{\mathcal{H}} \; + \\ & + \; \| \tilde{u}(t_1) - \tilde{u}(t_2) \|_{H_1}) \, \| y \|_X \end{aligned}$$

which implies

$$\text{(3.17)} \qquad \| F(t_1,x_1) - F(t_2,x_2) \|_X \leq C (\|x_1 - x_2\|_X + \|\tilde{\sigma}(t_1) - \tilde{\sigma}(t_2)\|_{H} + \|\tilde{u}(t_1) - \tilde{u}(t_2)\|_{H_1}) \quad .$$

Hence $F : [0,T] \times X \to X$ is a continuous operator and taking $t_1 = t_2 = t$ in (3.17) we get (3.16).

Let now denote by

$$\text{(3.18)} \qquad \bar{u} = u - \tilde{u}, \qquad \bar{\sigma} = \sigma - \tilde{\sigma} , \qquad x = (\bar{u}, \bar{\sigma}, \chi)$$

$$\text{(3.19)} \qquad \bar{u}_0 = u_0 - \tilde{u}(0) , \qquad \bar{\sigma}_0 = \sigma_0 - \tilde{\sigma}(0), \qquad x_0 = (\bar{u}_0, \bar{\sigma}_0, \chi_0) .$$

Lemma 3.2

The triplet $(u, \sigma, x) \in C^1(0,T,H_1 \times \mathcal{H}_1 \times Y)$ is a solution of (2.1) - (2.6) iff $x \in C^1(0,T,X)$ is a solution of the following Caudhy problem

$$\text{(3.20)} \qquad \dot{x}(t) = F(t, x(t)) \quad \text{for all} \quad t \in [0,T]$$

$$\text{(3.21)} \qquad x(0) = x_0 .$$

Proof : Using (3.11) - (3.14) , (3.18) , (3.19) it is easy to see that $(u,\sigma,\chi) \in C^1(0,T.H_1 \times \mathcal{H}_1 \times Y)$ is a solution of (2.1) - (2.6) iff $x = (\bar{u}, \bar{\sigma}, \chi) \in C^1(0,T,X)$ and

$$\left.\begin{aligned} &\text{(3.22)} \qquad \dot{\varepsilon}(\bar{u}) = A\dot{\bar{\sigma}} + G(\bar{\sigma} + \tilde{\sigma} , \varepsilon(\bar{u}) + \varepsilon(\tilde{u}), \chi) \\ &\text{(3.23)} \qquad \dot{x} = \phi(\bar{\sigma} + \tilde{\sigma} , \varepsilon(\bar{u}) + \varepsilon(\tilde{u}), \chi) \end{aligned}\right] \quad \text{in } \Omega \times (0,T)$$

$$\text{(3.24)} \qquad \bar{u}(0) = \bar{u}_0 , \quad \bar{\sigma}(0) = \bar{\sigma}_0 , \quad \chi(0) = \chi_0 \qquad \text{in } \Omega.$$

Let us suppose that (3.22) and (3.23) are fulfilled. Using (2.8) we have

$$< A^{-1}\dot{\varepsilon}(\bar{u}), \varepsilon(v) >_H \; = \; < A^{-1}G(\bar{\sigma} + \tilde{\sigma} , \varepsilon(\bar{u}) + \varepsilon(\tilde{u}), \chi) , \varepsilon(v) >_H$$

$$< A\dot{\bar{\sigma}}, \tau >_H \; = \; - < G(\bar{\sigma} + \tilde{\sigma}, \varepsilon(\bar{u}) + \varepsilon(\tilde{u}), \chi), \tau >_H$$

$$< \dot{\chi}, \eta >_Y \; = \; < \phi(\bar{\sigma} + \tilde{\sigma} , \varepsilon(\bar{u}) + \varepsilon(\tilde{u}), \chi), \eta >_Y$$

for all $y = (v, \tau, \eta) \in X$ and $t \in [0,T]$. Using now (3.10) and (3.15) we get

$$\text{(3.25)} \qquad < \dot{x}(t) , y >_X \; = \; < F(t, x(t)), y >_X$$

for all $y \in X$ and $t \in [0,T]$ which implies (3.20).

Conversely, let (3.20) hold and let

$$z(t) = \dot{\varepsilon}(\bar{u}(t)) - A\dot{\bar{\sigma}}(t) - G(\bar{\sigma}(t) + \tilde{\sigma}(t), \varepsilon(\bar{u}(t)) + \varepsilon(\tilde{u}(t)), \chi(t)) \tag{3.26}$$

for all $t \in [0,T]$. Taking $y = (\upsilon,0,0) \in X$ in (3.25) and using (2.8) we get

$$< A^{-1} z(t), \varepsilon(\upsilon) >_H \; = 0 \quad \text{for all} \quad \upsilon \in V \text{ and } t \in [0,T]. \tag{3.27}$$

Taking $y = (0,\tau,0) \in X$ in (3.25) and using again (2.8) we get

$$< z(t), \tau >_H \; = 0 \quad \text{for all} \quad \tau \in V \quad \text{and} \quad t \in [0,T]. \tag{3.28}$$

This implies $z(t) \in \varepsilon(V)$ for all $t \in [0,T]$ and taking $\varepsilon(\upsilon) = z(t)$ in (3.27) after using (3.1) we get $z(t) = 0$ for all $t \in [0,T]$ hence by (3.26) we get (3.22). In the same way (3.23) follows by (3.25) taking $y = (0,0,\eta) \in X$ where η is an arbitrary element of Y.

Hence we proved that (3.22), (3.23) are equivalent to (3.20) and we finish the proof with the remark that (3.24) is also equivalent to (3.21).

Proof of theorem 3.1

Using (3.5) and (3.6) we get $x_0 \in X$ and by lemma 3.1 and the classical Cauchy-Lipschitz existence theorem we get that (3.20), (3.21) has a unique solution $x \in C^1(0,T,X)$. Theorem 3.1 follows now from lemma 3.2.

4. The continuous dependence of the solution upon the input data

In this section two solutions of the problem (2.1) - (2.6) for two different input data are considered. An estimation of the difference of these solutions is given for finite time intervals that give the continuous dependence of the solution upon all input data (theorem 4.1). In this way, the finite-time stability of the solution is obtained (corollary 4.1).

Theorem 4.1

Let (3.1) - (3.3) *hold and let* (u_i, σ_i, χ_i) *be the solution of* (2.1) - (2.6) *for the data* b_i, f_i, g_i, u_{0i}, σ_{0i}, χ_{0i}, $i = 1,2$, *such that* (3.4) - (3.6) *hold. Then there exists* $C > 0$ *such that*

$$\|u_1 - u_2\|_{j,T,H_1} + \|\sigma_1 - \sigma_2\|_{j,T,\mathcal{H}_1} + \|\chi_1 - \chi_2\|_{j,T,Y} \le$$

(4.1) $$\le C\,(\|u_{01} - u_{02}\|_{H_1} + \|\sigma_{01} - \sigma_{02}\|_{\mathcal{H}_1} + \|\chi_{01} - \chi_{02}\|_Y +$$

$$+ \|b_1 - b_2\|_{j,T,H} + \|f_1 - f_2\|_{j,T,H_\Gamma} + \|g_1 - g_2\|_{j,T,V'_\Gamma})\,, \quad j = 0,1.$$

Proof : Let $(\tilde{u}_i, \tilde{\sigma}_i)$ be the solution of the elastic problem (3.11) - (3.14) for the data b_i, f_i, g_i, $i = 1,2$ and

(4.2) $$\bar{u}_i = u_i - \tilde{u}_i\,,\ \bar{\sigma}_i = \sigma_i - \tilde{\sigma}_i\,,\ x_i = (\bar{u}_i, \bar{\sigma}_i, \chi_i)$$

(4.3) $$\bar{u}_{0i} = u_{0i} - \tilde{u}_i(0)\,,\ \bar{\sigma}_{0i} = \sigma_{0i} - \tilde{\sigma}_i(0)\,,\ x_{0i} = (\bar{u}_{0i}, \bar{\sigma}_{0i}, \chi_{0i})\,, \qquad i = 1,2.$$

Using lemma 3.2 we have

(4.4) $$\dot{x}_i(t) = F_i(t, x_i(t)) \quad \text{for all} \quad t \in [0,T]$$

(4.5) $$x_i(0) = x_{0i}$$

where the operators F_i are defined by (3.15) replacing $(\tilde{u}, \tilde{\sigma})$ by $(\tilde{u}_i, \tilde{\sigma}_i)$, $i = 1,2$. Using (3.15) and (3.1) - (3.3) we get

(4.6) $$\|F_1(t, y_1) - F_2(t, y_2)\|_X \le C\,(\|y_1 - y_2\|_X + \|\tilde{\sigma}_1(t) - \tilde{\sigma}_2(t)\|_{\mathcal{H}} + \|\tilde{u}_1(t) - \tilde{u}_2(t)\|_{H_1}$$

for all $y_1, y_2 \in X$ and $t \in [0,T]$. Hence, by (4.4) we get

$$< \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t) >_X \;=\; < F_1(t, x_1(t)) - F_2(t, x_2(t)), x_1(t) - x_2(t) >_X \le$$
$$\le C\,(\|x_1(t) - x_2(t)\|_X + \|\tilde{\sigma}_1(t) - \tilde{\sigma}_2(t)\|_{\mathcal{H}} + \|\tilde{u}_1(t) - \tilde{u}_2(t)\|_{H_1})\,\|x_1(t) - x_2(t)\|_X$$

for all $t \in [0,T]$ and using (4.5) it results

$$\frac{1}{2}\|x_1(s) - x_2(s)\|^2 \le \frac{1}{2}\|x_{01} - x_{02}\|_X^2 + C\int_0^s (\|\tilde{\sigma}_1(t) - \tilde{\sigma}_2(t)\|_{\mathcal{H}} + \|\tilde{u}_1(t) - \tilde{u}_2(t)\|_{H_1})\,.$$
$$.\ (\|x_1(t) - x_2(t)\|_X)\,dt + C\int_0^s \|x_1(t) - x_2(t)\|_X^2\,dt$$

for all $s \in [0,T]$. Using a Gronwall-type lemma it follows

$$\|x_1(s) - x_2(s)\|_X \le C\,(\|x_{01} - x_{02}\|_X + \int_0^s (\|\tilde{\sigma}_1(t) - \tilde{\sigma}_2(t)\|_{\mathcal{H}} + \|\tilde{u}_1(t) - \tilde{u}_2(t)\|_{H_1})\,dt$$

for all $s \in [0,T]$ hence

$$\| x_1 - x_2 \|_{0,T,X} \le C (\| x_{01} - x_{02} \|_X + \| \tilde{\sigma}_1 - \tilde{\sigma}_2 \|_{0,T,\mathcal{H}} + \| \tilde{u}_1 - \tilde{u}_2 \|_{0,T,H_1}) \tag{4.7}$$

Using again (4.4) and (4.6) we have

$$\| \dot{x}_1(t) - \dot{x}_2(t) \|_X \le C (\| x_1(t) - x_2(t) \|_X + \| \tilde{\sigma}_1(t) - \tilde{\sigma}_2(t) \|_X + \| \tilde{u}_1(t) - \tilde{u}_2(t) \|_{H_1}) \quad \text{for all} \quad t \in [0,T]$$

and by (4.7) we get

$$\| \dot{x}_1 - \dot{x}_2 \|_{0,T,X} \le C (\| x_{01} - x_{02} \|_X + \| \tilde{\sigma}_1 - \tilde{\sigma}_2 \|_{0,T,\mathcal{H}} + \| \tilde{u}_1 - \tilde{u}_2 \|_{0,T,H_1}) . \tag{4.8}$$

Taking into account (4.2) and (4.3) inequalities (4.7) and (4.8) imply

$$\begin{aligned} & \| u_1 - u_2 \|_{0,T,H_1} + \| \sigma_1 - \sigma_2 \|_{0,T,\mathcal{H}_1} + \| \chi_1 - \chi_2 \|_{0,T,Y} \le \\ & \le C (\| u_{01} - u_{02} \|_{H_1} + \| \sigma_{01} - \sigma_{02} \|_{\mathcal{H}_1} + \| \chi_{01} - \chi_{02} \|_Y + \\ & + \| \tilde{\sigma}_1 - \tilde{\sigma}_2 \|_{0,T,\mathcal{H}} + \| \tilde{u}_1 - \tilde{u}_2 \|_{0,T,H_1}) \end{aligned} \tag{4.9}$$

$$\begin{aligned} & \| \dot{u}_1 - \dot{u}_2 \|_{0,T,H_1} + \| \dot{\sigma}_1 - \dot{\sigma}_2 \|_{0,T,\mathcal{H}_1} + \| \dot{\chi}_1 - \dot{\chi}_2 \|_{0,T,Y} \le \\ & \le C (\| u_{01} - u_{02} \|_{H_1} + \| \sigma_{01} - \sigma_{02} \|_{\mathcal{H}_1} + \| \chi_{01} - \chi_{02} \|_Y + \\ & + \| \tilde{\sigma}_1 - \tilde{\sigma}_2 \|_{1,T,\mathcal{H}} + \| \tilde{u}_1 - \tilde{u}_2 \|_{1,T,H_1}) . \end{aligned} \tag{4.10}$$

Using standard arguments from (3.11) - (3.14) we get

$$\| \tilde{u}_1 - \tilde{u}_2 \|_{j,T,H_1} + \| \tilde{\sigma}_1 - \tilde{\sigma}_2 \|_{j,T,\mathcal{H}} \le C (\| b_1 - b_2 \|_{j,T,H} + \| f_1 - f_2 \|_{j,T,H_\Gamma} + \| g_1 - g_2 \|_{j,T,V'_\Gamma}$$

$j = 0,1$ hence from (4.9) and (4.10) we deduce (4.1).

In particular, from theorem 4.1 we deduce

Corollary 4.1

Let the hypothesis of theorem 4.1 hold. If $b_1 = b_2$, $f_1 = f_2$, $g_1 = g_2$ then

$$\begin{aligned} & \| u_1 - u_2 \|_{j,T,H_1} + \| \sigma_1 - \sigma_2 \|_{j,T,\mathcal{H}_1} + \| \chi_1 - \chi_2 \|_{j,T,Y} \le \\ & \le C (\| u_{01} - u_{02} \|_{H_1} + \| \sigma_{01} - \sigma_{02} \|_{\mathcal{H}_1} + \| \chi_{01} - \chi_{02} \|_Y) , \qquad j = 0,1. \end{aligned} \tag{4.11}$$

In order to avoid misunderstanding we recall some definitions of stability theory following Hahn [17], ch. 5. A solution (u, σ, χ) of the problem (2.1) - (2.6) will be called :

i) stable if there exists $m : \mathbf{R}_+ \to \mathbf{R}_+$ a continuous function with $m(0) = 0$ such that

$$\|u(t) - u_1(t)\|_{H_1} + \|\sigma(t) - \sigma_1(t)\|_{\mathcal{H}_1} + \|\chi(t) - \chi_1(t)\|_Y \leq$$
$$\leq m(\|u_0 - u_{01}\|_{H_1} + \|\sigma_0 - \sigma_{01}\|_{\mathcal{H}_1} + \|\chi_0 - \chi_{01}\|_Y)$$

for all $t > 0$ and $u_{01}, \sigma_{01}, \chi_{01}$, satisfying (3.5), (3.6) where (u_1, σ_1, χ_1) is the solution of (2.1) - (2.6) for the initial data $u_{01}, \sigma_{01}, \chi_{01}$;

ii) finite time stable if (4.12) holds for all finite time intervals.

Remark 4.1

From (4.11) we deduce the finite time stability of every solution of (2.1) - (2.6) . Some unidimensional examples can be considered in order to prove that generally stability does not hold.

REFERENCES

[1] N. Cristescu and I. Suliciu, *Viscoplasticity*, Martinus Nijhoff, The Netherlands, Ed. Tehnica, Bucharest, 1982.

[2] G. Duvaut and J.L. Lions, *Les inéquations en mécanique et en physique,* Dunod, Paris, 1972.

[3] P. Suquet, *Sur les équations de la plasticité ; existence et régularité des solutions,* J. Méc. Téor. Appl. 20, 3-39 (1981).

[4] P. Suquet, *Evolution problems for a class of dissipative materials,* Quart. Appl. Math. 38, 391-414 (1981).

[5] P. Suquet, *Plasticité et homogénisation*, Thèses, Univ. de Paris VI, 1982.

[6] M. Djaoua and P. Suquet, *Evolution quasi-statique des milieux visco-plastique de Maxvell-Norton,* Math. Methods Appl. sci. 6, 192-205 (1984).

[7] I.R. Ionescu and M. Sofonea, *Quasistatic processes for elastic-viscoplastic materials,* Quart. Appl. Math. 2, 229-243 (1988).

[8] I.R. Ionescu , *Dynamic processes for a class of elastic-viscoplastic materials,* Preprint Series in Math., INCREST, Bucharest, 64 (1988).

[9] J. Necas ans J. Kratochvil, *On existence of the solution of boundary value problems for elastic-inelastic solids,* Comment. Math. Univ. Carolinae 14, 755-760 (1973).

[10] O. John, *On the solution of displacement boundary value problem for elastic-inelastic materials,* Appl. Math. 19 , 65-70 (1974).

[11] P. Laborde, *On viscoplasticity with hardening,* Numer. Funct. Anal. Optim. 1, 315-339 (1979).

[12] M. Sofonea, *Functional methodes in thermo-elasto-visco-plasticity*, Ph. D. Thesis, Univ. of Bucharest, 1988 (in Romanian).

[13] M. Sofonea, *Some remarks on the behaviour of the solution in dynamic processes for rate-type models*, to appear in ZAMP, 1990.

[14] I.R. Ionescu, *Functional and numerical methods in viscoplasticity*, Ph. D. Thesis, Univ. of Bucharest, 1990 (in Romanian).

[15] I. Suliciu and M. Sabac, *Energy estimates in onedimensional rate-type viscoplasticity*, J. Math. Anal. Appl., 131, 2, 354-372 (1988).

[16] N. Cristescu, *Elastic/Viscoplastic constitutive equations for rock*, Int : J. Rock Mech. Min. Sci. & Geomech. Abster 24 (5), 271-282 (1987).

[17] W. Hahn, *Stability of motion*, Springer-Verlag, Berlin, 1967.

Manuscrit reçu le 16 mars 1990.