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FAILURE OF COMPLETENESS PROPERTIES OF INTUITIONISTIC PREDICATE LOGIC FOR CONSTRUCTIVE MODELS

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ABSTRACT

We consider a principle of constructivity **RED** which states that every decidable predicate over the natural numbers is (weakly) recursively enumerable (r.e.). **RED** is easily seen to be derived from Church's thesis $\mathbf{CT}_0$ («every construction is given by a recursive function»).

Results :

(1) **RED** implies that the species of valid first order predicate schemata is not r.e., and hence - that intuitionistic first order predicate logic $\mathcal{L}_1$ is incomplete.

(2) We construct a specific schema of $\mathcal{L}_1$ which is valid if **RED**, but unprovable in $\mathcal{L}_1$.

(3) The two results above hold even when validity is generalized to validity with Kreisel-Troelstra [70]'s choice sequences as parameters.

(4) The method is used also to construct a schema of $\mathcal{L}_1$, unprovable in $\mathcal{L}_1$, but of whose all metasubstitutions with Σ_1^0 number theoretic predicates are provable in Heyting's arithmetic $\mathcal{G}$. This is a simple bound on possible improvements of the maximality result of Leivant [75] .

1. INTRODUCTION

For a general introduction to the subject matter of this note the reader is referred to Troelstra [76].

The main two results of this note are the following. Firstly, we show (along the line of Kreisel [70] p. 133) that the schema **RED** mentioned above in the abstract implies that the species of intuitionistically valid formulae of predicate logic is not r.e.. This we do by exhibiting a sequence $\{F_{m,n}\}$ of formulae, the validity of $F_{m,n}$ meaning intuitively : «the r.e. species W_m and W_n are not separable by any pair of decidable species». By **RED** decidable species are (weakly) r.e., so we may say instead «W_m and W_n are recursively inseparable». The proof is then concluded by showing (intuitionistically) that the species $\{\langle m,n\rangle \mid W_m \text{ and } W_n \text{ are recursively inseparable}\}$ is not r.e..

Secondly, we pick up m,n for which $F_{m,n}$ is not valid, but s.t. the recursive inseparability of W_m and W_n is provable in intuitionistic arithmetic. This last proof may be reshaped to apply to any given Σ^0_1 model of arithmetic ; we conclude that any Σ^0_1-metasubstitution of $F_{m,n}$ is provable in arithmetic, thus proving result (4) of the abstract.

1.1. NOTATIONAL CONVENTIONS.

We use the logical constants &, $\wedge$, $\rightarrow$, $\forall$, $\exists$ and $\perp$ (for absurdity ; negation $\neg$ is then definable). The letters P,Q,T,Z,S... are used for predicate parameters (in the language of first-order predicate logic $\mathcal{L}_1$), while bold face **T**,**Z**,**S**... are used for certain number theoretic predicates explicitely defined below.

For recursion theoretic notions we use the notations of Kleene [52] and [69] : T^n for the computation predicate for n-ary functions, U for the result-extracting function, $\{e\}$ for the e'th partial recursive function, $\simeq$ for partial equality, etc.

We assume given an encodement of sequences

$$\langle\ \rangle : \bigcup_k \mathbb{N}^k \rightarrow \mathbb{N}$$

i.e. - $(n_1, ..., n_k) \longmapsto \langle n_o, ..., n_k \rangle$, together with a length-function $\underline{lth}$ and inverse projection functions $(n)_j$ which satisfy

$$\underline{lth}(\langle n_o, ..., n_k\rangle) = k \quad ; \quad (\langle n_o, ..., n_k\rangle)_j = n_j \quad (j \leq k).$$

We also fix some Gödel-coding for syntactic objects, and write $\ulcorner \sigma \urcorner$ for the code of σ . For formal r.e. theories $\mathcal{T}$ we write $\text{Prov}_{\mathcal{T}}(p,r)$ for a canonical proof predicate for $\mathcal{T}$, and $\underline{Pr}_{\mathcal{T}}(r)$ for $\exists p\, \underline{Prov}_{\mathcal{T}}(p,r)$.

1.2. - 1.4. FORMAL SETTING.

1.2. LANGUAGE.

Our metamathematical setting is the intuitionistic theory of species, $\mathcal{L}_2$. An inspection on the proofs below shows that actually only a small fragment of $\mathcal{L}_2$ is used, but since it is of little value for our purpose to delimit this fragment precisely we do not bother to do so.

Prawitz ([65] p. 72) has shown that Heyting's Arithmetic $\mathfrak{A}$ is interpretable in $\mathcal{L}_2$; however, to avoid notational complications we shall use lower case a,b,c etc. for numeric variables, and lower case x,y,x_i etc. for unrestricted first order variables. A list z_0, ..., z_i, ... of unrestricted variables will be reserved for a specific use, and we shall assume that they do not occur in formulae otherwise. We also assume that $\mathfrak{A}$ as well as $\mathcal{L}_2$ contain symbols and defining equations for all primitive recursive functions.

The system we study is first order intuitionistic predicate logic $\mathcal{L}_1$, without equality and without function symbols. We shall find it convenient to refer also to $\mathcal{L}_1$ extended to the language with all constants of $\mathfrak{A}$ (= , numerals and prim. rec. functions). We write $\mathcal{L}_1\mathfrak{A}$ for the resulting system.

1.4. THE VALIDITY PREDICATE.

Tarski's definition of a Π^1_1 validity predicate $\underline{\mathrm{Val}}$ for $\mathcal{L}_1$ (analogously to the classical case) is standard (cf. e.g. Tarski [36]). The central property of $\underline{\mathrm{Val}}$ is of course

(1) $$\vdash_{\mathcal{L}_2} \underline{\mathrm{Val}}(\ulcorner G \urcorner) \leftrightarrow \forall D^1 \forall \vec{P}\ G^D$$

for each schema G of $\mathcal{L}_1$, where $\vec{P}$ is the list of all predicate parameters occurring in G, D^1 is a (unary) predicate variable and G^D comes from G by restricting all quantifiers to D.

To avoid confusion one should note that the predicate $\underline{\mathrm{Val}}$ does not express any analysis of the notion of constructive truth (contrary to truth definitions like Beth's and Kripke's semantics and the various realizability predicates). Here the constructive content of $\mathcal{L}_1$ shows only through the constructive meaning given (in $\mathcal{L}_2$) to the logical constants occurring in $\underline{\mathrm{Val}}$. Our discussion is therefore valid regardless of any specific analysis of constructivity.

1.4 THE SCHEMA **RED**.

We define the schema **RED**, for «recursive enumerability of decidable predicates» to be

$$\forall a\ [P(a) \vee \neg P(a)] \rightarrow \neg\neg\exists e\ \forall a\ [P(a) \leftrightarrow \exists b\ T^1(e,a,b)]$$

It is easily seen that **RED** is derivable from the following weak variant of Church's thesis $\mathbf{CT_0}$:

$CT_0^-!$: $\forall a \ \exists ! b\, P(a,b) \rightarrow \neg\neg \exists e \ \forall a \ \exists b \ [T^1(e,a,b) \ \& \ P(a,U(b))]$

V. Lifshits has sown (in 1974, unpublished) that even the positive version $CT_0!$ of $CT_0^-!$ is strictly weaker (in $\mathcal{G}$) than CT_0.

2. CHARACTERIZATION OF (WEAKLY) RECURSIVELY INSEPARABLE PAIRS OF R.E. SETS IN $\mathcal{L}_1$ VIA RED.

2.1. A FINITE AXIOMATIZATION OF KLEENE'S PREDICATE T.

In the language of $\mathcal{L}_1$, fix three predicates Eq(x,y), Z(x) and S(x,y). We think of these as representing equality, zero and the successor relation. Let A^S ($\equiv$ the axiom of the theory of successor) be the conjunction of the closure of the following formulae of $\mathcal{L}_1$.

(1) $Eq(x,x)$

(2) $Eq(x,y) \ \& \ Eq(x,z) \rightarrow Eq(y,z)$

(3) $Z(x) \rightarrow [Z(y) \longleftrightarrow Eq(x,y)]$

(4) $S(x,y) \rightarrow [S(x,z) \longleftrightarrow Eq(y,z)]$

(5) $S(x,y) \rightarrow [S(z,y) \longleftrightarrow Eq(x,z)]$

(6) $Z(x) \rightarrow \neg S(y,x)$

(7) $\exists z\, Z(z)$

(8) $\exists y\, S(x,y)$

Consider the familiar definig schemata for primitive recursive functions :

(1) (zero) $f_n(x) = 0$

(2) (successor) $f_n(x) = x^+$ (:= the successor of x)

$(3)_{q,i}$ (projection) $f_n(x_0, \ldots, x_q) = x_i$ $(i \leq q)$

$(4)_{q,r}$ (composition) $f_n(x_0, \ldots, x_q) = f_m(f_{i_0}(x_0, \ldots, x_q), \ldots, f_{i_r}(x_0, \ldots, x_q))$

(5) (recursion) $f_n(0,x) = f_m(x)$

$f_n(y^+, x) = f_\ell(y,x,f_n(y,x))$

These equations may be axiomatized by

(1^*) $F_n(x,v) \longleftrightarrow Z(v)$

(2^*) $F_n(x,v) \longleftrightarrow S(x,v)$

$(3^*)_{q,i}$ $F_n(x_0, \ldots, x_q, x_i)$ $(i \leq q)$

$(4^*)_{q,r}$ $\bigwedge_{j \leq r} F_{i_j}(x_0, \ldots, x_q, y_j) \rightarrow [F_n(x_0, \ldots, x_q, v) \longleftrightarrow F_m(y_0, \ldots, y_r, v)]$

(5^*) $Z(z) \rightarrow [F_n(z,x,v) \longleftrightarrow F_m(x,v)]$.&.

$S(y,w) \ \& \ F_n(y,x,z) \rightarrow [F_n(w,x,v) \longleftrightarrow F_\ell(y,x,z,v)]$

The characteristic function τ of Kleene's predicate T is primitive recursive ; there is therefore a list

$$\mathcal{E}_0, \ldots, \mathcal{E}_m$$

of equations, each of which is an instance of one of (1) - (5) with some auxiliary function letters $f_0, \ldots, f_k$, t , which defines τ . Taking the corresponding instances of $(1)^* - (5)^*$, with predicate letters $F_0, \ldots, F_k$, T corresponding to $f_0, \ldots, f_k$, t , we obtain a list $B_0, \ldots, B_m$ of formulae of $\mathcal{L}_1$ which defines T. We let

$$A :\equiv A^s \ \& \ \bigwedge_{i \leq m} C_i$$

where C_i is the closure of B_i $(i \leq m)$. Let $z_0, \ldots, z_i, \ldots$ be a list of variables (not occurring in A) ; define

$$A_n \equiv A_n(z_0, \ldots, z_n) :\equiv A \ \& \ Z(z_0) \ \& \ \bigwedge_{i < n} S(z_i, z_{i+1})$$

2.2. We refer below to the schema

$$G \equiv G[T,P,Q] :\equiv D[P,Q] \rightarrow E[T,P,Q]$$

where $D[P,Q] :\equiv \forall x\, [P(x) \vee \neg P(x)] \ \& \ \forall x\, [Q(x) \vee \neg Q(x)]$

$$E[T,P,Q](w_1, w_2) :\equiv \exists x\, [\exists y\, T(w_1,x,y) \ \& \ P(x)]$$
$$\vee \ \exists x\, [\exists y\, T(w_2,x,y) \ \& \ Q(x)]$$
$$\vee \ \exists x\, [\neg P(x) \ \& \ \neg Q(x)]$$

Write (in the language of $\mathcal{L}_2$)

$$\mathrm{Val}^{\omega}(\ulcorner G(a_1,a_2) \urcorner) :\equiv \forall P,Q\ G[T,P,Q]\,(a_1,a_2)$$

$$\mathrm{Val}^{\omega, \Sigma_1^0}(\ulcorner G(a_1,a_2) \urcorner) :\equiv \forall e_1,e_2\ G[T,W_{e_1},W_{e_2}]\,(a_1,a_2)$$

where W_e is the number theoretic predicate

$$W_e(a) :\equiv \exists b\, T(e,a,b)$$

$\mathrm{Val}^{\omega, \Sigma_1^0}(\ulcorner G(a_1,a_2) \urcorner)$ then reads : the r.e. sets W_{a_1} and W_{a_2} are not separable by any pair of decidable r.e. sets.

2.3. LEMMA. (in $\mathcal{L}_2$)

$$T(a,b,c) \ \& \ A_n(z_o, \ldots, z_n) \rightarrow T(z_a,z_b,z_c)$$

where $n := \max[a,b,c]$.

PROOF : One proves

$$F_i(a_o, ..., a_{n_i}) \& A_m \rightarrow F_i(z_{a_o}, ..., z_{a_{n_i}}) \qquad (m := \max[a_o, ..., a_{n_i}])$$

for each primitive recursive predicate F_i used in the definition of T, by induction on the length of definition of F_i. The details are routine. ■

2.4. LEMMA. (in $\mathcal{L}_2$)

$$\underline{Val}(\ulcorner A_n \rightarrow G(z_m, z_n)\urcorner) \longleftrightarrow \underline{Val}^{\omega}(\ulcorner G(\bar{m},\bar{n})\urcorner) \qquad (m < n)$$

PROOF : I. Assume $\underline{Val}(\ulcorner A_n \rightarrow G(z_m, z_n)\urcorner)$, i.e. -

$$\forall Z,S,Eq,F_1, ..., F_k,T,P,Q \ \{A_n[Z, ..., T](z_o, ..., z_n) \rightarrow G[T,P,Q](z_m, z_n)\}$$

Hence

(1) $\quad A_n[Z,S, ..., T](\bar{0}, ..., \bar{n}) \rightarrow \forall P,Q \ G[T,P,Q](\bar{m},\bar{n})$

where Z, ..., T are the number theoretic predicates for zero, successor etc.

Since the premise of (1) is trivially true, we obtain $\underline{Val}^{\omega}(\ulcorner G(\bar{m},\bar{n})\urcorner)$ as required.

II. Assume

(2) $\quad \underline{Val}^{\omega}(\ulcorner G(\bar{m},\bar{n})\urcorner) \equiv \forall P,Q \ \{D[P,Q] \rightarrow \neg\neg E[T,P,Q](\bar{m},\bar{n})\}$

Fix P and Q and assume

(3) $\quad A_n[Z,S,Eq,F_1, ..., F_k,T](z_o, ..., z_n)$

and

(4) $\quad D[P,Q]$

From (2) and (4) we then get

(5) $\quad \neg\neg E[T,P,Q](\bar{m},\bar{n})$

Assume

(6) $\quad E[T,P,Q](\bar{m},\bar{n})$

i.e. - either $T(\bar{m},a,b) \ \& \ P(a)$ for some a,b

or $T(\bar{n},a,b) \ \& \ Q(a)$ for some a,b

or $\neg P(a) \ \& \ \neg Q(a)$ for some a.

which, by lemma 2.3. and assumption (3) implies $E[T,P,Q](z_m,z_n)$. This being derived from (6), we get by intuitionistic propositional logic that (5) implies $\neg\neg E[T,P,Q](z_m, z_n)$ as required. ■

2.5. LEMMA. (in $\mathcal{L}_2$) RED implies

$$\underline{Val}^{\omega}(\ulcorner G(\bar{m},\bar{n})\urcorner) \longleftrightarrow \underline{Val}^{\omega,\Sigma^0_1}(\ulcorner G(\bar{m},\bar{n})\urcorner) \qquad (m < n)$$

PROOF : The implication from left to right is trivial. Assume on the other hand

(1) $\mathrm{Val}^{\omega,\Sigma^0_1}(\ulcorner G(\bar{m},\bar{n})\urcorner) :\equiv \forall e_1, e_2 \left\{ D[W_{e_1}, W_{e_2}] \to \neg\neg E[T, W_{e_1}, W_{e_2}](\bar{m},\bar{n}) \right\}$

and, fixing P and Q, assume D [P,Q]. By **RED** D[P,Q] implies the weak existence of some e_1, e_2 s.t.

$$P(a) \longleftrightarrow W_{e_1}(a) \quad ; \quad Q(a) \longleftrightarrow W_{e_2}(a)$$

which by (1) implies $\neg\neg E[T,P,Q](\bar{m},\bar{n})$ as required. ■

2.6. PROPOSITION. (in $\mathcal{L}_2$) **RED** implies

$$\underline{\mathrm{Val}}(\ulcorner A_n \to G(z_m, z_n)\urcorner) \longleftrightarrow \underline{\mathrm{Val}}^{\omega,\Sigma^0_1}(\ulcorner G(\bar{m},\bar{n})\urcorner)$$

PROOF : Immediate from 2.4 and 2.5. ■

3. WEAK INCOMPLETENESS OF $\mathcal{L}_1$ (UNDER RED).

3.1. PROPOSITION. The species $S := \left\{ \langle m,n \rangle \mid \underline{\mathrm{Val}}^{\omega,\Sigma^0_1}(\ulcorner G(\bar{m},\bar{n})\urcorner) \right\}$ is not r.e.

PROOF : Assume that the speciesS is r.e., i.e. - for some primitive recursive predicate R

$$\langle m, n \rangle \in S \longleftrightarrow \exists a\, R(a,m,n) .$$

Let

$$\mathcal{C}^+ := \mathcal{C}^+ + \{ V_{m,n} \mid \langle m,n \rangle \in S \}$$

where

$$V_{m,n} :\equiv \underline{\mathrm{Val}}^{\omega,\Sigma^0_1}(\ulcorner G(\bar{m},\bar{n})\urcorner)$$

Note that we may define a primitive recursive proof predicate $\underline{\mathrm{Prov}}_{\mathcal{C}^+}$ for $\mathcal{C}^+$ as follows :

$$\underline{\mathrm{Prov}}_{\mathcal{C}^+}(\langle p,q,m,n \rangle, \ulcorner F \urcorner) :\equiv \underline{\mathrm{Prov}}_{\mathcal{C}}(p, \underline{\mathrm{imp}}(m,n,\ulcorner F\urcorner)) \;\&\; R^*(q,m,n)$$

where $\underline{\mathrm{imp}}$ is a primitive recursive function which satisfies

$$\underline{\mathrm{imp}}(m,n,\ulcorner F\urcorner) = \ulcorner \bigwedge_{j < \underline{\mathrm{lth}}(m)} V_{(m)_j,(n)_j} \to F \urcorner$$

and where

$$R^*(q,m,n) :\equiv \underline{\mathrm{lth}}(q) = \underline{\mathrm{lth}}(m) = \underline{\mathrm{lth}}(n) \;\&\; \forall j < \underline{\mathrm{lth}}(q)\, R((q)_j,(m)_j,(n)_j).$$

$\underline{\mathrm{Prov}}_{\mathcal{C}^+}$ is easily proved in $\mathcal{C}$ to satisfy the elementary derivability conditions. Let

$$W_{a_1} = \{ q \mid \underline{\mathrm{Pr}}_{\mathcal{C}^+}(q) \}$$

(1) $$W_{a_2} = \{ q \mid \underline{\mathrm{Pr}}_{\mathcal{C}^+}(\mathrm{neg}(q)) \}$$

where $\underline{neg}$ is a primitive recursive function which satisfies $\underline{neg}(\ulcorner F \urcorner) = \ulcorner \neg F \urcorner$ for any formula F in the language of $\mathfrak{A}$. Since $\mathfrak{A}^+$ is r.e., the consistency of $\mathfrak{A}^+$ is equivalent to the statement «W_{a_1} and W_{a_2} are recursively inseparable» (see Smullyan [61] p. 63, thm. 27 ; notice that the proof there is intuitionistic).

Put formally :

(2) $\vdash_{\mathfrak{A}} \quad \underline{Con}_{\mathfrak{A}^+} \longleftrightarrow V_{a_1, a_2}$

where $\underline{Con}_{\mathfrak{A}^+} :\equiv \neg \underline{Pr}_{\mathfrak{A}^+}(\ulcorner \bot \urcorner)$. In $\mathcal{L}_2$ we know however that $\underline{Con}_{\mathfrak{A}^+}$ is true, and consequently V_{a_1,a_2} must be an axiom of $\mathfrak{A}^+$. Together with (2) this implies

$$\vdash_{\mathfrak{A}^+} \quad \underline{Con}_{\mathfrak{A}^+}$$

which contradicts Gödel second incompleteness theorem (since $\underline{Prov}_{\mathfrak{A}^+}$ satisfies the elementary derivability conditions). Thus the assumption that S is r.e. leads (in $\mathcal{L}_2$) to a contradiction. ■

REMARK : The reference to Smullyan's theorem is made for the sake of brevity.
One may instead pick up any pair W_{e_1}, W_{e_2} of r.e. sets which are proved in $\mathfrak{A}$ to be recursively inseparable (cf. e.g. Rogers [67] p. 94 thm. XII(c), whose proof is intuitionistic). Defining

$$W_{d_i} := \left\{ a \mid \exists b\, T(e_i,a,b) \;\&\; \forall c < a \;\; \neg \underline{Prov}_{\mathfrak{A}^+}(c, \ulcorner \bot \urcorner) \right\} \qquad i = 1, 2$$

we find that $\underline{Con}_{\mathfrak{A}^+}$ is equivalent (in $\mathfrak{A}$) to the recursive inseparability of W_{d_1} and W_{d_2}.

THEOREM I. (Weak incompleteness). In $\mathcal{L}_2$ + **RED** we can prove that not every valid formula of $\mathcal{L}_1$ is provable.

PROOF : If every valid formula of $\mathcal{L}_1$ is provable in $\mathcal{L}_1$ then the species of valid formulae is the same as the species of provable formulae, and so it must be r.e.. The formulae of the form $A_n \rightarrow G(z_m, z_n)$ are recursively recognizable, and thus the valid formulae of this form make up an r.e. species as well. By 2.6. S is then r.e., contradicting proposition 3.1. above.

3.2. The result of 3.1 can be classically improved by the following

PROPOSITION : $S := \{ \langle m,n \rangle \mid W_m, W_n \text{ rec. inseparable} \}$ is Π^0_2-complete.

PROOF. (essentially due to C. Jockusch) Fix a pair R_1, R_2 of recursively inseparable r.e. sets. Let $k(e), h_1(e), h_2(e)$ be prim. rec. functions defined (through the s-m-n theorem) by

$$x \in W_{k(e)} \equiv \exists y < x \; y \in W_e$$

$$W_{h_1(e)} = W_{k(e)} \cap R_1 \; ; \quad W_{h_2(e)} = W_{k(e)} \cap R_2$$

Then : (i) W_e finite $\Rightarrow$ $W_{k(e)}$ finite

$\Rightarrow$ $W_{h_1(e)}$ and $W_{h_2(e)}$ finite

$\Rightarrow$ $\langle h_1(e), h_2(e) \rangle \notin S$

(ii) W_e infinite $\Rightarrow$ $W_{k(e)} = \mathbb{N}$

$\Rightarrow$ $W_{h_i(e)} = R_i$ $\quad (i = 1, 2)$

$\Rightarrow$ $\langle h_1(e), h_2(e) \rangle \in S$

So the set

$$I := \{ e \mid W_e \text{ is infinite} \}$$

reduces to S. But I is known to be Π^0_2-complete (cf. Rogers [67] p. 326, 1.3), and hence S is also Π^0_2-complete. ∎

3.3. PROPOSITION. There are numbers m,n for which we can prove in $\mathcal{L}_2$ + **RED** that $A_n \to G(z_m, z_n)$ is valid but not provable in $\mathcal{L}_1$.

PROOF : By the proof of 3.1. there are m, n s.t. $V_{m,n}$, and so by 2.6 $A_n \to G(z_m, z_n)$ is valid, assuming **RED**. On the other hand, if

$$\vdash_{\mathcal{L}_1} A_n \to G[T,P,Q]\,(z_m, z_n)$$

then

(1) $\vdash_{\mathfrak{G}} G[T,\mathbf{P},\mathbf{Q}]\,(\bar{m},\bar{n})$ for any unary number theoretic predicates **P**,**Q**.

The Gödel translation G^o of G (see Kleene [52] p. 493) is therefore also provable in $\mathfrak{G}$; but D^o is provable (idem, p. 119 *51 a), and by intuitionistic predicate logic $E[T,\mathbf{P},\mathbf{Q}]^o$ implies $\neg\neg E[T,\mathbf{P},\mathbf{Q}]$ whenever **P** and **Q** do not contain $\mathbf{V}$, so (1) implies

$$\vdash_{\mathfrak{G}} \neg\neg E[T,\overline{\mathbf{W}}_m, \overline{\mathbf{W}}_n]\,(\bar{m},\bar{n})$$

which implies (in $\mathcal{L}_2$) that $\neg\neg E[T,\overline{\mathbf{W}}_m,\overline{\mathbf{W}}_n](\bar{m},\bar{n})$ is true. For the m,n considered W_m and W_n are however (provably) disjoint, and so by the definition of E we have $\neg E[T,\overline{\mathbf{W}}_m, \overline{\mathbf{W}}_n](m,n)$, a contradiction. ∎

REMARK : A simpler proof of the above proposition runs as follows. Classically the schema G is equivalent to E. E is obviously not valid classically, so it cannot be provable even in classical first order logic, by Gödel's completeness theorem. The formalization of this proof in the intuitionistic theory of species is however slightly problematic.

3.4. INCOMPLETENESS W.R.T. VALIDITY WITH CHOICE PARAMETERS

Let α be a variable for choice sequences, and assume that for the kind of choice sequences considered we have

(1) $\forall \alpha \quad \neg\neg\exists e \ \alpha \simeq \{e\}$

which is the case for the choice sequences investigated by Kreisel-Troelstra [70] (cf. 6.2.1 there).

(1) implies quite trivially for every negated formula $\neg A(\alpha)$

(2) $\forall e \ \neg A(\{e\}) \rightarrow \forall \alpha \ \neg A(\alpha)$.

Let us refer now to a notion of validity with choice parameters,

$$\underline{Val}^{CS}(\ulcorner G \urcorner) :\equiv \forall \alpha \ \underline{Val}(\ulcorner G^{\alpha} \urcorner)$$

where G^{α} comes from G by replacing each atomic subformula $P(t_1, ..., t_n)$ by $P(\alpha, t_1, ..., t_n)$. (This is not weaker than allowing the choice parameters to be distinct for each predicate letter, as can readily be seen.)

A straightforward observation shows now that the discussion above remains correct when Val and Val^{ω} are replaced by Val^{CS} and an analogue $Val^{\omega,CS}$, respectively, Val^{ω,Σ_1^0} remains unchanged and the schema **RED** is generalized to

(3) $\mathbf{RED}^{CS} : \forall \alpha \ \{ \forall a \, [P(\alpha, a) \vee \neg P(\alpha, a)] \rightarrow \neg\neg\exists e \ \forall a \ [P(\alpha \, . \, a) \leftrightarrow \exists b \ T(e,a,b)] \ \}$

But by (2) $\mathbf{RED}^{CS}$ is implied outright by **RED**, since **RED** is negative, and (3) with α varying over total recursive functions is just a special case of (the quantified variant of) **RED**.

To recapitulate, we have obtained

THEOREM II. In $\mathcal{L}_2$ + **RED** we can prove that the species of formulae (of $\mathcal{L}_1$) which are valid with choice parameters is not r.e., and we may exhibit a specific formula of $\mathcal{L}_1$ which is valid with choice parameters but not provable in $\mathcal{L}_1$.

4. $\mathcal{L}_1$ IS NOT Σ_1^0-MAXIMAL.

4.1. DEFINITION OF MAXIMALITY.

Let $\mathcal{C}$ be a class of number theoretic predicates and $\mathcal{S}$ be a formal system in a language extending the language of arithmetic. A formula $H[P_0, ..., P_k]$ of $\mathcal{L}_1$ is a $\mathcal{C}$-<u>schema of</u> $\mathcal{S}$ when

$$\vdash_{\mathcal{S}} \quad H\,[P_0^*, ..., P_k^*]$$

for every $P_0^*, ..., P_k^*$ in $\mathcal{C}$. We define formally for $\mathcal{S} = \mathcal{G}$, $\mathcal{C} =$ the class of Σ_1^0 predicates :

(1) $\underline{Sch}(\ulcorner H[P_o, ..., P_k]\urcorner) \; :\equiv \; \forall e \; \underline{Pr}_{\mathfrak{G}} (\ulcorner H[W_{(e)_o}, ..., W_{(e)_k}]\urcorner)$

where (for $m \geq 1$)

$$W_e(a_1, ..., a_m) \; :\equiv \; \exists b \, T^m(e, a_1, ..., a_m, b).$$

(In defining the predicate Sch we implicitly use a primitive recursive substitution function, the definition of which is routine).

A system $\mathfrak{L}$ of logic is said to be $\mathfrak{C}$-maximal for $\mathfrak{S}$ if

$$\mathfrak{L} = \{H \mid H \text{ in the language of } \mathfrak{L}, H \text{ is a } \mathfrak{C}\text{-schema of } \mathfrak{S}.\}$$

D.H.J. de Jongh has proved (in 1969) that the intuitionistic propositional logic is Σ^o_1-maximal for intuitionistic arithmetic $\mathfrak{G}$ (and certain extensions of $\mathfrak{G}$). In Leivant [75] (chs. B3-B5) it is proved that $\mathfrak{L}_1$ is Π^o_2-maximal for $\mathfrak{G}$ (and certain extensions of $\mathfrak{G}$). We shall now show that $\mathfrak{L}_1$ is not Σ^o_1-maximal even for a fragment of $\mathfrak{G}$.

4.2. For a schema $H \equiv H[Z,S,Eq,F_0, ..., F_r, T ; P_0, ..., P_t]$ of $\mathfrak{L}_1$ we define the schema

$$H^d \equiv H^d [P_0, ..., P_t] \; :\equiv \; H[W_{(d)_o}, ..., W_{(d)_{r+3}} ; P_0, ..., P_t]$$

of $\mathfrak{L}_1 \mathfrak{G}$, and the schema

$$H^\omega \equiv H^\omega[P_0, ..., P_t] \; :\equiv \; H[Z,S,Eq, ..., T ; P_0, ..., P_t]$$

of $\mathfrak{L}_1 \mathfrak{G}$, where $Z, ..., T$ are the intended number theoretic predicates, i.e. - $Z(x) :\equiv x = 0$, etc.

$$H^{d,e} \; :\equiv \; H^d [W_{(e)_0}, ..., W_{(e)_t}]$$

$$H^{\omega,e} \; :\equiv \; H^\omega [W_{(e)_0}, ..., W_{(e)_t}]$$

If the r.e. interpretations

$$Z^*(a) \; :\equiv \; W_{(d)_0} \; ; S^*(a,b) :\equiv W_{(d)_1} \; ; \; Eq^*(a,b) :\equiv W_{(d)_2}$$

of the zero, successor and equality predicates satisfy the axiom of successor A^6 (cf. 2.1), then Z^* and S^* generate a structure isomorphic to $\langle \omega, Z, S \rangle$ through a recursive (total) function defined as follows :

$$\nu(0) := (\mu a. T^1((d)_0, (a)_0, (a)_1))_0$$

$$\nu(n+1) := (\mu a. T^2((d)_1, \nu(n), (a)_0, (a)_1))_0$$

We let then

$$N_d(a) \; :\equiv \; \exists b \; \nu(b) \simeq a.$$

For a formula H of $\mathcal{L}_1 \mathcal{G}$ we now write $H^{[d]}$ for the schema of $\mathcal{L}_1 \mathcal{G}$ which comes from H by restricting all quantifiers to N_d.

4.3. LEMMA. (in $\mathcal{G}$) Let $H(b_0, ..., b_m)$ be a formula of $\mathcal{L}_1$.

(1) $\forall e\, \underline{Pr}_{\mathcal{G}} (\ulcorner H^{\omega, e}(\bar{n}_0, ..., \bar{n}_m)\urcorner) \rightarrow \forall d,e\, \underline{Pr}_{\mathcal{G}} (\ulcorner A_n^d \rightarrow H^{[d]d,e}(z_{n_0}, ..., z_{n_m})\urcorner)$

where $n := \max[n_i \mid i < m]$, and A, A_n are defined as in 2.1, but with defining formulae for all predicates T^j, where j ranges over the «dimensions» of the predicate letters occurring in H.

PROOF : Let $\mathcal{G}$ be given by a Gentzen natural deduction system (cf. e.g. Prawitz [71]).
For a formula $H \equiv H[Z,S,Eq,...,T]\ (n_0, ..., n_m)$ of $\mathcal{G}$ let us write H^d for $H[Z,S,Eq,..., T](z_{n_0}, ..., z_{n_m})^d$ as defined in 4.2. above. I.e. -

$$H^d :\equiv H[W_{(d)_0}, ..., W_{(d)_{r+3}}](z_{n_0}, ..., z_{n_m}) .$$

It is easy to verify that for each inference rule of the Gentzen system considered,

$$\frac{\{ J_i(\bar{n}_0, ..., \bar{n}_p) \}_i}{K(\bar{n}_0, ..., \bar{n}_p)}$$

(where $\bar{n}_0, ..., \bar{n}_p$ is the list of all numerals occurring in the formulae shown) the inference

$$\frac{\bigwedge_i J_i^{[d]d}(z_{n_0}, ..., z_{n_p})}{K^{[d]d}(z_{n_0}, ..., z_{n_p})}$$

is a derived rule of $\mathcal{G} + A_n^d$, where $n := \max[n_i \mid i \leq p]$. Hence if a formula $H(\bar{n}_0, ..., \bar{n}_p)$ of $\mathcal{G}$ is derivable by a natural deduction Δ then the formula $H^{[d]d}(z_{n_0}, ..., z_{n_p})$ is derivable in $\mathcal{G} + A_q^d$, where q is a bound on the value of all numerals occurring in Δ. By the existential conjuncts of the axiom A^S (formulae (.) and (8) in 2.1) q may be cut down to $n := \max[n_i \mid i \leq p]$.

Assume now the premise of (1),

(2) $\forall e\, \underline{Pr}_{\mathcal{G}}(\ulcorner H^{\omega, e}(\bar{n}_0, ..., \bar{n}_m)\urcorner)$

and fix d and e. For $j < t$ let the predicate letter P_j be m_j+1 place and define the number $e^* = \langle (e^*)_0, ..., (e^*)_t \rangle$ through the s-m-n theorem by

(3) $\{(e^*)_j\}(a_0, ..., a_{m_j}) :\simeq \{(e)_j\}(\nu(a_0), ..., \nu(a_{m_j}))\ .\quad j < t$

where the function ν , which depends on d, is defined as in 4.2. above.
Further, let $e_1 = \langle (e_1)_0, ..., (e_1)_t \rangle$ be defined by

(4) $(e_1)_j := \nu((e^*)_j)$, $j \leq t$.

By (2) now

$$\vdash_{\mathfrak{a}} J \qquad \text{where } J :\equiv H^{\omega, e_1}(\bar{n}_0, ..., \bar{n}_m)$$

and so by the above argument

(5) $$\vdash_{\mathfrak{a}} A_n{}^d \rightarrow J^{[d]\,d}(z_{n_0}, ..., z_{n_m})$$

where $n := \max [n_j \mid j \leq p]$. It is readily seen however that for

$$H \equiv H[Z, ..., T ; P_0, ..., P_t]$$

(6) $$J^{[d]\,d}(z_{n_0}, ..., z_{n_m}) \equiv H^{[d]}[W_{(d)_0}, ..., W_{(d)_{r+3}} ; W'_{(e_1)_0}, ..., W'_{(e_1)_t}](z_{n_0}, ..., z_{n_m})$$

where for $j \leq t$

$$W'_{(e_1)_j}(a_0, ..., a_{m_j}) :\equiv \exists b \in N_d \; T_d((e_1)_j, a_0, ..., a_{m_j}, b)$$

$$\leftrightarrow \exists b \quad T_d(\nu((e^*)_j), a_0, ..., a_{m_j}, \nu(b)) \quad \text{by (4)}$$

Here T_d is the d-interpretation of the predicate T^{m_j} i.e. - $T_d :\equiv W_{(d)_q}$ for some q.

If $a_0, ..., a_{m_j} \in N_d$ then

$a_i = \nu(c_i)$ for some c_i $(i \leq m_j)$. So for such $a_0, ..., a_{m_j}$

$$W'_{(e_1)_j}(a_0, ..., a_{m_j}) \leftrightarrow \exists b \; T_d(\nu((e^*)_j, \nu(c_0), ..., \nu(c_{m_j}), \nu(b))$$

$$\leftrightarrow \exists b \; T^{m_j}((e^*)_j, c_0, ..., c_{m_j}, b)$$

by the definition of the interpretation generated by d, provided $A_n{}^d$ holds

$$\leftrightarrow \exists b \; T^{m_j}((e)_j, a_0, ..., a_{m_j}, b) \qquad \text{by (3)}$$

$$\equiv : \quad W_{(e)_j}(a_0, ..., a_{m_j})$$

Since in $H^{[d]}$ all quantifiers are indeed restricted to N_d , we thus have from (6)

$$J^{[d]\,d}(z_{n_0}, ..., z_{n_m}) \leftrightarrow H^{[d]\,d,e}(z_{n_0}, ..., z_{n_m})$$

(provided A_n^d), and so (5) implies the provability in $\mathcal{G}$ of

$$A_n^d \rightarrow H^{[d]d,e}(z_{n_0}, \ldots, z_{n_m}) .$$

This being the case for arbitrary d and e we have obtained the conclusion of (1), as required. ∎

4.4. THEOREM III. $\mathcal{L}_1$ is not Σ^0_1-maximal for $\mathcal{G}$. Specifically - there are m and n for which $A_n \rightarrow E[T,P,Q](z_m, z_n)$ is a Σ^0_1-schema of $\mathcal{G}$, but is not provable in $\mathcal{L}_1$. (Here the schema E is defined as in 2.2.).

PROOF : Let W_m, W_n be a pair of r.e. sets, proved in $\mathcal{G}$ to be recursively inseparable (as in 3.1). I.e. - for each e_0, e_1

$$\vdash_{\mathcal{G}} E[T, W_{e_0}, W_{e_1}](\bar{m},\bar{n})$$

which by lemma 4.3 implies that for any d,e

(1) $$\vdash_{\mathcal{G}} A_n^d \rightarrow E^{[d]\, d,e}(z_m, z_n) .$$

But E is existential, and therefore $E^{[d]}$ implies (in predicate logic) E. So (1) for arbitrary d,e actually states that $A_n \rightarrow E(z_m, z_n)$ is a Σ^0_1-schema of $\mathcal{G}$.

On the other hand this formula cannot be proved in $\mathcal{L}_1$, as we have seen in 3.3. ∎

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