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THE THEORY OF BOOLEAN ALGEBRAS WITH A DISTINGUISHED SUBALGEBRA IS UNDECIDABLE

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§ 0. INTRODUCTION

We prove the following theorems :

Theorem 1** Let T_1 and T_2 be theories in the language $L = \{ \cup, \cap, -, 0, 1 \}$ such that there are infinite Boolean algebras (hereafter denoted by BA) B_1, B_2 such that $B_i \models T_i$ $i = 1, 2$, let P be a unary predicate and $S = T_1 \cup T_2^{(P)}$, where $T_2^{(P)}$ is the relativization of T_2 to P , then S is undecidable.

Theorem 2 : The theory of 1-dimensional cylindric algebras (denoted by CA_1) is undecidable. Theorems 1 and 2 answer a question of Henkin and Monk in [2] Problem 7 ; there they also point out that the decidability problems of theorems 1 and 2 are closely related, this relation is formulated in the following proposition :

Proposition : (a) Let $\langle B, c \rangle$ be a CA_1 where B is a BA and c a unary operation on B then $A = \{ b \mid b \in B \text{ and } c(b) = b \}$ is a subalgebra of B , and for every $b \in B$ $c(b)$ is the minimum of the set $\{ a \mid b \subseteq a \in A \}$.

(b) Let B be a BA and A be a subalgebra of B suppose that for every $b \in B$ $a_b = \min(\{ a \mid b \subseteq a \in A \})$ exists ; define $c(b) = a_b$, then $\langle B, c \rangle$ is a CA_1 .

Let T_C be the theory of CA_1 's and T_B be the theory of BA's with a distinguished subalgebra P , with the additional axiom that for every b there is a minimal a_b such that $P(a_b)$ and $b \subseteq a_b$, then certainly T_C and T_B are bi-interpretable.

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** R. McKenzie proved independently at about the same time, that the theory of Boolean algebras with a distinguished subalgebra is undecidable. The method of his proof is different from ours.

The classical result about the decidability of the theory of BA's appears in Tarski's [5], and in Ershov [1]. Ershov in [1] also proved that the theory of BA's with a distinguished maximal ideal is decidable, Rabin [4] proved the decidability of the theory of countable BA's with quantification over ideals.

Henkin proved that the equational theory of CA_2 's is decidable and Tarski proved the undecidability of the equational theory of CA_n 's for $n \geq 4$.

In our construction we interpret the theory of two equivalence relations in a model $\langle B, \cup, \cap, -, 0, 1, A \rangle$ but neither B nor A are complete BA's. We do not know the answer to the following question :

Let $K = \{ \langle B, \cup, \cap, -, 0, 1, A \rangle \mid B \text{ is a BA, } A \text{ is a subalgebra of } B, A \text{ and } B \text{ are complete} \}$ is $\text{Th}(K)$ decidable ?

We also do not know whether an analogue of theorem 1 for T_B holds. For instance let S be T_B together with the axioms that say that both the universe and P are atomic BA's is S decidable ?

§ 1. THE CONSTRUCTION

$\cup, \cap, -, 0, 1$ denote the operations and constants of a BA and \subseteq denotes its partial order. A, B, C denote BA's ; $At(B), A\ell(B), As(B)$ denote the set of atoms of B , the set of non-zero, non-maximal atomless elements of B and the set of non-zero, non-maximal atomic elements of B respectively. Let $I(B)$ be the ideal generated by $A\ell(B) \cup As(B)$, $B^{(1)} = B/I(B)$ and if $b \in B$ $b^{(1)} = b/I(B)$. If $D \subseteq B$ $c\ell(D)$ denotes the subalgebra of B generated by D . $B \times C$ denotes the direct product of B and C . $\prod_{j \in J} B_j$ denotes the direct product of $\{B_j \mid j \in J\}$, and we assume that for every $j_1 \neq j_2$ $B_{j_1} \cap B_{j_2} = \{0\}$, so we can identify the element c of B_{j_0} with the element $f_c \in \prod_{j \in J} B_j$ where $f_c(j) = 0$ if $j \neq j_0$ and $f_c(j_0) = c$. We denote by 1_B the maximal element of B .

Let B_T be the BA of finite and cofinite subsets of ω and B_L the countable atomless BA. Let F_1 be the non-principal ultrafilter of B_T and F_2 be an ultrafilter in B_L ; let B_M be the following subalgebra of $B_T \times B_L : B_M = \{(a,b) \mid a \in F_1 \text{ iff } b \in F_2\}$; notice that $B_M^{(1)} \cong \{0,1\}$. For every i let $B_i \cong B_M, B^> = \prod_{i \in \omega} B_i$ and $B^< = c\ell(\bigcup_{i \in \omega} B_i)$. We denote 1_{B_i} by 1_i .

Lemma 3 : Let E_0 and E_1 be equivalence relations on ω then there is a model $M = \langle B, \cup, \cap, -, 0, 1, A \rangle \models T_B$ such that $\langle \omega, E_0, E_1 \rangle$ is explicitly interpretable in M .

Proof : We denote by i/E_ε the E_ε -equivalence class of i and by ω/E_ε the set of E_ε -equivalence classes. For every $i \in \omega$ let

$$\{b_{\varepsilon, \sigma, j}^i \mid \varepsilon \in \{0,1\}, \sigma \in \omega/E_\varepsilon, j \in \omega\} \subseteq A\ell(B_i)$$

$$\langle \varepsilon, \sigma, j \rangle \neq \langle \varepsilon', \sigma', j' \rangle \Rightarrow b_{\varepsilon, \sigma, j}^i \cap b_{\varepsilon', \sigma', j'}^i = 0 \text{ and for every } b \in A\ell(B_i)$$

$1 \leq |\langle \varepsilon, \sigma, j \rangle \mid b \cap b_{\varepsilon, \sigma, j}^i \neq 0| < \aleph_0$. For every $i \in \omega$ let

$\{a_{\varepsilon, \sigma, j}^{-i} \mid \varepsilon \in \{0,1\}, \sigma \in \omega/E_\varepsilon, j \in \omega\}$ be a set of pairwise disjoint subsets of

$At(B_i)$ such that $At(B_i) = \cup \{a_{\varepsilon, \sigma, j}^{-i} \mid \varepsilon \in \{0,1\}, \sigma \in \omega/E_\varepsilon, j \in \omega\}$ and

$$|a_{\varepsilon, \sigma, j}^{-i}| = \begin{cases} 1 & \varepsilon = 0 \text{ and } i \in \sigma \\ 2 & \varepsilon = 0 \text{ and } i \notin \sigma \\ 3 & \varepsilon = 1 \text{ and } i \in \sigma \\ 4 & \varepsilon = 1 \text{ and } i \notin \sigma \end{cases}$$

For every ε, σ and j as above let $c_{\varepsilon, \sigma, j} \in B^>$ be
 $c_{\varepsilon, \sigma, j} = \bigcup \{b_{\varepsilon, \sigma, j}^i \cup \bigcup a_{\varepsilon, \sigma, j}^i \mid i \in \omega\}$ where $\bigcup D$ denotes the supremum of
 D in $B^>$. Let $A = \text{cl}(\{c_{\varepsilon, \sigma, j} \mid \varepsilon \in \{0, 1\}, \sigma \in \omega/E_\varepsilon, j \in \omega\})$, $B = \text{cl}(B^< \cup A)$ and
 $M = \langle B, \cup, \cap, -, 0, 1, A \rangle$. We show that $M \models T_B$. It suffices to show that
 $a_b = \min(\{a \mid b \subseteq a \in A\})$ exists for elements $b \in B$ of the following forms :
 $b \in \text{At}(B_i) \cup \text{Al}(B_i)$; $b \in B_i$ and $b^{(1)} = 1_i^{(1)}$; $b \in B^<$ and $1_i \subseteq b$ for almost
all $i \in \omega$; this follows from the fact that every $b \in B$ can be represented in the form
 $\bigcup_{i=1}^n (b_i \cap a_i)$ where each b_i is of the above form and $a_i \in A$. In each of the above cases
the existence of a_b is easily checked. Thus $M \models T_B$.

We now define formulas $\varphi_U(x)$, $\varphi_{Eq}(x,y)$, $\varphi_\varepsilon(x,y) \varepsilon \in \{0, 1\}$ such that
 $M \models \varphi_U[a]$ iff for some $i \in \omega$ $a^{(1)} = 1_i^{(1)}$, $M \models \varphi_{Eq}[a,b]$ iff $a^{(1)} = b^{(1)}$
and $M \models \varphi_\varepsilon[a_1, a_2]$ iff for some $i_j \in \omega$ $a_j^{(1)} = 1_{i_j}^{(1)}$ and $\langle i_1, i_2 \rangle \in E_\varepsilon$.
 $\varphi_U(x)$ says that $x^{(1)} \in \text{At}(B^{(1)})$ and for no $y \in \text{At}(A)$ $x^{(1)} = y^{(1)}$.

$\varphi_{Eq}(x,y)$ says that $x^{(1)} = y^{(1)}$. $\varphi_0(x,y)$ says: $\varphi_U(x) \wedge \varphi_U(y)$ and there are x_1, y_1
such that $x^{(1)} = x_1^{(1)}$, $y^{(1)} = y_1^{(1)}$ and for every $z \in \text{At}(A)$

$|\{u \mid z \cap x_1 \supseteq u \in \text{At}(B)\}| = 1$ iff $|\{u \mid z \cap y_1 \supseteq u \in \text{At}(B)\}| = 1$. φ_1 is
defined similarly. The desired properties of φ_U , φ_{Eq} and φ_ε are easily checked, and
the lemma is proved.

Since the theory of two equivalence relations is undecidable T_B and T_C are undecidable
and theorem 2 is proved.

Theorem 1 easily follows from the following lemma.

Lemma 4 : Let E_1, E_2 be equivalence relations on ω then there are models

$M_i = \langle B_i, \cup, \cap, -, 0, 1, A_i \rangle$ $i = 1, \dots, 4$ such that $\langle \omega, E_1, E_2 \rangle$ is explicitly
interpretable in M_i and B_1, A_1 are atomic, B_2, A_2 are atomless, B_3 is atomic A_3 is atomless,
and B_4 is atomless A_4 is atomic.

Proof : Let B_0, A_0, M_0 denote B, A and M of lemma 3 respectively. For $i = 1, 2$ M_i can
easily be constructed so that $\langle B_i/H_i, \cup, \cap, -, 0, 1, A_i/H_i \rangle \cong M_0$ where
 $H_i = \{b \mid b \in B_i \text{ and for every } a \subseteq b \ a \in A_i\}$. Since such an H_i is definable in M_i M_0
can be interpreted in M_i $i = 1, 2$.

For $i = 3$ a similar construction works. Let B be an atomic saturated countable BA and I a maximal non-principal ideal of B . Let A be an atomless subalgebra of B such that :

(a) for every $b \in B$ which contains infinitely many atoms there is a non-zero $a \in A$ such that $a \subseteq b$;

(b) for every $b \in A_s(B)$ there is an $a \in A$ such that $(a-b) \cup (b-a)$ contains only finitely many atoms of B . Let $J = I \cap A$. For every non-zero $a \in B_0$ let F_a be an ultrafilter in B which contains a , and $\langle B_a, A_a, I_a, J_a \rangle$ a copy of $\langle B, A, I, J \rangle$. Let $B^1 = \prod \{B_a \mid 0 \neq a \in B_0\}$ and let B_3 be the following subalgebra of B^1 :

$B_3 = \text{c}\ell(\cup\{I_a \mid 0 \neq a \in B_0\} \cup \{g_a \mid 0 \neq a \in B_0\})$ where $g_a(b) = 1_{B_b}$ iff $a \in F_b$ and $g_a(b) = 0$ otherwise. Let $A_3 = \text{c}\ell(\cup\{J_a \mid 0 \neq a \in B_0\} \cup \{g_a \mid a \in A_0\})$. Certainly B_3 is atomic and A_3 is atomless. Let $I = \{a \mid \{b \mid a \supseteq b \in \text{At}(B_3)\} \in \aleph_0\}$. I is an ideal in B_3 , and I is definable in M_3 by the formula $\varphi(x) \equiv \forall y(0 \neq y \subseteq x \rightarrow \sim P(y))$. Let $B_3^1 = B_3/I$ and $A_3^1 = \{a/I \mid a \in A_3\}$, then $\langle B_3^1, \cup, \cap, -, 0, 1, A_3^1 \rangle \cong M_2$, so M_2 is interpretable in M_3 and thus $\langle \omega, E_1, E_2 \rangle$ is interpretable in M_3 as desired.

In order to construct M_4 we assume that B_1 is a subalgebra of $P(\omega)$ and $\text{At}(B_1) = \{\{n\} \mid n \in \omega\}$. Let $B_L^i \cong B_L$ for every $i \in \omega$. B_4 is the following subalgebra of $\prod_{i \in \omega} B_L^i$: $B_4 = \text{c}\ell(\cup_{i \in \omega} B_L^i \cup \{f_a \mid a \in B_1\})$ where $f_a(n) = 1_{B_L^n}$ if $n \in a$ and $f_a(n) = 0$ otherwise. Let $A_4 = \text{c}\ell(\{f_a \mid a \in A_1\})$ and $M_4 = \langle B_4, \cup, \cap, -, 0, 1, A_4 \rangle$.

Certainly B_4 is atomless and A_4 is atomic. Let $B_4^1 = \{b \mid b \in B_4 \text{ and for every } a \in \text{At}(A_4) \text{ either } b \supseteq a \text{ or } -b \supseteq a\}$, then $\langle B_4^1, \cup, \cap, -, 0, 1, A_4 \rangle \cong M_1$ and B_4^1 is certainly definable in M_4 , thus $\langle \omega, E_1, E_2 \rangle$ is definable in M_4 and the lemma is proved.

We omit the proof of theorem 1 which follows easily from lemma 4, the fact that every countable BA can be embedded in e.g. B_L , and from [6] pp. 293-302.

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