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VIRGILIO MUŠKARDIN

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REPRESENTATION THEOREM FOR FINITE QUASI-BOOLEAN ALGEBRAS

Virgilio MUŠKARDIN

University of Bristol, BRISTOL, Great Britain

INTRODUCTION

In constrast to the classical logic it has been recognised that a statement may fail to be either true or false. This recognition has been clearly embeded in the notion of «inexact predicate» developed by Körner (7.). An associated logic, called the *logic of inexactness*, is a three-valued logic based on Kleene's «strong tables» (6.). Algebras related to this logic in the same way as boolean algebras (BA) are related to the classical logic are termed *quasi-boolean algebras* (QBA). By definition, $\underline{Q} = (Q; I, 0, \vee, \wedge, ')$ is a QBA if and only if (iff) $(Q; I, 0, \vee, \wedge)$ is a distributive lattice with unit I and zero 0, and ' is an unary operation such that $x'' = x$ and $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$ for any x, y in Q. If for every x, y in Q the additional condition $x \wedge x' \leq y \vee y'$ holds, $\underline{Q}$ is a *normal* QBA. Actually, QBAs accompaning the logic of inexactness are normal. The relation between the logic of inexactness and normal QBAs has been observed and studied by Cleave (3.), (4.). See also (5.). Quasi-boolean algebras were termed so by Bialynicki-Birula and Rasiowa (1.). In the same paper they gave the *general representation theorem* for QBAs which corresponds to the well known Stone's result for BAs. However, it is also well known that for finite BAs a stronger result is obtained. The aim of this paper is to give the corresponding *representation theorem for finite QBAs.*

Every finite BA is isomorphic with the field of all subsets of the set of its *atoms.* (The atoms of a finite BA are of course the join-irreducible elements.) For a finite QBA an analogous result is obtained by considering the partially ordered set (poset) of its join-irreducible elements. In order to state our representation theorem we need the notion of an involutive poset.

INVOLUTIVE POSETS.

DEFINITION 1. $\underline{P} = (P\ ;\ \leq,\ \sim)$ is an *involutive poset* iff $\forall\ a, b \in P$:

1. $a \leq a$,
2. $a \leq b\ \&\ b \leq a \rightarrow a = b$,
3. $a \leq b\ \&\ b \leq c \rightarrow a \leq c$,
4. $a \in P \rightarrow \tilde{a} \in P$,
5. $\tilde{\tilde{a}} = a$,
6. $a \leq b \rightarrow \tilde{b} \leq \tilde{a}$.

I.e. An involutive poset is a partially ordered set with an involution which reverses the ordering defined on it.

DEFINITION 2. Let $(P\ ;\ \leq)$ be any poset and $p \subseteq P$. p is an *initial subset* of P iff $\forall\ a, x \in P : a \in p\ \&\ x \leq a \rightarrow x \in p$.

DEFINITION 3. For any poset $(P\ ;\ \leq)$, Q(P) denotes the set of all initial subsets of P.

DEFINITION 4. Let $\underline{P} = (P\ ;\ \leq,\ \sim)$ be an involutive poset. For every $p \in Q(P)$ define $\tilde{p} \subseteq P$ by

$x \in p \longleftrightarrow \tilde{x} \in \tilde{p}$.

DEFINITION 5. Let $\underline{P}$ and Q(P) be as in Definition 4. Define a *quasi-complementation* ' on Q(P) by

$\forall\ p \in Q(P) : p' = P \setminus \tilde{p}$.

Here $\setminus$ is the set-theoretic difference operator.

One can easily verify from these definitions

LEMMA 1. For every $\underline{P} = (P\ ;\ \leq\ ,\ \sim)$, Q(P) defined by Definition 3. is closed under the set-theoretic union $\cup$ and intersection $\cap$, as well as under the quasi-complementation ' defined in Definition 5.

This result gives rise to the following definition.

DEFINITION 6. $\underline{Q}(\underline{P}) = (Q(P)\ ;\ P, \emptyset\ ,\ \cup\ ,\ \cap\ ,\ ')$ is called the *quasi-field of all initial subsets of an involutive poset* $\underline{P}$.

Note : The notion of a quasi-field of sets was first defined by Bialynicki-Birula and Rasiowa (1.).

THEOREM 2. Every quasi-field of all initial subsets of an involutive poset of m elements is a quasi-boolean algebra of dimension m.

The notions of dimension d(x) of an element x of a QBA $\underline{Q} = (Q\ ;\ I, 0,\ \vee, \wedge\ ,\ ')$ and of dimension $d(\underline{Q})$ of a QBA itself coincide with the corresponding notions defined on the distributive lattice $(Q\ ; I, 0, \vee, \wedge)$. For these and other fundamental lattice-theoretic notions see Birkhoff (2.).

PROOF. Let $\underline{Q}(\underline{P})$ be as in Definition 6. By Lemma 1. $(Q(P)\ ; P, \emptyset, \cup, \cap)$ is a ring of sets and therefore a distributive lattice where P, $\emptyset$ are its unit element, zero element respectively. Furthermore, by Definition 5. and Lemma 1. $p'' = P \setminus \tilde{p}'$. From Definition 4. and Definition 1(5). we get $\tilde{p}' = P \setminus p$. Thus $p'' = p$. One easily establishes

$$\widetilde{p \cap q} = \tilde{p} \cap \tilde{q} \quad \text{and} \quad \widetilde{p \cup q} = \tilde{p} \cup \tilde{q} \tag{1}$$

and using this result proves that ' fulfills De Morgan's laws. It remains to prove

$$d(\underline{Q}(\underline{P})) = \operatorname{card}(P)\ . \tag{2}$$

It is easy to see that $\operatorname{card}(p) = d(p)$ for every p in Q(P) because elements of Q(P) are ordered by $\subseteq$. Hence (2) follows. Q.E.D.

LEMMA 3. Let $\underline{Q} = (Q\ ; I, 0, \vee, \wedge, ')$ be any finite QBA and $(J(Q)\ ; \leq)$ the partially ordered set of all non-zero join-irreducible elements of Q, and Q(J) the set of all initial subsets of J(Q). Then $(Q\ ; I, O, \vee, \wedge) \cong (Q(J)\ ; J(Q), \emptyset, \cup, \cap)$ where $\cup$, $\cap$ are the set-theoretic union, intersection respectively.

Observe that in the case $\underline{Q}$ is a BA, J(Q) is the set of its atoms and consequently unordered set. Of course, Q(J) is then the set of all subsets of J(Q).

PROOF. Observe that both $(Q\ ; I, O, \vee, \wedge)$ and $(Q(J)\ ; J(Q), \emptyset, \cup, \cap)$ are distributive lattices. For any finite distributive lattice the mapping $J : a \mapsto J(a)$, $a \in Q$ is bijective. Recall that $J(a) = \{x \in J(Q) : x \leq a\}$. We show that

$$Q(J) = \{ J(a) : a \in Q \}\ . \tag{3}$$

Obviously, $\forall\ a \in Q$, $J(a) \in Q(J)$. Now take any $p \in Q(J)$. Then there exists an a in Q such that $a = \bigvee_{a_i \in p} a_i$. Using Lemma 1. in (2.) p 139 we deduce $J(a) \subseteq p$. But J(a) contains all join-irreducible elements $\leq$ a. Therefore $J(a) = p$. This establishes (3).

The mapping $J : Q \to Q(J)$ is obviously isotone (order preserving) and so a desired isomorphism. Q.E.D.

Since $J : Q \to Q(J)$ is an isomorphism, we can transfer $' : Q \to Q$ to $' : Q(J) \to Q(J)$ in an obvious unique way :

COROLLARY 4. Define the unary operation ' on Q(J) by

$$\forall a \in Q : \quad (J(a))' \equiv J'(a) = J(a') \tag{4}$$

Then

$$\underline{Q}(\underline{J}) = (Q(J)\ ; J(Q), \emptyset, \cup, \cap, ') \tag{5}$$

is a QBA and

$$\underline{Q} \cong \underline{Q}(\underline{J})\ . \tag{6}$$

Our main result can now be stated.

THEOREM 5. Let $\underline{Q}$ be any finite QBA and $\underline{Q}(\underline{J})$ the isomorphic QBA (5).
Then an involution $\sim$ on J(Q) can be defined uniquely such that
$\underline{J}(\underline{Q}) = (J(Q) ; \leq , \sim)$
is an involutive poset and

$$\forall J(a) \in Q(J) : J'(a) = J(Q) \setminus \tilde{J}(a) \qquad (7)$$

where $\tilde{J}(a) = \{\tilde{x} : x \in J(a)\}$.

THE REPRESENTATION THEOREM

In order to prove Theorem 5. we need some further definitions and lemmas. Recall :

DEFINITION 7. $a \in P$ is a *minimal element* in a poset $(P ; \leq)$ iff
$\forall x \in P : x \leq a \rightarrow x = a.$

Note. If $(P ; \leq)$ is a poset and $a \in P$ a minimal element, then $(P \setminus \{a\} ; \leq)$ is a poset.

DEFINITION 8. Definition of $\sim$ on J(Q).
Suppose card(J(Q)) = m. Order the elements of J(Q) in the following way :

Let a_1 be any minimal element of J(Q) ;

Let a_2 be any minimal element of $J(Q) \setminus \{a_1\}$;

etc.

Then $J(Q) = \{a_1, ..., a_m\}$. Clearly, by construction, every set $p_k = \{a_1, ..., a_k\}$, $k \leq m$, is an initial subset of J(Q) i.e. $p_k \in Q(J)$, and

$$\emptyset = p_0 \subsetneq p_1 \subsetneq ... \subsetneq p_m = J(Q) \qquad (8)$$

is a complete chain in the sense that for all $k < m$ $p_k \prec\!\!\subset p_{k+1}$, where « $\prec\!\!\subset$ » denotes the covering relation (cf. (2.)) for sets. The covering relation between elements of any poset we shall denote by $\prec$. Observe that

$$\overset{m}{\underset{1}{\forall}} p_k \exists x_k \in Q : p_k = J(x_k) \ \& \ x_k = \bigvee_1^k a_i . \quad \text{Thus}$$

$$a_i = J(x_i) \setminus J(x_{i-1}). \qquad (9)$$

By Corollary 4. $J(x_k) \prec\!\!\subset J(x_{k+1}) \rightarrow J(x'_k) \supset\!\!\succ J(x'_{k+1})$.

Note : $J(x'_0) = J(Q)$, $J(x'_m) = \emptyset$. Thus, ranging k from o to m-1, each difference $J(x'_k) \setminus J(x'_{k+1})$ introduces a new element of J(Q). For each $a_i \in J(Q)$ define $\tilde{a}_i$ by

$$\{\tilde{a}_i\} = J(x'_{i-1}) \setminus J(x'_i), \quad 1 \leq i \leq m \qquad (10)$$

Obviously, the mapping $\sim : J(Q) \rightarrow J(Q)$ so defined is <u>bijective</u>.

LEMMA 6. $\forall\ x, y, s, z \in Q$:

$J(x) \setminus J(y) = J(s) \setminus J(z) \rightarrow J(y') \setminus J(x') = J(z') \setminus J(s')$.

PROOF. Assume

$J(x) \setminus J(y) = J(s) \setminus J(z)$. (11)

Without loss of generality we can also assume

$J(y) \subseteq J(x)\ \&\ J(z) \subseteq J(s)$ (12)

for $J(x) \setminus J(y) = J(x) \setminus J(x \wedge y)$ and $J(x \wedge y) \subseteq J(x)$; similarly for s, z. Now, if $J(x) = \emptyset$ or $J(x) = J(y)$, then (11) in conunction with (12) yields $J(y') \setminus J(x') = J(z') \setminus J(s') = \emptyset$ and our lemma holds. Therefore from now on we assume

$J(x) \neq \emptyset\ , J(s) \neq \emptyset\ \&\ J(y) \subset J(x),\ J(z) \subset J(s)$. (13)

Let us firstly suppose

$J(y) \prec J(x)\ \&\ J(z) \prec J(s)$. (14)

Since J is an isomorphism : $\underline{Q} \rightarrow \underline{Q}(\underline{J})$ (cf. Lemma 3., Corollary 4.), (14) is equivalent to :

$y \prec x\ \&\ z \prec s$. (15)

In accordance with (14), let

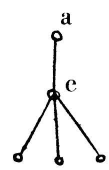
$J(x) = J(y) \cup \{a\}\ \&\ J(s) = J(z) \cup \{a\}$ (16)

Observe

$a \in J(Q) \rightarrow E\ e \in Q : e \prec a\ \&\ J(a) = J(e) \cup \{a\}$. (17)

Clearly, $a \leq x,\ e \leq y\ ;\ a \leq s,\ e \leq z$.

(i) e is meet-irreducible.



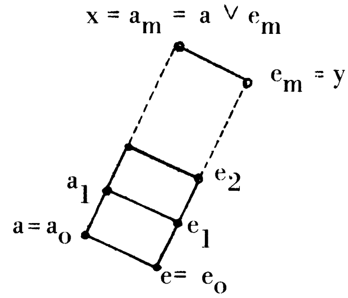
Since $a \not\leq y$ (16) and $e \prec a,\ e \leq y$

$a \wedge y = e$. Hence, by assumption (i), $y = e$.

Then, trivially, $x = a$. A similar argument yields $z = e, s = a$.

Hence the lemma follows.

(ii) e is meet-reducible.



Since $e \leq y$, there exists a complete chain

$e = e_o \prec e_1 \prec \ldots \prec e_{m-1} = y$.

Clearly, $\forall\ i = o, \ldots, m : a \not\leq e_i$.

By the covering conditions (Corollary 2. in (2.) p 66)

$a \prec a \vee e_1\ ;\ e_1 \prec a \vee e_1$.

Denote $a = a_o$, $a \vee e_1 = a_1$.

Applying the same argument on e_1, a_1, e_2 we get $a_2 \succ a_1$, etc., until eventually we reach $a_{m-1} \vee e_m = a_m = x$. It is easy to see that we shall reach x just as a_m , for $\underline{Q}$ satisfies

Jordan-Dedekind chain condition (see (2.) p 11 and Theorem 3. p 68) - 2 fixed points being e,x. Thus we have 2 complete chains

$$\left.\begin{array}{l} a = a_o \prec a_1 \prec \ldots \prec a_{m-1} \prec a_m = x \\ e = e_o \prec e_1 \prec \ldots \prec e_{m-1} \prec e_m = y \\ \text{s.t. } \forall\, i = o, \ldots, m : e_i \prec a_i,\ a_i = e_i \vee a_{i-1} \quad (a_{o-1} = a_o) \end{array}\right\} \quad (18)$$

It follows that $\forall\, i = o, \ldots, m,\quad a_i = e_i \vee a$.

Applying the same argument on s and z we get :

$$\left.\begin{array}{l} a = a_o = \bar{a}_o \prec \bar{a}_1 \prec \ldots \prec \bar{a}_{n-1} \prec \bar{a}_n = s \\ e = e_o = \bar{e}_o \prec \bar{e}_1 \prec \ldots \prec \bar{e}_{n-1} \prec \bar{e}_n = z \\ \text{s.t. } \forall\, i = 1, \ldots, n : \bar{e}_i \prec \bar{a}_i,\ \bar{a}_i = \bar{e}_i \vee \bar{a}_{i-1} \end{array}\right\} \quad (19)$$

Hence, $\bar{a}_i = \bar{e}_i \vee a$. In general, $m \not\leq n$.

Diagram I illustrates the preceeding analysis.

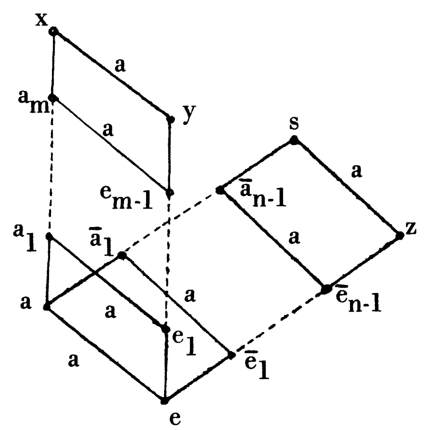


Diagram I

Note : In particular cases some of e_i may coincide with $\bar{e}_i$.

We know that ' reverses the ordering of elements in Q, i.e. converts $\prec$ into $\succ$. Thus we have :

$e \prec a \leftrightarrow e' \succ a' \leftrightarrow J(e') \supset J(a')$. Hence

$$\exists\, b \in J(Q) : J(e') \setminus J(a') = \{b\}. \quad (20)$$

From (18) and (20) one deduces

$$\forall\, i = o, \ldots, m : \quad b \notin J(a'_i). \quad (21)$$

Again by (18) and (20) one deduces inductively

$$\forall\, i = o, \ldots, m : \quad b \in J(e'_i). \quad (22)$$

Furthermore, $e'_i \succ a'_i \leftrightarrow J(e'_i) \supset J(a'_i)$, and by (21), (22) it follows $\forall\, i = o, \ldots, m$ $J(e'_i) \setminus J(a'_i) = \{b\}$. In particular, for $i = m$, $J(y') \setminus J(x') = \{b\}$, i.e. $J(e') \setminus J(a') = J(y') \setminus J(x') = \{b\}$.

The same argument applied to s', z' yields

$J(e') \setminus J(a') = J(z') \setminus J(s') = \{ b \}$, which establishes our lemma under the assumption (14).

Diagram II is obtained by applying ' on the elements of Diagram I.

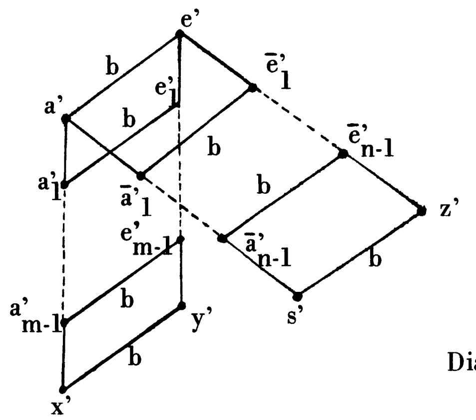


Diagram II

It remains to prove the lemma without restriction (14), i.e. when $y < x$ & $z < s$.
But then, there exist complete chains :

$y = y_o \prec y_1 \prec \dots \prec y_{k-1} \prec y_k = x,$

$z = z_o \prec z_1 \prec \dots \prec z_{\ell-1} \prec z_\ell = s.$

By (11), $\ell = k$, and for each i there exists j (i, j = 1, ..., k) s.t.

$$J(y_i) \setminus J(y_{i-1}) = J(z_j) \setminus J(z_{j-1}) \tag{23}$$

and vice versa. But (23) satisfies assumption (14). Hence

$J(y'_{i-1}) \setminus J(y'_i) = J(z'_{j-1}) \setminus J(z'_j)$. This completes the proof. Q.E.D.

LEMMA 7. Given a finite QBA $\underline{Q}$, the mapping $\sim$: $J(Q) \to J(Q)$ defined by Definitions 8. is *unique*, i.e. does not depend on the choice of a maximal chain (8) in $\underline{Q}(\underline{J})$.

PROOF. Suppose card(J(Q)) = m. Let $(a_1, \dots, a_m)$ be any ordering of the elements of J(Q) such that

a_1 is a minimal element of J(Q) and
a_i is a minimal element of $J(Q) \setminus \{a_1, \dots, a_{i-1}\}$
for $1 \leq i \leq m$. (*)

It is obvious that such an ordering exists.

Suppose $a_1, \dots, a_m$ and $b_1, \dots, b_m$ are two orderings of J(Q) satisfying (*). Define

$x_k = \bigvee_1^k a_i$, $y_k = \bigvee_1^k b_i$. By (10) $\{\tilde{a}_i\} = J(x'_{i-1}) \setminus J(x'_i)$, $\{\tilde{b}_j\} = J(y'_{j-1}) \setminus J(y'_j)$. (24)

We must prove that $a_i = b_j \rightarrow \tilde{a}_i = \tilde{b}_j$.

By (9) $\{a_i\} = J(x_i) \setminus J(x_{i-1})$, $\{b_j\} = J(y_j) \setminus J(y_{j-1})$.

If $a_i = b_j$, then from (24) by Lemma 6. follows $\tilde{a}_i = \tilde{b}_j$. Q.E.D.

LEMMA 8. $\sim$ defined by Definition 8. is an involution on $J(Q)$.

PROOF. Let a_i, $\tilde{a}_i$ be defined by (9), (10) respectively. By Lemma 7.

$\{\tilde{\tilde{a}}_i\} = J(x_i'') \setminus J(x_{i-1}'') = J(x_i) \setminus J(x_{i-1})$ by involutivity of '.

Thus $\forall\ a_i \in J(Q) : \tilde{\tilde{a}}_i = a_i$. Q.E.D.

LEMMA 9. $\forall\ a_i, a_j \in J(Q) : a_i \leq a_j \rightarrow \tilde{a}_j \leq \tilde{a}_i$.

PROOF. Let $J(Q)$ be ordered as in Definition 8. and a_i, a_j, $\tilde{a}_i$, $\tilde{a}_j$ defined accordingly by (9), (10). Obviously $a_i = a_j \longleftrightarrow \tilde{a}_i = \tilde{a}_j$.

Assume $a_i \leq a_j$. Then $J(x_i) \subseteq J(x_j)$ and since $a_j \notin J(x_i)$, $J(x_i) \subseteq J(x_{j-1})$. Hence

$x_{i-1} \prec x_i \leq x_{j-1} \prec x_j$ and equivalently

$$x'_{i-1} \succ x'_i \geq x'_{j-1} \succ x'_j \ . \tag{25}$$

Observe that

$$J(\tilde{a}_i) \subseteq J(x'_{i-1}) \tag{26}$$

i.e. $\tilde{a}_i \leq x'_{i-1}$.

If $\tilde{a}_i = x'_{i-1}$ the proof is trivial, for then $J(x'_{i-1}) = J(\tilde{a}_i)$ and by (25) $J(x'_{j-1}) \subseteq J(\tilde{a}_i)$. Hence, since $J(\tilde{a}_j) \subseteq J(x'_{j-1})$ (like (26)), $J(\tilde{a}_j) \subseteq J(\tilde{a}_i)$ i.e. $\tilde{a}_j \leq \tilde{a}_i$ as required.

Let us now consider the general case. Since $x'_i \geq x'_{j-1}$ (25), there exists a complete chain

$$x'_i = y_{oo} \succ y_{o1} \succ \dots \succ y_{on-1} = x'_{j-1} \succ x'_j = y_{on} \ . \tag{27}$$

By (26) $\tilde{a}_i \leq x'_{i-1}$. Thus there exists a complete chain

$$x'_{i-1} = c_o \succ c_1 \succ \dots \succ c_{k-1} \succ c_k = \tilde{a}_i \ .$$

(See Diagram III below).

It is easy to see that the elements

$$y_{oo} = x'_i\ ,\ y_{io} = y_{i-1,o} \wedge c_i \quad i = 1, \dots, k \tag{28}$$

form the chain in $\underline{Q}$ and

$$\forall\ i = 0, \dots, k\ :\ y_{io} \prec c_i. \tag{29}$$

Now, we also find the following chains in $\underline{Q}$:

$$\begin{array}{cccccc} y_{1o} & \succ & y_{11} \succ & \cdots & \succ & y_{1n} \\ \vdots & & \vdots & & & \vdots \\ y_{ko} & \succ & y_{k1} \succ & \cdots & \succ & y_{kn} \end{array} \tag{30}$$

defined recursively by

$$y_{ij} = y_{i-1,j} \wedge y_{i,j-1} \qquad \begin{array}{l} i = 1, \ldots, k \\ j = 1, \ldots, n \end{array} \tag{31}$$

where y_{io}, $i = 0, \ldots, k$ are defined by (28) and y_{oj}, $j = 0, \ldots, n$ are given by (27).

From (29), (30), (31) we conclude $y_{ij} \leq c_i$.

In particular,

$$y_{k,n-1} \leq \tilde{a}_i . \tag{32}$$

Now observe the chains :

$x'_{j-1} = y_{o,n-1} \succ y_{1,n-1} \succ \cdots \succ y_{k,n-1}$,

$x'_j = y_{on} \succ y_{1n} \succ \cdots \succ y_{kn}$.

Since $\tilde{a}_j \notin J(x'_j)$, $\tilde{a}_j \notin J(y_{in})$ $i = 0, \ldots, k$.

But $J(x'_{j-1}) = J(x'_j) \cup J(y_{1,n-1})$ and $\tilde{a}_j \in J(x'_{j-1})$. Therefore $\tilde{a}_j \in J(y_{1,n-1})$.

Proceeding in this way we get $\tilde{a}_j \in J(y_{i,n-1})$, $i = 0, \ldots, k$. In particular, $\tilde{a}_j \in J(y_{k,n-1})$.

Hence $J(\tilde{a}_j) \subseteq J(y_{k,n-1})$ or equivalently

$$\tilde{a}_j \leq y_{k,n-1} . \tag{33}$$

(32), (33) yield $\tilde{a}_j \leq \tilde{a}_i$, which completes the proof. Q.E.D.

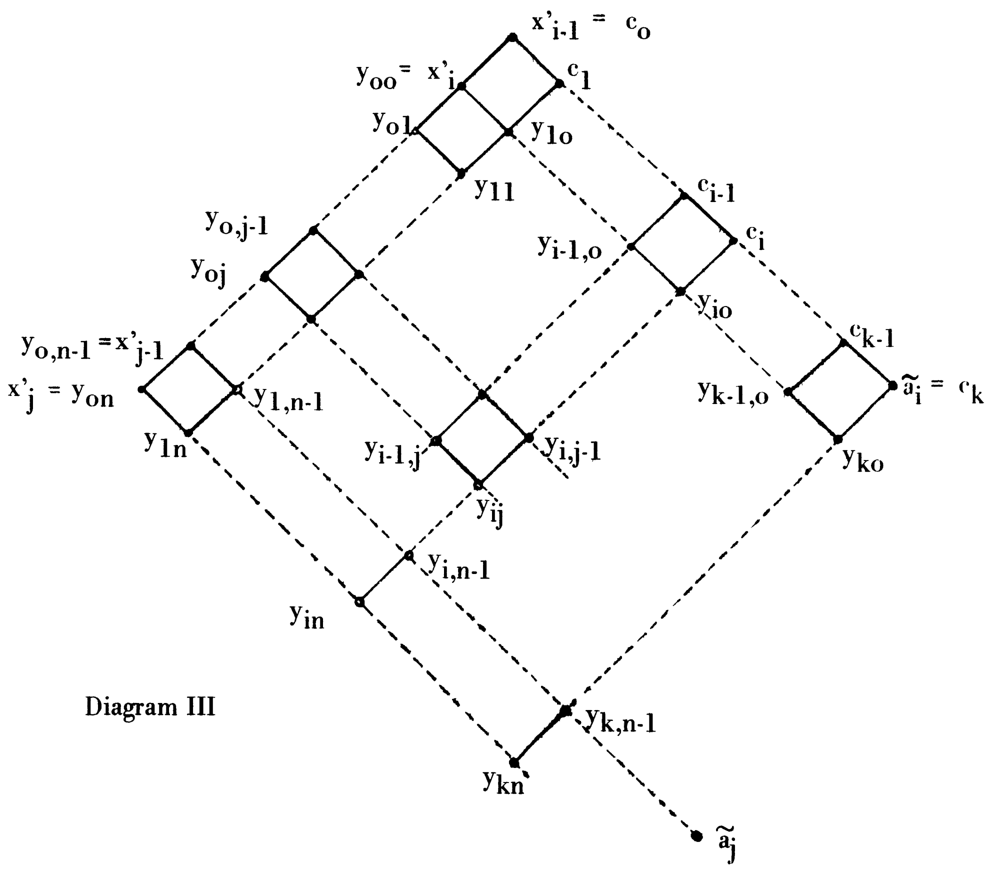


Diagram III

PROOF OF THEOREM 5. Since $\underline{J}(\underline{Q})$ is a partially ordered set, it satisfies 1., 2., 3. of Definition 1. By Definition 8. and lemmas 7. - 9. it also satisfies 4., 5., 6. of Definition 1. Hence, $\underline{J}(\underline{Q})$ is an involutive poset.

Definition of $\tilde{J}(x)$ in Theorem 5. is equivalent to

$$a \in J(x) \longleftrightarrow \tilde{a} \in \tilde{J}(x). \tag{34}$$

To complete the proof of Theorem 5. is to prove (7) ; by (4), this is to prove

$$J(x') = J(Q) \setminus \tilde{J}(x). \tag{35}$$

We firstly establish

$$a \in J(x) \longleftrightarrow \tilde{a} \notin J(x'). \tag{36}$$

Suppose $a \in J(x)$. By construction of $\underline{Q}(\underline{J})$:

$$a \in J(x) \rightarrow E\, y,z \in Q : J(z) \prec J(y) \subseteq J(x) \;\&\; \{a\} = J(y) \setminus J(z)$$

$$\longleftrightarrow E\, y', z' \in Q : J(z') \succ J(y') \supseteq J(x') \;\&\; \{\tilde{a}\} = J(z') \setminus J(y') \rightarrow \tilde{a} \notin J(x').$$

Conversely, if $\tilde{a} \in J(x')$, then the above argument would yield $a \notin J(x)$, for $\sim$, ' are involutive. Hence (36) holds.

Now (35) follows trivially. Q.E.D.

COROLLARY 10. *Representation theorem.*
Every finite quasi-boolean algebra is isomorphic with the quasi-field of all initial subsets of the involutive poset of its non-zero join-irreducible elements.

PROOF. By theorem 5. $' : Q(J) \rightarrow Q(J)$ is a quasi-complementation on $Q(J)$ (cf. Def. 5) and $\underline{Q}(\underline{J})$ is the quasi-field of all initial subsets of the involutive poset $\underline{J}(\underline{Q})$ (cf. Def. 6). Now, Corollary 10. follows from Corollary 4. Q.E.D.

COROLLARY 11. The number of (non isomorphic) quasi-boolean algebras of dimension m is equal to the number of (non-isomorphic) involutive posets of m elements.

PROOF. Recall : $d(\underline{Q}) = \text{card}(J(Q))$. The corollary follows from Theorem 2. and Corollary 10. Q.E.D.

Thus every involutive poset can be regarded as the involutive poset of all non-zero join-irreducible elements of a QBA.

Notice also

COROLLARY 12. $\underline{Q}_1 \cong \underline{Q}_2 \longleftrightarrow \underline{J}(\underline{Q}_1) \cong \underline{J}(\underline{Q}_2)$.

THEOREM 13. A QBA $\underline{Q}$ is normal iff

$$\forall a \in J(Q) : a \leq \tilde{a} \text{ or } \tilde{a} \leq a. \tag{37}$$

PROOF. Recall, by definition, $\underline{Q}$ is normal iff

$$\forall x, y \in Q : x \wedge x' \leq y \vee y'. \tag{38}$$

Since $J : \underline{Q} \to \underline{Q}(\underline{J})$ is an isomorphism, (38) is equivalent to

$\forall\, x, y \in Q: \quad J(x) \cap J(x') \subseteq J(y) \cup J(y').$ (39)

Assume (37). Suppose $a \in J(x) \cap J(x')$. Then $\tilde{a} \notin J(x) \cap J(x')$ by (36). This in conjunction with the assumption (37) yields :

$a \leq \tilde{a},$ (40)

for $J(x) \cap J(x')$ is an initial set.

Trivially, $a \in J(y)$ or $a \notin J(y)$. If $a \in J(y)$, then a fortiori $a \in J(y) \cup J(y')$. If $a \notin J(y)$, then $\tilde{a} \in J(y')$ by (36), and by (40) $a \in J(y')$. Thus again $a \in J(y) \cup J(y')$, as required by (39).

Conversely, assume the negation of (37), i.e.

$E\, a \in J(Q): \quad a \nleq \tilde{a} \;\&\; \tilde{a} \nleq a.$ (41)

Clearly $a \in J(a)$. But $\tilde{a} \notin J(a)$, for $\tilde{a} \in J(a) \to J(\tilde{a}) \subseteq J(a) \longleftrightarrow \tilde{a} \leq a$ contradicts (41). Also, by (36), $\tilde{a} \notin J(a')$. Hence, $\tilde{a} \notin J(a) \cup J(a')$. Again by (36)

$a \in J(a) \cap J(a').$ (42)

Similarly we get $\tilde{a} \in J(\tilde{a}) \cap J(\tilde{a}')$ and by (36)

$a \notin J(\tilde{a}) \cup J(\tilde{a}').$ (43)

By (42), (43) $\quad J(a) \cap J(a') \nsubseteq J(\tilde{a}) \cup J(\tilde{a}')$

what contradicts (39). Q.E.D.

THEOREM 14. Let $\circ$ be the identity function on $J(Q)$ i.e.

$\forall\, a \in J(Q): \quad \mathring{a} = a.$ (44)

Then $\underline{Q}$ is a boolean algebra iff

$\underline{J}(\underline{Q}) = (J(Q)\,;\, =,\ \circ\,).$ (45)

PROOF. Assume that $\underline{Q}$ is a BA. If for some $a, b \in J(Q)$, $a < b$, then there exists $c = a' \wedge b$ such that $c < b$ and $a \vee c = a \vee (a' \wedge b) = (a \vee a') \wedge (a \vee b) = I \wedge b = b$ what contradicts $b \in J(Q)$. Hence $J(Q)$ is an unordered set $(J(Q)\,;\, =)$. Since BA is a normal QBA, $\mathring{a} = a$ by Theorem 13. Thus (45).

Conversely, assume (45). Then by (36), (44), $J(x) \cap J(x') = \emptyset$, or equivalently,

$\forall\, x \in Q: \; x \wedge x' = 0.$

But this is just the condition which makes a QBA into a BA. Q.E.D.

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