ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 17, n° 2 (2010), p. 425-431.

<http://ambp.cedram.org/item?id=AMBP_2010__17_2_425_0>

Publication éditée par le laboratoire de mathématiques de l'université Blaise-Pascal, UMR 6620 du CNRS Clermont-Ferrand — France

Annales mathématiques Blaise Pascal **17**, 425-431 (2010)

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Abstract

In ultrametric Nevanlinna theory, the Nevanlinna's second main theorem for small functions has only been proved in the case of at most three small functions. In this paper, we prove a second main theorem for q small functions of a special type when the residue characteristic of the field is zero.

Le theoréme de Nevanlinna ultramétrique pour petites fonctions

Résumé

En théorie de Nevanlinna ultramétrique, le second théorème fondamental de Nevanlinna pour des petites fonctions a seulement été établi pour trois petites fonctions. Dans cet article, on montre un second théorème fondamental pour q petites fonctions d'un certain type quand la caractéristique résiduelle du corps est zero.

1. Introduction

The Nevanlinna Second Main Theorem on three small functions is well known in complex meromorphic functions. It is easily proved by using the so-called bi-ratio technique. However, there is no way to generalize such a technique to n small functions. In 2004, Yamanoi proved the Second Main Theorem on n small functions in [6] by a method that has no link with the elementary proof of the theorem on three small functions.

Now, let K be an algebraically closed field of characteristic zero, complete with respect to an ultrametric absolute value. The Nevanlinna Second Main Theorem does exist for meromorphic functions in the whole field and also for meromorphic functions in an open disk, see [1], [2], [3]. In the same way as for complex functions, an ultrametric version of Nevanlinna Second Main Theorem on three small functions can be proved by

Keywords: Ultrametric Nevanlinna theory, Second main theorem.
Math. classification: 12H99.

the technique of bi-ratio. But it is impossible to adapt to p-adic analysis Yamanoi's work that mainly uses typically archimedean properties in the complex field, particularly integrations on paths. Therefore it remains open how to obtain a similar result in an ultrametric field and also, whether it can be obtained.

This is why, here, we will try to obtain a kind of Nevanlinna Second Main Theorem on n small functions in ultrametric analysis, in a case where such a solution appears: we place ourselves in a field of residue characteristic zero, consider just analytic functions inside an open disk and assume that the "small functions" actually are bounded and satisfy an additional condition: intersections of two global images are empty. The hypothesis of zero residue characteristic is due to the behaviour of derivatives in a field of residue characteristic $p \neq 0$, which is explained after the proof of the theorem.

Let K be an algebraically closed field of characteristic zero and of residue characteristic zero, complete for an ultrametric absolute value $|\ |$.

Let $d(0, R^-)$ denote a disk $d(0, R^-) = \{x \in K|\ |x - 0| < R\}$ and let $d(0, R) = \{x \in K|\ |x - 0| \leq R\}$. Moreover, let $C(0, R)$ denote the circle $C(0, R) = \{x \in K|\ |x - 0| = R\}$.

For each $r \in]0, R[$, let $\eta(r, f)$ denote the number of zeros of f in the circle $C(0, r)$ counting multiplicity, and let $\overline{\eta}(r, f)$ be the number of zeros of f in $C(0, r)$ ignoring multiplicity.

Let $\mathcal{A}(d(0, R^-))$ denote the analytic functions in $d(0, R^-)$. Moreover, let $\mathcal{A}_b(d(0, R^-))$ denote the bounded analytic functions in $d(0, R^-)$ and let $\mathcal{A}_u(d(0, R^-)) = \mathcal{A}(d(0, R^-)) \setminus \mathcal{A}_b(d(0, R^-))$ be the unbounded analytic functions in $d(0, R^-)$.

Notation 1.1. For $f \in \mathcal{A}(d(0, R^-))$, let

$$\phi_{a,r}(f) = \lim_{|x-a| \to r, |x-a| < r} |f(x)|.$$

Moreover, given $f \in \mathcal{A}_b(d(0, R^-))$, let

$$||f|| = \sup\{|f(x)| \mid x \in d(0, R)\}.$$

Let $f \in \mathcal{A}(d(0, R^-))$. Let $(a_n)_{n \in \mathbb{N}^*}$ be the sequence of zeros of f with $0 < |a_n| \leq |a_{n+1}|$, with a_0 whenever $f(0) = 0$ and let k_n denote the order of the zero a_n. Moreover, if $f(0) = 0$, we denote by k_0 the order of the origine as a zero and we set $l_0 = 1$. And if $f(0) \neq 0$ we set $k_0 = l_0 = 0$.

Then we define the counting function of zeros of f as

$$Z(r,f) = k_0 \log r + \sum_{|a_n| \le r} k_n(\log r - \log |a_n|).$$

Respectively, let the counting function ignoring multiplicities be defined as $\overline{Z}(r,f) = l_0 \log r + \sum_{|a_n|\le r}(\log r - \log|a_n|)$.

Remark 1.2. As far as analytic functions are involved, it is now easy to state the second main theorem by only using the function $Z(r,f)$, this function being equivalent to the characteristic function $T(r,f)$ for analytic functions.

Definition 1.3. We call a function $\omega \in \mathcal{A}(d(0,R^-))$ *a small function with respect to* $f \in \mathcal{A}(d(0,R^-))$, if

$$\lim_{r\to R} \frac{Z(r,\omega)}{Z(r,f)} = 0.$$

Acknowledgement. The paper was begun during a short stay at Université Blaise Pascal.

2. Second main theorem for small functions

The following lemma is a special case of Theorem 2.6.1 in [5].

Lemma 2.1. *Let* $f \in \mathcal{A}(d(0,R^-))$ *and* $\omega \in \mathcal{A}(d(0,R^-))$ *and let* ω *be a small function of* f. *Then we have*

$$Z(r, f-\omega) = Z(r,f) + O(1).$$

Lemma 2.2. *For* $f \in \mathcal{A}(d(0,R^-))$ *we have*

$$Z(r,f') \le Z(r,f) - \log r.$$

Proof. For analytic functions, it is obvious that $Z(r,fg) = Z(r,f) + Z(r,g)$. Hence $Z(r,f) = Z(r,f/g) + Z(r,g)$. The lemma now follows from Corollary 2.4.14 in [5], which says that $Z(r,f'/f) \le -\log r$. □

By Theorem 23.7 in [4], we have

Lemma 2.3. *Let* $f, g \in \mathcal{A}(d(0,R^-))$, *let* $r > 0$ *and assume that* $\phi_{a,r}(f) > \phi_{a,r}(g)$. *Then* f *and* $f+g$ *have the same number of zeros (counting multiplicity) in* $d(a,r)$.

Theorem 2.4. *Let $f \in \mathcal{A}_u(d(0,R^-))$ and let $\omega_j \in \mathcal{A}_b(d(0,R^-))$ be of the form $\omega_j = a_j + \theta_j$, $j = 1, \ldots, q$, where $a_j \in K$ and*

$$||\theta_j|| < \min_{j \neq k} |a_j - a_k|.$$

Then

$$(q-1)Z(r,f) \leq \sum_{j=1}^{q} \overline{Z}(r, f - \omega_j) + O(1). \tag{2.1}$$

Proof. Let $\omega_j = a_j + \theta_j$, $1 \leq j \leq q$ and let

$$||\theta_j|| < \min_{k \neq j} |a_j - a_k| =: t_j. \tag{2.2}$$

Let $f \in \mathcal{A}_u(d(0,R^-))$ and let $\alpha \in d(0,R^-)$ be such that $f(\alpha) = \omega_j(\alpha)$ for some $1 \leq j \leq q$.

Since $\phi_{\alpha,r}(f)$ is an increasing function of r, and tends to infinity as r tends to R, there exists a unique radius $\rho_{\alpha,j}$ for which

$$\phi_{\alpha,\rho_{\alpha,j}}(f - a_j) = t_j.$$

By hypothesis (2.2), it follows that $f - \omega_k$ has no zeros in $d(\alpha, \rho_{\alpha,j}^-)$ for $k \neq j$. On the other hand, since $\phi_{\alpha,\rho_{\alpha,j}}(f - a_j) > ||\theta_j||$, the number of zeros of $f - a_j$ in $d(\alpha, \rho_{\alpha,j}^-)$ is equal to this of $f - \omega_j$, see Lemma 2.3.

Now consider the circle $C(0,r)$ for fixed r, and consider the possibly several points $\alpha_{m,j} \in C(0,r)$, $1 \leq m \leq u_j$, $1 \leq j \leq q$, such that $f(\alpha_{m,j}) = \omega_j(\alpha_{m,j})$. For each indices m and j, let $\sigma_{m,j}$ denote the number $\rho_{\alpha_{m,j},j}$ previously defined as $\rho_{\alpha,j}$ for α.

Now the disks $d(\alpha_{m,j}, \sigma_{m,j}^-)$ are pairwise disjoint.

For every $m = 1, \ldots, u_j$ and $j = 1, \ldots, q$, let $\nu_{m,j}$ be the number of zeros of $f - \omega_j$ in $d(\alpha_{m,j}, \sigma_{m,j}^-)$ counting multiplicities, and let $\overline{\nu}_{m,j}$ be this number ignoring multiplicities. Hence $\nu_{m,j}$ is also the number of zeros of $f - a_j$ in $d(\alpha_{m,j}, \sigma_{m,j}^-)$ counting multiplicities.

Since K has residue characteristic zero, it follows from the above that the number of zeros of f' in each disk $d(\alpha_{m,j}, \sigma_{m,j}^-)$ is exactly $\nu_{m,j} - 1$.

Hence for any $r \in]0, R[$, we have

$$\begin{aligned}\eta(r, f') &\geq \sum_{j=1}^{q}\left(\sum_{m=1}^{u_j}(\nu_{m,j} - 1)\right)\\ &\geq \sum_{j=1}^{q}\left(\sum_{m=1}^{u_j}(\nu_{m,j} - \overline{\nu}_{m,j})\right)\\ &\geq \sum_{j=1}^{q}(\eta(r, f - \omega_j) - \overline{\eta}(r, f - \omega_j)).\end{aligned}$$

Adding all the inequalities together, this finally yields

$$\sum_{j=1}^{q}(Z(r, f - \omega_j) - \overline{Z}(r, f - \omega_j)) \leq Z(r, f').$$

Since by Lemma 2.1 we have $Z(r, f - \omega_j) = Z(r, f) + O(1)$ and by Lemma 2.2 we have $Z(r, f') \leq Z(r, f) + O(1)$, we obtain

$$(q-1)Z(r, f) \leq \sum_{j=1}^{q} \overline{Z}(r, f - \omega_j) + O(1). \tag{2.3}$$

□

Writing (2.3) in the form

$$\frac{(q-1)Z(r, f)}{Z(r, f)} \leq \sum_{j=1}^{q} \frac{\overline{Z}(r, f - \omega_j)}{Z(r, f)} + o(1),$$

we can easily see that if, e.g., all the zeros of $f - \omega_j$ and $f - \omega_k$ are double for some $j \neq k$, then almost all the zeros of $f - \omega_i$ must be simple for any $i \neq j, k$.

Corollary 2.5. *At most two of the functions $f - \omega_i$ may be such that all their zeros are double.*

Remark 2.6. In any algebraically closed complete field, we have this important property on meromorphic functions:

$$|f'|(r) \leq \frac{|f|(r)}{r}.$$

Now, the equality holds when the residue characteristic is 0, provided the number of zeros and this of poles in $d(0, r)$ are not equal. However, such a property does not hold when the residue characteristic is $p \neq 0$. In the proof of the theorem, the equality is crucial. This is why we presently can't generalize the theorem to fields with a non-zero residue characteristic.

On the other hand, we have used the fact that $|f|(r)$ is an increasing property for analytic functions inside a disk. This is no longer true for meromorphic functions. Therefore, generalizing the theorem to meromorphic functions does not seem possible in this way.

Acknowledgement. The author would like to thank Alain Escassut for useful conversations.

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SECOND MAIN THEOREM FOR SMALL FUNCTIONS

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