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Abstract

We prove that Huygens' principle and the principle of equipartition of energy hold for the modified wave equation on odd dimensional Damek–Ricci spaces. We also prove a Paley–Wiener type theorem for the inverse of the Helgason Fourier transform on Damek–Ricci spaces.

1. Introduction

On a Riemannian manifold X, consider the Cauchy problem for the modified wave equation

$$\begin{cases} (\mathcal{L}+c)u = u_{tt} \\ u(\cdot,0) = f \\ u_t(\cdot,0) = g \end{cases} \qquad f,g \in C_c^\infty(X), \tag{1.1}$$

where $\mathcal{L}$ is the Laplace–Beltrami operator on X and c is a suitable constant. Huygens' principle is said to hold for (1.1) if the support of the solution $(x,t) \longmapsto u(x,t)$ is contained in the set

$$\{(x,t) \in X \times (0,+\infty) \ : \ t-R < d(x,x_0) < t+R\}$$

whenever f and g are supported in the geodesic ball of radius R centered in x_0. Hadamard [15] raised the question of finding all Riemannian manifolds X for which the Cauchy problem (1.1) satisfies Huygens' principle. He proved that the fact that the dimension of X is odd is a necessary condition. In this paper we give a simple proof of the converse for Damek–Ricci spaces and we show the exponential and strict version of the principle of equipartition of energy. We also generalize to the class

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of Damek–Ricci spaces a Paley–Wiener type theorem for the inverse Helgason Fourier transform proved by N. Andersen [1] for noncompact symmetric spaces.

We recall that Damek–Ricci spaces form a large class of harmonic Riemannian manifolds which includes all real rank one symmetric spaces of the noncompact type; except from these, Damek–Ricci spaces are nonsymmetric harmonic manifolds [13]. Although radial analysis is essentially the same for symmetric and nonsymmetric Damek–Ricci spaces (see [2] and the references therein), the study of nonradial analysis is often much more complicated in the nonsymmetric case. This is due to the lack of a group acting transitively by isometries on geodesic spheres [11].

T. Branson, G. Ólafsson, and H. Schlichtkrull [9] studied Huygens' principle and principle of equipartition of energy for the modified wave equation on noncompact symmetric spaces. We follow their approach applying the following result proved in [5, 4] (see Theorem 2.2 below): the Helgason Fourier transform of a smooth function with compact support on a Damek–Ricci space S is an entire function of exponential type. Until now the converse of this statement for nonsymmetric Damek–Ricci spaces is an open problem (see [16] for the proof of the Paley–Wiener theorem in the symmetric case and [3, 20] for partial results in the nonsymmetric case). We recall that the Helgason Fourier transform $\mathcal{F}f$ of an integrable function f on a Damek–Ricci space S is a scalar function defined on $\mathbb{R} \times \partial S$, where ∂S is the boundary of S. The Helgason Fourier transform reduces to the spherical transform for radial functions. In this paper we characterize the space of square integrable functions on S whose Helgason Fourier transform has bounded support in the real variable.

We also would like to mention that Huygens' principle for the radial part of the Laplace–Beltrami operator on Damek–Ricci spaces has been studied by J. El Kamel and C. Yacoub [14] and by Branson, Ólafsson and A. Pasquale [8] in the general context of Jacobi operators. Moreover, the case of the Dunkl-Cherednik Laplacian has been studied by F. Ayadi [6] and S. Ben Saïd [7].

In [18] M. Noguchi describes the solution of the modified wave equation on Damek–Ricci spaces in terms of means over geodesic spheres and the heat kernel. As an application, he shows that Huygens' principle holds on odd dimensional Damek–Ricci spaces. However, this approach requires the use of lengthy computations.

The paper is organized as follows. In Section 2 we introduce the basic notions on Damek–Ricci spaces and recall the most relevant facts about the Helgason Fourier transform. In Section 3 and 4 we deal respectively with Huygens' principle and the principle of equipartition of energy. In the last section we prove our Paley–Wiener type theorem for the inverse Fourier transform.

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2. Notation and preliminaries

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra, with an inner product $\langle\cdot,\cdot\rangle$. Write $\mathfrak{n}$ as an orthogonal sum $\mathfrak{n}=\mathfrak{v}\oplus\mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{n}$. For each Z in $\mathfrak{z}$, define the map $J_Z:\mathfrak{v}\longrightarrow\mathfrak{v}$ by the formula

$$\langle J_Z X, Y\rangle = \langle [X,Y], Z\rangle \qquad \forall X,Y\in\mathfrak{v}.$$

Following Kaplan [17], we say that the Lie algebra $\mathfrak{n}$ is of Heisenberg type, or H–type for short, if

$$J_Z^2 = -|Z|^2 I_{\mathfrak{v}} \qquad \forall Z\in\mathfrak{z}, \tag{2.1}$$

where $I_{\mathfrak{v}}$ is the identity on $\mathfrak{v}$. A connected and simply connected Lie group N whose Lie algebra is an H–type algebra is said to be a Heisenberg type group, or H–type for short. The Iwasawa N-groups associated to all real rank one simple groups are H–type. Note that from property (2.1), it follows that $\mathfrak{z}=[\mathfrak{v},\mathfrak{v}]$, and moreover the dimension of $\mathfrak{v}$ is even, $2m$ say. We denote by Q the number $m+k$, where k is the dimension of the centre $\mathfrak{z}$. Since N is a nilpotent Lie group, the exponential mapping is surjective. We denote by (X,Z), with X in $\mathfrak{v}$ and Z in $\mathfrak{z}$, the element $\exp(X+Z)$ of the group N. By the Baker–Campbell–Hausdorff formula the group law in N is given by

$$(X,Z)\,(X',Z') = \Big(X+X'\,,Z+Z'+\tfrac{1}{2}[X,X']\Big)$$

for every X,X' in $\mathfrak{v}$ and Z,Z' in $\mathfrak{z}$.

The Iwasawa N-groups are characterized, among all H–type groups, by an algebraic condition, called the J^2-condition [10].

Let $\mathfrak{n}$ be an H–type algebra, and $\mathfrak{a}$ be a one-dimensional real Lie algebra with an inner product, spanned by the unit vector H. We extend the Lie bracket to $\mathfrak{n} \oplus \mathfrak{a}$ by linearity and the requirement that

$$\begin{aligned} [H, X] &= \tfrac{1}{2} X \quad \forall X \in \mathfrak{v} \\ [H, Z] &= Z \quad\ \ \forall Z \in \mathfrak{z} \end{aligned}$$

We extend the inner products on $\mathfrak{n}$ and $\mathfrak{a}$ to an inner product on $\mathfrak{n} \oplus \mathfrak{a}$ by requiring that $\mathfrak{n}$ and $\mathfrak{a}$ be orthogonal. Note that $\mathfrak{n}$ is an ideal in $\mathfrak{n} \oplus \mathfrak{a}$.

Let S denote the connected, simply connected Lie group with Lie algebra $\mathfrak{n} \oplus \mathfrak{a}$. The subgroups $\exp(\mathfrak{a})$ and $\exp(\mathfrak{n})$ are closed, connected and simply connected, and will be denoted by $A \simeq \mathbb{R}^+$ and N respectively. Then S may and will be identified with the semidirect product $S = NA$ of N and A. As customary, we write (X, Z, a) for the element $na = \exp(X + Z)\, a$ of S. The product law in S is given by

$$\begin{aligned} (X, Z, a)\,(X', Z', a') &= n(an'a^{-1})\, aa' \\ &= \left(X + a^{\frac{1}{2}} X'\,, Z + Z' + \tfrac{1}{2} a^{\frac{1}{2}} [X, X'], aa'\right) \end{aligned}$$

for every $na = (X, Z, a)$ and $n'a' = (X', Z', a')$ in S.

We equip S with the left-invariant Riemannian metric which coincides with the inner product on $\mathfrak{n} \oplus \mathfrak{a}$ defined above when $\mathfrak{n} \oplus \mathfrak{a}$ is viewed as the tangent space to S at the identity o. The boundary ∂S of S, i.e., the set of endpoints of all geodesics, may be identified with (the one-point compactification of) N, see [4].

The geodesic distance of the point $x = (X, Z, a)$ from the identity $o = (0, 0, 1)$ of S is (see [13])

$$d(x, o) = \log \frac{1 + r(x)}{1 - r(x)}, \tag{2.2}$$

where $0 \leq r(x) < 1$ and $r(X, Z, a)^2 = 1 - 4a \left[\left(1 + a + \frac{|X|^2}{4}\right)^2 + |Z|^2 \right]^{-1}$.

A function f on S is said to be radial if, for all x in S, $f(x)$ depends only on the geodesic distance $d(x, o)$ from the identity o. We recall that a spherical function on S is a radial eigenfunction of the Laplace–Beltrami operator $\mathcal{L}$ normalized to take value 1 at the identity o. For λ in $\mathbb{C}$ we denote by Φ_λ the spherical function such that $\mathcal{L}\Phi_\lambda = -\left(\lambda^2 + \frac{Q^2}{4}\right) \Phi_\lambda$.

The expression of Poisson kernel $\mathcal{P}$ (see [12]) is given by the formula

$$\mathcal{P}(na, n') = P_a(n^{-1}n'), \qquad \forall na \in S,\ n' \in N,$$

where, for any $a > 0$, P_a is the function on N defined by the rule

$$P_a(n) = P_a(X,Z) = a^Q \left(\left(a + \tfrac{|X|^2}{4} \right)^2 + |Z|^2 \right)^{-Q} \qquad \forall n = (X,Z) \in N.$$

We define the normalized Poisson kernel via the formula

$$\mathcal{Q}(x,n) = \frac{\mathcal{P}(x,n)}{\mathcal{P}(o,n)} \qquad \forall x \in S \quad \forall n \in N,$$

and we use the notation

$$\mathcal{Q}_\lambda = \mathcal{Q}^{1/2 - i\lambda/Q} \qquad \lambda \in \mathbb{C}.$$

Denote by $\mathcal{L}$ the Laplace–Beltrami operator on S. One can verify that

$$\mathcal{L}\mathcal{Q}_\lambda(x,\cdot) = -\left(\lambda^2 + \tfrac{Q^2}{4} \right) \mathcal{Q}_\lambda(x,\cdot) \qquad \forall \lambda \in \mathbb{C} \quad \forall x \in S. \tag{2.3}$$

We shall use the following relation, proved in [5], between $\mathcal{Q}_\lambda$ and the spherical function Φ_λ

$$\Phi_\lambda(y^{-1}x) = \int_N \mathcal{Q}_\lambda(x,n)\, \mathcal{Q}_{-\lambda}(y,n)\, P_1(n)\, dn \qquad \forall x,y \in S, \quad \forall \lambda \in \mathbb{C}. \tag{2.4}$$

We define the Fourier transform of a Schwartz function f on S by the rule

$$\mathcal{F}f(\lambda,n) = \int_S f(x)\, \mathcal{Q}_\lambda(x,n)\, dx \qquad \forall \lambda \in \mathbb{C}, \forall n \in N.$$

By (2.4) and by $\Phi_\lambda = \Phi_{-\lambda}$ one derives the following parity condition

$$\int_N \mathcal{Q}_\lambda(x,n)\, \mathcal{F}f(-\lambda,n)\, P_1(n)\, dn = \int_N \mathcal{Q}_{-\lambda}(x,n)\, \mathcal{F}f(\lambda,n)\, P_1(n)\, dn, \tag{2.5}$$

for every x in S and λ in $\mathbb{C}$. Let $\mathbf{c}(\lambda)$ be the analogous of Harish-Chandra's function, i.e., the meromorphic function given by

$$\mathbf{c}(\lambda) = \frac{2^{Q-2i\lambda}\, \Gamma(2i\lambda)\, \Gamma\left(\frac{2m+k+1}{2}\right)}{\Gamma\left(\frac{Q}{2} + i\lambda\right)\, \Gamma\left(\frac{m+1}{2} + i\lambda\right)}$$

The inversion and the Plancherel formulas for the Fourier transform are (see [5]):

$$\begin{aligned} f(x) &= \frac{1}{2} \int_{\mathbb{R}} \int_N \mathcal{F}f(\lambda,n)\, \mathcal{Q}_{-\lambda}(x,n)\, d\nu(\lambda,n) \\ \int_S |f(x)|^2\, dx &= \frac{1}{2} \int_{\mathbb{R}} \int_N |\mathcal{F}f(\lambda,n)|^2\, d\nu(\lambda,n), \end{aligned} \tag{2.6}$$

where the Plancherel measure $d\nu$ on $\mathbb{R} \times N$ is given by

$$d\nu(\lambda, n) = \frac{1}{2\pi} P_1(n) \, |\mathbf{c}(\lambda)|^{-2} dn d\lambda.$$

We recall the easy part of the Paley–Wiener Theorem for Damek–Ricci spaces proved in [5, 3]. Denote by $B_R = \{x \in S \ : \ d(x, o) \leq R\}$ the closed geodesic ball of radius R.

Definition 2.1. We say that a function $\psi : \mathbb{C} \times N \to \mathbb{C}$ satisfying the parity condition (2.5) is an entire function of exponential type corresponding to R if for every nonnegative integer j there exists a positive constant C_j such that

$$|\psi(\lambda, n)| \leq C_j \, (1 + |\lambda|)^{-j} \, e^{R \, |\mathrm{Im}(\lambda)|} \qquad \forall \lambda \in \mathbb{C}, \, n \in N.$$

Theorem 2.2. [5] *Let f be a C^∞ function on S supported in B_R. Then $\mathcal{F}f$ is an entire function of exponential type corresponding to R.*

3. Huygens' principle

By left-invariance, we can reduce the study of Huygens' principle for (1.1) to the case where $x_0 = o$. Let u be the solution of the Cauchy problem for the modified wave equation:

$$\begin{cases} (\mathcal{L} + \frac{Q^2}{4}) u = u_{tt} & \\ u(\cdot, 0) = f & t > 0, \\ u_t(\cdot, 0) = g & \end{cases} \tag{3.1}$$

where the initial data f and g are in $C_c^\infty(S)$. By the principle of *finite propagation speed* [19] the function $u(\cdot, t)$ is compactly supported for each fixed t; moreover the support of the solution $(x, t) \longmapsto u(x, t)$ is contained in the set

$$\{(x, t) \in S \times (0, +\infty) \ : \ d(x, o) < t + R\},$$

whenever f and g are supported in the geodesic ball B_R. Thus in order to show that Huygens' principle holds for the Cauchy problem (3.1) we need to prove that $u(x, t)$ vanishes for $d(x, o) > t - R$ whenever f and g are supported in the geodesic ball B_R.

Theorem 3.1. *Let u be the solution of the Cauchy problem (3.1). If the initial data f and g are supported in the closed geodesic ball B_R, then for all x in S and $t > 0$,*

i) *if* $\dim S$ *is odd*

$$|u(x,t)| \leq C\,\Phi_0(x)\,e^{-\alpha(t-d(x,o)-R)} \qquad \forall \alpha > 0;$$

ii) *if* $\dim S$ *is even*

$$|u(x,t)| \leq C\,\Phi_0(x)\,\left(\tfrac{Q}{2}-\alpha\right)^{-1} e^{-\alpha(t-d(x,o)-R)} \qquad \forall \alpha \in \left(0, \tfrac{Q}{2}\right).$$

Proof. Let u be the solution of the Cauchy problem (3.1). Since $u(\cdot,t)$ is compactly supported, by (2.3) the Cauchy problem (3.1) is equivalent to the following

$$\begin{cases} \mathcal{F}u_{tt}((\lambda,n);t) = -\lambda^2 u((\lambda,n);t) \\ \mathcal{F}u((\lambda,n);0) = \mathcal{F}f(\lambda,n) \\ \mathcal{F}u_t((\lambda,n);0) = \mathcal{F}g(\lambda,n) \end{cases} \qquad \forall (\lambda,n) \in \mathbb{R}\times N, \quad \forall t > 0, \quad (3.2)$$

where we use the notation $\mathcal{F}u((\lambda,n);t) = \mathcal{F}(u(\cdot,t))(\lambda,n)$. Then

$$\mathcal{F}u((\lambda,n);t) = \mathcal{F}f(\lambda,n)\,\cos\lambda t + \mathcal{F}g(\lambda,n)\,\frac{\sin\lambda t}{\lambda}.$$

Applying the inversion formula we obtain

$$u(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}}\left[F(\lambda,x)\,\cos\lambda t + G(\lambda,x)\frac{\sin\lambda t}{\lambda}\right]d\lambda, \qquad (3.3)$$

where for λ in $\mathbb{C}$ and x in S

$$F(\lambda,x) = |\mathbf{c}(\lambda)|^{-2}\int_N \mathcal{F}f(\lambda,n)\,\mathcal{Q}_{-\lambda}(x,n)\,P_1(n)dn = F(-\lambda,x)$$

and

$$G(\lambda,x) = |\mathbf{c}(\lambda)|^{-2}\int_N \mathcal{F}g(\lambda,n)\,\mathcal{Q}_{-\lambda}(x,n)\,P_1(n)dn = G(-\lambda,x).$$

The function $\lambda \longmapsto |\mathbf{c}(\lambda)|^{-2}$ has a zero of the second order in $\lambda = 0$, moreover it is a polynomial when k is even (i.e., $\dim S = 2m+k+1$ is odd) and it is meromorphic with simple poles in $\lambda \in \pm i\left(\frac{Q}{2}+\mathbb{N}\right)$ when k is odd. Therefore for every x in S the even function

$$\lambda \longmapsto F(\lambda,x)\,\cos\lambda t + G(\lambda,x)\frac{\sin\lambda t}{\lambda}$$

is entire when dim$S$ is odd and is holomorphic in $\left\{\lambda \in \mathbb{C} \ : \ |\mathrm{Im}(\lambda)| < \frac{Q}{2}\right\}$ when dim$S$ is even. This allows us to shift the integration from $\mathbb{R}$ to $\mathbb{R}+i\alpha$ in (3.3) with $\alpha > 0$ if dim$S$ is odd and $0 < \alpha < Q/2$ if dim$S$ is even:

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi}\int_{\mathbb{R}} \left(F(\lambda,x) + \frac{G(\lambda,x)}{i\lambda}\right) e^{i\lambda t} d\lambda \\ &= \frac{1}{4\pi} e^{-\alpha t} \int_{\mathbb{R}} \left(F(\lambda+i\alpha,x) + \frac{G(\lambda+i\alpha,x)}{i\lambda-\alpha}\right) e^{i\lambda t} d\lambda \end{aligned}$$

for all x in S and $t > 0$.

By Theorem 2.2 and by formula (2.4), for every nonnegative integer j there exists a positive constant C_j such that, whenever the function $|\mathbf{c}(\lambda)|^{-2}$ is well defined,

$$\begin{aligned} \left|F(\lambda,x)\right| &\le C_j \, |\mathbf{c}(\lambda)|^{-2} (1+|\lambda|)^{-j} e^{R\,|\mathrm{Im}(\lambda)|} \int_N \mathcal{Q}_{i\mathrm{Im}(\lambda)}(x,n)\, P_1(n)\, dn \\ &= C_j \, |\mathbf{c}(\lambda)|^{-2} (1+|\lambda|)^{-j} e^{R\,|\mathrm{Im}(\lambda)|}\, \Phi_{i\mathrm{Im}(\lambda)}(x). \end{aligned}$$

for any x in S. The same inequality holds for the function G.

Notice that when dim$S$ is even

$$|\mathbf{c}(\lambda)|^{-2} \le C \left(\tfrac{Q}{2} - |\mathrm{Im}(\lambda)|\right)^{-1} |\lambda|^2 (1+|\lambda|)^{2m+k-2} \qquad |\mathrm{Im}(\lambda)| < Q/2$$

while when dim$S$ is odd

$$|\mathbf{c}(\lambda)|^{-2} \le C\,|\lambda|^2\,(1+|\lambda|)^{2m+k-2} \qquad \forall \lambda \in \mathbb{C}.$$

Therefore there exists a positive constant C, independent of α, such that if dim$S$ is even

$$\left|F(\lambda+i\alpha,x) + \frac{G(\lambda+i\alpha,x)}{i\lambda-\alpha}\right| \le C\, \frac{e^{\alpha R}\,\Phi_{i\alpha}(x)}{(Q/2-\alpha)\,(1+\lambda^2)} \qquad \forall \alpha \in (0, Q/2)$$

and if dim$S$ is odd

$$\left|F(\lambda+i\alpha,x) + \frac{G(\lambda+i\alpha,x)}{i\lambda-\alpha}\right| \le C\, \frac{e^{\alpha R}\,\Phi_{i\alpha}(x)}{(1+\lambda^2)} \qquad \forall \alpha > 0,$$

for every λ in $\mathbb{R}$.

We claim that

$$\left|\Phi_\lambda(x)\right| \le e^{|\mathrm{Im}(\lambda)| d(x,o)}\,\Phi_0(x) \qquad \forall \lambda \in \mathbb{C}, \quad \forall x \in S\,.$$

From this the thesis follows easily, so we now prove the claim. It was noted in [4] that

$$\mathcal{Q}(x,n) \le e^{Q\,d(x,o)} \qquad \forall x \in S,\, n \in N,$$

from which we obtain that, when $\mathrm{Im}(\lambda) > 0$,

$$|\mathcal{Q}_\lambda(x,n)| \leq \mathcal{Q}_0(x,n)\, e^{\mathrm{Im}(\lambda)\, d(x,o)} \qquad \forall x \in S,\, n \in N. \tag{3.4}$$

Since $\mathcal{Q}_\lambda(\cdot, n)$ is an eigenfunction of the Laplace–Beltrami operator (see formula (2.3)) and $\mathcal{Q}(o,n) = 1$ for every n in N, we conclude that $\Phi_\lambda = \mathcal{R}\mathcal{Q}_\lambda(\cdot, n)$ for every n in N and λ in $\mathbb{C}$, where $\mathcal{R}$ is the "averaging projector" over the geodesic spheres introduced in [13]. The claim now follows from (3.4) and the parity relation $\Phi_\lambda = \Phi_{-\lambda}$. □

Since $0 \leq \Phi_0(x) \leq 1$ for every x in S, Theorem 3.1 implies that

if $\dim S$ is odd

$$|u(x,t)| \leq C\, e^{-\alpha(t-d(x,o)-R)} \qquad \forall \alpha > 0$$

if $\dim S$ is even

$$|u(x,t)| \leq C\, \left(\tfrac{Q}{2} - \alpha\right)^{-1} e^{-\alpha(t-d(x,o)-R)} \qquad \forall \alpha \in \left(0, \tfrac{Q}{2}\right).$$

Corollary 3.2. *Let* $\dim S$ *be odd and let* u *be the solution of the Cauchy problem (3.1). If the initial data* f *and* g *are supported in the closed geodesic ball* B_R*, then*

$$|u(x,t)| = 0 \quad \text{if} \quad d(x,o) > t - R.$$

Proof. We let α tend to infinity in $|u(x,t)| \leq C\,\Phi_0(x)\, e^{\alpha(t-d(x,o)-R)}$. □

As already remarked, if the initial data f and g are supported in the closed geodesic ball B_R, the support of the solution $(x,t) \longmapsto u(x,t)$ is contained in the set $\{(x,t) \in S \times (0,+\infty) \,:\, d(x,o) < t + R\}$. So Corollary 3.2 shows that Huygens' principle holds for the Cauchy problem (3.1) on odd dimensional Damek–Ricci spaces.

4. Equipartition of energy

In this section we show that the difference between the kinetic $\mathbf{K}[u]$ and the potential energy $\mathbf{P}[u]$ of the solution u of the Cauchy problem (3.1) decays exponentially when the time grows. The kinetic and the potential energies are defined as follows

$$\mathbf{K}[u](t) = \tfrac{1}{2}\int_S |u_t(x,t)|^2\, dx$$

$$\mathbf{P}[u](t) = -\tfrac{1}{2}\int_S \left[\left(\mathcal{L} + \tfrac{Q^2}{4}\right) u\right](x,t)\; \overline{u(x,t)}\, dx.$$

By formula (3.3) and by Parseval's formula we obtain

$$\begin{aligned}\mathbf{K}[u](t) &= \tfrac{1}{4}\int_{\mathbb{R}\times N} |-\lambda\,\mathcal{F}f(\lambda,n)\,\sin\lambda t + \mathcal{F}g(\lambda,n)\cos\lambda t|^2\,d\nu(\lambda,n)\\ \mathbf{P}[u](t) &= \tfrac{1}{4}\int_{\mathbb{R}\times N} |\lambda\,\mathcal{F}f(\lambda,n)\,\cos\lambda t + \mathcal{F}g(\lambda,n)\sin\lambda t|^2\,d\nu(\lambda,n).\end{aligned} \tag{4.1}$$

Observe that the total energy $\mathbf{K}[u](t)+\mathbf{P}[u](t)$ is independent of the time t, is non negative and vanishes if and only if u vanishes identically.

Theorem 4.1. *Let u be the solution of the Cauchy problem (3.1). If the initial data f and g are supported in the closed geodesic ball B_R, then*

i) *if dim$S$ is odd*

$$|\mathbf{K}[u](t)-\mathbf{P}[u](t)| \le C\,e^{-2\alpha(t-R)} \qquad \forall t>0 \quad \forall\alpha>0;$$

ii) *if dim$S$ is even*

$$|\mathbf{K}[u](t)-\mathbf{P}[u](t)| \le C\,\left(\tfrac{Q}{2}-\alpha\right)^{-1} e^{-2\alpha(t-R)} \qquad \forall t>0 \quad \forall\alpha\in\left(0,\tfrac{Q}{2}\right).$$

Proof. By (4.1) an easy computation gives

$$|\mathbf{K}[u](t)-\mathbf{P}[u](t)| = \int_{\mathbb{R}} \left(\Theta(\lambda)+i\lambda\Psi(\lambda)\right)\,e^{i2\lambda t}d\lambda \qquad \forall t>0,$$

where

$$\begin{aligned}\Theta(\lambda) &= \frac{1}{8\pi}|\mathbf{c}(\lambda)|^{-2}\int_N \left(-|\lambda\mathcal{F}f(\lambda,n)|^2+|\mathcal{F}g(\lambda,n)|^2\right)\,P_1(n)dn\\ \Psi(\lambda) &= \frac{1}{8\pi}|\mathbf{c}(\lambda)|^{-2}\int_N \mathrm{Re}\left(\mathcal{F}f(\lambda,n)\overline{\mathcal{F}g(\lambda,n)}\right)\,P_1(n)dn.\end{aligned}$$

The thesis follows using the same arguments as in Theorem 3.1. □

Corollary 4.2. *Let* dim$S$ *be odd and let u be the solution of the Cauchy problem (3.1). If the initial data f and g are supported in the closed geodesic ball B_R, then*

$$|\mathbf{K}[u](t)-\mathbf{P}[u](t)| = 0 \qquad \forall t>R.$$

5. **Paley–Wiener type Theorem**

In [5] with R. Camporesi we proved that the Fourier transform $\mathcal{F}$ extends to an isometry from $L^2(S)$ onto the space $L^2(\mathbb{R}^+ \times N, d\nu)$, therefore the inverse Fourier transform $\mathcal{F}^{-1}$ is well defined from $L^2(\mathbb{R}^+ \times N, d\nu)$ onto $L^2(S)$. In this section we determine the image under $\mathcal{F}^{-1}$ of the space of functions in $L^2(\mathbb{R}^+ \times N, d\nu)$ with bounded support in the real variable.

We define the support, $\operatorname{supp} g$, of a ν–measurable function g on $\mathbb{R}^+ \times N$ to be the smallest closed set in $\mathbb{R}^+ \times N$, outside which the function g vanishes almost everywhere and we write

$$R_g = \sup_{(\lambda,n)\in \operatorname{supp} g} |\lambda|.$$

Note that we may have $R_g = +\infty$.

Lemma 5.1. *Let g be a function on $\mathbb{R}^+ \times N$ such that $(\lambda, n) \longmapsto \lambda^j\, g(\lambda, n)$ belongs to $L^2(\mathbb{R}^+ \times N, d\nu)$, for every nonnegative integer j. Then*

$$R_g = \lim_{j\to+\infty} \left\{ \int_0^{+\infty} \int_N \lambda^{2j} |g(\lambda, n)|^2 \, d\nu(\lambda, n) \right\}^{1/2j}.$$

Proof. First suppose that R_g is finite and let $0 < \epsilon < R_g$. Then

$$\begin{aligned} \liminf_{j\to+\infty} &\left\{ \int_0^{+\infty} \int_N \lambda^{2j} |g(\lambda, n)|^2 \, d\nu(\lambda, n) \right\}^{1/2j} \\ &\geq \liminf_{j\to+\infty} \left\{ \int_{R_g-\epsilon}^{R_g} \int_N \lambda^{2j} |g(\lambda, n)|^2 \, d\nu(\lambda, n) \right\}^{1/2j} \\ &\geq R_g - \epsilon. \end{aligned}$$

Moreover

$$\begin{aligned} \limsup_{j\to+\infty} \left\{ \int_0^{+\infty} \int_N \lambda^{2j} |g(\lambda, n)|^2 \, d\nu(\lambda, n) \right\}^{1/2j} &\leq R_g \limsup_{j\to+\infty} \|g\|^{1/j}_{L^2(\mathbb{R}^+\times N, d\nu)} \\ &= R_g. \end{aligned}$$

Thus the thesis follows in the case where R_g is finite.

Suppose now that $R_g = +\infty$. Then for every $M > 0$ we have

$$\int_M^{\infty} \int_N \lambda^{2j} |g(\lambda, n)|^2 \, d\nu(\lambda, n) > 0$$

and

$$\begin{aligned}\liminf_{j\to+\infty}&\left\{\int_0^{+\infty}\int_N \lambda^{2j}|g(\lambda,n)|^2\,d\nu(\lambda,n)\right\}^{1/2j}\\&\geq \liminf_{j\to+\infty}\left\{\int_M^{\infty}\int_N \lambda^{2j}|g(\lambda,n)|^2\,d\nu(\lambda,n)\right\}^{1/2j}\\&\geq M.\end{aligned}$$

The thesis follows in the case where $R_g=+\infty$. □

Definition 5.2. Let R be a positive real number. We say that a C^∞ function f on S is in the space $PW_R^2(S)$ if the following two conditions are satisfied

(1) $\mathcal{L}^j f$ belongs to $L^2(S)$ for every nonnegative integer j;

(2) $\lim_{j\to+\infty}\left\|\left(\mathcal{L}+\frac{Q^2}{4}\right)^j f\right\|_2^{1/2j}=R.$

Definition 5.3. Let R be a positive real number. We say that a function g in $L^2(\mathbb{R}^+\times N,d\nu)$ belongs to $L^2_R(\mathbb{R}^+\times N,d\nu)$ if $R_g=R$.

Theorem 5.4. *Let R be a positive real number. The inverse Fourier transform $\mathcal{F}^{-1}$ is a bijection of $L^2_R(\mathbb{R}^+\times N,d\nu)$ onto $PW^2_R(S)$.*

Proof. Let g be in $L^2_R(\mathbb{R}^+\times N,d\nu)$. Then $f=\mathcal{F}^{-1}g$ is smooth by the Lebesgue dominated convergence theorem. In order to prove that f belongs to $PW^2_R(S)$ we need to verify that the two conditions in Definition 5.2 are satisfied.

By the inversion formula (2.6) and by (2.3)

$$\mathcal{L}^j f(x)=(-1)^j\int_0^{+\infty}\int_N\left(\lambda^2+\tfrac{Q^2}{4}\right)^j g(\lambda,n)\,\mathcal{Q}_{-\lambda}(x,n)\,d\nu(\lambda,n).$$

Since $(\lambda,n)\longmapsto\left(\lambda^2+\frac{Q^2}{4}\right)^j g(\lambda,n)$ is in $L^2_R(\mathbb{R}^+\times N,d\nu)$, the function $\mathcal{L}^j f$ belongs to $L^2(S)$ for every non negative integer j. Moreover by the Plancherel Theorem and by Lemma 5.1 we have

$$\begin{aligned}\lim_{j\to+\infty}\left\|(\mathcal{L}+\tfrac{Q^2}{4})^j f\right\|_2^{1/2j}&=\lim_{j\to+\infty}\left\{\int_0^{+\infty}\int_N|(-\lambda^2)^j g(\lambda,n)|^2\,d\nu(\lambda,n)\right\}^{1/4j}\\&=R.\end{aligned}$$

This shows that f belongs to $PW^2_R(S)$.

Viceversa, let f be in $PW_R^2(S)$ and $g = \mathcal{F}f$. From the Plancherel Theorem it follows that the function $(\lambda, n) \in \mathbb{R}^+ \times N \longmapsto \lambda^{2j}\, g(\lambda, n)$ is in $L^2(\mathbb{R}^+ \times N, d\nu)$ for every nonnegative integer j, so that $R_g = R$ by Lemma 5.1. This shows that g belongs to $L_R^2(\mathbb{R}^+ \times N, d\nu)$. $\square$

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