ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 17, n° 1 (2010), p. 153-163.

<http://ambp.cedram.org/item?id=AMBP_2010__17_1_153_0>

Publication éditée par le laboratoire de mathématiques de l'université Blaise-Pascal, UMR 6620 du CNRS Clermont-Ferrand — France

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http://www.cedram.org/

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A note on uniform or Banach density

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Abstract

In this note we present and comment three equivalent definitions of the so called *uniform* or *Banach* density of a set of positive integers.

Introduction

The concept of density of an infinite set of positive integers appeared to be a basic tool to obtain an idea of the magnitude and of the structure of subsets of positive integers. During the last decades several types of densities have been introduced and studied. In light of this, both questions on relationships between various kinds of densities and various, but equivalent, ways to define a given type of density are interesting.

The most frequently used density concept is that of *asymptotic* (or *natural*) density. The *upper* (resp. *lower*) asymptotic density of a set A of positive integers, denoted by $\overline{d}(A)$ (resp. $\underline{d}(A)$), is the upper (lower) limit, as n tends to infinity, of the "local density" of A in the initial interval $[1,n]$, that is of

$$\frac{1}{n}\ |A \cap [1,n]\ |.$$

N.B.- If M is a finite subset of $\mathbb{N}$, then the symbol $|M|$ stands for the cardinality of M.

The aim of this note is to present and comment three equivalent definitions of the so called *uniform* or *Banach* density of a set of positive integers. Given a set $A \subseteq \mathbb{N}$, its upper and lower Banach densities, denoted by $\overline{b}(A)$ and $\underline{b}(A)$ (see definitions in sections 1 and 3, respectively), satisfy the relation

$$0 \leq \underline{b}(A) \leq \underline{d}(A) \leq \overline{d}(A) \leq \overline{b}(A) \leq 1. \tag{0.1}$$

Keywords: Banach density, uniform density.
Math. classification: 11B05.

It is easy to prove that these densities can have any prescribed values, belonging to the interval [0,1] and satisfying the inequalities in relation (0.1).

The principal characteristic of the uniform density is that it is more sensitive to local density in any interval, not necessarily initial, than the asymptotic density. For instance the set

$$A = \cup_{n\geq 1}\{n! + 1, n! + 2, \ldots, n! + n\},$$

having rare but sufficiently long blocks of consecutive integers, has asymptotic density zero while its upper uniform density equals to 1 and so its uniform density does not exist.

In what follows $\mathbb{N}$ denotes the set of all non negative integers and $\mathbb{N}^*$ the set of all positive integers. If s, t are integers, $0 \leq s \leq t$, then $I = [s, t] \cap \mathbb{N}$ denotes an interval in $\mathbb{N}$.

1. **The definitions and our setting**

1.1. **Banach density**

In [1], [2], [11] and [12], for instance, the notion of Banach density is used. If A is a subset of $\mathbb{N}$, then we put

$$E = E(A) = \{x \in [0,1]; \forall l \in \mathbb{N}^* \exists I \subseteq \mathbb{N}^* : |I| \geq l \wedge \frac{|A \cap I|}{|I|} \geq x\}.$$

Obviously, the set E is a subinterval of $[0,1]$ containing 0. The number $\overline{b}(A) = \sup E$ is called the (**upper**) **Banach density** of the set A. See also [13], [7, page 235] and [9, page 72] for a slightly different formulation of the above definition. For simplicity, we shall write b instead of $\overline{b}(A)$ in sections 1 and 2.

1.2. **Uniform density**

In the papers [3], [4] and [5], the following measure of magnitude of a set $A \subseteq \mathbb{N}$, based on translating intervals $[k, k+h] \cap \mathbb{N}$ of $\mathbb{N}$, was introduced. If k, h are integers, $0 \leq k, 0 \leq h$, then we put

$$A(k, k+h) = |A \cap [k, k+h]|.$$

For an integer s, $s \geq 1$, we set

$$\alpha^s = \limsup_{n\to\infty} A(n+1, n+s),\ \ \gamma^s = \sup_{n\geq 0} A(n+1, n+s).$$

The numbers α^s, γ^s are integers from the set $\{0, 1, \ldots, s\}$, satisfying the inequality $\alpha^s \leq \gamma^s$. The **upper uniform density** of A is defined either as (see [4])

$$a = \lim_{s\to\infty} \frac{\alpha^s}{s}$$

or as (see [3], [5])

$$c = \lim_{s\to\infty} \frac{\gamma^s}{s}.$$

The existence of these limits is often stated without proof or any reference to a proof (see [4], [5], [16]).

In [15] and [17] it is proved that the limit a exists. Proofs in those papers are of similar nature, each one using an euclidean division. P. Ribenboim [15] observes that the function $s \mapsto \alpha^s$ is subadditive and then proves the existence of the limit a. The existence of each of the limits a and c (but not their equality) follows directly from the subadditivity of each of the functions $s \mapsto \alpha^s$ and $s \mapsto \gamma^s$ by the following lemma, attributed to Fekete [8] (see also [14, 18, 6]).

Lemma 1.1. *If the real sequence* $(a_n)_{n\geq 1}$ *satisfies*

$$a_{m+n} \leq a_m + a_n$$

for all m *and* n, *then* $\lim_{n\to\infty} \frac{a_n}{n}$ *exists and is equal to* $\inf_{n\in\mathbb{N}^*} \frac{a_n}{n}$. *This value is either a real number or* $-\infty$.

In the paper [10] the equality $a = c$ is stated without proof. As it was said in section 1.1, some authors prefer the Banach density b. The advantage is that no proof of existence is needed. Of course in the definitions of a and c one can put $\limsup$ (or $\liminf$) instead of $\lim$. But for applications it is convenient to know that the limit exists. We haven't seen anywhere the statements $a = b$ and $b = c$. However, the last one is implicit in Bergelson' papers [1, 2, 3]).

1.3. **Our contribution**

We present in this note a *single* proof of the existence of a, of the existence of c and of the equalities

$$a = c = b = \inf_{s \in \mathbb{N}^*} \frac{\alpha^s}{s} = \inf_{s \in \mathbb{N}^*} \frac{\gamma^s}{s}.$$

Our proof does not use Lemma 1.1. It is partly inspired from a draft by the late Professor Tibor Šalát where he proved the existence of a.

Here is the approach. Let us introduce the following upper and lower limits :

$$\overline{a} = \limsup_{s \to \infty} \frac{\alpha^s}{s}, \ \underline{a} = \liminf_{s \to \infty} \frac{\alpha^s}{s},$$

$$\overline{c} = \limsup_{s \to \infty} \frac{\gamma^s}{s}, \ \underline{c} = \liminf_{s \to \infty} \frac{\gamma^s}{s}.$$

Clearly, the numbers defined above are related by

$$0 \leq \underline{a} \leq \min\{\overline{a}, \underline{c}\} \leq \max\{\overline{a}, \underline{c}\} \leq \overline{c} \leq 1.$$

Further let

$$\underline{\underline{a}} = \inf_{s \in \mathbb{N}^*} \frac{\alpha^s}{s}, \ \underline{\underline{c}} = \inf_{s \in \mathbb{N}^*} \frac{\gamma^s}{s}.$$

Obviously we have

$$0 \leq \underline{\underline{a}} \leq \min\{\underline{a}, \underline{\underline{c}}\} \leq \max\{\underline{a}, \underline{\underline{c}}\} \leq \underline{c} \leq \overline{c} \leq 1.$$

The main idea in our proof is to compare the above introduced quantities to b. We shall prove in the next section the following proposition.

Proposition 1.2. *We have* $b \leq \underline{\underline{a}}$ *and* $\overline{c} \leq b$.

Proposition 1.2 straightforwardly implies the following theorem.

Theorem 1.3. *We have* $\overline{a} = \underline{a}$ *and* $\overline{c} = \underline{c}$*, so both limits* a *and* c *exist and moreover* $a = b = c = \underline{\underline{a}} = \underline{\underline{c}}$.

Remark 1.4. The limit a is usually denoted by $\overline{u}$ (or $\overline{u}(A)$) and called the upper uniform density (of the set A) (see [5]). In the literature we have examined, the upper Banach density is also denoted by $BD(A)$ (see [12]).

2. Proof of Proposition 1.2

2.1. Demonstration of $\overline{c} \le b$

The case $\overline{c} = 0$ is trivial. Suppose $\overline{c} > 0$. It suffices to show that $[0, \overline{c}) \subseteq E$, i.e. that for each $x \in [0, 1]$, the following implication holds

$$0 \le x < \overline{c} \Longrightarrow x \in E.$$

Since zero belongs to E we may assume that $0 < x < \overline{c}$. By the definition of $\overline{c}$, there exists a sequence $(s_1 < s_2 < \dots)$ of integers such that for every $i \in \mathbb{N}^*$ we have

$$\frac{\gamma^{s_i}}{s_i} > x,$$

i.e.

$$\gamma^{s_i} = \sup_{n \ge 0} A(n+1, n+s_i) > x s_i.$$

The supremum being effectively a maximum, there exists $n_i \in \mathbb{N}$ satisfying

$$A(n_i + 1, n_i + s_i) > x s_i.$$

The last inequality implies that x belongs to E. $\square$

2.2. Demonstration of $b \le \underline{a}$

We need to prove that

$$b \le \inf_{s \ge 1} \frac{\alpha^s}{s}.$$

In other words, that for any $s \ge 1$, we have

$$b \le \frac{\alpha^s}{s}.$$

We proceed indirectly. Suppose that for some integer $s_0 \ge 1$, we have

$$\frac{\alpha^{s_0}}{s_0} < b.$$

Fix two reals $x_1, x_2 \in [0, 1]$ such that

$$\frac{\alpha^{s_0}}{s_0} < x_1 < x_2 < b.$$

We shall show that x_2 does not belong to E, which gives a contradiction.

There exists a multiple of s_0, say n_0, such that the following implication is true

$$n \geq n_0 \Longrightarrow A(n+1, n+s_0) < x_1 s_0. \tag{2.1}$$

In order to get the conclusion $x_2 \notin E$, it suffices to find an $l_0 \in \mathbb{N}^*$ such that for each interval $I \subseteq \mathbb{N}^*$ the following implication holds

$$|I| \geq l_0 \Longrightarrow |A \cap I| < x_2 |I|.$$

To this end we shall consider two cases depending on the form of the interval $I \subseteq \mathbb{N}^*$. First, let $I = [n+1, n+h]$, where $n \geq n_0$, $h \geq 1$. Taking into account that s_0 is a positive integer, h can be written in the form

$$h = qs_0 + r, \qquad \text{with} \qquad 0 \leq r < s_0.$$

Consequently

$$qs_0 \leq h < (q+1)s_0.$$

Owing to (2.1) and to the above inequality we get

$$\begin{aligned} |A \cap I| &\leq A(n+1, n+s_0) + A(n+s_0+1, n+2s_0) + \cdots + \\ &\quad A(n+qs_0+1, n+h) \leq \sum_{k=1}^{q+1} A(n+(k-1)s_0+1, n+ks_0) \\ &\leq (q+1)x_1 s_0. \end{aligned}$$

Second, if $n < n_0$ then the interval $I = [n+1, n+h]$ can be partitioned in at most $\frac{n_0+s_0}{s_0}$ subintervals with the same length s_0, each of which fails the upper bound in (2.1). Hence, in general we have

$$|A \cap I| \leq (q+1)x_1 s_0 + n_0 + s_0. \tag{2.2}$$

The right-hand side of (2.2) can be rewritten in the form

$$qx_1 s_0 + x_1 s_0 + s_0 + n_0 = hx_1 - rx_1 + x_1 s_0 + n_0 + s_0,$$

therefore

$$|A \cap I| \leq hx_1 + x_1 s_0 + n_0 + s_0 < hx_2,$$

under the condition

$$h > \frac{x_1 s_0 + n_0 + s_0}{x_2 - x_1}.$$

Putting $l_0 = \lfloor \frac{x_1 s_0 + n_0 + s_0}{x_2 - x_1} \rfloor + 1$ we obtain

$$|I| = h \geq l_0 \Longrightarrow |A \cap I| < x_2 |I|,$$

which finishes the proof. $\square$

3. **Lower Banach density**

Similar "dual" results to those we derived in the previous section are valid for lower density. The property dual to Theorem 1.3 results from the fact that the lower density of a set $A \subseteq \mathbb{N}$ is equal to 1 minus the upper density of the set $\widetilde{A} = \mathbb{N} \setminus A$. This is rather obvious for the uniform density (see Propositions 3.1 and 3.2 below) and it needs a short proof for the Banach density (see Proposition 3.3).

The concept of lower uniform density $\underline{u}$ can be defined for any set $A \subseteq \mathbb{N}$ using the numbers

$$\alpha_s = \liminf_{n\to\infty} A(n+1, n+s),\ \ \gamma_s = \inf_{n\geq 0} A(n+1, n+s).$$

Obviously, $\gamma_s \leq \alpha_s$ and these are integers belonging to $\{0, 1, \dots, s\}$.

Proposition 3.1. *The limits* $\lim_{s\to\infty} \frac{\alpha_s}{s}$, $\lim_{s\to\infty} \frac{\gamma_s}{s}$ *exist and are equal.*

Proof. We have

$$\widetilde{A}(n+1, n+s) = s - A(n+1, n+s).$$

If we denote by $\widetilde{\alpha}^s, \widetilde{\gamma}^s$ the values defined in section 1.2, corresponding to the set $\widetilde{A}$, then we get

$$\frac{\widetilde{\alpha}^s}{s} = 1 - \frac{\alpha_s}{s},\quad \frac{\widetilde{\gamma}^s}{s} = 1 - \frac{\gamma_s}{s}.$$

This gives

$$\lim_{s\to\infty} \frac{\alpha_s}{s} = 1 - \lim_{s\to\infty} \frac{\widetilde{\alpha}^s}{s} = 1 - \overline{u}(\widetilde{A}),$$

$$\lim_{s\to\infty} \frac{\gamma_s}{s} = 1 - \lim_{s\to\infty} \frac{\widetilde{\gamma}^s}{s} = 1 - \overline{u}(\widetilde{A}).$$

This completes the proof. □

The common value $\lim_{s\to\infty} \frac{\alpha_s}{s} = \lim_{s\to\infty} \frac{\gamma_s}{s} =: \underline{u}(A)$ is called the **lower uniform density** of the set A.

In a similar way one can verify the following property.

Proposition 3.2. *We have*

$$\underline{u}(A) = \sup_{s\geq 1} \frac{\alpha_s}{s} = \sup_{s\geq 1} \frac{\gamma_s}{s}.$$

Let us turn now to the Banach density. To define the lower Banach density $\underline{b}(A)$ of the set A, we introduce the set

$$F = F(A) = \{y \in [0,1]; \forall l \in \mathbb{N}^* \exists I \subseteq \mathbb{N}^* : |I| \geq l \wedge |A \cap I| \leq y|I|\}.$$

Obviously, the set F is a subinterval of $[0,1]$ containing 1.

Define the **lower Banach density** by

$$\underline{b} = \underline{b}(A) = \inf F.$$

The following proposition together with Proposition 3.1 and Theorem 1.3 show that the above definition of lower Banach density gives the equality $\underline{b}(A) = \underline{u}(A)$, as well.

Proposition 3.3. *For every $A \subseteq \mathbb{N}$ we have $\underline{b}(A) = 1 - \overline{b}(\mathbb{N} \setminus A) = 1 - \overline{b}(\widetilde{A})$.*

Proof. Let $A \subseteq \mathbb{N}$. For an interval I in $\mathbb{N}$ we set

$$\nu(I) = \frac{|A \cap I|}{|I|}, \qquad \widetilde{\nu}(I) = \frac{|\widetilde{A} \cap I|}{|I|}.$$

We have

$$\widetilde{\nu}(I) = 1 - \nu(I).$$

So the condition $\nu(I) \leq y$ in the definition of the set F is equivalent to $1 - \widetilde{\nu}(I) \leq y$, that is $\widetilde{\nu}(I) \geq 1 - y$. Recall that

$$\overline{b}(\widetilde{A}) = \sup \widetilde{E},$$

where

$$\widetilde{E} = \{x \in [0,1]; \forall l \in \mathbb{N}^* \exists I \subseteq \mathbb{N}^* : |I| \geq l \wedge \widetilde{\nu}(I) \geq x\}.$$

The above remark gives

$$y \in F \Longrightarrow 1 - y \in \widetilde{E}.$$

Similarly

$$x \in \widetilde{E} \Longrightarrow 1 - x \in F.$$

We conclude that

$$\inf F = 1 - \sup \widetilde{E}$$

which is the required assertion. $\square$

Corollary 3.4. *For every $A \subseteq \mathbb{N}$ we have $\underline{b}(A) = \underline{u}(A)$.*

Once the upper and lower densities are defined, the notion of density can be introduced as usual: A set A has **Banach density** provided that $\overline{b}(A) = \underline{b}(A)$, this common value is denoted by $b(A)$. We can conclude from (0.1) that whenever the Banach density of a set A exists it is equal to its asymptotic density $d(A)$. However, the concept of Banach density is interesting even if only the upper and lower Banach densities exist. It brings more precise information about the spacing of the elements in the set and also about the extremal distribution of its elements in intervals of arbitrary length.

Remark 3.5. The term "uniform density" is redundant in the light of proved results. Since the notion of Banach density has been introduced prior to the term "uniform density", we propose to use the term "Banach density".

Acknowledgements. We would like to thank the referee for his suggestions and constructive comments. The second and the third author are grateful to the Jean Monnet University in Saint-Etienne for hospitality and financial support during their visit that led to the completion of this work.

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