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Satoshi Ishiwata

Abstract

We obtain another proof of a Gaussian upper estimate for a gradient of the heat kernel on cofinite covering graphs whose covering transformation group has a polynomial volume growth. It is proved by using the temporal regularity of the discrete heat kernel obtained by Blunck [2] and Christ [3] along with the arguments of Dungey [7] on covering manifolds.

1. **Introduction**

Let $X = (V, E)$ be an oriented locally finite connected graph. We consider the reversible random walk on X defined by functions $p : E \to \mathbb{R}_{>0}$ and $m : V \to \mathbb{R}_{>0}$ satisfying

$$p(e)m(o(e)) = p(\overline{e})m(t(e))$$

and

$$\sum_{e \in E_x} p(e) = 1 \quad x \in V,$$

where $o(e)$ is the origin of e, $t(e)$ is the end of e, $\overline{e}$ is the inverse edge of e and $E_x = \{e \in E \,|\, o(e) = x\}$. Here, $p(e)$ is the probability that a particle at $o(e)$ moves to $t(e)$ in a unit time. The function m on V is a measure on V. Then the transition probability $p_n(x, y)$, $x, y \in V$ is given by

$$p_n(x, y) = \sum_{(e_1, e_2, \ldots, e_n) \in C_{x,n}, t(e_n) = y} p(e_1)p(e_2) \cdots p(e_n),$$

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where $C_{x,n}$ is the set of paths from x with length n. The transition operator P associated with the random walk generates a discrete semigroup $\{P^n\}_{n\in\mathbb{N}}$ acting on functions on V defined by

$$P^n f(x) = \sum_{y\in V} p_n(x,y) f(y).$$

Then the kernel of P^n on the weighted graph (X,m) is given by

$$k_n(x,y) = p_n(x,y)m(y)^{-1}.$$

The purpose of this paper is to obtain a Gaussian upper estimate for the gradient of k_n on X assuming that the latter admits a cofinite group action with polynomial volume growth. By the results of Gromov [9], without loss of generality, we can always assume that the covering transformation group Γ of X is a nilpotent group of order D. Moreover, we also assume that the random walk on such X is Γ-invariant, namely, $p : E \to \mathbb{R}_{>0}$ and $m : V \to \mathbb{R}_{>0}$ are Γ-invariant.

Under our assumptions, the Gaussian upper estimate for k_n

$$k_n(x,y) \leq C n^{-D/2} e^{-C' d(x,y)^2/n} \quad \forall x,y \in V \tag{1.1}$$

is known (see [12] and also [10]). Here $d(x,y)$ is the graph distance, the length of the shortest path from x to y and C, C' are some positive constants.

Moreover, the following theorem for the Gaussian upper estimate for the gradient of k_n has been proved by [12] along the method of [10]. Let ∇_1 be the gradient with respect to the first variable given by

$$\begin{aligned}
\nabla_1 k_n(x,y) &= \left(\sum_{d(x,\omega)\leq 1} |k_n(\omega,y) - k_n(x,y)|^2 p_1(x,\omega) \right)^{1/2} \\
&= \left(\sum_{e\in E_x} |dk_n(e,y)|^2 p(e) \right)^{1/2},
\end{aligned}$$

where d is the exterior derivative defined by

$$df(e) = f(t(e)) - f(o(e)), \quad e \in E$$

for a function f on V. Similarly we denote by $\nabla_2 k_n(x,y)$ the gradient with respect to the second variable, namely

$$\nabla_2 k_n(x,y) = \left(\sum_{d(y,\omega)\le 1} |k_n(x,\omega) - k_n(x,y)|^2 p_1(y,\omega) \right)^{1/2} = \left(\sum_{e\in E_y} |dk_n(x,e)|^2 p(e) \right)^{1/2}.$$

Then we have

Theorem 1.1. *Let $X = (V,E)$ be a non-bipartite covering graph whose covering transformation group has polynomial volume growth of order D. Then there exist C, $C' > 0$ such that*

$$\nabla_1 k_n(x,y) \le Cn^{-(D+1)/2} e^{-C' d(x,y)^2/n} \tag{1.2}$$

for $n \in \mathbb{N}^*$ and $x, y \in V$.

It should be noted that the estimate (1.2) is closely related to the boundedness of the Riesz transform (see [1], [13] and [11]). Let Δ be a discrete Laplacian on X given by $\Delta = I - P$. Then we have

Theorem 1.2. *Let X be as above. For $1 < p < \infty$, there exists $C_p > 0$ such that*

$$\|\nabla \Delta^{-1/2} f\|_{L^p} \le C_p \|f\|_{L^p}$$

for all finitely supported functions f on V. Here $\|\cdot\|_{L^p}$ is the L^p norm with respect to the measure m on V.

On the other hand, Dungey proved (1.2) for the heat kernel on covering manifolds with polynomial volume growth in [6]. Recently, in [7], he gave a new proof of (1.2) using the well-known temporal regularity of the heat kernel (see for instance [5]). In this paper, we give a shorter proof of Theorem 1.1 along with the arguments of the recent result by Dungey [7]. Let

$$\partial_1 k_n(x,y) = k_{n+1}(x,y) - k_n(x,y).$$

We use the following discrete version of the temporal regularity proved by Blunck [2], Christ [3]:

Theorem 1.3. *Let X be an oriented non-bipartite graph satisfying*

$$C^{-1} r^D \le m(B(x,r)) \le Cr^D \quad x \in V, r \ge 1$$

and (1.1), where $B(x,r)$ is the ball centered at $x \in V$ with radius r. Then there exist positive constants C and C' such that

$$\partial_1 k_n(x,y) \le C n^{-(D+2)/2} e^{-C' d(x,y)^2/n} \tag{1.3}$$

for $n \in \mathbb{N}^*$ and $x, y \in V$.

We remark that Dungey gave a short proof of (1.3) recently in [8].

2. Proof of Theorem 1.1

Let $\nu = m+n$ be a positive integer, where we choose $m = n$ or $m = n+1$ depending on whether ν is even or odd. By the Cauchy-Schwarz inequality, it is easy to see that

$$e^{rd^2(u,z)/\nu} \nabla_1 k_\nu(u,z) \le \|e^{2rd^2(u,\cdot)/n} \nabla_1 k_n(u,\cdot)\|_{L^2} \|e^{2rd^2(\cdot,z)/m} k_m(\cdot,z)\|_{L^2}$$

for $r > 0$.

By the Gaussian upper bound (1.1) for k_n, for small $r > 0$, there exists $C_r > 0$ such that

$$\|e^{2rd^2(\cdot,z)/m} k_m(\cdot,z)\|_{L^2} \le C_r m^{-D/4}. \tag{2.1}$$

Let $F \subset V$ be a fundamental domain for the action of the transformation group Γ on V. Namely, F is a subset in V such that for all $x \in V$, there exists a unique pair $\gamma_x \in \Gamma$ and $x_0 \in F$ satisfying $x = \gamma_x x_0$. Then we denote $F_x = \gamma_x F$. The following lemma gives a comparison of the weighted integral for $\nabla_1 k_n$ and $\nabla_2 k_n$. Similar arguments can be found in [12].

Lemma 2.1. *There exists a positive constant $C > 0$ such that*

$$\|e^{2rd^2(u,\cdot)/n} \nabla_1 k_n(u,\cdot)\|_{L^2}^2 \le C \sum_{y_0 \in F_u} \|e^{2rd^2(y_0,\cdot)/n} \nabla_2 k_n(y_0,\cdot)\|_{L^2}^2 m(y_0)$$

for all $u \in X$.

Proof. Since m is Γ-invariant, there exists $C > 0$ such that $C^{-1} < \min\{m(x) \mid x \in V\}$. Then we have

$$\begin{aligned}
&\|e^{2rd^2(u,\cdot)/n} \nabla_1 k_n(u,\cdot)\|_{L^2}^2 \\
&\quad \le C \sum_{v_0 \in F_u} \sum_{y \in V} \left| e^{2rd^2(v_0,y)/n} \nabla_1 k_n(v_0,y) \right|^2 m(y) m(v_0) \\
&\quad = C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} \left| e^{2rd^2(v_0,\gamma y_0)/n} \nabla_1 k_n(v_0,\gamma y_0) \right|^2 m(\gamma y_0) m(v_0).
\end{aligned}$$

From the Γ-invariance of the distance function d and k_n, the latter is

$$C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} \left| e^{2rd^2(\gamma^{-1} v_0, y_0)/n} \nabla_1 k_n(\gamma^{-1} v_0, y_0) \right|^2 m(y_0) m(\gamma^{-1} v_0).$$

By replacing γ^{-1} with γ in the sum of Γ, we get

$$C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} \left| e^{2rd^2(\gamma v_0, y_0)/n} \nabla_1 k_n(\gamma v_0, y_0) \right|^2 m(y_0) m(\gamma v_0).$$

Since $\nabla_1 k_n(\gamma v_0, y_0) = \nabla_2 k_n(y_0, \gamma v_0)$, this is also

$$\begin{aligned} &C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} \left| e^{2rd^2(y_0, \gamma v_0)/n} \nabla_2 k_n(y_0, \gamma v_0) \right|^2 m(\gamma v_0) m(y_0) \\ &= C \sum_{y_0 \in F_u} \sum_{v \in V} \left| e^{2rd^2(y_0, v)/n} \nabla_2 k_n(y_0, v) \right|^2 m(v) m(y_0) \\ &= C \sum_{y_0 \in F_u} \| e^{2rd^2(y_0, \cdot)/n} \nabla_2 k_n(y_0, \cdot) \|_{L^2}^2 m(y_0). \end{aligned}$$

□

Remark 2.2. In this proof, we use only the Γ-invariance of d, $\nabla_1 k_n$ and $\nabla_1 k_n(x,y) = \nabla_2 k_n(y,x)$. Therefore, there are other definitions of ∇ so that the previous lemma holds. For example, we can obtain the same results for

$$\nabla_1^p k_n(x,y) := \Big(\sum_{d(\omega,x) \le 1} |k_n(\omega,y) - k_n(x,y)|^p p_1(x,\omega) \Big)^{1/p}, \quad 1 < p < \infty,$$

$$\nabla_1^\infty k_n(x,y) := \sup_{d(\omega,x) \le 1} |k_n(\omega,y) - k_n(x,y)|,$$

which are comparable with each other.

By the same arguments as the continuous case ([4]), we obtain the following by "discrete integration by parts". Similar arguments can be found in [10] and [12].

Lemma 2.3. *For $e \in E$, let $m(e)$ be a weight of e defined by $m(e) = p(e) m(o(e))$. Then we have*

$$\begin{aligned} \| e^{2rd^2(y_0, \cdot)/n} \nabla_2 k_n(y_0, \cdot) \|_{L^2}^2 = &- \sum_{e \in E} d e^{4rd^2(y_0, e)/n} k_n(y_0, o(e)) d k_n(y_0, e) m(e) \\ &- 2 \sum_{v \in V} e^{4rd^2(y_0, v)/n} k_n(y_0, v) \partial_1 k_n(y_0, v) m(v). \end{aligned}$$

Proof.

$$\begin{aligned}
&\sum_{v\in V}\left|e^{2rd^2(y_0,v)/n}\nabla_2 k_n(y_0,v)\right|^2 m(v)\\
&=\sum_{v\in X} e^{4rd^2(y_0,v)/n}\sum_{e\in E_v}|dk_n(y_0,e)|^2 p(e)m(v)\\
&=\sum_{e\in E} e^{4rd^2(y_0,o(e))/n}|k_n(y_0,t(e))-k_n(y_0,o(e))|^2 p(e)m(o(e))\\
&=\sum_{e\in E} e^{4rd^2(y_0,o(e))/n}\Big(k_n(y_0,t(e))^2-2k_n(y_0,t(e))k_n(y_0,o(e))\\
&\qquad\qquad\qquad\qquad\qquad\qquad+k_n(y_0,o(e))^2\Big)m(e)\\
&=\sum_{e\in E} e^{4rd^2(y_0,o(e))/n}k_n(y_0,t(e))\left(k_n(y_0,t(e))-k_n(y_0,o(e))\right)m(e)\\
&\quad+\sum_{e\in E} e^{4rd^2(y_0,o(e))/n}k_n(y_0,o(e))\left(k_n(y_0,o(e))-k_n(y_0,t(e))\right)m(e).
\end{aligned}$$

Since $m(e)=m(\overline{e})$, by replacing e with $\overline{e}$ in the sum of E in the first term,

$$\begin{aligned}
&=\sum_{e\in E} e^{4rd^2(y_0,t(e))/n}k_n(y_0,o(e))\left(k_n(y_0,o(e))-k_n(y_0,t(e))\right)m(e)\\
&\quad+\sum_{e\in E} e^{4rd^2(y_0,o(e))/n}k_n(y_0,o(e))\left(k_n(y_0,o(e))-k_n(y_0,t(e))\right)m(e)\\
&=\sum_{e\in E}\left(e^{4rd^2(y_0,t(e))/n}-e^{4rd^2(y_0,o(e))/n}\right)k_n(y_0,o(e))\\
&\qquad\qquad\qquad\qquad\cdot\left(k_n(y_0,o(e))-k_n(y_0,t(e))\right)m(e)\\
&\quad+2\sum_{e\in E} e^{4rd^2(y_0,o(e))/n}k_n(y_0,o(e))\left(k_n(y_0,o(e))-k_n(y_0,t(e))\right)m(e)\\
&=-\sum_{e\in E} de^{4rd^2(y_0,e)/n}k_n(y_0,o(e))dk_n(y_0,e)m(e)\\
&\quad-2\sum_{v\in V} e^{4rd^2(y_0,v)/n}k_n(y_0,v)\sum_{e\in E_v}(k_n(y_0,t(e))-k_n(y_0,o(e)))p(e)m(v).
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{e\in E_v}(k_n(y_0,t(e))-k_n(y_0,o(e)))p(e)\\
&=\sum_{e\in E_v}k_n(y_0,t(e))p(e)-k_n(y_0,v)\\
&=k_{n+1}(y_0,v)-k_n(y_0,v)\\
&=\partial_1 k_n(y_0,v),
\end{aligned}$$

the lemma is proved. □

Finally, we apply the temporal regularity for k_n along with the argument of Lemma 2.3 in [4]. Let

$$\begin{aligned}
I(n,y_0)&=\sum_{v\in V}\left|e^{2rd^2(y_0,v)/n}\nabla_2 k_n(y_0,v)\right|^2 m(v),\\
I_1(n,y_0)&=-2\sum_{v\in V}e^{4rd^2(y_0,v)/n}k_n(y_0,v)\partial_1 k_n(y_0,v)m(v),\\
I_2(n,y_0)&=-\sum_{e\in E}de^{4rd^2(y_0,e)/n}k_n(y_0,o(e))dk_n(y_0,e)m(e).
\end{aligned}$$

Lemma 2.3 says that $I(n,y_0)=I_1(n,y_0)+I_2(n,y_0)$. Using (1.1) and (1.3), for sufficiently small $r>0$, there exists $C_r>0$ such that

$$|I_1(n,y_0)|\le C_r n^{-D/2-1}.$$

By the Cauchy-Schwarz inequality, we have that $|I_2(n,y_0)|$ is less than

$$\begin{aligned}
&\sum_{v\in V}\left(\sum_{e\in E_v}|de^{4rd^2(y_0,e)/n}|^2p(e)\right)^{1/2}\left(\sum_{e\in E_v}|dk_n(y_0,e)|^2p(e)\right)^{1/2}k_n(y_0,v)m(v)\\
&\quad=\sum_{v\in V}\left(\sum_{e\in E_v}|de^{4rd^2(y_0,e)/n}|^2p(e)\right)^{1/2}\nabla_2 k_n(y_0,v)k_n(y_0,v)m(v).
\end{aligned}$$

Using the mean value theorem for $f(x) = e^{4rx^2/n}$, there exists $d(y_0, o(e)) < s < d(y_0, t(e))$ such that

$$\begin{aligned}
e^{4rd^2(y_0,t(e))/n} - e^{4rd^2(y_0,o(e))/n} &= \frac{8rs}{n} e^{4rs^2/n}(d(y_0,t(e)) - d(y_0,o(e))) \\
&\leq \frac{8r(d(y_0,o(e))+1)}{n} e^{4r(d(y_0,o(e))+1)^2/n} \\
&\leq \frac{C_r}{\sqrt{n}} e^{16rd^2(y_0,o(e))/n}
\end{aligned}$$

for some $C_r > 0$. Then we have that $|I_2(n, y_0)|$ is less than

$$\begin{aligned}
&\frac{C_r}{\sqrt{n}} \sum_{v\in V} e^{16rd^2(y_0,v)/n} \nabla_2 k_n(y_0,v) k_n(y_0,v) m(v) \\
&\qquad \leq \frac{C_r}{\sqrt{n}} \left(\sum_{v\in V} e^{28rd^2(y_0,v)/n} |k_n(y_0,v)|^2 m(v) \right)^{1/2} \\
&\qquad\qquad \cdot \left(\sum_{v\in V} |e^{2rd^2(y_0,v)/n} \nabla_2 k_n(y_0,v)|^2 m(v) \right)^{1/2}.
\end{aligned}$$

For sufficiently small $r > 0$, we obtain

$$|I_2(n, y_0)| \leq C_r' n^{-D/4-1/2} I(n, y_0)^{1/2}.$$

Then we conclude that

$$I(n, y_0) \leq C_r n^{-D/2-1} + C_r' n^{-D/4-1/2} I(n, y_0)^{1/2},$$

namely,

$$I(n, y_0) \leq C n^{-D/4-1/2}$$

for some $C > 0$. Together with (2.1), this completes the proof of Theorem 1.1.

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