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Discrete version of Dungey’s proof for the gradient heat
kernel estimate on coverings


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Abstract

We obtain another proof of a Gaussian upper estimate for a gradient of the heat kernel on cofinite covering graphs whose covering transformation group has a polynomial volume growth. It is proved by using the temporal regularity of the discrete heat kernel obtained by Blunck [2] and Christ [3] along with the arguments of Dungey [7] on covering manifolds.

1. Introduction

Let $X = (V, E)$ be an oriented locally finite connected graph. We consider the reversible random walk on $X$ defined by functions $p : E \rightarrow \mathbb{R}_{>0}$ and $m : V \rightarrow \mathbb{R}_{>0}$ satisfying

$$p(e)m(o(e)) = p(\overline{e})m(t(e))$$

and

$$\sum_{e \in E_x} p(e) = 1 \quad x \in V,$$

where $o(e)$ is the origin of $e$, $t(e)$ is the end of $e$, $\overline{e}$ is the inverse edge of $e$ and $E_x = \{ e \in E \mid o(e) = x \}$. Here, $p(e)$ is the probability that a particle at $o(e)$ moves to $t(e)$ in a unit time. The function $m$ on $V$ is a measure on $V$. Then the transition probability $p_n(x, y)$, $x, y \in V$ is given by

$$p_n(x, y) = \sum_{(e_1, e_2, \ldots, e_n) \in C_{x, n}, t(e_n) = y} p(e_1)p(e_2) \cdots p(e_n),$$

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where $C_{x,n}$ is the set of paths from $x$ with length $n$. The transition operator $P$ associated with the random walk generates a discrete semigroup $\{P^n\}_{n \in \mathbb{N}}$ acting on functions on $V$ defined by

$$P^n f(x) = \sum_{y \in V} p_n(x, y) f(y).$$

Then the kernel of $P^n$ on the weighted graph $(X, m)$ is given by

$$k_n(x, y) = p_n(x, y)m(y)^{-1}.$$

The purpose of this paper is to obtain a Gaussian upper estimate for the gradient of $k_n$ on $X$ assuming that the latter admits a cofinite group action with polynomial volume growth. By the results of Gromov [9], without loss of generality, we can always assume that the covering transformation group $\Gamma$ of $X$ is a nilpotent group of order $D$. Moreover, we also assume that the random walk on such $X$ is $\Gamma$-invariant, namely, $p : E \to \mathbb{R}_{>0}$ and $m : V \to \mathbb{R}_{>0}$ are $\Gamma$-invariant.

Under our assumptions, the Gaussian upper estimate for $k_n$

$$k_n(x, y) \leq Cn^{-D/2}e^{-C'd(x,y)^2/n} \quad \forall x, y \in V$$

(1.1)

is known (see [12] and also [10]). Here $d(x, y)$ is the graph distance, the length of the shortest path from $x$ to $y$ and $C$, $C'$ are some positive constants.

Moreover, the following theorem for the Gaussian upper estimate for the gradient of $k_n$ has been proved by [12] along the method of [10]. Let $\nabla_1$ be the gradient with respect to the first variable given by

$$\nabla_1 k_n(x, y) = \left( \sum_{d(x, \omega) \leq 1} |k_n(\omega, y) - k_n(x, y)|^2 p_1(x, \omega) \right)^{1/2}$$

$$= \left( \sum_{e \in E_x} |dk_n(e, y)|^2 p(e) \right)^{1/2},$$

where $d$ is the exterior derivative defined by

$$df(e) = f(t(e)) - f(o(e)), \quad e \in E.$$
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for a function \( f \) on \( V \). Similarly we denote by \( \nabla_2 k_n(x, y) \) the gradient with respect to the second variable, namely

\[
\nabla_2 k_n(x, y) = \left( \sum_{d(y, \omega) \leq 1} |k_n(x, \omega) - k_n(x, y)|^2 p_1(y, \omega) \right)^{1/2}
\]

\[
= \left( \sum_{e \in E_y} |d k_n(x, e)|^2 p(e) \right)^{1/2}.
\]

Then we have

**Theorem 1.1.** Let \( X = (V, E) \) be a non-bipartite covering graph whose covering transformation group has polynomial volume growth of order \( D \). Then there exist \( C, C' > 0 \) such that

\[
\nabla_1 k_n(x, y) \leq C n^{-(D+1)/2} e^{-C' d(x, y)^2/n}
\]

(1.2)

for \( n \in \mathbb{N}^* \) and \( x, y \in V \).

It should be noted that the estimate (1.2) is closely related to the boundedness of the Riesz transform (see [1], [13] and [11]). Let \( \Delta \) be a discrete Laplacian on \( X \) given by \( \Delta = I - P \). Then we have

**Theorem 1.2.** Let \( X \) be as above. For \( 1 < p < \infty \), there exists \( C_p > 0 \) such that

\[
\|\nabla (\Delta^{-1/2} f)\|_{L^p} \leq C_p \|f\|_{L^p}
\]

for all finitely supported functions \( f \) on \( V \). Here \( \| \cdot \|_{L^p} \) is the \( L^p \) norm with respect to the measure \( m \) on \( V \).

On the other hand, Dungey proved (1.2) for the heat kernel on covering manifolds with polynomial volume growth in [6]. Recently, in [7], he gave a new proof of (1.2) using the well-known temporal regularity of the heat kernel (see for instance [5]). In this paper, we give a shorter proof of Theorem 1.1 along with the arguments of the recent result by Dungey [7]. Let

\[
\partial_1 k_n(x, y) = k_{n+1}(x, y) - k_n(x, y).
\]

We use the following discrete version of the temporal regularity proved by Blunck [2], Christ [3]:

**Theorem 1.3.** Let \( X \) be an oriented non-bipartite graph satisfying

\[
C^{-1} r^D \leq m(B(x, r)) \leq C r^D \quad x \in V, r \geq 1
\]
and (1.1), where \( B(x,r) \) is the ball centered at \( x \in V \) with radius \( r \). Then there exist positive constants \( C \) and \( C' \) such that
\[
\partial_1 k_n(x,y) \leq C n^{-(D+2)/2} e^{-C' d(x,y)^2/n}
\]
for \( n \in \mathbb{N}^* \) and \( x, y \in V \).

We remark that Dungey gave a short proof of (1.3) recently in [8].

2. Proof of Theorem 1.1

Let \( \nu = m + n \) be a positive integer, where we choose \( m = n \) or \( m = n + 1 \) depending on whether \( \nu \) is even or odd. By the Cauchy-Schwarz inequality, it is easy to see that
\[
e^{rd^2(u,z)/\nu} \nabla_1 k_\nu(u,z) \leq \left( \sum_{y \in V} \frac{e^{rd^2(v_0,y)/\nu} \nabla_1 k_\nu(v_0,y)}{m(y)} \right)^{1/2} \leq \left( \sum_{y \in V} \frac{e^{rd^2(v_0,y)/\nu} \nabla_1 k_\nu(v_0,y)}{m(y)} \right)^{1/2} \]
for \( r > 0 \).

By the Gaussian upper bound (1.1) for \( k_n \), for small \( r > 0 \), there exists \( C_r > 0 \) such that
\[
\| \nabla_1 k_n(u, \cdot) \|_{L^2} \leq C_r m^{-D/4}.
\]
Let \( F \subset V \) be a fundamental domain for the action of the transformation group \( \Gamma \) on \( V \). Namely, \( F \) is a subset in \( V \) such that for all \( x \in V \), there exists a unique pair \( \gamma x_0 \in F \) satisfying \( x = \gamma x_0 \). Then we denote \( \gamma x_0 \). The following lemma gives a comparison of the weighted integral for \( \nabla_1 k_n \) and \( \nabla_2 k_n \). Similar arguments can be found in [12].

**Lemma 2.1.** There exists a positive constant \( C > 0 \) such that
\[
\| e^{rd^2(u,\cdot)/n} \nabla_1 k_n(u, \cdot) \|_{L^2}^2 \leq C \sum_{y_0 \in F_u} \| e^{rd^2(y_0,\cdot)/n} \nabla_2 k_n(y_0, \cdot) \|_{L^2}^2 m(y_0)
\]
for all \( u \in X \).

**Proof.** Since \( m \) is \( \Gamma \)-invariant, there exists \( C > 0 \) such that \( C^{-1} \leq \min \{ m(x) \mid x \in V \} \). Then we have
\[
\| e^{rd^2(u,\cdot)/n} \nabla_1 k_n(u, \cdot) \|_{L^2}^2 \leq C \sum_{v_0 \in F_u} \sum_{y \in V} \left| e^{rd^2(v_0,y)/n} \nabla_1 k_n(v_0, y) \right|^2 m(y) m(v_0)
\]
\[
= C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} \left| e^{rd^2(v_0,\gamma y_0)/n} \nabla_1 k_n(v_0, \gamma y_0) \right|^2 m(\gamma y_0) m(v_0).
\]
From the $\Gamma$-invariance of the distance function $d$ and $k_n$, the latter is
\[
C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} |e^{2rd^2(\gamma^{-1}v_0, y_0)/n} \nabla_1 k_n(\gamma^{-1}v_0, y_0)|^2 m(y_0)m(\gamma^{-1}v_0).
\]
By replacing $\gamma^{-1}$ with $\gamma$ in the sum of $\Gamma$, we get
\[
C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} |e^{2rd^2(\gamma v_0, y_0)/n} \nabla_1 k_n(\gamma v_0, y_0)|^2 m(y_0)m(\gamma v_0).
\]
Since $\nabla_1 k_n(\gamma v_0, y_0) = \nabla_2 k_n(y_0, \gamma v_0)$, this is also
\[
C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} |e^{2rd^2(\gamma v_0, v)/n} \nabla_2 k_n(y_0, v)|^2 m(v)m(y_0)
\]
\[
= C \sum_{y_0 \in F_u} \sum_{v \in V} |e^{2rd^2(y_0, v)/n} \nabla_2 k_n(y_0, v)|^2 m(v)m(y_0)
\]
\[
= C \sum_{y_0 \in F_u} \|e^{2rd^2(y_0, \cdot)/n} \nabla_2 k_n(y_0, \cdot)\|_{L^2}^2 m(y_0).
\]

\[\square\]

**Remark 2.2.** In this proof, we use only the $\Gamma$-invariance of $d$, $\nabla_1 k_n$ and $\nabla_1 k_n(x, y) = \nabla_2 k_n(y, x)$. Therefore, there are other definitions of $\nabla$ so that the previous lemma holds. For example, we can obtain the same results for
\[
\nabla_1^p k_n(x, y) := \left( \sum_{d(\omega, x) \leq 1} |k_n(\omega, y) - k_n(x, y)|^p p_1(x, \omega) \right)^{1/p}, \quad 1 < p < \infty,
\]
\[
\nabla_1^\infty k_n(x, y) := \sup_{d(\omega, x) \leq 1} |k_n(\omega, y) - k_n(x, y)|,
\]
which are comparable with each other.

By the same arguments as the continuous case ([4]), we obtain the following by “discrete integration by parts”. Similar arguments can be found in [10] and [12].

**Lemma 2.3.** For $e \in E$, let $m(e)$ be a weight of $e$ defined by $m(e) = p(e)m(o(e))$. Then we have
\[
\|e^{2rd^2(y_0, \cdot)/n} \nabla_2 k_n(y_0, \cdot)\|_{L^2}^2 = - \sum_{e \in E} de^4rd^2(y_0, e)/(m_k(y_0, o(e))dk_n(y_0, e)m(e))
\]
\[-2 \sum_{v \in V} e^{4rd^2(y_0, v)/n}k_n(y_0, v)\partial k_n(y_0, v)m(v).
\]
Since $m(e) = m(\bar{e})$, by replacing $e$ with $\bar{e}$ in the sum of $E$ in the first term,

$$
\sum_{e \in E} e^{4rd^2(y_0, o(e))} k_n(y_0, o(e)) (k_n(y_0, o(e)) - k_n(y_0, t(e))) m(e)
$$

$$
+ \sum_{e \in E} e^{4rd^2(y_0, o(e))} k_n(y_0, o(e)) (k_n(y_0, o(e)) - k_n(y_0, t(e))) m(e)
$$

$$
\sum_{e \in E} \left( e^{4rd^2(y_0, o(e))} - e^{4rd^2(y_0, o(e))} \right) k_n(y_0, o(e))
$$

$$
\cdot (k_n(y_0, o(e)) - k_n(y_0, t(e))) m(e)
$$

$$
+ 2 \sum_{e \in E} e^{4rd^2(y_0, o(e))} k_n(y_0, o(e)) (k_n(y_0, o(e)) - k_n(y_0, t(e))) m(e)
$$

$$
= - \sum_{e \in E} d e^{4rd^2(y_0, e)} k_n(y_0, o(e)) d k_n(y_0, e) m(e)
$$

$$
- 2 \sum_{v \in V} e^{4rd^2(y_0, v)} k_n(y_0, v) \sum_{e \in E_v} (k_n(y_0, t(e)) - k_n(y_0, o(e))) p(e) m(v).
$$
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Since
\[
\sum_{e \in E_v} (k_n(y_0, t(e)) - k_n(y_0, o(e))) p(e)
\]
\[
= \sum_{e \in E_v} k_n(y_0, t(e)) p(e) - k_n(y_0, v)
\]
\[
= k_{n+1}(y_0, v) - k_n(y_0, v)
\]
\[
= \partial_1 k_n(y_0, v),
\]
the lemma is proved. □

Finally, we apply the temporal regularity for \(k_n\) along with the argument of Lemma 2.3 in [4]. Let

\[
I(n, y_0) = \sum_{v \in V} \left| e^{2rd^2(y_0,v)/n} \nabla_2 k_n(y_0, v) \right|^2 m(v),
\]
\[
I_1(n, y_0) = -2 \sum_{v \in V} e^{4rd^2(y_0,v)/n} k_n(y_0, v) \partial_1 k_n(y_0, v) m(v),
\]
\[
I_2(n, y_0) = - \sum_{e \in E} d e^{4rd^2(y_0,e)/n} k_n(y_0, o(e)) d k_n(y_0, e) m(e).
\]

Lemma 2.3 says that \(I(n, y_0) = I_1(n, y_0) + I_2(n, y_0)\). Using (1.1) and (1.3), for sufficiently small \(r > 0\), there exists \(C_r > 0\) such that

\[
|I_1(n, y_0)| \leq C_r n^{-D/2-1}.
\]

By the Cauchy-Schwarz inequality, we have that \(|I_2(n, y_0)|\) is less than
\[
\sum_{v \in V} \left( \sum_{e \in E_v} |d e^{4rd^2(y_0,e)/n}|^2 p(e) \right)^{1/2} \left( \sum_{e \in E_v} |d k_n(y_0, e)|^2 p(e) \right)^{1/2} k_n(y_0, v) m(v)
\]
\[
= \sum_{v \in V} \left( \sum_{e \in E_v} |d e^{4rd^2(y_0,e)/n}|^2 p(e) \right)^{1/2} \nabla_2 k_n(y_0, v) k_n(y_0, v) m(v).
\]
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Using the mean value theorem for \( f(x) = e^{4rx^2/n} \), there exists \( d(y_0, o(e)) < s < d(y_0, t(e)) \) such that

\[
e^{4rd^2(y_0, t(e))/n} - e^{4rd^2(y_0, o(e))/n} = \frac{8rs}{n} e^{4rs^2/n} (d(y_0, t(e)) - d(y_0, o(e))) - \frac{8r(d(y_0, o(e)) + 1)}{n} e^{4r(d(y_0, o(e)) + 1)^2/n} \leq \frac{C_r}{\sqrt{n}} e^{16rd^2(y_0, o(e))/n}
\]

for some \( C_r > 0 \). Then we have that \( |I_2(n, y_0)| \) is less than

\[
\frac{C_r}{\sqrt{n}} \sum_{v \in V} e^{16rd^2(y_0, v)/n} \nabla^2 k_n(y_0, v) k_n(y_0, v) m(v)
\leq \frac{C_r}{\sqrt{n}} \left( \sum_{v \in V} e^{28rd^2(y_0, v)/n} |k_n(y_0, v)|^2 m(v) \right)^{1/2} \cdot \left( \sum_{v \in V} e^{2rd^2(y_0, v)/n} \nabla^2 k_n(y_0, v)^2 m(v) \right)^{1/2}.
\]

For sufficiently small \( r > 0 \), we obtain

\[
|I_2(n, y_0)| \leq C'_r n^{-D/4 - 1/2} I(n, y_0)^{1/2}.
\]

Then we conclude that

\[
I(n, y_0) \leq C_r n^{-D/2 - 1} + C'_r n^{-D/4 - 1/2} I(n, y_0)^{1/2},
\]

namely,

\[
I(n, y_0) \leq C n^{-D/4 - 1/2}
\]

for some \( C > 0 \). Together with (2.1), this completes the proof of Theorem 1.1.

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References


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