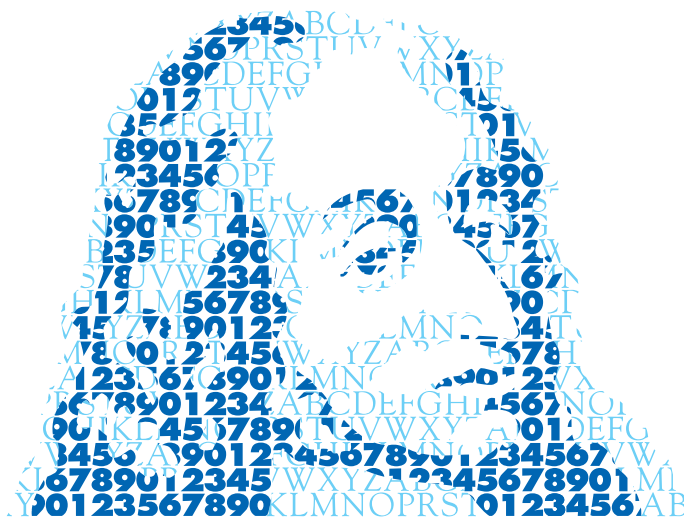


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Cyclically valued rings and formal power series

GÉRARD LELOUP

Abstract

Rings of formal power series $k[[C]]$ with exponents in a cyclically ordered group C were defined in [2]. Now, there exists a “valuation” on $k[[C]]$: for every σ in $k[[C]]$ and c in C , we let $v(c, \sigma)$ be the first element of the support of σ which is greater than or equal to c . Structures with such a valuation can be called cyclically valued rings. Others examples of cyclically valued rings are obtained by “twisting” the multiplication in $k[[C]]$. We prove that a cyclically valued ring is a subring of a power series ring $k[[C, \theta]]$ with twisted multiplication if and only if there exist invertible monomials of every degree, and the support of every element is well-ordered. We also give a criterion for being isomorphic to a power series ring with twisted multiplication. Next, by the way of quotients of cyclic valuations, it follows that any power series ring $k[[C, \theta]]$ with twisted multiplication is isomorphic to a $R'[[C', \theta']]$, where C' is a subgroup of the cyclically ordered group of all roots of 1 in the field of complex numbers, and $R' \simeq k[[H, \theta]]$, with H a totally ordered group. We define a valuation $v(\epsilon, \cdot)$ which is closer to the usual valuations because, with the topology defined by $v(a, \cdot)$, a cyclically valued ring is a topological ring if and only if $a = \epsilon$ and the cyclically ordered group is indeed a totally ordered one.

1. Introduction.

The formal power series with exponents in a cyclically ordered group gave rise to cyclically valued rings. Recall that $(C, +, (\cdot, \cdot, \cdot))$ (or more simply $(C, (\cdot, \cdot, \cdot))$, resp. C) is a *cyclically ordered* abelian group, if $(C, +)$ is an abelian group and (\cdot, \cdot, \cdot) satisfies for every a, b, c, d :

- $(a, b, c) \Rightarrow a \neq b \neq c \neq a \ \& \ (b, c, a)$
- $(a, b, c) \Rightarrow (a + d, b + d, c + d)$ (compatibility).
- (c, \cdot, \cdot) is a strict total order on $C \setminus \{c\}$.

For every $c \in C$, we will denote by \leq_c the associate order on C with first element c . For $\emptyset \neq X \subset C$, $\min_c X$ will denote the minimum of (X, \leq_c) , if it exists.

Definition 1.1. ([2]) Let C be a cyclically ordered group, R be a commutative ring, v a mapping from $C \times R$ onto $C \cup \{\infty\}$, where for every

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a, b in C , $a <_b \infty$, and let $\sigma \in R$.

The *support* of σ is the set $\text{Supp}(\sigma) := \{v(a, \sigma) \mid a \in C\}$.

σ is a *monomial* if the support of σ is a singleton. If $\text{Supp}(\sigma) = \{c\}$, c will be called the *degree* of σ .

σ is a *constant* if either $\sigma = 0$ or $\text{Supp}(\sigma) = \{0\}$.

(R, v) is a *cyclically valued ring* if the following five conditions hold.

(1) For every $a \in C$, $(R, +, v(a, \cdot))$ is a valued group.

(2) For every $\sigma \in R$ and $a \in C$, if $v(a, \sigma) = a$, then there exists a unique monomial $\mu_{a, \sigma}$ such that $v(a, \sigma - \mu_{a, \sigma}) \neq a$. If $v(a, \sigma) \neq a$, we set $\mu_{a, \sigma} = 0$.

(3) For every $\sigma \in R$ and $a \in C$, $\min_a(\text{Supp}(\sigma))$ exists and is equal to $v(a, \sigma)$.

(4) For every σ and σ' in R , $\text{Supp}(\sigma\sigma') \subset \text{Supp}(\sigma) + \text{Supp}(\sigma')$.

(5) For every $n \in \mathbb{N}^*$, $a \in C$, $\sigma \in R$ and $\sigma' \in R$, if $\text{card}(\text{Supp}(\sigma) \cap (a - \text{Supp}(\sigma'))) = n$, say $\text{Supp}(\sigma) \cap (a - \text{Supp}(\sigma')) = \{a_1, \dots, a_n\}$, then $\mu_{a, \sigma\sigma'} = \mu_{a_1, \sigma} \mu_{a-a_1, \sigma'} + \dots + \mu_{a_n, \sigma} \mu_{a-a_n, \sigma'}$.

Notation 1.2. M will denote the set of all monomials of (R, v) , and for $c \in C$, M_c will denote the set of all monomials of degree c .

One can prove that condition (3) is equivalent to :

(3') For every σ in R and a, b in C , $a \leq_a b \leq_a v(a, \sigma) \Rightarrow v(b, \sigma) = v(a, \sigma)$ (see [5]).

Furthermore, if (R, v) satisfies (1) and (3), then for every σ, τ in R , $\text{Supp}(\sigma + \tau) \subset \text{Supp}(\sigma) \cup \text{Supp}(\tau)$.

Let S be a subset of C . We say that $(S, (\cdot, \cdot, \cdot))$ is *well-ordered* if there exists $c \in C$, such that the totally ordered set (S, \leq_c) is well-ordered. This implies that for every $c \in C$, the totally ordered set (S, \leq_c) is well-ordered. We know that the sum of any two well-ordered subsets of C is well-ordered (see [2]). If σ is a mapping from C to k , the support of σ is the subset of all c in C such that $\sigma(c) \neq 0$ (we will denote σ_c instead of $\sigma(c)$). Let $k[[C]]$ (resp. $k[C]$) be the subset of all mappings from C to k with well-ordered (resp. finite) support. For any $a \in C$, $\sigma \in k[[C]]$ (resp. $\sigma \in k[C]$) let $v(a, \sigma)$ be the lowest element of the support of σ ordered by $<_a$. We define an addition and a multiplication on $k[[C]]$ as usual. Then $(k[[C]], v)$ and $(k[C], v)$ are cyclically valued rings (see [2]).

We know that if the support of every element of a cyclically valued ring (R, v) is well-ordered and if M contains a group which is canonically isomorphic to C , then (R, v) embeds in a ring of formal power series

with cyclically ordered exponents $k[[C]]$. In Section 2 we prove that, by “twisting” the multiplication of $k[[C]]$, we can take the condition “ M contains invertible elements of every degree”, instead of “ M contains a group canonically isomorphic to C ” (Theorem 1). We will denote by $k[[C, \theta]]$ these formal power series rings with “twisted” multiplication. In Theorem 2, we give necessary and sufficient conditions for being isomorphic to a twisted ring of formal power series. These conditions imply that the support of every element is well-ordered. Notice that any valued field of equal characteristic embeds in some $k[[C, \theta]]$, whence the usual valuations can be seen as particular cases of cyclic valuations. If R contains elements σ such that $\text{nor Supp}(\sigma)$ nor $-\text{Supp}(\sigma)$ is well-ordered, then these theorems fail. We give examples of such rings.

In Section 3, we define and we characterize quotients of cyclically valued rings (Theorems 3 and 4). By means of these quotients, we prove that power series rings with cyclically ordered exponents are indeed power series rings with cyclically ordered exponents such that the group of exponents is archimedean, i.e. it embeds in the group of all roots of 1 in the field of complex numbers.

It is well-known that there exist at most one element $\epsilon \neq 0$ in C such that $\epsilon = -\epsilon$. If C doesn't contain such an element, then for every a, b in C set $a <_{\epsilon} b$ if and only if either $(a, b) \in -P \times P \cup \{0\}$, or $(a, b) \notin -P \times P \cup \{0\}$ and $a <_0 b$, where $P := \{c \in C \mid (0, c, 2c)\} = \{c \in C \mid (0, c, -c)\}$. P is called the *positive cone* of C . Note that we have $: P <_0 -P$ and $P \cup -P \cup \{0, \epsilon\} = C$.

Assume that for every $\sigma \in R$, $\min_{\epsilon}(\text{Supp}(\sigma))$ exists. Then we set

$$v(\epsilon, \sigma) := \min_{\epsilon}(\text{Supp}(\sigma)) \text{ if } \sigma \neq 0, \text{ and } v(\epsilon, 0) := \infty.$$

The *linear part* of C is the largest subgroup $l(C)$ such that $(l(C), \leq_{\epsilon})$ is a totally ordered group. $l(C)$ is a convex subset of (C, \leq_{ϵ}) and $C/l(C)$ embeds in the cyclically ordered group of all roots of 1 in the field of complex numbers (see [1]). C is a *linear cyclically ordered group* if $C = l(C)$.

In Section 4, we show that, if C is a linear cyclically ordered group, then $v(\epsilon, \cdot)$ satisfies the usual rules $\forall \sigma, \forall \tau, v(\epsilon, \sigma\tau) = v(\epsilon, \sigma) + v(\epsilon, \tau)$. Furthermore, if the product of any two monomials is not 0, then for every $a \in C \cup \{\epsilon\}$, $(R, v(a, \cdot))$ is a topological ring if and only if either C is a linear cyclically ordered group and $a = \epsilon$ or $(C, <_0)$ has a greatest element (Theorem 5).

Various people asked whether cyclically valued rings are definable in a relational language. Section 5 gives a positive answer to this question. The goal is to make easier a model theoretic approach to cyclically valued groups. However, it remains an open question to characterize elementarily equivalent cyclically valued rings. A first step was made in [5] : by dropping the multiplication symbol, one can get classes of existentially equivalent additive groups (R, v) .

2. Cyclically valued rings with C interpretable.

Definition 2.1. Let (R, v) be a cyclically valued ring. We will say that C is *interpretable* if for every $c \in C$, there exists an invertible monomial of degree c .

First let us prove some basic facts about the unit element and monomials.

Lemma 2.2. *Assume that (R, v) is a cyclically valued ring, and let $\mu \in M$ such that, for every τ in M , $\mu\tau \neq 0$. Then for every σ in R , $\text{Supp}(\mu\sigma) = \text{Supp}(\mu) + \text{Supp}(\sigma)$.*

Proof. Let d be the degree of μ . We already know : $\text{Supp}(\mu\sigma) \subset \text{Supp}(\mu) \cup \text{Supp}(\sigma) = d + \text{Supp}(\sigma)$. Let $a \in C$ such that $a - d \in \text{Supp}(\sigma)$. Then $\{a - d\} = \text{Supp}(\sigma) \cap (a - \text{Supp}(\mu))$, hence by (5) of Definition 1.1 $\mu_{a, \mu\sigma} = \mu\mu_{a-d, \sigma} \neq 0$, and $a \in \text{Supp}(\mu\sigma)$. \square

Now, assume that μ satisfies conditions of Lemma 2.2, and that R contains a unit element 1. Then $\{d\} = \text{Supp}(\mu) = \text{Supp}(\mu \cdot 1) = \text{Supp}(\mu) + \text{Supp}(1) = \{d\} + \text{Supp}(1)$. It follows that $\text{Supp}(1) = \{0\}$ i.e. 1 is a monomial of degree 0.

Furthermore, assume that μ is invertible. Then by Lemma 2.2 $\{0\} = \text{Supp}(\mu\mu^{-1}) = \text{Supp}(\mu) + \text{Supp}(\mu^{-1}) = d + \text{Supp}(\mu^{-1})$. Whence $\text{Supp}(\mu^{-1}) = \{-d\}$ i.e. μ^{-1} is a monomial of degree $-d$.

Note that if $(M \cup \{0\}, \cdot)$ contains a unit element 1, then 1 is the unit element of R . Indeed, let $\sigma \in R$ and $a \in C$. $1 \cdot \sigma - \mu_{a, \sigma} = 1 \cdot \sigma - 1 \cdot \mu_{a, \sigma} = 1 \cdot (\sigma - \mu_{a, \sigma})$. By Lemma 2.2, $\text{Supp}(1 \cdot (\sigma - \mu_{a, \sigma})) = \{0\} + \text{Supp}(\sigma - \mu_{a, \sigma})$. Hence $v(a, 1 \cdot \sigma - \mu_{a, \sigma}) = v(a, \sigma - \mu_{a, \sigma}) \neq a$. Now, $\mu_{a, 1 \cdot \sigma}$ being unique, we have : $\mu_{a, 1 \cdot \sigma} = \mu_{a, \sigma}$. It follows : $\forall a \in C$, $\mu_{a, 1 \cdot \sigma - \sigma} = \mu_{a, 1 \cdot \sigma} - \mu_{a, \sigma} = 0$. Hence $1 \cdot \sigma - \sigma = 0$.

Proposition 2.3. *Let (R, v) be a cyclically valued ring. Assume that the product of any two monomials is not 0. Then the following (i) and (ii) are equivalent.*

(i) *The ring of all constants $M_0 \cup \{0\}$ is a field.*

(ii) *(M, \cdot) is a group.*

If this holds, then $C \simeq M/M_0$.

Proof. It's not difficult to check that $M_0 \cup \{0\}$ is a subring of R (see [2]). Now, recall that the hypothesis “the product of any two monomials is not 0” is equivalent to “ M is closed under multiplication”. Hence, so is M_0 , and $M_0 \cup \{0\}$ is an integral subring of R .

Assume that (M, \cdot) is a group, and let $\mu \in M_0$. We have already proved that $\deg(\mu^{-1}) = -\deg(\mu)$, hence $\mu^{-1} \in M_0$. It follows that $M_0 \cup \{0\}$ is a field.

Assume that $M_0 \cup \{0\}$ is a field. Let μ be a monomial of degree d . By definition of cyclically valued rings, there exists a monomial μ' of degree $-d$ (because v is onto). M is closed under multiplication, so $\mu\mu' \neq 0$. It follows that $\mu\mu'$ is a constant different from 0. Let μ_1 be the inverse of $\mu\mu'$, then $\mu\mu'\mu_1 = 1$, i.e. $\mu'\mu_1$ is the inverse of μ . \square

Definition 2.4. Let k be a commutative ring with 1, and θ be a mapping from $C \times C$ to k . We will say that $\theta(C, C)$ is a *commutative factor set* if it enjoys the following :

$$\forall (d_1, d_2) \in C \times C, \theta(d_1, d_2) = \theta(d_2, d_1),$$

$$\forall d \in C, \theta(0, d) = 1,$$

$$\forall (d_1, d_2, d_3) \in C \times C \times C, \theta(d_1, d_2)\theta(d_1 + d_2, d_3) = \theta(d_1, d_2 + d_3)\theta(d_2, d_3).$$

Let k be a commutative ring with 1, and $\theta : C \times C \rightarrow k$ be a mapping.

Any element σ of the Hahn product $\prod_{d \in C} k$, will be denoted by $\sigma = \sum_{d \in C} \sigma_d X^d$ instead of $\sigma = (\sigma_d)_{d \in C}$.

Let $\sigma = \sum_{d \in C} \sigma_d X^d$ and $\tau = \sum_{d \in C} \tau_d X^d$ in $\prod_{d \in C} k$. $\text{Supp}(\sigma)$ and $\text{Supp}(\tau)$ are well-ordered, hence for every $d \in C$, the set $\text{Supp}(\sigma) \cap (d - \text{Supp}(\tau))$ is finite, we set

$$\sigma\tau = \sum_{d \in C} \left(\sum_{c \in C} \sigma_c \tau_{d-c} \theta(c, d-c) \right) X^d$$

In the same way as any of [6], [7], [3] or [8], [9], we can prove that, with the multiplication $(\sigma, \tau) \mapsto \sigma\tau$ defined above, the Hahn product $\prod_{d \in C} k$ is a commutative ring with unit element $1 = X^0$ if and only if $\theta(C, C)$ is a commutative factor set. Furthermore, if k is a field, C is a linear cyclically ordered group and $\theta(C \times C) \subset k \setminus \{0\}$, then this Hahn product is a field.

Proposition 2.5. *Let k be a commutative ring with 1, and $\theta : C \times C \rightarrow k$ be a mapping such that $\theta(C, C)$ is a commutative factor set. For every $a \in C$ and $\sigma \in \prod_{d \in C} k$, we let $v(a, \sigma)$ be the first element of the support of σ , ordered by $<_a$. Then :*

- 1) $\prod_{d \in C} k$ is a cyclically valued ring.
- 2) The set of all polynomials is a cyclically valued subring of $\prod_{d \in C} k$.
- 3) k is naturally isomorphic to a subring of $\prod_{d \in C} k$.
- 4) If for every d in C , $\theta(-d, d)$ is a unit in k , then C is interpretable in $\prod_{d \in C} k$.

Proof. We let the reader check 1) and 2).

3) The embedding of k into $\prod_{d \in C} k$ is given in the following way. Let $x \in k \setminus \{0\}$, the support of its image is $\{0\}$, and the corresponding coefficient is x , so we will assume $k \subset \prod_{d \in C} k$.

4) If, for every d in C , $\theta(-d, d)$ is a unit in k , then $X^d \theta(-d, d)^{-1} X^{-d} = X^0 = 1$, hence X^d is a unit, and C is interpretable. \square

In [3], Kaplansky proved that any perfect henselian valued field of equal characteristic with value group G and residue field k embeds in some $k[[G, \theta]]$. Now, G can be cyclically ordered by setting for every a, b, c in G , (a, b, c) if and only if either $a < b < c$ or $b < c < a$ or $c < a < b$. The usual valuation is the valuation $v(\epsilon, \cdot)$. So, in the case of equal characteristic, the usual valuation is a particular case of a cyclic valuation.

Notation 2.6. Let k be a commutative ring with 1, and $\theta : C \times C \rightarrow k \setminus \{0\}$ be a mapping such that $\theta(C, C)$ is a commutative factor set. $k[[C, \theta]]$ will be the Hahn product $\prod_{d \in C} k$ together with the mapping $(\sigma, \tau) \mapsto \sigma\tau$. We set $k[C, \theta] = \{\sigma \in k[[C, \theta]] \mid \text{Supp}(\sigma) \text{ is finite}\}$.

Remark 2.7. The proofs of some results of [2] extend to the “twisted” power series rings :

- if $k[[C, \theta]]$ is a field, then k is a field, $\theta(C \times C) \subset k \setminus \{0\}$, and $C/l(C)$ embeds in the group of all roots of 1 in the field of complex numbers,
- if k is a field, $\theta(C \times C) \subset k \setminus \{0\}$ and $C/l(C)$ is finite, then $k[[C, \theta]]$ is a field,
- $k[C, \theta]$ is integral if and only if k is integral, C is torsion-free and $\theta(C \times C) \subset k \setminus \{0\}$.

Theorem 1. *Let (R, v) be a cyclically valued ring such that C is interpretable and let k be the ring of constants of R .*

For every $d \in C \setminus \{0\}$, fix an invertible monomial μ_d , and let $\mu_0 = 1$.

CYCLICALLY VALUED RINGS

For d_1 and d_2 in C , let $\theta(d_1, d_2) = (\mu_{d_1}\mu_{d_2})\mu_{d_1+d_2}^{-1}$ (we can assume that $\mu_{-d} = \mu_d^{-1}$, so, for all d in C , $\theta(-d, d) = 1$) then we have the following.

1) $\theta(C, C)$ is a commutative factor set.

2) For σ in R , d in C , set $\Psi_d(\sigma) := \mu_{d,\sigma}\mu_d^{-1}$ (with $\mu_{d,\sigma}$ the only element of k such that $v(d, \sigma - \mu_{d,\sigma}) \neq d$), and let $\Psi(\sigma) := (\Psi_d(\sigma))_{d \in C} \in \prod_{d \in C} k$ (cartesian product).

If the support of every element of R is well-ordered, then Ψ is an isomorphism from the cyclically valued ring (R, v) into the cyclically valued ring $k[[C, \theta]]$.

Proof. .

1) From the definition, it follows : $\forall (d_1, d_2) \in C \times C$, $\theta(d_1, d_2) = \theta(d_2, d_1)$, and $\forall d \in C$, $\theta(0, d) = 1$. Let $(d_1, d_2, d_3) \in C \times C \times C$,

$$\begin{aligned} \theta(d_1, d_2)\theta(d_1 + d_2, d_3) &= \mu_{d_1}\mu_{d_2}\mu_{d_1+d_2}^{-1}\mu_{d_1+d_2}\mu_{d_3}\mu_{d_1+d_2+d_3}^{-1} \\ &= \mu_{d_1}\mu_{d_2}\mu_{d_3}\mu_{d_1+d_2+d_3}^{-1} \\ &= \mu_{d_1}\mu_{d_2+d_3}\mu_{d_1+d_2+d_3}^{-1}\mu_{d_2}\mu_{d_3}\mu_{d_2+d_3}^{-1} \\ &= \theta(d_1, d_2 + d_3)\theta(d_2, d_3). \end{aligned}$$

Therefore $\theta(C, C)$ is a commutative factor set.

2) Assume that the support of every element is well-ordered. Hence : $\forall \sigma \in R$, $\Psi(\sigma) \in \prod_{a \in C} k$. We let the reader check that for every $d \in C$, the mapping

$$\begin{cases} (R, +) & \rightarrow & (M_d \cup \{0\}, +) \\ \sigma & \mapsto & \mu_{d,\sigma} \end{cases}$$

is a morphism of groups. We deduce that

$$\begin{cases} (R, +) & \rightarrow & k \\ \sigma & \mapsto & \Psi_d(\sigma) = \mu_{d,\sigma}\mu_d^{-1} \end{cases} \quad \text{and} \quad \begin{cases} (R, +) & \rightarrow & (\prod_{d \in C} k, +) \\ \sigma & \mapsto & \Psi(\sigma) \end{cases}$$

are morphisms of groups. If $\sigma \neq 0$, then $\text{Supp}(\sigma) \neq \emptyset$. Let $a \in \text{Supp}(\sigma)$. We deduce from (3') that $v(a, \sigma) = a$, hence $\mu_{a,\sigma} \neq 0$. Hence $\Psi(\sigma) \neq 0$. Now, straightforward checkings show that Ψ is an isomorphism of cyclically valued groups.

Let σ and τ be elements of R , d in C , and $(d'_1, d''_1), \dots, (d'_n, d''_n)$ be all

the elements of $\text{Supp}(\sigma) \times \text{Supp}(\tau)$ such that $d'_i + d''_i = d$. Then

$$\begin{aligned} \Psi_d(\sigma\tau) &= \mu_{d,\sigma\tau}\mu_d^{-1} \\ &= \sum_{i=1}^n \mu_{d'_i,\sigma}\mu_{d''_i,\tau}\mu_d^{-1} \\ &= \sum_{i=1}^n \mu_{d'_i,\sigma}\mu_{d'_i}^{-1}\mu_{d''_i,\tau}\mu_{d''_i}^{-1}\mu_{d'_i}\mu_{d''_i}\mu_d^{-1} \\ &= \sum_{i=1}^n \Psi_{d'_i}(\sigma)\Psi_{d''_i}(\tau)\theta(d'_i, d''_i). \end{aligned}$$

Then Ψ is an isomorphism of rings. \square

If we drop the hypothesis : “the support of every element is well-ordered”. Ψ is an isomorphism from $(R, +)$ to the cartesian product $\prod_{c \in C} k$. Let σ and σ' in R . If, for every $a \in C$, $\text{card}(\text{Supp}(\sigma) \cap (a - \text{Supp}(\sigma')))$ is finite, then $\sigma\sigma'$ is defined by the rule : if $\text{Supp}(\sigma) \cap (a - \text{Supp}(\sigma')) = \{a_1, \dots, a_n\}$, then $\mu_{a,\sigma\sigma'} = \mu_{a_1,\sigma}\mu_{a-a_1,\sigma'} + \dots + \mu_{a_n,\sigma}\mu_{a-a_n,\sigma'}$. We can define $\Psi(\sigma)\Psi(\sigma')$ in the same way. Otherwise, we can't say anything about $\sigma\sigma'$.

Before going further, we give examples of cyclically valued rings containing elements σ such that nor $\text{Supp}(\sigma)$ nor $-\text{Supp}(\sigma)$ is well-ordered. Let k be a field such that the transcendence degree of k over the field \mathbb{Q} of rational numbers is infinite, let C be the cyclically ordered group \mathbb{Z} of all integers, and let $(\alpha_c)_{c \in C}$ be a family of element of k which is algebraically independent over \mathbb{Q} . Let $\alpha := (\alpha_c)_{c \in C} \in \prod_{c \in C} k$. The elements of $\prod_{c \in C} k$ will be denoted by $\sigma = \sum_{c \in C} \sigma_c X^c$ instead of $\sigma = (\sigma_c)_{c \in C}$. We let $k[C]$ be the subgroup of all polynomials of $\prod_{c \in C} k$. $k[C]$ is a ring. For every $\sigma \in k[C]$, the support of σ is finite, so $\text{card}(\text{Supp}(\sigma) \cap (a - \text{Supp}(\alpha)))$ is finite, and we define the product $\sigma\alpha$ by the usual rules : $\sigma\alpha := \sum_{a \in C} (\sum_{b \in C} \sigma_b \alpha_{a-b}) X^a$.

First example. We set $R := \{\sigma + \tau\alpha \mid \sigma \in k[C], \tau \in k[C]\}$, and we let $\alpha\alpha = 0$. We will prove in the second example that $\sigma + \tau\alpha = 0 \Rightarrow \sigma = \tau = 0$. Thus, we can define a multiplication on R by setting, for all $\sigma_1, \tau_1, \sigma_2, \tau_2$ in $k[C]$, $(\sigma_1 + \tau_1\alpha)(\sigma_2 + \tau_2\alpha) = \sigma_1\sigma_2 + (\sigma_1\tau_2 + \sigma_2\tau_1)\alpha$. For every $a \in C$, we set $v(a, \sigma + \tau\alpha) = \min \text{Supp}(\sigma + \tau\alpha)$. (R, v) is a cyclically valued ring, and $\text{Supp}(\alpha) = C$.

Second example. For every positive integer n , we set $\alpha^n := \sum_{c \in C} \alpha_c^n X^c \in \prod_{c \in C} k$. For every $\sigma = \sum_{a \in C} \sigma_a X^a \in k[C]$, $\text{Supp}(\sigma)$ is finite so we can set $\sigma\alpha^n := \sum_{a \in C} (\sum_{b \in C} \sigma_b \alpha_{a-b}^n) X^a$. R is the additive subgroup generated by the $\sigma\alpha^n$, with $\sigma \in k[C]$ and n a positive integer. For every $a \in C$, we define $v(a, \cdot)$ in the same way as in the first example. Then $(R, +, v)$ satisfies conditions (1), (2), (3) of Definition 1.1.

CYCLICALLY VALUED RINGS

For every $\sigma_0, \dots, \sigma_m$ in $k[C]$, the support of $\sigma_0 + \sigma_1\alpha + \dots + \sigma_m\alpha^m$ is either cofinite or empty, and if this sum is equal to 0, then $\sigma_0 = \dots = \sigma_m = 0$. Indeed,

$$\sigma_0 + \sigma_1\alpha + \dots + \sigma_m\alpha^m = \sum_{a \in C} \left(\sum_{b \in C} \sum_{k=0}^m \sigma_{k,b} \alpha_{a-b}^k \right) X^a.$$

The σ_k being polynomials, there is a finite number of b in C such that there exists $k \in \{0, \dots, m\}$ with $\sigma_{k,b} \neq 0$. Let b_1, \dots, b_n be these elements. For $i \in \{1, \dots, n\}$, let $Q_i(T)$ be the polynomial $\sum_{k=0}^m \sigma_{k,b_i} T^k$. Therefore

$$\sigma_0 + \sigma_1\alpha + \dots + \sigma_m\alpha^m = \sum_{a \in C} \left(\sum_{i=1}^n Q_i(\alpha_{a-b_i}) \right) X^a.$$

The transcendence degree of $\mathbb{Q}(\sigma_{k,b_i} \mid 0 \leq k \leq m, 1 \leq i \leq n)$ over \mathbb{Q} is finite. If some σ_k 's are not equal to 0, i.e. some Q_i 's are not equal to 0, then there is at most a finite number of n -tuples $(\alpha_{a_1}, \dots, \alpha_{a_n})$ such that $Q_1(\alpha_{a_1}) + \dots + Q_n(\alpha_{a_n}) = 0$. Hence there is a finite number of $a \in C$ such that $\sum_{i=1}^n Q_i(\alpha_{a-b_i}) = 0$, i.e. the support of $\sigma_0 + \sigma_1\alpha + \dots + \sigma_m\alpha^m$ is cofinite.

Hence, if $\sigma_0 + \sigma_1\alpha + \dots + \sigma_m\alpha^m = 0$, then $\sigma_0 = \sigma_1 = \dots = \sigma_m = 0$. It follows that : $\sigma_0 + \sigma_1\alpha + \dots + \sigma_m\alpha^m = \sigma'_0 + \sigma'_1\alpha + \dots + \sigma'_{m'}\alpha^{m'} \Rightarrow m = m'$ and $\sigma_0 = \sigma'_0, \dots, \sigma_m = \sigma'_m$. We can define a multiplication on R by setting :

$$(\sigma_0 + \sigma_1\alpha + \dots + \sigma_m\alpha^m)(\tau_0 + \sigma_1\alpha + \dots + \tau_n\alpha^n) = \sum_{i=0}^{m+n} \left(\sum_{j=0}^i \sigma_j \tau_{i-j} \right) \alpha^i$$

(if $i > n$ (resp. $j > m$), we set $\tau_i = 0$ (resp. $\sigma_j = 0$)). So $(R, +, \cdot)$ is a commutative ring.

Let σ and σ' in R .

If σ and σ' belong to $k[C]$, then (4) and (5) hold, by properties of $k[C]$.

If σ and σ' belong to $R \setminus k[C]$, then their supports are cofinite, hence $\text{Supp}(\sigma) + \text{Supp}(\sigma') = C$, and (4) follows. Now, hypothesis of (5) are not satisfied, hence (5) holds.

Assume that $\sigma' = \tau \in k[C]$, and $\sigma = \sigma_0 + \sigma_1\alpha + \dots + \sigma_m\alpha^m$, with $\sigma_0, \dots, \sigma_m$ in $k[C]$. We have

$$\sigma = \sum_{a \in C} \left(\sigma_{0,a} + \sum_{i=1}^m \sum_{b \in C} \sigma_{i,b} \alpha_{a-b}^i \right) X^a.$$

Hence, for every $a \in C$,

$$\mu_{a,\sigma} = \sigma_{0,a} + \sum_{i=1}^m \sum_{b \in C} \sigma_{i,b} \alpha_{a-b}^i.$$

Now,

$$\begin{aligned} \tau\sigma &= \tau\sigma_0 + \tau\sigma_1\alpha + \cdots + \tau\sigma_m\alpha^m \\ &= \sum_{a \in C} (\sum_{b \in C} \tau_b \sigma_{0,a-b}) X^a + \sum_{i=1}^m (\sum_{a \in C} (\sum_{b \in C} \tau_b \sigma_{i,a-b}) X^a) \alpha^i \\ &= \sum_{a \in C} (\sum_{b \in C} \tau_b \sigma_{0,a-b}) X^a + \sum_{i=1}^m (\sum_{c \in C} (\sum_{a \in C} (\sum_{b \in C} \tau_b \sigma_{i,a-b}) \alpha_{c-a}^i) X^c) \\ &= \sum_{c \in C} (\sum_{b \in C} \tau_b (\sigma_{0,c-b} + \sum_{i=1}^m \sum_{a \in C} \sigma_{i,a-b} \alpha_{c-a}^i)) X^c \\ &= \sum_{c \in C} (\sum_{b \in C} \tau_b \mu_{c-b,\sigma}) X^c. \end{aligned}$$

(R, v) satisfies (5) and (4).

In order to give a criterion for Ψ being onto, we need some definitions.

Definition 2.8. ([10]) Let (R, v) be a cyclically valued ring.

(a) Let $a \in C$, I be an initial segment of (C, \leq_a) , σ, τ be elements of R . We say that τ is a *section of σ by I* if $\text{Supp}(\tau) = \text{Supp}(\sigma) \cap I$, and $v(a, \sigma - \tau) > I$.

(b) We say that R is *closed under section* if, for every $a \in C$, every initial segment I of (C, \leq_a) and every $\sigma \in R$, R contains a section of σ by I .

Remark 2.9. We see that the section of σ by I is unique because if τ_1 and τ_2 are sections of σ by I , then $v(a, \tau_1 - \tau_2) = v(a, \tau_1 - \sigma + \sigma - \tau_2) \geq_a \min_a(v(a, \tau_1 - \sigma), v(a, \sigma - \tau_2)) >_a I$. Hence $v(a, \tau_1 - \tau_2) \notin \text{Supp}(\tau_1) \cup \text{Supp}(\tau_2) \subset I$. It follows $\tau_1 - \tau_2 = 0$.

Definition 2.10. Let (R, v) be a cyclically valued ring, and $a \in C$.

(a) A sequence $(\sigma_s)_{s \in S}$ of R , with S a well-ordered set, is a *pseudo-Cauchy sequence* of $(R, v(a, \cdot))$ if for every $s_1 < s_2 < s_3$ in S , $v(a, \sigma_{s_1} - \sigma_{s_2}) <_a v(a, \sigma_{s_2} - \sigma_{s_3})$ (see [3]).

(b) $(R, v(a, \cdot))$ is *spherically complete* if for every pseudo-Cauchy sequence $(\sigma_s)_{s \in S}$ of $(R, v(a, \cdot))$ there exists $\sigma \in R$ such that for every $s_1 < s_2 < s_3$ in S , $v(a, \sigma_{s_1} - \sigma_{s_2}) = v(a, \sigma_{s_1} - \sigma)$ (we say that σ is a *pseudo-limit* of $(\sigma_s)_{s \in S}$) (see [4]).

(c) (R, v) is *spherically complete* if, for every $a \in C$, $(R, v(a, \cdot))$ is spherically complete.

Example 2.11. Assume that R is a any of $k[[C]]$ or $k[[C, \theta]]$, with k a ring. In the same way as in the case of usual valuations, one can check that (R, v) is closed under section and spherically complete.

Proposition 2.12. *Let (R, v) be a cyclically valued ring. If, (R, v) is closed under section, then the support of every element of E is well-ordered.*

Proof. Let $\sigma \in R$ and assume that $\text{Supp}(\sigma)$ is not well-ordered. Then there exists $a \in C$ and a final segment F in (C, \leq_a) such that $F \cap \text{Supp}(\sigma)$ has no lowest element. Let σ' be the section of σ by $C \setminus F$, then $\{v(c, \sigma - \sigma') \mid c \in C\} = F \cap \text{Supp}(\sigma)$ has no lowest element, which contradicts (3) of Definition 1.1. \square

Proposition 2.13. *Let (R, v) be a cyclically valued ring.*

(a) *(R, v) is closed under section if and only if there exists $a \in C$ such that for every $\sigma \in R$ and for every initial segment I of (C, \leq_a) , R contains a section of σ by I .*

(b) *Assume that (R, v) is closed under section. Then (R, v) is spherically complete if and only if there exists $a \in C$ such that $(R, v(a, \cdot))$ is spherically complete.*

Proof. .

(a) Assume that there exists $a \in C$ such that for every $\sigma \in R$ and for every initial segment I of (C, \leq_a) , R contains a section of σ by I . Now, let $b \neq a$, $\sigma \in R$, J be an initial segment of (C, \leq_b) , and let σ_1 be the section of σ by $[a, b[$.

If $J \cap [a, b[= \emptyset$, then let $\sigma_2 \in R$ be the section of σ by $\{c \in C \mid c \leq_a J\}$. Then $\sigma' := \sigma_2 - \sigma_1$ is the section of σ by J in $(G, v(b, \cdot))$.

If $J \cap [a, b[\neq \emptyset$, then $J \cap [a, b[$ is an initial segment of (C, \leq_a) . Let $\sigma_3 \in R$ be the section of σ by $J \cap [a, b[$. Then $\sigma' = \sigma - \sigma_1 + \sigma_3$ is the section of σ by J in $(R, v(b, \cdot))$.

The converse is trivial.

(b) Let a, b in C , assume that $(R, v(a, \cdot))$ is spherically complete and closed under section. Let $(\sigma_s)_{s \in S}$ be a pseudo-Cauchy sequence of $(G, v(b, \cdot))$, with S a well-ordered set. By general properties of pseudo-Cauchy sequences, we may assume that S is a well-ordered subset of (C, \leq_b) such that for all $s_1 <_b s_2$ in S , $v(b, \sigma_{s_1} - \sigma_{s_2}) = s_1$.

i) Assume that $[a, b[\cap S = \emptyset$, i.e. $S \subset [b, a[$.

For every $s \in S$, let τ_s be the section of σ_s by $[b, a[$. It follows : $\text{Supp}(\sigma_s - \tau_s) \subset [a, b[$, and $v(b, \sigma_s - \tau_s) \geq_b a$ (by (3) of Definition 1.1).

For every $s_1 <_b s_2$ in S , we have $\text{Supp}(\tau_{s_1} - \tau_{s_2}) \subset \text{Supp}(\tau_{s_1}) \cup \text{Supp}(\tau_{s_2}) \subset [b, a[$. Hence $v(a, \tau_{s_1} - \tau_{s_2}) \geq_a b$, and $v(a, \tau_{s_1} - \tau_{s_2}) = v(b, \tau_{s_1} - \tau_{s_2})$. Now $s_1 = v(b, \sigma_{s_1} - \sigma_{s_2}) = v(b, \sigma_{s_1} - \tau_{s_1} + \tau_{s_1} - \tau_{s_2} + \tau_{s_2} - \sigma_{s_2}) = v(b, \tau_{s_1} - \tau_{s_2})$,

because $v(b, \sigma_{s_1} - \tau_{s_1}) >_b a >_b s_1$ and $v(b, \tau_{s_2} - \sigma_{s_2}) >_b a >_b s_1$.

It follows that $(\tau_s)_{s \in S}$ is a pseudo-Cauchy sequence of the spherically complete group $(R, v(a, \cdot))$; let τ be a pseudo-limit. By properties of pseudo-Cauchy sequences, $\forall s \in S, v(a, \tau - \tau_s) = s$.

Let $s \in S$, then $\min_a(\text{Supp}(\tau - \tau_s)) = v(a, \tau - \tau_s) = s \geq_a b$, hence $v(b, \tau - \tau_s) = \min_b(\text{Supp}(\tau - \tau_s)) = s <_b v(b, \sigma_s - \tau_s)$. Consequently $v(b, \tau - \sigma_s) = \min_b(v(b, \tau - \tau_s), v(b, \tau_s - \sigma_s)) = s$. We have proved that τ is a pseudo-limit of $(\sigma_s)_{s \in S}$ in $(R, v(b, \cdot))$.

ii) Assume that $S \cap [a, b[\neq \emptyset$, and let $S' = S \cap [a, b[$. Then S' is a final segment of S and $(\sigma_s)_{s \in S'}$ is a pseudo-Cauchy sequence of $(R, v(b, \cdot))$. Let $s_1 <_b s_2$ in S' . We have $a \leq_b v(b, \sigma_{s_1} - \sigma_{s_2}) = \min_b(\text{Supp}(\sigma_{s_1} - \sigma_{s_2}))$. Hence $\min_a(\text{Supp}(\sigma_{s_1} - \sigma_{s_2})) = \min_b(\text{Supp}(\sigma_{s_1} - \sigma_{s_2}))$, and $v(a, \sigma_{s_1} - \sigma_{s_2}) = v(b, \sigma_{s_1} - \sigma_{s_2})$. It follows that $(\sigma_s)_{s \in S'}$ is a pseudo-Cauchy sequence of $(R, v(a, \cdot))$. Let τ be a pseudo-limit of $(\sigma_s)_{s \in S'}$ in $(R, v(a, \cdot))$.

Let τ' be the section of τ by $[b, a[$, s_0 be the lowest element of (S', \leq_a) and σ' be the section of σ_{s_0} by $[b, a[$. Set $\tau'' = \tau - \tau' + \sigma'$. Note that, by properties of pseudo-Cauchy sequences, and by (3) of Definition 1.1, for every $s >_b s_0$ in S , σ' is the section of σ_s by $[b, a[$, because $v(b, \sigma_s - \sigma_{s_0}) >_b a$.

Let $s >_b s_0$ in S' . We have $v(b, \sigma' - \sigma_s) \geq_b a$, hence $v(b, \tau'' - \sigma_s) = v(b, \tau - \tau' + \sigma' - \sigma_s) \geq_b a$, it follows : $v(b, \tau'' - \sigma_s) = v(a, \tau'' - \sigma_s)$.

By definition of σ' and τ' , we have $v(a, \sigma') \geq_a b$ and $v(a, \tau') \geq_a b$. Hence $v(a, \sigma' - \tau') \geq_a b$. Now, $v(a, \tau - \sigma_s) = s <_a b$, $v(b, \tau'' - \sigma_s) = v(a, \tau'' - \sigma_s) = v(a, \tau - \sigma_s + \sigma' - \tau') = \min_a(v(a, \tau - \sigma_s), v(a, \sigma' - \tau')) = s$. So τ'' is a pseudo-limit of $(\sigma_s)_{s \in S'}$ in $(R, v(b, \cdot))$.

We have proved that $(R, v(b, \cdot))$ is spherically complete.

The converse is trivial. □

Theorem 2. *Assume that (R, v) is a cyclically valued ring. Let $k := M_0 \cup \{0\}$ and assume (a), (b) below :*

(a) C is interpretable.

(b) (R, v) is spherically complete and closed under section.

Then (R, v) is isomorphic to a ring $(k[[C, \theta]], v)$, for some $\theta : C \times C \rightarrow k$ such that $\theta(C \times C)$ is a commutative factor set.

Proof. By Proposition 2.12 and Theorem 1, there exists an isomorphism Ψ from (R, v) into a ring $(k[[C, \theta]], v)$, for some $\theta : C \times C \rightarrow k$ such that $\theta(C \times C)$ is a commutative factor set. It remains to prove that Ψ is onto. We identify R with its image in $k[[C, \theta]]$. Let σ be an element of $k[[C, \theta]]$. We prove by induction on the initial segment I of the support $S(\sigma)$ of σ

that R contains all the sections of σ . It will follow that σ belongs to R .

If I contains only a finite number of elements of the support of σ , this is true because R contains all the polynomials.

Assume that the property is true for all $I' < I$.

If $I = I' \cup \{c\}$, there exists a section $\tau' \in R$ of σ with support I' . Let $\tau := \tau' + \sigma_c X^c$, $\tau \in R$ because $(R, +)$ is a group, τ is a section of σ with support I .

If $I = \bigcup_{I' < I} I'$, let (s) be an increasing sequence of I , cofinal in I . For every s , set $I_s := \{a \in I \mid a \leq s\}$, and let $\tau_s \in R$ such that τ_s is a section of σ with support I_s , i.e. $\tau_s := \sum_{c \in I_s} \sigma_c X^c$. Then (τ_s) is a pseudo-Cauchy sequence of R , hence it has a pseudo-limit τ'' in R (because R is spherically complete), and the restrictions of τ'' and σ to I are equal. Let τ be the section of τ'' with support I , $\tau \in R$ because R is closed under section, and τ is a section of σ , too. \square

3. Quotients and extensions of cyclic valuations.

Theorem 3. *Let H be a subgroup of C .*

1) *Let R' be the set of all elements of R with supports contained in H , and v' be the restriction of v to $R' \times H$. Then (R', v') is a cyclically valued subring of (R, v) .*

2) *Assume that H is a convex subgroup of the linear part of C .*

a) *For every σ in $R \setminus \{0\}$ and every a in C , set $v''(a+H, \sigma) := \min_{a+H} \{v(c, \sigma) + H \mid c \in C\}$ and $v''(a+H, 0) := \infty$. Then (R, v'') satisfies (1), (3), (4) of Definition 1.1, and the set of all monomials of degree $a+H$ is $\{\rho \in R \mid \text{Supp}_v(\rho) \subset a+H\}$.*

b) *Let k be a ring, $\theta : C \times C \rightarrow k$ be a mapping such that $\theta(C, C)$ is a commutative factor set, and assume that $R = k[[C, \theta]]$ or $R = k[C, \theta]$. Then (R, v'') is a cyclically valued ring.*

Proof. .

1) $(R', +)$ is a subgroup of $(R, +)$. Indeed, $0 \in R'$, and if σ and τ belong to R' , then $\text{Supp}(-\sigma) = \text{Supp}(\sigma) \subset H$, and $\text{Supp}(\sigma + \tau) \subset \text{Supp}(\sigma) \cup \text{Supp}(\tau) \subset H$.

Any monomial with degree in H belongs to R' , hence $v' : R' \times H \rightarrow H \cup \{\infty\}$ is onto; furthermore, (R', v') satisfies condition (2) of Definition 1.1.

Clearly, for every $a \in H$, the restriction of $v(a, \cdot)$ to R' is a valuation of

groups, hence (R', v') enjoys (1) of Definition 1.1.

Let $\sigma \in R'$ and $a \in H$. $\min_a(\text{Supp}(\sigma))$ exists and is equal to $v(a, \sigma)$. Now, $\text{Supp}(\sigma) \subset H$, hence $v(a, \sigma) \in H$. It follows that $v'(a, \sigma) = v(a, \sigma) = \min_a(\text{Supp}(\sigma)) \cap H$ i.e. (R', v') satisfies (3) of Definition 1.1.

Let σ and τ in R' , then $\text{Supp}(\sigma\tau) \subset \text{Supp}(\sigma) + \text{Supp}(\tau) \subset H$, hence R' is a subring of R . Now, (R', v) satisfies (4) and (5) of Definition 1.1 because so does (R, v) .

2) a)

In order to prove that the definition of v'' is consistent, let $a \in C$ and $\sigma \neq 0$ in R . If $\text{Supp}_v(\sigma) \cap (a+H) \neq \emptyset$, then $\min_{a+H}\{v(c, \sigma) + H \mid c \in C\} = a + H$. Otherwise, let $b := v(a, \sigma) = \min_a\{v(c, \sigma) \mid c \in C\}$ and $a_1 \in a + H$. Then $\forall s \in \text{Supp}_v(\sigma) \setminus \{b\}$, (a, b, s) . Now, H is a convex subgroup of $l(C)$, hence (a_1, b, s) , therefore $v(a_1, \sigma) = b = \min_a\{v(c, \sigma) \mid c \in C\}$. It follows that $b + H = \min_{a+H}\{v(c, \sigma) + H \mid c \in C\}$.

Let $a \in C$, σ, τ in R . By definition, $v''(a + H, \sigma) = \infty \Leftrightarrow \sigma = 0$. Assume that $v''(a + H, \sigma - \tau) = a + H$, then there exists $a_1 \in a + H$ such that $v(a_1, \sigma - \tau) = a_1$. Now $v(a_1, \sigma - \tau) \geq_{a_1} \min_{a_1}(v(a_1, \sigma), v(a_1, \tau))$, hence $v(a_1, \sigma) = a_1$ or $v(a_1, \tau) = a_1$, in any case, $\min_{a+H}(v''(a + H, \sigma), v''(a + H, \tau)) = a + H$. Assume that $v''(a + H, \sigma - \tau) \neq a + H$, and let $b = v(a, \sigma - \tau)$. We have already proved that $v''(a + H, \sigma - \tau) = b + H$. We have $b = v(a, \sigma - \tau) \geq_a \min_a(v(a, \sigma), v(a, \tau))$, hence $\min_{a+H}(v''(a + H, \sigma), v''(a + H, \tau)) \leq_{a+H} b + H$. So (R, v'') satisfies (1) of Definition 1.1.

By hypothesis, for every σ in R , $\text{Supp}_{v''}(\sigma) \subset \{v(c, \sigma) + H \mid c \in H\}$. Let $b \in \text{Supp}_v(\sigma)$, then $v''(b + H, \sigma) = \min_{b+H}\{v(c, \sigma) \mid c \in C\} = b + H$, hence $\text{Supp}_{v''}(\sigma) = \{v(c, \sigma) + H \mid c \in C\}$. So for every $a \in C$, $v''(a + H, \sigma) = \min_{a+H} \text{Supp}_{v''}(\sigma)$ and (R, v'') satisfies (3) of Definition 1.1.

Let σ, τ in R . We have $\text{Supp}_v(\sigma\tau) \subset \text{Supp}_v(\sigma) + \text{Supp}_v(\tau)$, hence $\{v(c, \sigma\tau) + H \mid c \in C\} \subset \{v(c, \sigma) + H \mid c \in C\} + \{v(c, \tau) + H \mid c \in C\}$, i.e. $\text{Supp}_{v''}(\sigma\tau) \subset \text{Supp}_{v''}(\sigma) + \text{Supp}_{v''}(\tau)$. This proves (4) of Definition 1.1.

Trivially, the set of v'' -monomials of degree a is the set of all elements with v -support non-empty and contained in $a + H$.

2) b)

i) Let $\sigma \in R$, $a \in \text{Supp}_v(\sigma)$ and $\mu_{a+H, \sigma}$ be the restriction of σ to $a + H$. Then $\text{Supp}_v(\mu_{a+H, \sigma}) \subset a + H$ and $\text{Supp}_v(\sigma - \mu_{a+H, \sigma}) \cap (a + H) = \emptyset$, hence $v''(a + H, \sigma - \mu_{a+H, \sigma}) \neq a + H$. Trivially, if μ is a v'' -monomial such that $v''(a + H, \sigma - \mu) \neq a + H$, then $\mu = \mu_{a+H, \sigma}$. So (R, v'') satisfies (2) of Definition 1.1.

ii) Let σ, τ in R , $a + H \in \text{Supp}_{v''}(\sigma\tau)$ and $\{a_1 + H, \dots, a_n + H\} =$

$\text{Supp}_{v''}(\sigma) \cap ((a + H) - \text{Supp}_{v''}(\tau))$. In order to prove that

$$\mu_{a+H,\sigma\tau} = \mu_{a_1+H,\sigma}\mu_{a-a_1+H,\tau} + \cdots + \mu_{a_n+H,\sigma}\mu_{a-a_n+H,\tau},$$

it is sufficient to prove that

$$\text{Supp}_v(\sigma\tau - \mu_{a_1+H,\sigma}\mu_{a-a_1+H,\tau} - \cdots - \mu_{a_n+H,\sigma}\mu_{a-a_n+H,\tau}) \cap (a + H) = \emptyset.$$

Let $\sigma = \sum_{c \in C} \sigma_c X^c$, $\tau = \sum_{c \in C} \tau_c X^c$. For every i , $1 \leq i \leq n$, we have $\mu_{a_i+H,\sigma} = \sum_{c \in a_i+H} \sigma_c X^c$ and $\mu_{a-a_i+H,\tau} = \sum_{c \in a-a_i+H} \tau_c X^c$. Set

$$E := [(C/H) \times (C/H)] \setminus \{(a_1+H) \times ((a-a_1)+H)\} \cup \cdots \cup \{(a_n+H) \times (a-a_n+H)\},$$

then

$$\sigma\tau - \mu_{a_1+H,\sigma}\mu_{a-a_1+H,\tau} - \cdots - \mu_{a_n+H,\sigma}\mu_{a-a_n+H,\tau} = \sum_{(c_1, c_2) \in E} \sigma_{c_1} \tau_{c_2} X^{c_1+c_2} \theta(c_1, c_2).$$

Now, by hypothesis, $((c_1, c_2) \in E \text{ and } c_1 + c_2 \in a + H) \Rightarrow \sigma_{c_1} = 0 \text{ or } \tau_{c_2} = 0$, hence

$$\text{Supp}(\sigma\tau - \mu_{a_1+H,\sigma}\mu_{a-a_1+H,\tau} - \cdots - \mu_{a_n+H,\sigma}\mu_{a-a_n+H,\tau}) \cap (a + H) = \emptyset.$$

□

Definition 3.1. Let k be a ring, $\theta : C \times C \rightarrow k$ be a mapping such that $\theta(C, C)$ is a commutative factor set, $R = k[[C, \theta]]$ or $R = k[C, \theta]$, and let H be a convex subgroup of C . If v'' is the cyclic valuation defined by $v''(a + H, \sigma) = \min_{a+H} \{v(c, \sigma) + H \mid c \in C\}$ and $v(a + H, 0) = \infty$, we will say that v'' is a *quotient* of v .

Remark 3.2. Assume that $R = k[[C, \theta]]$ (or $k[C, \theta]$) and that H is the linear part of C . H being a totally ordered group, $k[[H, \theta]]$ is a “classical” power series ring with twisted multiplication. Set $R' := k[[H, \theta]]$, and $C' := C/H$. Then R' is the set of all constants of (R, v'') , and C' is interpretable in (R, v'') , because for every a in C , X^a is an invertible v'' -monomial. Hence there exists θ' such that $k[[C, \theta]] \simeq R'[[C', \theta']]$. So, any power series ring with twisted multiplication can be seen as a power series ring with twisted multiplication such that the cyclically ordered group is archimedean.

Theorem 4. *Let (R_1, v_1) be a cyclically valued ring such that C_1 is a linear cyclically ordered group. Let (R_2, v_2) be a cyclically valued ring such that C_2 is interpretable, the v_2 -support of every element of R is well-ordered, and the ring of all v_2 -constants is isomorphic to R_1 . Assume that R_2 contains a subset of v_2 -monomials $M_0 := \{X^{c_2} \mid c_2 \in C_2, \deg_{v_2}(X^{c_2}) = c_2\}$ such that $X^0 = 1$, and for every c_2, c'_2 in C_2 ,*

$X^{c_2} X^{c'_2} (X^{c_2+c'_2})^{-1} \in k \setminus \{0\}$, with k the ring of constants of (R_1, v_1) . Then there exists a cyclic valuation v_3 on R_2 such that v_2 is a quotient of v_3 , and the group of v_3 is the lexicographically ordered product $C_3 = C_1 \overleftarrow{\times} C_2$.

Proof. .

First we explain the notation $C_1 \overleftarrow{\times} C_2$. By a theorem of Rieger (see [1]) there exist a totally ordered abelian group G_2 and z_2 cofinal in the positive cone of G_2 such that $C_2 \simeq G_2/\mathbb{Z}z_2$. Then $C_1 \overleftarrow{\times} C_2 = (C_1 \overleftarrow{\times} G_2)/\mathbb{Z}(0, z_2)$.

In the following, for every σ in R_2 and a_2 in C_2 , we let $\sigma_{a_2} := \mu_{a_2, \sigma} (X^{a_2})^{-1}$ in R_1 be the only v_2 -constant such that $v_2(a_2, \sigma - \sigma_{a_2} X^{a_2}) \neq a_2$.

Let $(a_1, a_2) \in C_1 \times C_2$, $\sigma \in R_2$, and set $b_2 := v_2(a_2, \sigma)$. If $b_2 = a_2$ (i.e. $\sigma_{a_2} \neq 0$), set $b_1 := v_1(a_1, \sigma_{b_2})$. If $b_2 \neq a_2$, set $b_1 := v_1(\epsilon_1, \sigma_{b_2})$. Now, set $v_3((a_1, a_2), \sigma) := (b_1, b_2)$.

We show that the set of all v_3 -monomials is

$$\{\mu_1 X^{b_2} \mid \mu_1 \text{ is a } v_1\text{-monomial and } b_2 \in C_2\}.$$

Let μ_1 be a v_1 -monomial, b_1 be the v_1 -degree of μ_1 , and let $b_2 \in C_2$, $(a_1, a_2) \in C_1 \times C_2$. $v_2(a_2, \mu_1 X^{b_2}) = b_2$ because μ_1 is a v_2 -constant. If $b_2 \neq a_2$, then $v_3((a_1, a_2), \mu_1 X^{b_2}) = (v_1(\epsilon_1, \mu_1), b_2) = (b_1, b_2)$. If $b_2 = a_2$, then $v_3((a_1, a_2), \mu_1 X^{b_2}) = (v_1(a_1, \mu_1), b_2) = (b_1, b_2)$. Hence $\mu_1 X^{a_2}$ is a v_3 -monomial. Let τ be a v_3 -monomial and (b_1, b_2) be the v_3 -degree of τ . For every $(a_1, a_2) \in C_1 \times C_2$, $v_3((a_1, a_2), \tau) = (b_1, b_2)$, hence $v_2(a_2, \tau) = b_2$. Hence τ is a v_2 -monomial. It follows that $\tau = \mu_{b_2, \tau}$, and $\tau_{b_2} = \tau (X^{b_2})^{-1}$. So $\tau = \tau_{b_2} X^{b_2}$ (with τ_{b_2} a v_2 -constant). If $b_2 \neq a_2$, then $v_1(\epsilon_1, \tau_{b_2}) = b_1$. If $b_2 = a_2$, then for every a_1 , $v_1(a_1, \tau_{b_2}) = b_1$. Hence τ_{b_2} is a v_1 -monomial.

Consequently, the set of all v_3 -monomials is closed under multiplication, the degree of the product of any two v_3 -monomials is the sum of their degrees, and if τ is a v_3 -monomial such that $(\tau)^{-1}$ exists, then $(\tau)^{-1}$ is a v_3 -monomial.

(1) of Definition 1.1. Let σ and τ in R_2 , (a_1, a_2) in $C_1 \overleftarrow{\times} C_2$ and $(b_1, b_2) = v_3((a_1, a_2), \sigma - \tau)$. By hypothesis, we have

$$b_2 = v_2(a_2, \sigma - \tau) \geq_{a_2} \min_{a_2}(v_2(a_2, \sigma), v_2(a_2, \tau)).$$

If $b_2 = a_2$, then

$$b_1 = v_1(a_1, (\sigma - \tau)_{a_2}) = v_1(a_1, \sigma_{a_2} - \tau_{a_2}) \geq_{a_1} \min_{a_1}(v_1(a_1, \sigma_{a_1}), v_1(a_1, \tau_{a_1})).$$

If $\sigma_{a_2} \neq 0 \neq \tau_{a_2}$, then

$$\begin{aligned} (b_1, b_2) &\geq_{(a_1, a_2)} \min_{(a_1, a_2)}((v_1(a_1, \sigma_{a_2}), a_2), (v_1(a_1, \tau_{a_2}), a_2)) \\ &\geq_{(a_1, a_2)} \min_{(a_1, a_2)}(v_3((a_1, a_2), \sigma), v_3((a_1, a_2), \tau)). \end{aligned}$$

If $\tau_{a_2} = 0$, then $\sigma_{a_2} \neq 0$, and $v_2(a_2, \tau) >_{a_2} a_2 = v_2(a_2, \sigma)$. Hence

$$v_3((a_1, a_2), \sigma) <_{(a_1, a_2)} v_3((a_1, a_2), \tau).$$

Now, $v_1(a_1, \tau_{a_2}) = \infty$, hence

$$b_1 = v_1(a_1, \sigma_{a_2} - \tau_{a_2}) = v_1(a_1, \sigma_{a_2}).$$

It follows :

$$(b_1, b_2) = v_3((a_1, a_2), \sigma) = \min_{(a_1, a_2)}(v_3((a_1, a_2), \sigma), v_3((a_1, a_2), \tau)).$$

The case $\sigma_{a_2} = 0$ is similar.

If $b_2 \neq a_2$, then :

$$b_1 = v_1(\epsilon_1, (\sigma - \tau)b_2) = v_1(\epsilon_1, \sigma b_2 - \tau b_2) \geq_{\epsilon_1} \min_{\epsilon_1}(v_1(\epsilon_1, \sigma b_2), v_1(\epsilon_1, \tau b_2)).$$

If $b_2 >_{a_2} \min_{a_2}(v_2(a_2, \sigma), v_2(a_2, \tau))$, then :

$$v_3((a_1, a_2), \sigma - \tau) \geq_{(a_1, a_2)} \min_{(a_1, a_2)}(v_3((a_1, a_2), \sigma), v_3((a_1, a_2), \tau)).$$

If $b_2 = \min_{a_2}(v_2(a_2, \sigma), v_2(a_2, \tau))$, say $b_2 = v_2(a_2, \sigma)$, then $(\tau b_2 \neq 0 \Leftrightarrow v_2(a_2, \tau) = b_2)$. Hence :

$$v_3((a_1, a_2), \sigma - \tau) \geq_{(a_1, a_2)} \min_{(a_1, a_2)}(v_3((a_1, a_2), \sigma), v_3((a_1, a_2), \tau)).$$

(1) is proved.

(3) of Definition 1.1. Let $(a_1, a_2) \in C_1 \times C_2$, $\sigma \in R_2$, $(b_1, b_2) := v_3((a_1, a_2), \sigma)$ and $(c_1, c_2) \in C_1 \times C_2$ with $(c_1, c_2) \leq_{(a_1, a_2)} (b_1, b_2)$. We have

$$b_2 = v_2(a_2, \sigma) \text{ and } ((c_1, c_2) \leq_{(a_1, a_2)} (b_1, b_2) \Rightarrow c_2 \leq_{a_2} b_2),$$

hence $b_2 = v_2(c_2, \sigma)$, because v_2 satisfies (3').

If $b_2 \neq c_2$, then $v_3((c_1, c_2), \sigma) = (v_1(\epsilon_1, \sigma b_2), b_2) = (b_1, b_2)$.

If $b_2 = c_2 \neq a_2$, then $v_3((c_1, c_2), \sigma) = (v_1(c_1, \sigma b_2), b_2) = (b_1, b_2)$ (because v_1 satisfies (3')).

If $b_2 = a_2$, then $c_2 = a_2$ and $c_1 \leq_{a_1} b_1 = v_1(a_1, \sigma_{a_2})$. Hence $v_3((c_1, c_2), \sigma) = (v_1(c_1, \sigma_{a_2}), b_2) = (v_1(a_1, \sigma_{a_2}), b_2) = (b_1, b_2)$.

(2) of Definition 1.1. If $v_3((a_1, a_2), \sigma) \neq (a_1, a_2)$, take $\mu_{(a_1, a_2), \sigma} = 0$. If $v_3((a_1, a_2), \sigma) = (a_1, a_2)$, then by the definition of v_3 , we have $\sigma_{a_2} \neq 0$, $v_2(a_2, \sigma) = a_2$, $v_3(a_2, \sigma - \sigma_{a_2} X^{a_2}) \neq a_2$ and $v_1(a_1, \sigma_{a_2}) = a_1$. We know

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that $\mu_{a_1, \sigma_{a_2}} X^{a_2}$ is a v_3 -monomial, where $\mu_{a_1, \sigma_{a_2}}$ is the only v_1 -monomial such that $v_1(a_1, \sigma_{a_2} - \mu_{a_1, \sigma_{a_2}}) \neq a_1$. Now :

$$\begin{aligned} v_3((a_1, a_2), \sigma - \mu_{a_1, \sigma_{a_2}} X^{a_2}) &= v_3((a_1, a_2), \sigma - \sigma_{a_2} X^{a_2} + (\sigma_{a_2} \\ &\quad - \mu_{a_1, \sigma_{a_2}}) X^{a_2}) \\ &\geq_{(a_1, a_2)} \min_{(a_1, a_2)}(v_3((a_1, a_2), \sigma - \sigma_{a_2} X^{a_2}), \\ &\quad v_3((a_1, a_2), (\sigma_{a_2} - \mu_{a_1, \sigma_{a_2}}) X^{a_2})) \\ &>_{(a_1, a_2)} (a_1, a_2). \end{aligned}$$

(2) is proved.

(4) of Definition 1.1. Let $(a_1, a_2) \in \text{Supp}_{v_3}(\sigma\tau)$. The v_2 -support of every element being well-ordered, $\text{card}(\{(b, c) \in \text{Supp}_{v_2}(\sigma) \times \text{Supp}_{v_2}(\tau)\})$ is finite, say equal to n . Let $b_{2,i} \in \text{Supp}_{v_2}(\sigma)$, $c_{2,i} \in \text{Supp}_{v_2}(\tau)$, $1 \leq i \leq n$, such that $a_2 = b_{2,i} + c_{2,i}$ ($1 \leq i \leq n$). Then :

$$\begin{aligned} (\sigma\tau)_{a_2} &= \mu_{a_2, \sigma\tau} (X^{a_2})^{-1} \\ &= \mu_{b_{2,1}, \sigma} \mu_{c_{2,1}, \tau} (X^{a_2})^{-1} + \dots + \mu_{b_{2,n}, \sigma} \mu_{c_{2,n}, \tau} (X^{a_2})^{-1} \\ &= \sigma_{b_{2,1}} \tau_{c_{2,1}} X^{b_{2,1}} X^{c_{2,1}} (X^{a_2})^{-1} + \dots + \sigma_{b_{2,n}} \tau_{c_{2,n}} X^{b_{2,n}} X^{c_{2,n}} (X^{a_2})^{-1}, \end{aligned}$$

$$\text{with } X^{b_{2,i}} X^{c_{2,i}} (X^{a_2})^{-1} \in k \setminus \{0\}.$$

Recall that the support of the sum of any two elements is contained in the union of the supports of these elements (see [5]). Hence :

$$\begin{aligned} \text{Supp}_{v_1}(\sigma\tau)_{a_2} &\subset \bigcup_{1 \leq i \leq n} \text{Supp}_{v_1}(\sigma_{b_{2,i}} \tau_{c_{2,i}}) X^{b_{2,i}} X^{c_{2,i}} (X_2^a)^{-1} \\ &\subset \bigcup_{1 \leq i \leq n} \text{Supp}_{v_1}(\sigma_{b_{2,i}} \tau_{c_{2,i}}) \\ &\subset \bigcup_{1 \leq i \leq n} \text{Supp}_{v_1}(\sigma_{b_{2,i}}) + \text{Supp}_{v_1}(\tau_{c_{2,i}}). \end{aligned}$$

It follows : $a_1 \in \bigcup_{1 \leq i \leq n} \text{Supp}_{v_1}(\sigma_{b_{2,i}}) + \text{Supp}_{v_1}(\tau_{c_{2,i}})$ and

$$\begin{aligned} (a_1, a_2) &\in \bigcup_{1 \leq i \leq n} (\text{Supp}_{v_1}(\sigma_{b_{2,i}}) + \text{Supp}_{v_1}(\tau_{c_{2,i}})) \times \{b_{2,i} + c_{2,i}\} \\ &= \bigcup_{1 \leq i \leq n} (\text{Supp}_{v_1}(\sigma_{b_{2,i}}) + \text{Supp}_{v_1}(\tau_{c_{2,i}})) \times \{a_2\}. \end{aligned}$$

We deduce : $\text{Supp}_{v_3}(\sigma\tau) \subset \text{Supp}_{v_3}(\sigma) + \text{Supp}_{v_3}(\tau)$.

(5) of Definition 1.1. Let $(a_1, a_2) \in \text{Supp}_{v_3}(\sigma\tau)$ and assume that $(b_{1,ij}, b_{2,i}) \in \text{Supp}_{v_3}(\sigma)$, $(c_{1,ij}, c_{2,i}) \in \text{Supp}_{v_3}(\tau)$, $1 \leq i \leq n$, $1 \leq j \leq p_i$, are the only elements such that $(b_{1,ij}, b_{2,i}) + (c_{1,ij}, c_{2,i}) = (a_1, a_2)$ (so $a_1 = b_{1,ij} + c_{1,ij}$, $a_2 = b_{2,i} + c_{2,i}$). Then for every i, j , $v((b_{1,ij}, b_{2,i}), \sigma - \mu_{b_{1,ij}, \sigma_{b_{2,i}}} X^{b_{2,i}}) \neq (b_{1,ij}, b_{2,i})$ and $v((c_{1,ij}, c_{2,i}), \tau - \mu_{c_{1,ij}, \tau_{c_{2,i}}} X^{c_{2,i}}) \neq (c_{1,ij}, c_{2,i})$.

Hence :

$$\begin{aligned}
 & v_3((a_1, a_2), \sigma\tau - \mu_{b_{1,11}, \sigma_{b_{2,1}}} X^{b_{2,1}} \mu_{c_{1,11}, \tau_{c_{2,1}}} X^{c_{2,1}} - \dots \\
 & - \mu_{b_{1,1p_1}, \sigma_{b_{2,1}}} X^{b_{2,1}} \mu_{c_{1,1p_1}, \tau_{c_{2,1}}} X^{c_{2,1}} - \dots - \mu_{b_{1,n1}, \sigma_{b_{2,n}}} X^{b_{2,n}} \mu_{c_{1,n1}, \tau_{c_{2,n}}} X^{c_{2,n}} \\
 & - \dots - \mu_{b_{1,np_n}, \sigma_{b_{2,ni}}} X^{b_{2,n}} \mu_{c_{1,np_n}, \tau_{c_{2,n}}} X^{c_{2,n}}) \\
 & = v_3((a_1, a_2), \sigma\tau - (\sigma_{b_{2,1}} \tau_{c_{2,1}} X^{b_{2,1}} X^{c_{2,1}} + \dots + \sigma_{b_{2,n}} \tau_{c_{2,n}} X^{b_{2,n}} X^{c_{2,n}}) \\
 & + (\sigma_{b_{2,1}} \tau_{c_{2,1}} - \mu_{b_{1,11}, \sigma_{b_{2,1}}} \mu_{c_{1,11}, \tau_{c_{2,1}}} - \dots - \mu_{b_{1,1p_1}, \sigma_{b_{2,1}}} \mu_{c_{1,1p_1}, \tau_{c_{2,1}}}) X^{b_{2,1}} X^{c_{2,1}} + \dots \\
 & + (\sigma_{b_{2,ni}} \tau_{c_{2,n}} - \mu_{b_{1,n1}, \sigma_{b_{2,i}}} \mu_{c_{1,n1}, \tau_{c_{2,n}}} - \dots - \mu_{b_{1,np_n}, \sigma_{b_{2,n}}} \mu_{c_{1,np_n}, \tau_{c_{2,n}}}) X^{b_{2,n}} X^{c_{2,n}}) \\
 & \geq_{(a_1, a_2)} \min_{(a_1, a_2)} (v_3((a_1, a_2), \sigma\tau - (\sigma_{b_{2,1}} \tau_{c_{2,1}} X^{b_{2,1}} X^{c_{2,1}} + \dots \\
 & + \sigma_{b_{2,n}} \tau_{c_{2,n}} X^{b_{2,n}} X^{c_{2,n}})), v_3((a_1, a_2), (\sigma_{b_{2,1}} \tau_{c_{2,1}} - \mu_{b_{1,11}, \sigma_{b_{2,1}}} \mu_{c_{1,11}, \tau_{c_{2,1}}} - \dots \\
 & - \mu_{b_{1,1p_1}, \sigma_{b_{2,1}}} \mu_{c_{1,1p_1}, \tau_{c_{2,1}}}) X^{b_{2,1}} X^{c_{2,1}}), \dots, v_3((a_1, a_2), (\sigma_{b_{2,n}} \tau_{c_{2,n}} \\
 & - \mu_{b_{1,n1}, \sigma_{b_{2,n}}} \mu_{c_{1,n1}, \tau_{c_{2,n}}} - \dots - \mu_{b_{1,np_n}, \sigma_{b_{2,n}}} \mu_{c_{1,np_n}, \tau_{c_{2,n}}}) X^{b_{2,n}} X^{c_{2,n}})).
 \end{aligned}$$

Now, $v_2(a, \sigma\tau - (\sigma_{b_{2,1}} \tau_{c_{2,1}} X^{b_{2,1}} X^{c_{2,1}} + \dots + \sigma_{b_{2,n}} \tau_{c_{2,n}} X^{b_{2,n}} X^{c_{2,n}})) \neq a_2$, and for $1 \leq i \leq n$,

$$v_1(a_1, \sigma_{b_{2,i}} \tau_{c_{2,i}} - \mu_{b_{1,i1}, \sigma_{b_{2,i}}} \mu_{c_{1,i1}, \tau_{c_{2,i}}} - \dots - \mu_{b_{1,ip_i}, \sigma_{b_{2,i}}} \mu_{c_{1,ip_i}, \tau_{c_{2,i}}}) \neq a_1.$$

It follows :

$$\begin{aligned}
 & v_3((a_1, a_2), \sigma\tau - \mu_{b_{1,11}, \sigma_{b_{2,1}}} X^{b_{2,1}} \mu_{c_{1,11}, \tau_{c_{2,1}}} X^{c_{2,1}} - \dots \\
 & - \mu_{b_{1,np_n}, \sigma_{b_{2,n}}} X^{b_{2,n}} \mu_{c_{1,np_n}, \tau_{c_{2,n}}} X^{c_{2,n}}) >_{(a_1, a_2)} (a_1, a_2).
 \end{aligned}$$

We conclude by proving that v_2 is a quotient of v_3 . By construction, $H := C_1 \overleftarrow{\times} \{0\}$ is a convex subgroup of the linear part of $C_3 = C_1 \overleftarrow{\times} C_2$. We have $C_2 \simeq (C_1 \overleftarrow{\times} C_2) / (C_1 \overleftarrow{\times} \{0\})$, hence for every (a_1, a_2) in C_3 , we can set $(a_1, a_2) + H = a_2$. Let $\sigma \in R_2$ and $a_2 \in C_2$. By definition, for every a_1 in C_1 , we have $v_3((a_1, a_2), \sigma) + H = v_2(a_2, \sigma)$. Therefore $\min_{a_2} \{v_3(a_1, a_2) + H \mid a_1 \in C_1\} = v_2(a_2, \sigma) : v_2$ is a quotient of v_3 . \square

Corollary 3.3. *Let (R_2, v_2) be a cyclically valued ring such that C_2 is a summand in the set of monomials, and the support of every element of R_2 is well-ordered. Assume that the ring R_1 of all constants is a cyclically valued one, with a cyclic valuation v_1 such that the group C_1 is a linear cyclically ordered group. Then there exists a cyclic valuation v_3 on R_2 such that v_2 is a quotient of v_3 , and the group of v_3 is the lexicographically ordered product $C_1 \overleftarrow{\times} C_2$.*

Corollary 3.4. *Let S_1 be a cyclically valued ring with ring of constants k and cyclically ordered group C_1 , such that C_1 is a linear cyclically ordered group. Let C_2 be an abelian cyclically ordered group, $\theta : C_2 \times C_2 \rightarrow S_1$ be*

a mapping such that $\theta(C_2, C_2)$ is a commutative factor set, and the image of θ is a subset of k . Then there exists a cyclic valuation v_3 on $S_1[[C_2, \theta]]$ such that v_2 is a quotient of v_3 .

4. Valuation ϵ .

The valuation $v(\epsilon, \cdot)$ is first order definable in the language $(+, 0, \cdot, v, C, (\cdot, \cdot, \cdot))$, because $v(\epsilon, \cdot)$ exist if and only if :

$\forall \sigma \in R, (\exists a \in C, (-a <_0 a) \wedge v(a, \sigma) = a) \Rightarrow \exists a \in C, -a <_0 a \wedge \forall b \in C, (-b <_0 b \Rightarrow v(a, \sigma) \leq_0 v(b, \sigma))$,

and if that is the case then we define $v(\epsilon, \cdot)$ by setting for every $\sigma \in R$,

if $\exists a \in C, -a <_0 a$, then $v(\epsilon, \sigma)$ is the element $a \in C$ such that $-a <_0 a$ and $\forall b \in C, -b <_0 b \Rightarrow v(a, \sigma) \leq_0 v(b, \sigma)$;

otherwise, $v(\epsilon, \sigma) = v(0, \sigma)$.

If $\epsilon \in C$, (i.e. $C \setminus \{0\}$ contains an element ϵ such that $-\epsilon = \epsilon$), then $v(\epsilon, \cdot)$ exists.

Now, $<_\epsilon$ defines a Dedekind cut of C . Hence, by [5], we know that if the cyclically valued ring (R, v) enjoys : $\forall \sigma \in R, \min_\epsilon(\text{Supp}(\sigma))$ exists, then $(R, v(\epsilon, \cdot))$ is a valued group.

The reader can check that, if the support of every element is well-ordered, then $v(\epsilon, \cdot)$ exists.

Proposition 4.1. *Assume that the product of any two monomials of R is different from 0, that C is a linear cyclically ordered group and that $v(\epsilon, \cdot)$ exists. Then for all σ and τ in R , $v(\epsilon, \sigma\tau) = v(\epsilon, \sigma) + v(\epsilon, \tau)$, and R is integral.*

Proof. Let $a := v(\epsilon, \sigma)$, $b := v(\epsilon, \tau)$. Then $a = \min_\epsilon \text{Supp}(\sigma)$ and $b = \min_\epsilon \text{Supp}(\tau)$. Now, by hypothesis, (C, \leq_ϵ) is a totally ordered group, hence $\text{Supp}(\sigma) \cap (a + b - \text{Supp}(\tau)) = \{a\}$. It follows that $\mu_{a+b, \sigma\tau} = \mu_{a, \sigma} \mu_{b, \tau} \neq 0$, and $v(a + b, \sigma\tau) = a + b$, so $a + b \in \text{Supp}(\sigma\tau)$. Now, $\text{Supp}(\sigma\tau) \subset \text{Supp}(\sigma) + \text{Supp}(\tau)$, hence $v(\epsilon, \sigma) + v(\epsilon, \tau) = a + b = \min_\epsilon \text{Supp}(\sigma\tau)$, and $v(\epsilon, \sigma\tau) = v(\epsilon, \sigma) + v(\epsilon, \tau)$. In particular, $\text{Supp}(\sigma\tau) \neq \emptyset$ i.e. $\sigma\tau \neq 0$. \square

Theorem 5. *Assume that the product of any two monomials of the cyclically valued ring (R, v) is not 0, and let $a \in C \cup \{\epsilon\}$. Then $(R, v(a, \cdot))$ is a topological ring if and only if either C is a linear cyclically ordered group and $a = \epsilon$ or $(C, <_0)$ has a greatest element.*

Proof. .

First, we note that, by properties of valued groups, for any $a \in C \cup \{\epsilon\}$, $(R, +, v(a, \cdot))$ is a topological group. Hence $(R, +, v(a, \cdot))$ is a topological ring if and only if the multiplication is continuous.

In the following, we set, for any a, b in C and $\sigma \in R$, $B_{a,>}(\sigma, b) := \{\tau \in R \mid v(a, \tau - \sigma) >_a b\}$.

Assume that $(C, <_0)$ has a greatest element, say m . By compatibility, $a + m$ is the greatest element of $(C, <_a)$. It follows that for every $\sigma \in R$, $B_{a,>}(\sigma, a + m) = \{\sigma\}$. Hence the topology is discrete, and the multiplication is continuous.

From now on, $(C, <_0)$ has no greatest element.

Assume that $a \in C$.

Let $b \in P$, and let σ_0 be such that $\text{Supp}(\sigma_0) = \{0, b\}$ (i.e. σ is the sum of a monomial of degree b and of a monomial of degree 0). In order to prove that the multiplication is not continuous, we prove that for every $c \in C$, there exists a monomial $\mu \in B_{a,>}(0, c)$ such that $\mu \cdot \sigma_0 \notin B_{a,>}(0 \cdot \sigma_0, a - b) = B_{a,>}(0, a - b)$. Indeed, let $d \in C$ such that $d >_a c$, and let μ be a monomial of degree d . We have $\text{Supp}(\mu \cdot \sigma_0) = \{d, d + b\}$. If $d \leq_a a - b$, then $v(a, \mu \cdot \sigma_0) \leq_a a - b$, and $\mu \cdot \sigma_0 \notin B_{a,>}(0, a - b)$. If $d >_a a - b$, then $-b <_0 d - a$ and by general properties of cyclically ordered groups, $b + d - a <_0 b$. Now, $b \in P$ hence $b <_0 -b$, it follows $b + d - a <_0 -b$, i.e. $b + d <_a a - b$. Therefore $v(a, \mu \cdot \sigma_0) = b + d <_a a - b$, and $\mu \cdot \sigma_0 \notin B_{a,>}(0, a - b)$.

Assume that $a = \epsilon \notin C$, and that C is not linear cyclically ordered. Then, there exist c and d in the positive cone P of C such that $c + d \in -P$.

First, we point out that for every $e >_\epsilon c$ (that is $e \in P$ and $e >_0 c$), $d + e \in -P$. Indeed, $e \in P$, $-d \in -P$, hence $e <_0 -d$ (because $P <_0 -P$). We have $c <_0 e <_0 -d$, hence $c + d <_d d + e <_d 0$. By general properties of cyclically ordered groups, $0 <_{d+e} d <_{d+e} c + d$, hence $0 <_{d+e} c + d$. Consequently $c + d <_0 d + e$, hence $d + e \in -P$.

Now let μ_d be a monomial of degree d . Let $x >_\epsilon \max_\epsilon(b, c)$ (so $x \in P$) and let μ_e be a monomial of degree e . The support of $\mu_e \cdot (1 + \mu_d)$ is $\{e, d + e\}$ and is not contained in P , hence $\mu_e \cdot (1 + \mu_d)$ is not in $B_{\epsilon,>}(0, 0)$. We have $\mu_e \in B_{\epsilon,>}(0, b)$, but $\mu_e \cdot (1 + \mu_d) \notin B_{\epsilon,>}(0, 0)$: the multiplication is not continuous.

If $a = \epsilon$ and C is a linear cyclically ordered group, then by Proposition 4.1, for all σ and τ in R , $v(\epsilon, \sigma\tau) = v(\epsilon, \sigma) + v(\epsilon, \tau)$. It follows that $(R, v(\epsilon, \cdot))$ is a valued ring, so $(R, v(\epsilon, \cdot))$ is a topological ring. \square

Notice that “ $(C, <_0)$ has a greatest element” is equivalent to saying that “ $(C, <_0)$ is discretely ordered”. Indeed, if $(C, <_0)$ is discretely ordered, then $(C \setminus \{0\}, <_0)$ contains a lowest element e . Let $b \in C \setminus \{0, e, -e\}$, then $(0, e, b+e)$. It follows : $(-e, 0, b)$, hence $(0, b, -e)$, i.e. $b <_0 -e$. Therefore $-e$ is the greatest element of $(C, <_0)$. Now, assume that $(C, <_0)$ has a greatest element m . Let $b \notin \{0, -m\}$. We have $(0, b+m, m)$, and by compatibility, $(-m, b, 0)$, hence $-m <_0 b$. It follows that $-m$ is the successor of 0 in $(C, <_0)$. Now, let $a \in C, b \in C \setminus \{a, a-m\}$. We have $-m <_0 b-a$, hence $a-m <_a b$. Assume that $a <_0 b$. Hence $b <_a 0$, so $a-m <_a b <_a 0$. It follows : $a <_b a-m$, and $0 <_b a$, hence $0 <_b a-m$. So $a-m <_0 b$. Thus $a-m$ is the successor of a in $(C, <_0)$. Symmetrically, we can prove that every non 0 element is the successor of an other element in $(C, <_0)$.

5. Definition of cyclic valuations in a relational language.

It is well-known that a language for valued fields is the language of fields augmented with a new unary symbol for being an element of the valuation ring. Indeed, if K is a valued field with valuation ring A , then the value group is isomorphic to $(K \setminus \{0\})/A^*$, and the valuation mapping is the canonical mapping from $K \setminus \{0\}$ to $(K \setminus \{0\})/A^*$. Thank to this language, one can prove the theorems of Ax-Kochen-Ershov, which define classes of elementarily equivalent valued rings. In the case of cyclically valued rings, we will see that we can construct v with the group of all invertible monomials, the subgroup of all invertible constants, the positive cone and the subset $\{\sigma \in R \mid v(0, \sigma) = 0\}$. Now, defining classes of elementarily equivalent cyclically valued rings remains an open question.

Assume that P is the positive cone of a cyclically ordered group C (i.e. $P = \{c \in C \mid (0, c, -c)\}$). It is well-known that P satisfies (a), (b), (c), (d) below.

- (a) $P \cap -P = \emptyset$
- (b) There exists at most one $\epsilon \neq 0$ such that $\epsilon = -\epsilon$, and if this holds then $-P = \epsilon + P$,
- (c) $\forall c \in C, c \notin P \cup -P \Leftrightarrow c = -c$,
- (d) $\forall a \in P, \forall b \in P, \forall c \in P, (b-a \in P \text{ and } c-b \in P) \Rightarrow c-a \in P$.

Conversely, if C is an abelian group and P is a subset of C which satisfies (a), (b), (c), (d), then there is a cyclic order (\cdot, \cdot, \cdot) on C such that $P = \{c \in C \mid (0, c, -c)\}$.

This cyclic order is defined in the following way.

$(0, a, b)$ if and only if one of the three following conditions is satisfied :

$$a \in P, b \in P \cup \{\epsilon\}, b - a \in P$$

$$-a \in P \cup \{\epsilon\}, -b \in P, b - a \in P$$

$$a \in P, -b \in P.$$

And in general, (a, b, c) if and only if $(0, b - a, c - a)$ or $(0, c - b, a - b)$ or $(0, a - c, b - c)$.

Now, let (R, v) be a cyclically valued ring such that the cyclically ordered group C is interpretable. Let IM be the group of all invertible monomials, $IM_0 := IM \cap M_0$ be the subgroup of all invertible constants, and let IM_P be the subset of all invertible monomials μ such that the degree of μ is an element of the positive cone of C . We have $C \simeq IM/IM_0$, and IM_P gives rise to the cyclic order on C . For every μ in IM , we denote by $\bar{\mu}$ the class of μ modulo IM_0 . Let $V_0 := \{\sigma \in R \mid v(0, \sigma) = 0\}$. For every $\sigma \in R \setminus \{0\}$, there exists μ_σ such that $\sigma \mu_\sigma^{-1} \in V_0$, and $\forall \mu \in IM$, $\sigma \mu^{-1} \in V_0 \Rightarrow \bar{\mu}_\sigma \leq_0 \bar{\mu}$. The support of σ is the set $\{\bar{\mu} \mid \sigma \mu^{-1} \in V_0\}$, and $v(\bar{\mu}, \sigma) = \bar{\mu}_{\sigma \mu^{-1}}$.

Conversely, let R be an abelian ring with 1, IM be a subgroup of the group of all units of R , IM_0 be a subgroup of IM and IM_P be a subset of IM such that $IM_P \cdot IM_0 \subset IM_P$. Assume the following.

(a) $IM_P \cap IM_P^{-1} = \emptyset$.

(b) $\forall \mu_1 \in IM \setminus IM_0, \forall \mu_2 \in IM \setminus IM_0, (\mu_1^2 \in IM_0 \text{ and } \mu_2^2 \in IM_0 \Rightarrow \mu_1 \mu_2^{-1} \in IM_0)$, and if such a μ_1 exists, then $IM_P^{-1} = \mu_1 IM_P$.

(c) $\forall \mu \in IM, \mu \notin IM_P \cup IM_P^{-1} \Leftrightarrow \mu^2 \in IM_0$.

(d) $\forall \mu_1 \in IM, \forall \mu_2 \in IM, \forall \mu_3 \in IM, ((\mu_1 \in IM_P, \mu_2 \in IM_P, \mu_3 \in IM_P, \mu_2 \mu_1^{-1} \in IM_P, \mu_3 \mu_2^{-1} \in IM_P) \Rightarrow \mu_3 \mu_1^{-1} \in IM_P)$.

Then $\{\mu \cdot IM_0 \mid \mu \in IM_P\}$ is the positive cone of a cyclic order (\cdot, \cdot, \cdot) of the quotient group $C = IM/IM_0$. So, we will say that IM, IM_0 and IM_P define a cyclically ordered group in R .

For μ in IM , let $\bar{\mu}$ be the class of μ modulo IM_0 .

Assume that R contains a subset V_0 which satisfies :

(e) $V_0 \cap IM = IM_0, V_0 \cdot IM_0 = V_0$, and

(f) $\forall \sigma \in R \setminus \{0\}, \exists c_\sigma \in C, \forall \mu \in IM, (\bar{\mu} = c_\sigma \Rightarrow \sigma \mu^{-1} \in V_0)$ and $\sigma \mu^{-1} \in V_0 \Rightarrow c_\sigma \leq_0 \bar{\mu}$.

Then there exists a mapping v from $C \times R$ onto $C \cup \{\infty\}$ such that for every $\mu \in IM$, and every $\sigma \in R$:

if $\sigma = 0$, then $v(\bar{\mu}, \sigma) = \infty$,
if $\sigma \neq 0$, then $v(\bar{\mu}, \sigma) = \bar{\mu}c_{\mu^{-1}\sigma}$.

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