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Jamel El Kamel, Chokri Yacoub

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Huygens' principle and equipartition of energy for the modified wave equation associated to a generalized radial Laplacian

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Abstract

In this paper we consider the modified wave equation associated with a class of radial Laplacians L generalizing the radial part of the Laplace–Beltrami operator on hyperbolic spaces or Damek–Ricci spaces. We show that the Huygens' principle and the equipartition of energy hold if the inverse of the Harish–Chandra $\mathbf{c}$–function is a polynomial and that these two properties hold asymptotically otherwise. Similar results were established previously by Branson, Olafsson and Schlichtkrull in the case of noncompact symmetric spaces.

1 Introduction

In Euclidean space $X = \mathbb{R}^n$ of odd dimension, any solution $u(x,t)$ to the wave equation

$$\Delta_x u(x,t) = \frac{\partial^2}{\partial t^2} u(x,t)$$

is determined by the value of its initial data in an arbitrarily thin shell around the sphere $S(x,|t|)$. This is Huygens' principle. Moreover the total energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left|\frac{\partial}{\partial t} u(x,t)\right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla_x u(x,t)|^2 \, dx \qquad (1.1)$$

splits eventually equally into its kinetic and potential components if the initial data are compactly supported. See Duffin [7] and Branson [3].

Similar results were established by Branson, Helgason, Olafsson, Schlichtkrull in [4], [8], [12] for the modified wave equation on Riemannian symmetric spaces of noncompact type $X = G/K$, under the assumptions that $\dim X$ is odd and G has only one conjugacy class of Cartan subgroups. Otherwise these phenomena may not hold strictly speaking, but they do asymptotically, as shown by Branson, Olafsson and Schlichtkrull in [5].

This paper is devoted to another setting, which is known to share features with the previous one. Specifically we consider the modified wave equation

$$L_x u(x,t) = \frac{\partial^2}{\partial t^2} u(x,t) \tag{1.2}$$

with initial data

$$u(x,0) = f_0(x), \ \left.\frac{\partial}{\partial t}\right|_{t=0} u(x,t) = f_1(x) \tag{1.3}$$

associated to certain second order differential operators

$$Lu = \frac{d^2u}{dx^2} + \frac{A'(x)}{A(x)}\frac{du}{dx} + \rho^2 u \tag{1.4}$$

on $(0,+\infty)$. Following Chébli and Trimèche, we assume that the function $A(x)$ behaves as follows :

• $A(x) \sim x^{2\alpha+1}$ as $x \searrow 0$, where $\alpha > -\frac{1}{2}$. More precisely

$$A(x) = x^{2\alpha+1} B(x) \tag{1.5}$$

where $B : \mathbb{R} \to (0,+\infty)$ is a smooth even function with $B(0) = 1$.

• $A(x) \nearrow +\infty$ and $\frac{A'(x)}{A(x)} \searrow 2\rho > 0$ as $x \nearrow +\infty$. More precisely

$$\frac{A'(x)}{A(x)} = 2\rho + e^{-2\delta x} D(x) \tag{1.6}$$

where $\delta > 0$ and $D : (0,+\infty) \to \mathbb{R}$ is a smooth function which is bounded at infinity together with its derivatives.

Typical examples are given by the Jacobi operators (see the survey [9])

$$L_{\alpha,\beta} = \frac{d^2}{dx^2} + [\,(2\alpha+1)\coth x + (2\beta+1)\tanh x\,]\,\frac{d}{dx} + \rho^2 \tag{1.7}$$

with $\alpha \geq \beta > -\frac{1}{2}$. Here $A(x) = (\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}$, $\rho = \alpha + \beta + 1$ and $\delta = 1$. Recall that (1.7) includes the radial part of the Laplace–Beltrami operator on hyperbolic spaces and more generally on Damek–Ricci spaces [1].

Our paper is organized as follows. In Section 2, we recall some basic harmonic analysis associated to L. This theory was developed initially by Chébli [6] and Trimèche [13] (see also Trimèche [14] and Yacoub [15]) and was resumed by Bloom and Xu in the framework of hypergroups (see for instance [2]). We apply it next to solve the Cauchy problem (1.2).

In Section 3, we resume the analysis carried out in the symmetric space case by Branson, Olafsson and/or Schlichtkrull ([4], [12]) and we establish the following two properties of the wave equation (1.2), under the assumption that the inverse $\mathbf{c}(\lambda)^{-1}$ of the Harish–Chandra $\mathbf{c}$–function is a polynomial : On one hand, Huygens' principle holds and, on the other hand, the potential and kinetic energies contribute eventually equally to the total energy.

In Section 4, we show that these properties hold in general asymptotically, resuming again the analysis carried out in the symmetric space case by Branson, Olafsson and Schlichtkrull, this time in [5] .

2 Harmonic analysis associated to L

2.1 Eigenfunctions of L

(See [2], [6], [13], [14], [15]).

For every $\lambda \in \mathbb{C}$, the equation

$$L\varphi = -\lambda^2 \varphi \tag{2.1}$$

has a unique solution on $[\,0,+\infty)$ such that $\varphi(0)=1$ and $\varphi'(0)=0$. It is denoted by φ_λ. If $\lambda \neq 0$, the equation (2.1) has two other linearly independent solutions $\Phi_{\pm\lambda}$ on $(0,\infty)$ with the following behaviour at infinity :

$$\Phi_{\pm\lambda}(x) \sim e^{(\pm i\lambda-\rho)\,x} \quad \text{as} \quad x \to +\infty.$$

Moreover there exists a function $\mathbf{c}(\lambda)$ (the so–called Harish–Chandra $\mathbf{c}$–function) such that

$$\varphi_\lambda(x) = \mathbf{c}(\lambda)\Phi_\lambda(x) + \mathbf{c}(-\lambda)\Phi_{-\lambda}(x).$$

In the Jacobi setting (1.7), everything can be expressed in terms of classical special functions :

$$\varphi_\lambda(x) = {}_2F_1\left(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -(\sinh x)^2\right),$$

$$\Phi_\lambda(x) = (2\sinh x)^{(i\lambda-\rho)}\,{}_2F_1\left(\frac{\beta-\alpha+1-i\lambda}{2}, \frac{\rho-i\lambda}{2}; 1-i\lambda; (\sinh x)^{-2}\right),$$

$$\mathbf{c}(\lambda) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\,\frac{\Gamma(i\lambda)}{\Gamma(\alpha-\beta+i\lambda)}\,\frac{\Gamma(\frac{\alpha-\beta+i\lambda}{2})}{\Gamma(\frac{\rho+i\lambda}{2})}\,. \tag{2.2}$$

Properties of $\varphi_\lambda(x)$:
• $\varphi_\lambda(x)$ is a smooth even function in $x \in \mathbb{R}$ and an analytic even function in $\lambda \in \mathbb{C}$.
• Integral representation of Mehler type :

$$\varphi_\lambda(x) = \int_0^x K(x,y)\,\cos\lambda y\,dy \quad \forall\,\lambda\in\mathbb{C},\ \forall\,x\geq 0 \tag{2.3}$$

where $K(x,.)$ is an even nonnegative function on $\mathbb{R}$, which is supported in $[-x,x]$ and which is smooth in $(-x,x)$.
• $|\varphi_\lambda(x)| \leq \varphi_{i\,\mathrm{Im}\lambda}(x) \leq e^{|\mathrm{Im}\lambda||x|}\varphi_0(x) \quad \forall\,\lambda\in\mathbb{C},\ \forall\,x\in\mathbb{R}$.
In particular $|\varphi_\lambda(x)| \leq \varphi_0(x) \quad \forall\,\lambda\in\mathbb{R},\ \forall\,x\in\mathbb{R}$.
• $|\varphi_\lambda(x)| \leq 1 \quad \forall\,\lambda\in\mathbb{C}$ with $|\mathrm{Im}\lambda|\leq\rho$, $\forall\,x\in\mathbb{R}$.
• There exists a constant $c_0>0$ such that $\varphi_0(x)\sim c_0\,x\,e^{-\rho x}$ as $x\to+\infty$.
Hence there are positive constants C_1, C_2 such that

$$C_1\,(1+|x|)\,e^{-\rho|x|} \leq \varphi_0(x) \leq C_2\,(1+|x|)\,e^{-\rho|x|} \quad \forall\,x\in\mathbb{R}.$$

Properties of $\mathbf{c}(\lambda)$:
• The function $\lambda\mapsto\mathbf{c}(\lambda)^{-1}$ is holomorphic in the horizontal strip $|\mathrm{Im}\lambda|<\delta$, where δ is the constant occurring in (1.6).
• $\mathbf{c}(-\lambda)^{-1} = \overline{\mathbf{c}(\overline{\lambda})^{-1}}$.
• For every $0<\gamma<\delta$, there exist positive constants C_1, C_2 such that

$$C_1\,|\lambda|\,(1+|\lambda|)^{\alpha-\frac{1}{2}} \leq\ |\mathbf{c}(\lambda)|^{-1} \leq\ C_2\,|\lambda|\,(1+|\lambda|)^{\alpha-\frac{1}{2}} \tag{2.4}$$

$\forall\,\lambda\in\mathbb{C}$ with $|\mathrm{Im}\lambda|\leq\gamma$. Hence $\mathbf{c}(\lambda)^{-1}$ and its derivatives have polynomial growth on $\mathbb{R}$. More precisely

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\,\mathbf{c}(\lambda)^{-1}\right| \leq C_k\,(1+|\lambda|)^{\alpha+\frac{1}{2}} \quad \forall\,\lambda\in\mathbb{R}.$$

2.2 Generalized Fourier and Weyl transforms

(See [2], [6], [13], [14], [15]).

Recall some classical function spaces : $\mathcal{D}(\mathbb{R})$ denotes the space of smooth functions on $\mathbb{R}$ with compact support, $\mathcal{S}(\mathbb{R})$ the space of Schwartz functions on $\mathbb{R}$, and $\mathcal{H}(\mathbb{C})$ the space of entire functions h on $\mathbb{C}$, which are of exponential type and rapidly decreasing. This means that there exists $R \geq 0$ such that

$$\sup_{z\in\mathbb{C}} (1+|z|)^m \, e^{-R\,|\mathrm{Im}z|} \, |\frac{d^n}{dz^n} h(z)| < +\infty, \forall\, m,n \in \mathbb{N}. \tag{2.5}$$

A less familiar function space is the L^2 Schwartz space $\mathcal{S}_*^2(\mathbb{R}) = \varphi_0(x)\,\mathcal{S}_*(\mathbb{R})$. The subscript $*$ means that we restrict our attention to even functions.

The generalized Fourier transform and the generalized Weyl transform are defined, let say for $f \in \mathcal{S}_*^2(\mathbb{R})$, by the converging integrals

$$\mathcal{F}f(\lambda) = \int_0^{+\infty} \varphi_\lambda(x)\, f(x)\, A(x)\, dx \quad \forall\, \lambda \in \mathbb{R} \tag{2.6}$$

and

$$\mathcal{W}f(y) = \int_y^{+\infty} K(x,y)\, f(x)\, A(x)\, dx \quad \forall\, y \geq 0, \tag{2.7}$$

where $K(x,y)$ is the kernel occurring in (2.3). The terminology comes from Jacobi function theory, where $\mathcal{W}$ is expressed in terms of Weyl fractional transforms. In particular, let us recall that (2.7) is the Abel transform of radial functions on hyperbolic spaces and more generally on Damek–Ricci spaces.

These transforms are related by means of the classical Fourier transform

$$\mathcal{F}_0(g)(\lambda) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{i\lambda y}\, g(y)\, dy = \int_0^{+\infty} \cos\lambda y \; g(y)\, dy\,.$$

Specifically, $\mathcal{F}$ factorizes as $\mathcal{F}_0 \circ \mathcal{W}$, let say on $\mathcal{S}_*^2(\mathbb{R})$.

Properties of $\mathcal{W}$:
- $\mathcal{W}$ is a topological isomorphism between $\mathcal{S}_*^2(\mathbb{R})$ and $\mathcal{S}_*(\mathbb{R})$.
- $\mathcal{W}$ is a topological automorphism of $\mathcal{D}_*(\mathbb{R})$. More precisely, f is supported in $[-R,+R\,]$ if and only if $g = \mathcal{W}f$ is supported in $[-R,+R\,]$
- $\mathcal{W}$ is a transmutation operator between L and $\frac{d^2}{dx^2}$: $\mathcal{W}(Lf) = \frac{d^2}{dx^2}\,\mathcal{W}f$.

Properties of $\mathcal{F}$:
• $\mathcal{F}$ is a topological isomorphism between $\mathcal{S}^2_*(\mathbb{R})$ and $\mathcal{S}_*(\mathbb{R})$.
• $\mathcal{F}$ is a topological isomorphism between $\mathcal{D}_*(\mathbb{R})$ and $\mathcal{H}_*(C)$. More precisely, f is supported in $[-R,+R]$ if and only if $h=\mathcal{F}f$ is of exponential type R in the sense of (2.5).
• Inversion formula : There exists a constant $c_1>0$ such that

$$\mathcal{F}^{-1}h\,(x)=c_1\int_0^{+\infty}\varphi_\lambda(x)\,h(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\quad\forall\,x\in\mathbb{R}. \tag{2.8}$$

• Plancherel theorem : $\mathcal{F}$ extends to an isometric isomorphism between $\mathcal{L}^2((0,\infty),\,A(x)\,dx)$ and $\mathcal{L}^2((0,\infty),c_1\,|\mathbf{c}(\lambda)|^{-2}d\lambda)$:

$$\int_0^{+\infty}|f(x)|^2\,A(x)\,dx\;=c_1\int_0^{+\infty}|\mathcal{F}f(\lambda)|^2\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,. \tag{2.9}$$

2.3 Modified Weyl transform

Recall that $\lambda\longmapsto\mathbf{c}(\lambda)^{-1}$ is a smooth function on $\mathbb{R}$, which is tempered. Thus $\mathbf{c}^{-1}$ is a pointwise mutiplier of $\mathcal{S}(\mathbb{R})$ and we may consider the corresponding convolution operator

$$\mathcal{J}g=\mathcal{F}_0^{-1}(\frac{\mathcal{F}_0 g}{\mathbf{c}})=g*\mathcal{F}_0^{-1}(\mathbf{c}^{-1})$$

on $\mathcal{S}(\mathbb{R})$. Similarly, let us denote by $\overline{\mathcal{J}}$ the convolution operator corresponding to the multiplier $\overline{\mathbf{c}(\lambda)}^{\,-1}=\mathbf{c}(-\lambda)^{-1}$. One may modify the Weyl transform $\mathcal{W}$ by composing it with $\mathcal{J}$:

$$\widetilde{\mathcal{W}}=\mathcal{J}\circ\mathcal{W}\,.$$

We shall mostly do so when $\mathbf{c}^{-1}$ is a polynomial i.e. when $\mathcal{J}$ is a differential operator. Such modified transforms were considered by Lax and Phillips ([10], chap. IV; [11]), both for (odd–dimensional) Euclidean and for (3–dimensional) hyperbolic spaces, and more generally by Olafsson and Schlichtkrull [12] for the class of symmetric spaces $X=G/K$ where G has only one conjugacy class of Cartan subgroups.

Properties of $\widetilde{\mathcal{W}}$:
• $\widetilde{\mathcal{W}}$ is a continuous linear map of $\mathcal{S}^2_*(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$.
• $\widetilde{\mathcal{W}}(Lf)=\frac{d^2}{dx^2}\,\widetilde{\mathcal{W}}f$.

• Inversion formula :

$$f(x) = c_2 \int_0^x K(x,y)\,(\overline{\mathcal{J}} \circ \widetilde{\mathcal{W}})f(y)\,dy\,. \tag{2.10}$$

• Plancherel formula :

$$\int_0^{+\infty} |f(x)|^2 A(x)\,dx = c_2 \int_0^{+\infty} |\widetilde{\mathcal{W}} f(y)|^2\,dy\,. \tag{2.11}$$

Here $c_2 = \frac{c_1 \pi}{2}$.

2.4 Modified wave equation

Consider the Cauchy problem

$$\begin{cases} L_x u(x,t) = \frac{\partial^2}{\partial t^2} u(x,t) \\ u(x,0) = f_0(x), \ \frac{\partial}{\partial t}\big|_{t=0} u(x,t) = f_1(x) \end{cases} \tag{2.12}$$

with initial data $f_0, f_1 \in \mathcal{D}_*(\mathbb{R})$.

Properties :

• Finite propagation speed : If the initial data are supported in $[-R,+R]$, then u is supported in $\{(x,t) \mid |x| \leq |t| + R\}$

• Huygens' principle is said to hold if, in addition, u vanishes in the lacuna $\{(x,t) \mid |x| \leq |t| - R\}$.

• Conservation ot the total energy :

$$\mathcal{E}u(t) = \overbrace{\frac{1}{2} \int_0^{+\infty} |\frac{\partial}{\partial t} u(x,t)|^2 \, A(x)\,dx}^{\mathcal{K}u(t) \text{ kinetic energy}} + \overbrace{\frac{1}{2} \int_0^{+\infty} (-L_x) u(x,t)\, \overline{u(x,t)}\, A(x)\,dx}^{\mathcal{P}u(t) \text{ potential energy}}$$

is independent of t.

By applying the Weyl transform $\mathcal{W}$ or $\widetilde{\mathcal{W}}$ to (2.12), one gets the classical wave equation on $\mathbb{R}$:

$$\begin{cases} \frac{\partial^2}{\partial x^2} v(x,t) = \frac{\partial^2}{\partial t^2} v(x,t), \\ v(x,0) = g_0(x), \ \frac{\partial}{\partial t}\big|_{t=0} v(x,t) = g_1(x), \end{cases} \tag{2.13}$$

whose solution is well known :

$$v(x,t) = \frac{g_0(x+t) + g_0(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} g_1(y)dy\,. \tag{2.14}$$

3 Strict Huygens' principle and equipartition of energy

Throughout this section, we assume that $\mathbf{c}(\lambda)^{-1}$ is a polynomial. In the Jacobi setting (1.7), this happens exactly when $\alpha,\beta\in\frac{1}{2}+\mathbb{N}$. Specifically,

$$\mathbf{c}(\lambda)^{-1}=\frac{(\alpha+\frac{1}{2})!}{(2\alpha+1)!}\,2^{-\beta-\frac{1}{2}}\prod_{j=0}^{\alpha-\beta}(i\lambda+j)\prod_{k=1}^{\beta-\frac{1}{2}}(i\lambda+\alpha-\beta+2k)\,.$$

Among Damek–Ricci spaces $X=NA$, our assumption holds if and only if the center of the nilpotent component N has even dimension. In particular, among hyperbolic spaces, it holds only for real hyperbolic spaces of odd dimension. On the other hand, our assumption holds in many cases which don't correspond to Damek–Ricci spaces, the first one being $(\alpha,\beta)=(\frac{3}{2},\frac{1}{2})$.

3.1 Strict Huygens' principle

Huygens' principle holds obviously for the classical wave equation (2.13) on $\mathbb{R}$ with initial data $g_0\in\mathcal{D}(\mathbb{R})$ and $g_1\in\frac{d}{dx}\,\mathcal{D}(\mathbb{R})$. Arguying as in [12], we shall transfer this property to (2.12) via the modified Weyl transform $\widetilde{\mathcal{W}}$.

Proposition 3.1: *Let $f\in\mathcal{D}_*(\mathbb{R})$ and $R>0$. Then*
(a) $\operatorname{supp}f\subset[-R,+R\,]$ *if and only if* $\operatorname{supp}(\widetilde{\mathcal{W}}f)\subset[-R,+R\,]$,
(b) $\operatorname{supp}f\cap[-R,+R\,]=\emptyset$ *if* $\operatorname{supp}(\widetilde{\mathcal{W}}f)\cap[-R,+R\,]=\emptyset$.

Proof: (a) Recall from the general properties of $\mathcal{W}$ that $g=\mathcal{W}f\in\mathcal{D}_*(\mathbb{R})$ and that $\operatorname{supp}f\subset[-R,+R\,]\iff\operatorname{supp}g\subset[-R,+R\,]$. In this section, we assume that $\mathbf{c}(\lambda)^{-1}$ is a polynomial i.e. $\mathcal{J}$ is a differential operator. Consequently $\widetilde{\mathcal{W}}f=\mathcal{J}g\in\mathcal{D}(\mathbb{R})$ and $\operatorname{supp}g\subset[-R,+R\,]\iff\operatorname{supp}(\mathcal{J}g)\subset[-R,+R\,]$. While $\Longrightarrow$ is immediate, $\Longleftarrow$ follows from the classical Paley–Wiener theorem.
(b) may be obtained either from the inversion formula (2.10) or by duality. Indeed, let $\phi\in\mathcal{D}_*(\mathbb{R})$ such that $\operatorname{supp}\phi\subset[-R,+R\,]$. According to (a), $\widetilde{\mathcal{W}}\phi\in\mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\widetilde{\mathcal{W}}\phi)\subset[-R,+R\,]$. By the Plancherel formula (2.11), one gets

$$\int_0^{+\infty}f(x)\,\phi(x)\,A(x)\,dx=c_2\int_0^{+\infty}\widetilde{\mathcal{W}}f(y)\,\widetilde{\mathcal{W}}\phi(y)\,dy=0\,.$$

Hence f vanishes on $[-R,+R]$. □

Theorem 3.2: *Huygens' principle holds for the Cauchy problem (2.12).*

Proof: Let $u(x,t)$ be a solution to (2.12) with initial data supported in $[-R,+R]$. Then $v(x,t)=\widetilde{\mathcal{W}}_x u(x,t)$ is a solution to (2.13) with initial data supported in $[-R,+R]$. Notice that $g_1=\widetilde{\mathcal{W}}f_1=\mathcal{J}(\mathcal{W}f_1)$ is the derivative of a function in $\mathcal{D}_*(\mathbb{R})$ with support in $[-R,+R]$, since $\mathbf{c}(\lambda)^{-1}$ is a polynomial with no constant term, according to (2.4). Hence $v(x,t)$ vanishes outside the region

$$\{\,(x,t)\in\mathbb{R}^2 \mid |t|-R\le|x|\le|t|+R\,\}\,.$$

By Proposition 3.1, this holds true for $u(x,t)$ too. □

3.2 Strict equipartition of energy

It is well known (see [7]) that equipartition of energy holds (for $|t|\ge R$) for the classical wave equation (2.13) on $\mathbb{R}$ with initial data compactly supported in $[-R,R]$. We shall transfer this property to (2.12) using again the modified Weyl transform $\widetilde{\mathcal{W}}$.

Theorem 3.3: *Let $u(x,t)$ be the solution to (2.12) with initial data $f_0,f_1\in\mathcal{D}_*(\mathbb{R})$ supported in $[-R,+R]$. Then the total energy $\mathcal{E}u$ splits equally into its kinetic and potential components $\mathcal{K}u(t)$ and $\mathcal{P}u(t)$, as soon as $|t|\ge R$.*

Proof: Consider again the solution $v(x,t)=\widetilde{\mathcal{W}}_x u(x,t)$ to (2.13) with initial data $g_0=\widetilde{\mathcal{W}}f_0$ and $g_1=\widetilde{\mathcal{W}}f_1$. Notice again that $g_1=\mathcal{J}(\mathcal{W}f_1)$ is the derivative of a function in $\mathcal{D}(\mathbb{R})$ with support in $[-R,+R]$. Using the properties of $\widetilde{\mathcal{W}}$, we have for $|t|\ge R$

$$\begin{aligned}2\,\mathcal{P}u(t)&=-\int_0^{+\infty}L_x u(x,t)\,\overline{u(x,t)}\,A(x)\,dx\\&=-\frac{c_2}{2}\int_{-\infty}^{+\infty}\frac{\partial^2}{\partial x^2}v(x,t)\,\overline{v(x,t)}\,dx=\frac{c_2}{2}\int_{-\infty}^{+\infty}|\frac{\partial}{\partial x}v(x,t)|^2\,dx\\&=\frac{c_2}{2}\int_{-\infty}^{+\infty}|\frac{\partial}{\partial t}v(x,t)|^2\,dx=\int_0^{+\infty}|\frac{\partial}{\partial t}u(x,t)|^2\,A(x)\,dx=2\,\mathcal{K}u(t)\,.\end{aligned}$$ □

4 Asymptotic Huygens' principle and equipartition of energy

In this section we drop the assumption that $\mathbf{c}(\lambda)^{-1}$ is a polynomial and we show that the properties investigated in Section 3 hold asymptotically, for lack of holding strictly speaking. This will be achieved by resuming the analysis carried out in [5] and by using the generalized Fourier transform $\mathcal{F}$ instead of the modified Weyl transform $\widetilde{\mathcal{W}}$. Specifically, (2.12) is transformed by $\mathcal{F}$ into the ordinary differential equation

$$\begin{cases} \frac{\partial^2}{\partial t^2}\,\mathcal{F}u(\lambda,t) = -\lambda^2\,\mathcal{F}u(\lambda,t), \\ \mathcal{F}u(\lambda,0) = \mathcal{F}f_0(\lambda),\ \frac{\partial}{\partial t}\Big|_{t=0}\mathcal{F}u(\lambda,0)) = \mathcal{F}f_1(\lambda), \end{cases}$$

whose solution is given by

$$\mathcal{F}u(\lambda,t) = (\cos t\lambda)\,\mathcal{F}f_0(\lambda) + \frac{\sin t\lambda}{\lambda}\,\mathcal{F}f_1(\lambda)\,.$$

Transforming backwards by $\mathcal{F}^{-1}$, one gets

$$\begin{aligned} u(x,t) &= c_1\int_0^{+\infty}[\,(\cos t\lambda)\,\mathcal{F}f_0(\lambda) + \frac{\sin t\lambda}{\lambda}\,\mathcal{F}f_1(\lambda)\,]\,\varphi_\lambda(x)\,|\mathbf{c}(\lambda)|^{-2}d\lambda \\ &= \int_{-\infty}^{+\infty} e^{it\lambda}\,[\,h_0(x,\lambda) + \frac{h_1(x,\lambda)}{i\lambda}\,]\,d\lambda \qquad (4.1) \end{aligned}$$

where

$$h_j(x,\lambda) = \frac{c_1}{2}\mathcal{F}f_j(\lambda)\,\varphi_\lambda(x)\,\mathbf{c}(\lambda)^{-1}\mathbf{c}(-\lambda)^{-1} \quad (j=0,1).$$

Notice that $h_j(x,\lambda)$ is an even function both in x and in λ, which is smooth in $x\in\mathbb{R}$ and analytic in the strip $|\mathrm{Im}\lambda|<\delta$. Moreover there is no actual singularity in (4.1), since $\mathbf{c}(\lambda)^{-1}$ hence $h_1(x,\lambda)$ vanish at $\lambda=0$.

4.1 Asymptotic Huygens' principle

Theorem 4.1: *Let $u(x,t)$ be a solution to (2.12) with initial data supported in $[-R,+R\,]$ and let $0<\gamma<\delta$, where δ is the constant occurring in (1.6). Then there exists a constant $C\geq 0$ such that*

$$|u(x,t)| \leq C\,\varphi_0(x)\,e^{\gamma\,(|x|-|t|)}.$$

Proof: We may assume that x and t are nonnegative. Since

$$|\varphi_\lambda(x)| \leq \varphi_0(x)\, e^{|\mathrm{Im}\lambda||x|} \quad \forall\, x \in \mathbb{R}\,,\ \forall\, \lambda \in \mathbb{C}\,,$$

$$(2.4)' \qquad |\mathbf{c}(\lambda)|^{-1} \leq C\,|\lambda|\,(1+|\lambda|)^{\alpha-\frac{1}{2}} \quad \text{in the strip } |\mathrm{Im}\lambda| \leq \gamma\,,$$

$$|\mathcal{F}f_j(\lambda)| \leq C_N\,(1+|\lambda|)^{-N} e^{\,R\,|\mathrm{Im}\lambda|} \quad \forall\, \lambda \in \mathbb{C}\,,$$

one may shift the integral

$$u(x,t) = e^{-\gamma t} \int_{-\infty}^{+\infty} e^{it\lambda}\,[\,h_0(x,\lambda+i\gamma) + \frac{h_1(x,\lambda+i\gamma)}{i\lambda-\gamma}\,]\,d\lambda \tag{4.2}$$

and estimate

$$|u(x,t)| \leq C\,\varphi_0(x)\,e^{\,\gamma\,(R+x-t)}\,.$$

□

Remarks 4.2:

• The condition on γ can be improved, provided $\mathbf{c}(\lambda)^{-1}$ extends holomorphically to a larger strip. This is the case for hyperbolic spaces and more generally for Damek–Ricci spaces, where δ can be replaced by ρ.

• If $\mathbf{c}(\lambda)^{-1}$ is a polynomial, the estimate (2.4)' holds in $\mathbb{C}$. Thus we may shift the integral (4.2) as far as we wish and estimate

$$|u(x,t)| \leq C\,\varphi_0(x)\,e^{\,\gamma\,(R+|x|-|t|)}$$

with a constant $C \geq 0$ independent of $\gamma > 0$. Letting $\gamma \nearrow +\infty$, we obtain this way a new proof of Theorem 3.2.

4.2 Asymptotic equipartition of energy

Our aim in this subsection is to estimate in general the gap

$$\mathcal{G}u(t) = \mathcal{K}u(t) - \mathcal{P}u(t)$$

between the kinetic and the potential energies.

Theorem 4.3: *Under the same assumptions, there exists a constant $C \geq 0$ such that*

$$|\mathcal{G}u(t)| \leq C\,e^{-2\,\gamma\,|t|} \quad \forall\, t \in \mathbb{R}\,.$$

Proof: We have

$$\mathcal{G}u(t) = \mathcal{K}u(t) - \mathcal{P}u(t) = \frac{1}{2}\frac{\partial}{\partial t}\int_0^{+\infty} \frac{\partial}{\partial t} u(x,t)\,\overline{u(x,t)}\,A(x)\,dx$$

hence, via the Fourier transform,

$$\mathcal{G}u(t) = \frac{c_1}{4}\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} \frac{\partial}{\partial t}\mathcal{F}u(\lambda,t)\,\overline{\mathcal{F}u(\lambda,t)}\,|\mathbf{c}(\lambda)|^{-2} d\lambda\,.$$

After expanding

$$\begin{cases} \mathcal{F}u(\lambda,t) = (\cos t\lambda)\,\mathcal{F}f_0(\lambda) + \frac{\sin t\lambda}{\lambda}\,\mathcal{F}f_1(\lambda) \\ \frac{\partial}{\partial t}\mathcal{F}u(\lambda,t) = (\cos t\lambda)\,\mathcal{F}f_1(\lambda) - \lambda\,(\sin t\lambda)\,\mathcal{F}f_0(\lambda) \end{cases}$$

and differentiating with respect to t, one gets

$$\begin{aligned} \mathcal{G}u(t) &= \int_{-\infty}^{+\infty} [\,(\cos 2\,t\lambda)\,A(\lambda) - \lambda\,(\sin 2\,t\lambda)\,B(\lambda)\,]\,d\lambda \\ &= \int_{-\infty}^{+\infty} e^{i\,2\,t\lambda}\,[\,A(\lambda) + i\,\lambda\,B(\lambda)\,]\,d\lambda \end{aligned}$$

where

$$\begin{cases} A(\lambda) = \frac{c_1}{4}[-\lambda^2\,\mathcal{F}f_0(\lambda)\,\mathcal{F}\overline{f_0}(\lambda) + \mathcal{F}f_1(\lambda)\,\mathcal{F}\overline{f_1}(\lambda)\,]\,\mathbf{c}(\lambda)^{-1}\mathbf{c}(-\lambda)^{-1} \\ B(\lambda) = \frac{c_1}{4}[\,\mathcal{F}f_0(\lambda)\,\mathcal{F}\overline{f_1}(\lambda) + \mathcal{F}\overline{f_0}(\lambda)\,\mathcal{F}f_1(\lambda)\,]\,\mathbf{c}(\lambda)^{-1}\mathbf{c}(-\lambda)^{-1} \end{cases}$$

are even analytic functions in the strip $|\mathrm{Im}\lambda| < \delta$. Proceeding now as in the proof of Theorem 4.1, one obtains the estimate

$$|\mathcal{G}u(t)| \le C\,e^{2\,\gamma\,(R-t)} \quad \forall\, t \ge 0\,.$$

□

Remark 4.4: As observed in Subsection 4.1, one may on one hand improve the condition on γ and on the other hand reprove this way Theorem 3.3.

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Jamel El Kamel
Faculty of Science of Monastir
Department of Mathematics
Boulevard de l'environnement
Monastir
Tunisia
jamel.elkamel@fsm.rnu.tn

Chokri Yacoub
Faculty of Science of Monastir
Department of Mathematics
Boulevard de l'environnement
Monastir
Tunisia
chokri.yacoub@fsm.rnu.tn