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Abstract

We prove endpoint estimates for operators given by oscillating spectral multipliers on Riemannian manifolds with C^∞-bounded geometry and nonnegative Ricci curvature.

Keywords: spectral multipliers, wave equation, Riesz means
AMS Subject Classification: 58G03

1 Introduction and statement of the results

Let M be an $n-$dimensional, complete, noncompact Riemannian manifold with nonnegative Ricci curvature and let us assume that it has C^∞-bounded geometry, that is, the injectivity radius is positive and every covariant derivative of the curvature tensor is bounded (cf. [25]). Let $d(.,.)$ denote the Riemannian distance on M, dx its volume element. Let us denote by $B(x,r)$ the ball of radius $r>0$ centered at $x\in M$ and by $|B(x,r)|$ its volume. By the Bishop comparison theorem (cf. [5]), the assumption that M has nonnegative Ricci curvature implies that

$$\frac{|B(x,r)|}{|B(x,t)|}\leq\left(\frac{r}{t}\right)^n,\quad r\geq t>0, \tag{1.1}$$

and hence

$$|B(x,2r)|\leq 2^n\left|B(x,r)\right|,\quad r>0.$$

This is the so called 'doubling volume property' and makes M a 'space of homogeneous type' in the sense of Coifman and Weiss [8]. Thus we can define the atomic Hardy space $H^1(M)$ and $BMO(M)$, the space of functions of bounded mean oscillation, in the standard way (cf. [8]). Further, by Theorem B of [8], $BMO(M)$ is the dual of $H^1(M)$.

Let L be the Laplace-Beltrami operator. It admits a selfadjoint extension on $L^2(M)$, also denoted by L and hence the spectral resolution

$$L = \int_0^\infty \lambda dE_\lambda.$$

Given a bounded measurable function $m(\lambda)$, we can define, by the spectral theorem, the operator

$$m(L) = \int_0^\infty m(\lambda) dE_\lambda.$$

This operator is bounded on $L^2(M)$. The function $m(\lambda)$ is called multiplier.

Oscillating multipliers are multipliers of the type

$$m_{\alpha,\beta}(\lambda) = \psi(|\lambda|)\, |\lambda|^{-\beta/2}\, e^{i|\lambda|^{\alpha/2}}, \quad \alpha > 0,\ \beta \geq 0. \tag{1.2}$$

with ψ a smooth function which is 0 for $|\lambda| \leq 1$ and 1 for $|\lambda| \geq 2$.

In this article we shall prove some endpoint results concerning the L^p boundedness of the family of operators

$$m_{\alpha,\beta}(L) = \int_0^\infty m_{\alpha,\beta}(\lambda) dE_\lambda.$$

We have the following:

Theorem 1.1: *Let $m_{\alpha,\beta}$ be as above and let $\alpha \in (0,1)$. The following hold:*

(i). If $\beta = \frac{\alpha n}{2}$, then $m_{\alpha,\beta}(L)$ is bounded from $H^1(M)$ to $L^1(M)$, on $L^p(M)$, $1 < p < \infty$ and from $L^\infty(M)$ to $BMO\,(M)$.

(ii). If $0 \leq \beta < \frac{n\alpha}{2}$, then $m_{\alpha,\beta}(L)$ is bounded on $L^p(M)$, for $\beta \geq \alpha n \left|\frac{1}{p} - \frac{1}{2}\right|$, $1 < p < \infty$.

(iii). If $\beta > \frac{\alpha n}{2}$, then $m_{\alpha,\beta}(L)$ is bounded on $L^p(M)$ for $1 \leq p \leq \infty$.

Oscillating multipliers fall outside the scope of Calderón-Zygmund theory and they have been studied extensively. See for example [31, 14, 10, 11, 21, 22, 23, 28, 26] for $\mathbb{R}^n$ and [9, 1, 20, 12] for more abstract settings.

The above result, in the context of $\mathbb{R}^n$ and for $0 \leq \beta \leq \alpha n/2$, has been proved by Fefferman and Stein in [11]. In the context of Riemannian manifolds of nonnegative Ricci curvature, Alexopoulos [1], has proved that for any $\alpha > 0$, $m_{\alpha,\beta}(L)$ is bounded on L^p for $\beta > \alpha n \left|\frac{1}{p} - \frac{1}{2}\right|$, $1 \leq p \leq \infty$. According to [11], the results above, for $0 \leq \beta \leq \alpha n/2$, are optimal.

For the proof of the $H^1 - L^1$ boundedness of $m_{\alpha,\frac{\alpha n}{2}}(L)$, we follow the strategy that Alexopoulos sketches at the end of the paper [1]. The idea, which is due to M. Taylor, is to express $m_{\alpha,\beta}(L)$ in terms of the wave operator $\cos t\sqrt{L}$ and then use the Hadamard parametrix method to get very precise estimates of its kernel near the diagonal. Away from the diagonal, we use the finite propagation speed property of $\cos t\sqrt{L}$ and the fast decay of the multiplier at infinity to obtain that $m_{\alpha,\beta}(L)$ is bounded on L^p, $p \geq 1$.

To prove that the operator $m_{\alpha,\beta}(L)$ is bounded on L^p for $\beta = \alpha n \left|\frac{1}{p} - \frac{1}{2}\right|$, $1 < p < \infty$, we compose $m_{\alpha,\frac{n\alpha}{2}}(L)$ with the imaginary powers of the Laplacian, which are bounded on H^1, (cf. [19]), and then use the $H^1 - L^1$ boundedness of $m_{\alpha,\frac{\alpha n}{2}}(L)$ and complex interpolation.

We shall apply Theorem 1.1 in order to obtain similar results for the Riesz means associated with the Schrödinger type group $e^{isL^{\alpha/2}}$ i.e. for the family of operators

$$I_{k,\alpha}(L) = kt^{-k} \int_0^t (t-s)^{k-1} e^{isL^{\alpha/2}} ds, \quad 0 < \alpha < 1,\ k > 0.$$

We have the following

Theorem 1.2: *For any $\alpha \in (0,1)$, the following hold:*

(i). If $k = \frac{n}{2}$, then $I_{k,\alpha}(L)$ is bounded from $H^1(M)$ to $L^1(M)$, on $L^p(M)$, $1 < p < \infty$, and from $L^\infty(M)$ to $BMO\,(M)$.

(ii). If $k < \frac{n}{2}$, then $I_{k,\alpha}(L)$ is bounded on $L^p(M)$, for $k \geq n\left|\frac{1}{p} - \frac{1}{2}\right|$, $1 < p < \infty$.

(iii). If $k > \frac{n}{2}$, then $I_{k,\alpha}(L)$ is bounded on $L^p(M)$, $1 \leq p \leq \infty$.

In the context of $\mathbb{R}^n$, the operators $I_{k,\alpha}(L)$ are studied for example in [27] and [22]. According to [27], the results above, for $k \leq n/2$, are optimal. The operators $I_{k,\alpha}(L)$ have also been studied in more abstract contexts, see for example [1, 2, 17, 18, 4, 6].

It is worth mentioning that our approach is valid only for $\alpha \in (0,1)$. This is due to the fact that the estimates of the multiplier $m_{\alpha,\beta}(\lambda)$ are available only for $\alpha \in (0,1)$, (cf. [31] and Section 5).

The paper is organized as follows. In Section 2 we recall some known facts about the Hardy space H^1 and BMO (Subsection 2.1), the wave operator and the construction of its parametrix (Subsection 2.2). In Section 3 the estimates of the Fourier transform of the derivatives of the multiplier $m_{\alpha,\beta}(\lambda)$ are given. In Section 4 we give the estimates of the kernel of the operator $m_{\alpha,\beta}(L)$ near the diagonal and in Section 5 we establish its L^p-boundedness when $\beta > n/2$. In Section 6 we prove the $H^1 - L^1$ boundedness of the operator $m_{\alpha,\frac{\alpha n}{2}}(L)$ and in Section 7 we finish the proofs of Theorems 1.1 and 1.2.

Throughout this article the different constants will always be denoted by the same letter c. When their dependence or independence is significant, it will be clearly stated.

2 Preliminaries

2.1 The Hardy space H^1 and BMO

Let us recall that a complex-valued function a on M is an atom if it is supported in a ball $B(y_0,r)$ and satisfies

$$\|a\|_\infty \leq |B(y_0,r)|^{-1} \quad \textit{and} \quad \textstyle\int_M a(x)\,dx = 0.$$

A function f on M belongs to the Hardy space $H^1(M)$ if there exist $(\lambda_m)_{m\in\mathbb{N}} \in \ell^1$ and a sequence of atoms $(a_m)_{m\in\mathbb{N}}$ such that

$$f = \sum_{m\in\mathbb{N}} \lambda_m a_m,$$

where the series converges in $L^1(M)$. The norm $\|f\|_{H^1}$ is the infimum of $\sum_{m\in\mathbb{N}} |\lambda_m|$ for all such decompositions of f.

A function f belongs to $BMO(M)$, if there exists a constant $c>0$ such that for all balls $B(x,r)$,

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)-f_B|\,dy < c,$$

where

$$f_B = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy.$$

The smallest of all such constants c is the BMO norm of f.

Finally we note that the dual of $H^1(M)$ is $BMO(M)$, (cf. [8], Theorem B, p. 593).

2.2 The wave operator

Let $G_t(x,y)$ be the kernel of the wave operator $\cos t\sqrt{L}$. Note that $G_t(x,y)$ is also the solution of the wave equation

$$\begin{aligned} (\partial_t^2 + L_y)\, u(t,x,y) &= 0, \\ u(0,x,y) &= \delta_x(y), \\ \partial_t u(0,x,y) &= 0. \end{aligned} \tag{2.1}$$

In this article we shall exploit the fact that $G_t(x,y)$ propagates with finite propagation speed (cf. [7, 29]):

$$supp(G_t) \subseteq \{(x,y) : d(x,y) \leq |t|\}. \tag{2.2}$$

Next we shall recall some facts about the Hadamard parametrix construction for the kernel $G_t(x,y)$, (cf. [3, 4, 15]).

Let $\delta \in (0, r_0)$, to be fixed later, and let us consider, for every ball $B(x,\delta)$, $x \in M$, the exponential normal coordinates centered at x. Let $g_{ij}(x,y)$, $y \in B(x,\delta)$, be the metric tensor expressed in these coordinates and let us denote by $(g^{ij}(x,y))$ its inverse matrix. We have the following Taylor expansion of g_{ij}:

$$\begin{aligned} g_{ij}(x,y) = \;& \delta_{ij} +^2 A_{ijkl}(y_k - x_k)(y_l - x_l) \\ & +^3 A_{ijklm}(y_k - x_k)(y_l - x_l)(y_m - x_m) + ... \end{aligned} \tag{2.3}$$

where the ${}^kA_{ij...}$ are universal polynomials in the components of the curvature tensor and its first $k-2$ covariant derivatives at the point x, (cf. [24], p. 85). By the term "universal" we mean that the coefficients of the polynomials ${}^kA_{ij...}$ depend only on the dimension of the manifold.

It follows from (2.3) and the assumption of C^∞-bounded geometry that for any multi-index α there exists a positive constant c_α such that

$$\left|\partial_y^\alpha g_{ij}(x,y)\right| \leq c_\alpha, \; x \in M, \; y \in B(x,\delta). \tag{2.4}$$

Since $g_{ij}(x,x)=\delta_{ij}$, there is $c>0$ and $\delta\in(0,r_0)$ such that

$$c^{-1}\leq \det(g_{ij}(x,y))\leq c. \tag{2.5}$$

for all $x\in M$ and $y\in B(x,\delta)$.

In what follows, we shall fix a $\delta\in(0,\min(1,r_0))$ such that (2.5) is satisfied.

From (2.4) and (2.5) we also have that there is $c'_\alpha>0$ such that

$$\left|\partial_y^\alpha g^{ij}(x,y)\right|\leq c'_\alpha. \tag{2.6}$$

for all $x\in M$, $y\in B(x,\delta)$.

Let $\Theta(x,y)=\det\left(g_{ij}(x,y)\right)$. Then, the Laplace-Beltrami operator L can be written as follows:

$$L=\frac{1}{(\Theta(x,y)))^{1/2}}\sum_{i,j}\frac{\partial}{\partial y_i}(\Theta(x,y)))^{1/2}g^{ij}(x,y)\frac{\partial}{\partial y_j}.$$

Note that by (2.4), (2.5) and (2.6), the Laplacian can also be written as

$$L=\sum_{|\alpha|\leq 2}c_a(y)\partial_y^\alpha$$

with the coefficients satisfying

$$\left|\partial_y^\beta c_\alpha(y)\right|\leq c_{\alpha,\beta}, \tag{2.7}$$

for all $x\in M$, $y\in B(x,\delta)$ and any multi-index β.

Let us consider the following smooth functions:

$$U_0(x,y)=\Theta^{-1/2}(x,y)$$

and

$$U_{k+1}(x,y)=\Theta^{-1/2}(x,y)\int_0^1 s^k\Theta^{1/2}(x,y_s)L_2U_k(x,y_s)ds,$$

where y_s, $s\in[0,1]$, is the geodesic from x to y and L_2 denotes the Laplacian acting on the second variable. Note that $U_0(x,x)=1$.

In what follows, we always assume that $|t|\leq\delta$ and $y\in B(x,\delta)$, $x\in M$.

Let us consider the kernels

$$E_N(t,x,y)=C_0\sum_{k=0}^{N}(-1)^kU_k(x,y)\,|t|\,\frac{(t^2-d(x,y)^2)_+^{k-\frac{n+1}{2}}}{4^k\Gamma\left(k-\frac{n-1}{2}\right)}, \tag{2.8}$$

where C_0 is a normalizing constant.

They satisfy (cf. [3])

$$\begin{aligned}(\partial_t^2 + L_y)\, E_N(t,x,y) = \tfrac{C_0(-1)^N}{4^N\Gamma\left(N-\frac{n-1}{2}\right)}\, |t|\, (t^2 - d(x,y)^2)_+^{N-\frac{n+1}{2}}\, L_y U_N(x,y),\\ E_N(0,x,y) = \delta_x(y),\\ \partial_t E_N(0,x,y) = 0.\end{aligned} \tag{2.9}$$

Now, let us observe that by (2.4), (2.5) and (2.7) there exists a $c > 0$ such that

$$|U_0(x,y)| \le c_0 \quad \textit{and} \quad |L_y U_0(x,y)| \le c_1. \tag{2.10}$$

These also imply that for any $k \in \mathbb{N}$ there is $c > 0$ such that

$$|U_k(x,y)| \le \tfrac{c_1^k}{k!}, \quad |L_y U_k(x,y)| \le \tfrac{c_1^{k+1}}{k!} \quad \textit{and} \quad \|\nabla_y U_k(x,y)\| \le c\tfrac{c_1^k}{k!}, \tag{2.11}$$

for $x \in M$ and $y \in B(x,\delta)$.

If $k \ge \frac{n+1}{2}$, then (2.11) and the fact that

$$\Gamma\left(k - \tfrac{n+1}{2}\right) \sim k!, \quad \textit{as} \quad k \to \infty,$$

imply that

$$\left| U_k(x,y)\, |t|\, \frac{(t^2 - d(x,y)^2)_+^{k-\frac{n+1}{2}}}{4^k\Gamma\left(k - \frac{n-1}{2}\right)} \right| \le \frac{c_1^k}{k!}\delta\frac{\delta^{2k-(n+1)}}{4^k k!} \le \frac{c_1^k}{k!}\frac{\delta^{2k-n}}{4^k k!}. \tag{2.12}$$

From (2.8) and (2.12) we get that $E_N(t,x,y)$ converges uniformly as $N \to \infty$ and (2.9), (2.11) and (2.1) that the limit is $G_t(x,y)$. Thus we have the expansion

$$G_t(x,y) = C_0 \sum_{k=0}^{\infty} (-1)^k U_k(x,y)\, |t|\, \frac{(t^2 - d(x,y)^2)_+^{k-\frac{n+1}{2}}}{4^k\Gamma\left(k - \frac{n-1}{2}\right)}, \tag{2.13}$$

the convergence being uniform for $|t| \le \delta$ and $y \in B(y,\delta)$.

3 Estimates of the multiplier and of its derivatives

In this section we shall give some estimates for the derivatives of the Fourier transform of the multiplier $m_{\alpha,\beta}$.

Let us consider the function

$$f_{\alpha,\beta}(t) = m_{\alpha,\beta}(t^2) = \psi(t^2)\,|t|^{-\beta}\,e^{i|t|^\alpha}.$$

Let r_0 be the injectivity radius of M and us fix $\delta \in (0, r_0)$. Let $\chi_\delta(t)$ be a smooth and nonnegative function such that $\chi_\delta(t) = 1$ for $|t| \leq \delta/2$ and 0 for $|t| \geq \delta$. Set

$$\hat{f}^0_{\alpha,\beta}(t) = \hat{f}_{\alpha,\beta}(t)\chi_\delta(t), \qquad \hat{f}^\infty_{\alpha,\beta}(t) = \hat{f}_{\alpha,\beta}(t)(1 - \chi_\delta(t)). \tag{3.1}$$

In this article we shall need the following:

Lemma 3.1: *Let $\alpha \in (0,1)$ and $\beta = \frac{\alpha n}{2} + \varepsilon$, $\varepsilon \geq 0$. Then for all $m, N \in \mathbb{N}$ and $t \in \mathbb{R}$,*

$$\left|\partial_t^m \hat{f}^0_{\alpha,\beta}(t)\right| \leq c\,|t|^{-\left(1+m-\varepsilon-\frac{\alpha(n+1)}{2}\right)/(1-\alpha)}, \tag{3.2}$$

and

$$\left|\partial_t^m \hat{f}^\infty_{\alpha,\beta}(t)\right| \leq c\,|t|^{-N}. \tag{3.3}$$

Before proceed to the proof of Lemma 3.1, let us recall the following estimates from Wainger [31], Theorem 9. For any $\alpha \in (0,1)$ and $\epsilon > 0$, consider the function

$$f_{\epsilon,\alpha,b}(x) = e^{-\epsilon\|x\|}\psi\left(\|x\|^2\right)\|x\|^{-b}\,e^{i\|x\|^\alpha}, \qquad x \in \mathbb{R}^k.$$

We have that

$$\hat{f}_{\epsilon,\alpha,b}(\|x\|) = \|x\|^{\frac{2-k}{2}} \int_0^\infty e^{-\epsilon u}\psi(u^2)u^{-b+\frac{k}{2}}e^{iu^\alpha}J_{\frac{k-2}{2}}(u\,\|x\|)du \tag{3.4}$$

where $J_m(z)$ is the Bessel function.

Making use of this formula, Wainger proved that the limit

$$\hat{f}_{\alpha,b}(\|x\|) = \lim_{\epsilon\to 0}\hat{f}_{\epsilon,\alpha,b}(\|x\|)$$

exists and it is continuous for $x \neq 0$. Further, if $b > k\left(1 - \frac{\alpha}{2}\right)$, then $\hat{f}_{\alpha,b}$ is continuous also at $x = 0$, while if $b \leq k\left(1 - \frac{\alpha}{2}\right)$ and $M \in \mathbb{N}$, then

$$\begin{aligned}\hat{f}_{\alpha,b}(\|x\|) = \;& \|x\|^{-\left(k-b-\frac{\alpha k}{2}\right)/(1-\alpha)}\,e^{i\xi_\alpha\|x\|^{-\alpha/(1-\alpha)}}\textstyle\sum_{m=0}^M a_m\,\|x\|^{m\alpha/(1-\alpha)} \\ & + O\left(\|x\|^{(M+1)\alpha/(1-\alpha)}\right) + C(\|x\|),\end{aligned} \tag{3.5}$$

where $a_0 \neq 0$, ξ_α is real and $\xi_\alpha \neq 0$; C is a continuous function.

Furthermore

$$\left|\hat{f}_{\alpha,b}(\|x\|)\right| = O(\|x\|^{-N}), \quad as \quad \|x\| \to \infty, \tag{3.6}$$

for any $N \in \mathbb{N}$.

Proof of Lemma 3.1: If $m = 0$, then (3.2) and (3.3) are an immediate consequence of (3.5), with $k = 1$, and (3.6).

If $m = 2l$, $l \geq 1$, then $\partial^{2l} \hat{f}_{\alpha,\beta}$ is the Fourier transform of the function

$$(-i\lambda)^{2l} f_{\alpha,\beta}(\lambda) = (-i)^{2l} \psi(|\lambda|^2)\, |\lambda|^{-\beta+2l}\, e^{i|\lambda|^\alpha} = (-i)^{2l} f_{\alpha,\beta-2l}(\lambda).$$

Hence (3.2) and (3.3) follow again from (3.5) and (3.6) with $b = \beta - 2l$.

If $m = 2l+1$, then $\partial^{2l+1} \hat{f}_{\alpha,\beta}$ is the Fourier transform of the function

$$\varphi(\lambda) = (-i)^{2l+1} \psi(|\lambda|^2) \lambda\, |\lambda|^{-\beta+2l}\, e^{i|\lambda|^\alpha}.$$

Since this function is odd, we have

$$\begin{aligned} \partial^{2l+1} \hat{f}_{\alpha,\beta}(t) &= -2i \int_0^{+\infty} \varphi(x) \sin(tx) dx \\ &= -2i \lim_{\epsilon \to 0} \int_0^{+\infty} e^{-\epsilon x} \varphi(x) \sin(tx) dx. \end{aligned}$$

Since

$$\sin x = \sqrt{\frac{\pi x}{2}} J_{\frac{1}{2}}(x),$$

we have

$$\begin{aligned} \partial^{2l+1} \hat{f}_{\alpha,\beta}(t) &= c\sqrt{2\pi t} \lim_{\epsilon \to 0} \int_0^{\infty} e^{-\epsilon x} \psi(x^2) x^{-\beta+2l+3/2} e^{ix^\alpha} J_{\frac{1}{2}}(tx) dx \\ &= ct \lim_{\epsilon \to 0} \left\{ t^{-\frac{1}{2}} \int_0^{\infty} e^{-\epsilon x} \psi(x^2) x^{-\beta+2l+3/2} e^{ix^\alpha} J_{\frac{1}{2}}(tx) dx \right\}. \end{aligned}$$

The integral in brackets above is the same as the integral $\hat{f}_{\epsilon,\alpha,b}(t)$ in formula (3.4), with $k = 3$ and $b = \beta - 2l$. This gives, as $\epsilon \to 0$, the Fourier transform of the multiplier $f_{\alpha,b}(\lambda)$ in $\mathbb{R}^3$. Therefore, the estimates $\partial^{2l+1} \hat{f}_{\alpha,\beta}(t)$ follow again from (3.5) and (3.6).

4 The estimates of the kernel near the diagonal

Let us express the operator $m_{\alpha,\beta}(L)$ in terms of the wave operator $\cos t\sqrt{L}$. If $f_{\alpha,\beta}(t) = m_{\alpha,\beta}(t^2)$, then $m_{\alpha,\beta}(L) = f_{\alpha,\beta}(\sqrt{L})$ and since $f_{\alpha,\beta}$ is an even function, by the Fourier inversion formula we have that

$$m_{\alpha,\beta}(L) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{f}_{\alpha,\beta}(t) \cos t\sqrt{L} dt.$$

Let $m_{\alpha,\beta}(x,y)$ be the kernel of $m_{\alpha,\beta}(L)$. Then by the finite propagation speed property (2.2)

$$m_{\alpha,\beta}(x,y) = (2\pi)^{-1/2} \int_{|t|\geq d(x,y)} \hat{f}_{\alpha,\beta}(t) G_t(x,y) dt.$$

This kernel is singular near the diagonal and integrable at infinity. We want to split $m_{\alpha,\beta}(x,y)$ into these two parts and treat them separately. This can be done by considering the operators

$$m^0_{\alpha,\beta}(L) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{f}^0_{\alpha,\beta}(t) \cos t\sqrt{L} dt$$

and

$$m^\infty_{\alpha,\beta}(L) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{f}^\infty_{\alpha,\beta}(t) \cos t\sqrt{L} dt,$$

where $f^0_{\alpha,\beta}$ and $f^\infty_{\alpha,\beta}$ are defined in (3.1). We have

$$m_{\alpha,\beta}(L) = m^0_{\alpha,\beta}(L) + m^\infty_{\alpha,\beta}(L).$$

Let $m^0_{\alpha,\beta}(x,y)$ and $m^\infty_{\alpha,\beta}(x,y)$ denote the kernels of $m^0_{\alpha,\beta}(L)$ and $m^\infty_{\alpha,\beta}(L)$, respectively. Then

$$m^0_{\alpha,\beta}(x,y) = (2\pi)^{-1/2} \int_{\delta\geq|t|\geq d(x,y)} \hat{f}^0_{\alpha,\beta}(t) G_t(x,y) dt \tag{4.1}$$

and

$$m^\infty_{\alpha,\beta}(x,y) = (2\pi)^{-1/2} \int_{|t|>\delta} \hat{f}^\infty_{\alpha,\beta}(t) G_t(x,y) dt.$$

In the present section we deal with the kernel $m^0_{\alpha,\beta}(x,y)$. This kernel contains the singular part of the kernel $m_{\alpha,\beta}(x,y)$ and from (4.1) it follows that

$$supp(m^0_{\alpha,\beta}) \subset \{(x,y) \in M \times M:\ d(x,y) \leq \delta\}. \tag{4.2}$$

We shall obtain very good L^∞ estimates for $m^0_{\alpha,\beta}(x,y)$ by using the Hadamard parametrix construction for $G_t(x,y)$. These estimates allow us to prove in Section 6 that $m_{\alpha,\beta}(L)$ is bounded from H^1 to L^1 for $\beta = n\alpha/2$.

We have the following:

Lemma 4.1: *Let $\alpha \in (0,1)$. Then for all $\varepsilon \geq 0$, there exists a constant $c > 0$ such that for all $x, y \in M$*

$$\left| m^0_{\alpha,\frac{\alpha n}{2}+\varepsilon}(x,y) \right| \leq c d(x,y)^{-n+\frac{\varepsilon}{1-\alpha}} \tag{4.3}$$

and

$$\left\| \nabla_y m^0_{\alpha,\frac{\alpha n}{2}}(x,y) \right\| \leq c d(x,y)^{-(n+1)+\alpha'}, \tag{4.4}$$

where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$.

For $\beta = \frac{\alpha n}{2} + \varepsilon$ and $k = -1, 0, 1, ...$, we set

$$I_k(x,y) = \int_{\mathbb{R}} \hat{f}^0_{\alpha,\beta}(t)\, |t|\, \frac{(t^2 - d(x,y)^2)_+^{k-\frac{n+1}{2}}}{\Gamma\left(k - \frac{n-1}{2}\right)} dt.$$

Lemma 4.1 is a consequence of the expansion (2.13) of $G_t(x,y)$ and of the following:

Lemma 4.2: *(i). If $0 \leq k \leq \frac{n+1}{2}$, then there is a $c > 0$ such that*

$$|I_k(x,y)| \leq c d(x,y)^{-n+\frac{\varepsilon}{1-\alpha}}, \qquad \forall x, y \in M. \tag{4.5}$$

(ii). If $k > \frac{n+1}{2}$, then there is a $c > 0$ such that

$$|I_k(x,y)| \leq c \frac{\delta^{2k}}{\Gamma\left(k - \frac{n-1}{2}\right)}, \qquad \forall x, y \in M. \tag{4.6}$$

(iii). If $k = -1$ and $\varepsilon = 0$, then there is a $c > 0$ such that

$$|I_k(x,y)| \leq c d(x,y)^{-(n+2)+\alpha'}, \qquad \forall x, y \in M. \tag{4.7}$$

Proof: The proof is given in steps. Let us set, for simplicity, $d = d(x, y)$.
Proof of (4.5) for $n = 2p + 1$. This is the simpler case. If we put $t = ud$, then we have

$$\begin{aligned} I_k(x,y) &= d^{2k-n+1}\int_{\mathbb{R}} |u|\, \hat{f}^0_{\alpha,\beta}(ud)\frac{\left(u^2-1\right)_+^{k-\frac{n+1}{2}}}{\Gamma\left(k-\frac{n-1}{2}\right)}du \\ &= d^{2k-n+1}\int_{\mathbb{R}} |u|\, \hat{f}^0_{\alpha,\beta}(ud)\,(u+1)^{k-p-1}\,\frac{(u-1)_+^{k-p-1}}{\Gamma(k-p)}du. \end{aligned}$$

Since

$$\frac{(u-1)_+^{k-p-1}}{\Gamma\,(k-p)} = \delta^{(p-k)}(u-1), \quad for \quad k \le p+1, \tag{4.8}$$

(cf. [13], p. 56), we have

$$\begin{aligned} I_k &= d^{2k-n+1}\left(\partial_u^{p-k}\,|u|\,\hat{f}^0_{\alpha,\beta}(ud)\,(u+1)^{k-p-1}\right)\Big|_{u=1} \\ &= d^{2k-n+1}\sum_{m=0}^{p-k} c_{m,p,k}\,\left(\partial_u^m \hat{f}^0_{\alpha,\beta}(ud)\partial_u^{p-k-m}\left(|u|\,(u+1)^{k-p-1}\right)\right)\Big|_{u=1} \\ &= d^{2k-n+1}\sum_{m=0}^{p-k} c'_{m,p,k}\,\left(\partial_u^m \hat{f}^0_{\alpha,\beta}(ud)\right)\Big|_{u=1}. \end{aligned}$$

Making use of Lemma 3.1, we get that for all $m = 0,...,p-k$,

$$\begin{aligned} \left|\partial_u^m \hat{f}^0_{\alpha,\beta}(ud)_{u=1}\right| &\le \frac{cd^m}{d^{\left(1+m-\varepsilon-\frac{n+1}{2}\alpha\right)/(1-\alpha)}} \\ &= \frac{cd^m d^{\varepsilon/(1-\alpha)}}{d^{(1+m-(p+1)\alpha)/(1-\alpha)}} \\ &= \frac{d^m d^{\varepsilon/(1-\alpha)}}{dd^{(m-p\alpha)/(1-\alpha)}} \\ &= cd^{-1}d^{\varepsilon/(1-\alpha)}d^{\alpha(p-m)/(1-\alpha)} \\ &\le cd^{-1}d^{\varepsilon/(1-\alpha)}d^{\alpha k(1-\alpha)}. \end{aligned}$$

This implies that for all $k \ge 0$,

$$|I_k| \le cd^{2k-n+1}d^{-1}d^{\varepsilon/(1-\alpha)}d^{\alpha k(1-\alpha)} \le cd^{-n}d^{\varepsilon/(1-\alpha)}$$

which proves (4.5), when $n = 2p + 1$.

Proof of (4.5), for $n = 2p$. In this case we have

$$I_k(x,y) = \int_{\mathbb{R}} |t|\, \hat{f}^0_{\alpha,\beta}(t) \frac{(t^2-d^2)_+^{k-p-\frac{1}{2}}}{\Gamma\left(k-p+\frac{1}{2}\right)} dt.$$

The calculations now are more complicated because $k-p-\frac{1}{2}$ is no more an integer. If we put $t = du$ and $v = u+1$, then

$$\begin{aligned} I_k &= cd^{2k-2p+1} \int_{|u|>1} |u|\, \hat{f}^0_{\alpha,\beta}(du)\, (u^2-1)_+^{k-p-\frac{1}{2}}\, du \\ &= cd^{2k-2p+1} \int_{u>1} u \hat{f}^0_{\alpha,\beta}(du)\, (u+1)^{k-p-\frac{1}{2}}\, (u-1)_+^{k-p-\frac{1}{2}}\, du \\ &+ cd^{2k-2p+1} \int_{u<-1} (-u) \hat{f}^0_{\alpha,\beta}(du)\, |u-1|^{k-p-\frac{1}{2}}\, (-(u+1))_+^{k-p-\frac{1}{2}}\, du \\ &= cd^{2k-2p+1} \int_{v>0} (v+1) \hat{f}^0_{\alpha,\beta}(d(v+1))(v+2)^{k-p-\frac{1}{2}} v_+^{k-p-\frac{1}{2}} dv \\ &+ cd^{2k-2p+1} \int_{v>0} (v+1) \hat{f}^0_{\alpha,\beta}(-d(v+1))(v+2)^{k-p-\frac{1}{2}} w_+^{k-p-\frac{1}{2}} dv. \end{aligned}$$

Since $\hat{f}^0_{\alpha,\beta}$ is an even function

$$I_k = 2cd^{2k-2p+1} \int_{v>0} (v+1) \hat{f}^0_{\alpha,\beta}(d(v+1))(v+2)^{k-p-\frac{1}{2}} v_+^{k-p-\frac{1}{2}} dv.$$

We shall only treat the term I_0 which is the most singular near $v = 0$. The integrals I_k, $k > 0$, can be treated similarly. We have

$$I_0 = cd^{-2p+1} \int_0^\infty (v+1) \hat{f}^0_{\alpha,\beta}(d(v+1))(v+2)^{-p-\frac{1}{2}} v_+^{-p-\frac{1}{2}} dv. \tag{4.9}$$

By replacing the term $(v+2)^{-p-\frac{1}{2}}$ by its Taylor's expansion at $v = 0$, we can see that the most singular part of I_0 is the integral

$$J_0 := d^{-2p+1} \int_0^\infty \hat{f}^0_{\alpha,\beta}(d(v+1)) v_+^{-p-\frac{1}{2}} dv.$$

Let us observe that $\hat{f}_{\alpha,\beta}(d(v+1))$ is the Fourier transform of the function

$$\frac{1}{d} f_{\alpha,\beta}\left(\frac{t}{d}\right) e^{it} = \frac{1}{d} \psi\left(\left|\frac{t}{d}\right|^2\right) \left|\frac{t}{d}\right|^{-\varepsilon-\alpha n/2} e^{i\left|\frac{t}{d}\right|^\alpha} e^{it}.$$

Also, the Fourier transform of the distribution $v_+^{-p-\frac{1}{2}}$ is equal to

$$i\Gamma\left(-p+\frac{1}{2}\right)\left[e^{-i\frac{\pi}{2}\left(p+\frac{1}{2}\right)}t_+^{p-\frac{1}{2}}-e^{+i\frac{\pi}{2}\left(p+\frac{1}{2}\right)}t_-^{p-\frac{1}{2}}\right],$$

(cf. [13], p. 172). So,

$$\begin{aligned}
J_0 &= d^{-2p+1}\int_{-\infty}^{\infty}\tfrac{1}{d}\psi\left(\left|\tfrac{t}{d}\right|^2\right)\left|\tfrac{t}{d}\right|^{-\varepsilon-\alpha n/2}e^{i\left|\frac{t}{d}\right|^{\alpha}}e^{it}\left[c_1t_+^{p-\frac{1}{2}}-c_2t_-^{p-\frac{1}{2}}\right]dt\\
&= d^{-2p+1}d^{p-\frac{1}{2}}\int_{-\infty}^{\infty}\psi\left(u^2\right)|u|^{-\varepsilon-\alpha n/2}e^{i|u|^{\alpha}}e^{iud}\left[c_1u_+^{p-\frac{1}{2}}-c_2u_-^{p-\frac{1}{2}}\right]du\\
&= J_{0,1}+J_{0,2}.
\end{aligned}\tag{4.10}$$

We shall only treat $J_{0,1}$. The term $J_{0,2}$ can be treated similarly. We have

$$\begin{aligned}
J_{0,1} &= c_1d^{-2p+1}d^{p-\frac{1}{2}}\int_0^{\infty}\psi\left(u^2\right)u^{-\frac{\alpha n}{2}-\varepsilon+p-\frac{1}{2}}e^{iu^{\alpha}}\cos(ud)du\\
&+ic_1d^{-2p+1}d^{p-\frac{1}{2}}\int_0^{\infty}\psi\left(u^2\right)u^{-\frac{\alpha n}{2}-\varepsilon+p-\frac{1}{2}}e^{iu^{\alpha}}\sin(ud)du\\
&= d^{-2p+1}d^{p-\frac{1}{2}}c_1(L_1+iL_2).
\end{aligned}\tag{4.11}$$

Now L_1 is the Fourier transform of the even function

$$f_{\alpha,b}(u)=\psi\left(|u|^2\right)|u|^{-\frac{\alpha n}{2}-\varepsilon+p-\frac{1}{2}}e^{i|u|^{\alpha}},$$

with $b=\frac{\alpha n}{2}+\varepsilon-p+\frac{1}{2}$. So, by (3.5), with $k=1$, we get that

$$\begin{aligned}
|L_1| &\le cd^{-\left(1-\frac{\alpha n}{2}-\varepsilon+p-\frac{1}{2}-\frac{\alpha}{2}\right)/(1-\alpha)}\\
&= cd^{-\left(\frac{1-\alpha}{2}+p(1-\alpha)\right)/(1-\alpha)}d^{\frac{\varepsilon}{(1-\alpha)}}=d^{-p-\frac{1}{2}}d^{\frac{\varepsilon}{(1-\alpha)}}.
\end{aligned}\tag{4.12}$$

By the formula $\sin x=\sqrt{\frac{\pi x}{2}}J_{\frac{1}{2}}(x)$, we have

$$\begin{aligned}
L_2 &= \int_0^{\infty}\psi\left(u^2\right)u^{-\frac{\alpha n}{2}-\varepsilon+p-\frac{1}{2}}e^{iu^{\alpha}}\sin(ud)du\\
&= c\sqrt{d}\int_0^{\infty}\psi\left(u^2\right)u^{-\frac{\alpha n}{2}-\varepsilon+p}e^{iu^{\alpha}}J_{\frac{1}{2}}(ud)du\\
&= cd\lim_{0<\rho\to 0}\left\{d^{-\frac{1}{2}}\int_0^{\infty}e^{-\rho u}\psi\left(u^2\right)u^{-\left(\frac{\alpha n}{2}+\varepsilon-p+\frac{3}{2}\right)+\frac{3}{2}}e^{iu^{\alpha}}J_{\frac{1}{2}}(ud)du\right\}.
\end{aligned}$$

The integral in the brackets above is the same as the integral $\hat{f}_{\epsilon,\alpha,b}$ in (3.4) with $k=3$ and $b=\frac{\alpha n}{2}+\varepsilon-p+\frac{3}{2}$. Therefore, by (3.5), with $k=3$, we get that, for

$$\begin{aligned} |L_2| &\leq cdd^{-\left(3-\frac{\alpha n}{2}-\varepsilon+p-\frac{3}{2}-\frac{3\alpha}{2}\right)/(1-\alpha)} \\ &= cdd^{-\left(\frac{3}{2}(1-\alpha)+p(1-\alpha)\right)/(1-\alpha)} d^{\varepsilon/(1-\alpha)} \\ &= cdd^{-\frac{3}{2}} d^{-p} d^{\varepsilon/(1-\alpha)} = cd^{-\frac{1}{2}} d^{-p} d^{\varepsilon/(1-\alpha)}. \end{aligned} \tag{4.13}$$

It follows from (4.11), (4.12) and (4.13) that

$$|J_{0,1}| \leq cd^{-2p+1} d^{p-\frac{1}{2}} d^{-p-\frac{1}{2}} = cd^{-n} d^{\frac{\varepsilon}{(1-\alpha)}}. \tag{4.14}$$

Putting all together, from (4.9) to (4.14), we get

$$|I_k(x,y)| \leq cd^{-n} d^{\frac{\varepsilon}{(1-\alpha)}}$$

which proves (4.5), for $n=2p$.

Proof of (4.6). If $k>\frac{n+1}{2}$, then by (3.2) and (3.3) we get

$$\begin{aligned} |I_k(x,y)| &\leq c\int_{d\leq|t|\leq\delta} \left|\hat{f}^0_{\alpha,\beta}(t)\right| |t| \frac{\left(t^2-d^2\right)_+^{k-\frac{n+1}{2}}}{\Gamma\left(k-\frac{n-1}{2}\right)} dt \\ &\leq \frac{c}{\Gamma\left(k-\frac{n-1}{2}\right)} \int_{d\leq|t|\leq\delta} |t|^{-\left(1-\varepsilon-\frac{\alpha(n+1)}{2}\right)/(1-\alpha)} |t|^{2k-n}\, dt. \end{aligned}$$

But, if $k>\frac{n+1}{2}$, then

$$2k-n-\frac{1-\varepsilon-\frac{\alpha(n+1)}{2}}{(1-\alpha)} \geq \frac{2\varepsilon+\alpha(n-1)}{2(1-\alpha)} > 0,$$

so,

$$|I_k(x,y)| \leq c\frac{\delta^{2k-n+1-\left(1-\varepsilon-\frac{\alpha(n+1)}{2}\right)/(1-\alpha)}}{\Gamma\left(k-\frac{n-1}{2}\right)} \leq c\frac{\delta^{2k}}{\Gamma\left(k-\frac{n-1}{2}\right)}.$$

Proof of (4.7). We shall only treat the case $n=2p+1$. The case $n=2p$ can be treated similarly. As in the proof of (4.5), we have to estimate the integral

$$
\begin{aligned}
I_{-1}(x,y) &= \int_{\mathbb{R}} \hat{f}^0_{\alpha,\beta}(t)\,|t|\,\frac{\left(t^2-d(x,y)^2\right)_+^{-1-\frac{n+1}{2}}}{\Gamma\left(-1-\frac{n-1}{2}\right)}dt \\
&= d^{-n-1}\int_{\mathbb{R}} \hat{f}^0_{\alpha,\beta}(du)\,|u|\,\frac{\left(u^2-1\right)_+^{-1-\frac{n+1}{2}}}{\Gamma\left(-1-\frac{n-1}{2}\right)}du \\
&= d^{-n-1}\int_{\mathbb{R}} \hat{f}^0_{\alpha,\beta}(du)\,|u|\,(u+1)^{-p-2}\,\frac{(u-1)_+^{-p-2}}{\Gamma(-p-1)}dt \\
&= d^{-n-1}\partial_u^{p+1}\left(|u|\,\hat{f}^0_{\alpha,\beta}(ud)\,(u+1)^{-p-2}\right|_{u=1}.
\end{aligned}
$$

So,

$$
\begin{aligned}
|I_{-1}(x,y)| &\le cd^{-n-1}\sum_{m=0}^{p+1} c'_{m,p}\frac{d^m}{d^{(1+m-(p+1)\alpha)/(1-\alpha)}} \\
&= cd^{-n-1}\sum_{m=0}^{p+1} c'_{m,p}\frac{d^m}{dd^{(m-p\alpha)/(1-\alpha)}} \\
&= cd^{-n-2}\sum_{m=0}^{p+1} c'_{m,p}d^{\frac{m-m\alpha-m+p\alpha}{1-\alpha}} \\
&= cd^{-n-2}\sum_{m=0}^{p+1} c'_{m,p}d^{\frac{\alpha}{1-\alpha}(p-m)} \le cd^{-n-2}d^{-\alpha/(1-\alpha)} = cd^{-n-2}d^{\alpha'}.
\end{aligned}
$$

□

Proof of Lemma 4.1: (i). It is a consequence of (2.11) and Lemma 4.2. (ii) Making use of (2.13), we have

$$
\begin{aligned}
\nabla_y G_t(x,y) &= \sum_{k=0}^{\infty}(-1)^k\nabla_y U_k(x,y)\,|t|\,\frac{\left(t^2-d(x,y)^2\right)_+^{k-\frac{n+1}{2}}}{4^k\Gamma\left(k-\frac{n-1}{2}\right)} \\
&-\sum_{k=0}^{\infty}U_k(x,y)\,|t|\left(k-\tfrac{n+1}{2}\right)\frac{\left(t^2-d(x,y)^2\right)_+^{k-\frac{n+1}{2}-1}}{4^k\Gamma\left(k-\frac{n-1}{2}\right)}2d\nabla_y(d) \\
&= I + II.
\end{aligned}
$$

Now, it follows from (2.11) and the estimates (4.5), (4.6) for $\varepsilon = 0$, that

$$|I| \le cd(x,y)^{-n}.$$

To deal with II we first note that $\|\nabla_y d(x,y)\| \leq 1$ for $d(x,y) \leq 1$. Then, by (4.6) and (4.7) we have

$$|II| \leq c d(x,y)^{-(n+1)+\alpha'}.$$

5 The L^p boundedness of $m_{\alpha,\beta}(L)$ for $\beta > \frac{\alpha n}{2}$

In this Section we prove claim (iii) of Theorem 1.1 which states that for all $\alpha \in (0,1)$ and $\beta > \frac{\alpha n}{2}$, $m_{\alpha,\beta}(L)$ is bounded on L^p, $p \geq 1$.

We note that the L^p boundedness of $m^{\infty}_{\alpha,\beta}(L)$ for $\beta \geq \frac{\alpha n}{2}$, can be extracted from [1]. We shall give below a simple proof of this result by adapting an argument from [29].

Proposition 5.1: *If $\alpha \in (0,1)$ and $\beta \geq \frac{\alpha n}{2}$, then $m^{\infty}_{\alpha,\beta}(L)$ is bounded on L^p, $p \geq 1$.*

Proof: We have that

$$m^{\infty}_{\alpha,\beta}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}^{\infty}_{\alpha,\beta}(t) \cos t\sqrt{\lambda} dt$$

and by the estimate (3.3) of $\hat{f}^{\infty}_{\alpha,\beta}(t)$ we get that $m^{\infty}_{\alpha,\beta}$ is bounded. Thus $m^{\infty}_{\alpha,\beta}(L)$ is bounded on L^2. Therefore, the Proposition will be a consequence of the following:

$$\sup_{x\in M} \int_M \left| m^{\infty}_{\alpha,\beta}(x,y) \right| dy < \infty. \tag{5.1}$$

Let us first notice that the Dirac mass δ_x at x can be written as $\delta_x = L^k \varphi_x + \psi_x$, where $k = \left[\frac{n}{4}\right] + 1$ and where the functions φ_x and ψ_x are in $L^2(B(x,r_0))$, with r_0 the injectivity radius of M (cf. [29], p. 776). Also by the assumption of C^{∞}-bounded geometry, we can assume that there is $c > 0$ such that $\|\varphi_x\|_2 \leq c$ and $\|\psi_x\|_2 \leq c$ for all $x \in M$. We have

$$\begin{aligned} m^{\infty}_{\alpha,\beta}(x,y) &= m^{\infty}_{\alpha,\beta}(L)\delta_x(y) = L^k m^{\infty}_{\alpha,\beta}(L)\varphi_x(y) + m^{\infty}_{\alpha,\beta}(L)\psi_x(y) \\ &= (\sqrt{L})^{2k} f^{\infty}_{\alpha,\beta}(\sqrt{L})\varphi_x(y) + f^{\infty}_{\alpha,\beta}(\sqrt{L})\psi_x(y) \\ &= (-i)^{-2k} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \partial^{2k} \hat{f}^{\infty}_{\alpha,\beta}(t) \cos t\sqrt{L}\varphi_x(y) dt \\ &+ (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{f}^{\infty}_{\alpha,\beta}(t) \cos t\sqrt{L}\psi_x(y) dt \\ &= I_1(x,y) + I_2(x,y). \end{aligned} \tag{5.2}$$

By the estimates (3.3) of $\partial_t^m \hat{f}^\infty_{\alpha,\beta}(t)$ and the finite propagation speed property we have that

$$\begin{aligned} |I_1(x,y)| &\leq c\int_{-\infty}^{\infty}\left|\partial^{2k}\hat{f}^\infty_{\alpha,\beta}(t)\cos t\sqrt{L}\varphi_x(y)\right|dt \\ &= c\sum_{j\geq 1}\int_{j\leq|t|\leq j+1}\left|\partial^{2k}\hat{f}^\infty_{\alpha,\beta}(t)\right|\left|\cos t\sqrt{L}\varphi_x(y)\right|dt \\ &\leq c\sum_{j\geq 1}\tfrac{1}{j^N}\int_{j\leq|t|\leq j+1}\left|\mathbf{1}_{B(x,r_0+j+1)}(y)\cos t\sqrt{L}\varphi_x(y)\right|dt. \end{aligned} \tag{5.3}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned} \int_M\left|\mathbf{1}_{B(x,R)}(y)\cos t\sqrt{L}\varphi_x(y)\right|dy &\leq |B(x,R)|^{\frac{1}{2}}\left\|\cos t\sqrt{L}\varphi_x\right\|_2 \\ &\leq cR^{n/2}\left\|\cos t\sqrt{L}\right\|_2\|\varphi_x\|_2 \\ &\leq cR^{n/2} \end{aligned} \tag{5.4}$$

since $\left\|\cos t\sqrt{L}\right\|_2 \leq 1$ and $\|\varphi_x\|_2 \leq c$ for all $x \in M$.

Let $N > 2+\frac{n}{2}$. Then, it follows from (5.2), (5.3) and (5.4) that

$$\int_M |I_1(x,y)|\,dy \leq c\sum_{j\geq 1}(r_0+j+1)^{\frac{n}{2}}\frac{1}{j^N}\int_{j\leq|t|\leq j+1}dt \leq c\sum_{j\geq 1}\frac{1}{j^{N-\frac{n}{2}}}$$

and hence

$$\sup_{x\in M}\int_M |I_1(x,y)|\,dy < \infty.$$

The term $I_2(x,y)$ can be treated similarly. □

Proposition 5.2: *If $\alpha \in (0,1)$ and $\beta > \frac{\alpha n}{2}$, then $m^0_{\alpha,\beta}(L)$ is bounded on L^p, $p \geq 1$.*

Proof: Since $m^0_{\alpha,\beta}(L) = m_{\alpha,\beta}(L) - m^\infty_{\alpha,\beta}(L)$, Proposition 5.1 implies that $m^0_{\alpha,\beta}(L)$ is bounded on L^2. If $\beta = \frac{\alpha n}{2}+\varepsilon$, $\varepsilon > 0$, then from (4.2) and (4.3) we have that

$$\begin{aligned} \sup_{x\in M}\int_M\left|m^0_{\alpha,\beta}(x,y)\right|dy &= \sup_{x\in M}\int_{B(x,\delta)}\left|m^0_{\alpha,\beta}(x,y)\right|dy \\ &\leq c\sup_{x\in M}\int_{B(x,\delta)}d(x,y)^{-n+\frac{\varepsilon}{1-\alpha}}dy \\ &= c\sup_{x\in M}\int_0^\delta r^{-n+\frac{\varepsilon}{1-\alpha}}r^{n-1}dr = c\delta^{\frac{\varepsilon}{1-\alpha}} \end{aligned}$$

and the Proposition follows. $\square$

6 H^1–L^1 boundedness of the operator $m_{\alpha,\frac{\alpha n}{2}}(L)$

In this section we prove claim (i) of Theorem 1.1. By the duality of H^1 with BMO, the $H^1 - L^1$ boundedness of $m_{\alpha,\frac{\alpha n}{2}}(L)$ is a consequence of the following

Proposition 6.1: *If $\alpha \in (0,1)$, then the operator $m_{\alpha,\frac{\alpha n}{2}}(L)$ is bounded from $L^\infty(M)$ to $BMO(M)$.*

The L^p-boundedness of $m_{\alpha,\frac{\alpha n}{2}}(L)$ for $p \in (1,\infty)$, follows from the L^2 boundedness and Proposition 6.1 by interpolation and duality.

The strategy of the proof of Proposition 6.1 is inspired from [11]. It is based on the following Lemmata.

Lemma 6.2: *There is a constant $A > 0$ such that*

$$\int_{d(x,y_1)>2d(y,y_1)^{1-\alpha}} \left| m^0_{\alpha,\frac{\alpha n}{2}}(x,y) - m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1) \right| dx < A, \tag{6.1}$$

for all $y_1 \in M$ and $y \in B(y_1,\delta)$.

Proof: Let us fix $y_1 \in M$ and $y \in B(y_1,\delta)$. Let $y(s)$, $s \in [0, d(y,y_1)]$, be the geodesic segment from y to y_1. Then

$$m^0_{\alpha,\frac{\alpha n}{2}}(x,y) - m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1) = \int_0^{d(y,y_1)} \nabla_y m^0_{\alpha,\frac{\alpha n}{2}}(x,y(s))ds.$$

By (4.4) and the mean value theorem, we get that

$$\left| m^0_{\alpha,\frac{\alpha n}{2}}(x,y) - m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1) \right| \leq c\frac{d(y,y_1)}{d(x,y^*)^{n+1-\alpha'}}, \tag{6.2}$$

for some y^* on $y(s)$.

Let us set $d = d(y,y_1)$, $A_k = B(y_1, 2^{k+1}d^{1-\alpha}) \setminus B(y_1, 2^k d^{1-\alpha})$ and

$$I_k = \int_{A_k} \left| m^0_{\alpha,\frac{\alpha n}{2}}(x,y) - m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1) \right| dx.$$

Then

$$\int_{d(x,y_1)>2d(y,y_1)^{1-\alpha}} \left|m^0_{\alpha,\frac{\alpha n}{2}}(x,y) - m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1)\right| dx$$
$$= \sum_{k\geq 1} \int_{A_k} \left|m^0_{\alpha,\frac{\alpha n}{2}}(x,y) - m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1)\right| dx = \sum_{k\geq 1} I_k.$$

Since $d \leq \delta \leq 1$, we have

$$d(x,y^*) \geq 2^k d^{1-\alpha} - d \geq 2^{k-1} d^{1-\alpha}, \qquad \forall x \in A_k, \ \ \forall k \geq 1.$$

Now, by (6.2) and since $(1-\alpha)(1-\alpha') = 1$, we have

$$\begin{aligned} I_k &\leq c \int_{A_k} \tfrac{d(y,y_1)dx}{d(x,y^*)^{n+1-\alpha'}} \leq c \int_{A_k} \tfrac{ddx}{\left(2^{k-1}d^{1-\alpha}\right)^{n+1-\alpha'}} \\ &\leq \tfrac{cd|A_k|}{\left(2^k d^{1-\alpha}\right)^{n+1-\alpha'}} \leq \tfrac{cd\left(2^{k+1}d^{1-\alpha}\right)^n}{\left(2^k d^{1-\alpha}\right)^{n+1-\alpha'}} \\ &= \tfrac{cd}{(2^k)^{1-\alpha'} d^{(1-\alpha)(1-\alpha')}} = \tfrac{c}{(2^k)^{1-\alpha'}}. \end{aligned}$$

It follows that

$$\int_{d(x,y_1)>d(y,y_1)^{1-\alpha}} \left|m^0_{\alpha,\frac{\alpha n}{2}}(x,y) - m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1)\right| dx$$
$$= \sum_{k=1}^{\infty} I_k \leq c \sum_{k=1}^{\infty} \tfrac{1}{(2^{k-1})^{1-\alpha'}} < \infty$$

since $1 - \alpha' > 0$ for $\alpha \in (0,1)$. □

The following Lemma is based on a local version of a generalization of Hardy-Littlewood-Sobolev theorem due to Varopoulos, (cf. [30], p. 12).

Lemma 6.3: *For any $\alpha \in (0,1)$, $m_{\alpha,\frac{\alpha n}{2}}(L)$ is bounded from L^2 to $L^{\frac{2}{1-\alpha}}$.*

Proof: We write

$$\begin{aligned} m_{\alpha,\frac{\alpha n}{2}}(L) &= \psi(|L|)\,|L|^{-\alpha n/4}\, e^{i|L|^{\alpha/2}} \\ &= (1+L)^{-\alpha n/4}\psi(|L|)\,|L|^{-\alpha n/4}\,(1+L)^{\alpha n/4} e^{i|L|^{\alpha/2}} \\ &= (1+L)^{-\alpha n/4}\Phi(L), \end{aligned}$$

where $\Phi(\lambda) = \psi(|\lambda|)\,|\lambda|^{-\alpha n/4}\,(1+\lambda)^{\alpha n/4} e^{i|\lambda|^{\alpha/2}}$. Since $\Phi(\lambda)$ is bounded, it suffices to show that the potential operator $(1+L)^{-\alpha n/4}$ is bounded from L^2

to $L^{\frac{2}{1-\alpha}}$. To this end, let $q_t(x,y)$ be the kernel of the semigroup $e^{-t(1+L)}$ and $p_t(x,y)$ the heat kernel of M. Then

$$q_t(x,y) = e^{-t} p_t(x,y).$$

By the Li-Yau estimate of p_t:

$$p_t(x,y) \leq c \frac{e^{-d(x,y)^2/ct}}{\left|B\left(x,\sqrt{t}\right)\right|},$$

for all $t > 0$ and $x, y \in M$, (cf. [16]), it follows that

$$q_t(x,y) \leq \begin{cases} ct^{-n/2}, & \forall t \leq 1, \\ ce^{-t} \leq ct^{-n/2}, & \forall t \geq 1. \end{cases} \tag{6.3}$$

From (6.3) it follows that

$$\left\| e^{-t(1+L)} f \right\|_\infty \leq ct^{-n/2} \left\| f \right\|_1, \qquad \forall f \in L^1, \qquad \forall t > 0.$$

As it is shown by Varopoulos, (cf. [30], p. 12), this estimate implies that the operators $(1+L)^{-\gamma/2}$, $\gamma > 0$, are bounded from L^p to L^q for $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$ and $1 < p < \infty$. The Lemma follows by taking $\gamma = \alpha n/2$ and $p = 2$. □

Proof of Proposition 6.1: In order to prove that $m_{\alpha, \frac{\alpha n}{2}}(L)$ is bounded from L^∞ to BMO it enough to show that there is a constant $c > 0$, such that for every ball $B(y_1, r) = B$ and every $f \in C_0^\infty(M)$

$$\int_B \left| m_{\alpha, \frac{\alpha n}{2}}(L) f(x) - (m_{\alpha, \frac{\alpha n}{2}}(L) f)_B \right| dx \leq c \left\| f \right\|_\infty |B|, \tag{6.4}$$

where $(m_{\alpha, \frac{\alpha n}{2}}(L) f)_B$ is the mean value of $m_{\alpha, \frac{\alpha n}{2}}(L) f$ on B.

Let us then fix a ball $B(y_1, r) = B$ and let us set, in order to simplify the notation, $B_\alpha = B(y_1, 2r^{1-\alpha})$. If $f \in C_0^\infty(M)$, then we shall write $f = f\chi_{B_\alpha} + f\chi_{B_\alpha^c} := f_1 + f_2$.

To prove (6.4), we shall show that

$$\int_B \left| m_{\alpha, \frac{\alpha n}{2}}(L) f_1(x) \right| dx \leq c \left\| f \right\|_\infty |B|, \tag{6.5}$$

and

$$\int_B \left| m_{\alpha, \frac{\alpha n}{2}}(L) f_2(x) - (m_{\alpha, \frac{\alpha n}{2}}(L) f)_B \right| dx \leq c \left\| f \right\|_\infty |B|. \tag{6.6}$$

Proof of (6.5). If $r>1$, then $r^{1-\alpha}\leq r$ and hence

$$\begin{aligned}\int_B \left|m_{\alpha,\frac{\alpha n}{2}}(L)f_1(x)\right|dx \leq\ & \left\|m_{\alpha,\frac{\alpha n}{2}}(L)f_1\right\|_2 |B|^{1/2} \leq c\left\|f_1\right\|_2 |B|^{1/2}\\ &= c\left\|f\chi_{B_\alpha}\right\|_2 |B|^{1/2} \leq c\left\|f\right\|_\infty |B_\alpha|^{1/2}|B|^{1/2}\\ &= c\left\|f\right\|_\infty \left|B(y_1,2r^{1-\alpha})\right|^{1/2}|B|^{1/2}\\ &\leq c\left\|f\right\|_\infty \left|B(y_1,2r)\right|^{1/2}|B|^{1/2}\leq c\left\|f\right\|_\infty |B|\,.\end{aligned}$$

In the case when $r\leq 1$, we proceed by arguing as in [11], Theorem 1, p. 143 (see also [9], Theorem 2.1). Let $p=2/(1-\alpha)$ and let p' be its conjugate exponent. Then by Lemma 6.3 and Hölder's inequality

$$\begin{aligned}\int_B \left|m_{\alpha,\frac{\alpha n}{2}}f_1(x)\right|dx\ &\leq |B|^{1/p'}\left\|m_{\alpha,\frac{\alpha n}{2}}f_1\right\|_p \leq c\,|B|^{1/p'}\left\|f_1\right\|_2\\ &\leq c\,|B|^{1/p'}\left\|f_1\right\|_2 = c\,|B|^{1/p'}\left\|f\chi_{B_\alpha}\right\|_2\\ &\leq c\,|B|^{1/p'}\left\|f\right\|_\infty \left|B(y_1,2r^{1-\alpha})\right|^{1/2}\\ &\leq c\left\|f\right\|_\infty r^{\frac{n}{p'}+(1-\alpha)\frac{n}{2}} = cr^n\left\|f\right\|_\infty \leq c\,|B|\left\|f\right\|_\infty,\end{aligned}$$

since $\frac{n}{p'}+(1-\alpha)\frac{n}{2}=\frac{n}{p'}+\frac{n}{p}=n$. This completes the proof of (6.5).
Proof of (6.6). We have

$$\begin{aligned}&\left|m_{\alpha,\frac{\alpha n}{2}}(L)f_2(x)-(m_{\alpha,\frac{\alpha n}{2}}(L)f)_B\right|\\ &\leq \left|m^0_{\alpha,\frac{\alpha n}{2}}(L)f_2(x)-(m^0_{\alpha,\frac{\alpha n}{2}}(L)f_2)_B\right|\\ &+\left|(m^0_{\alpha,\frac{\alpha n}{2}}(L)f_2)_B-(m_{\alpha,\frac{\alpha n}{2}}(L)f)_B\right|+\left|m^\infty_{\alpha,\frac{\alpha n}{2}}(L)f_2(x)\right|.\end{aligned}\tag{6.7}$$

We write

$$m_{\alpha,\frac{\alpha n}{2}}(L)f=m^0_{\alpha,\frac{\alpha n}{2}}(L)f_1+m^0_{\alpha,\frac{\alpha n}{2}}(L)f_2+m^\infty_{\alpha,\frac{\alpha n}{2}}(L)f,$$

and we recall that the operator $m^0_{\alpha,\frac{\alpha n}{2}}(L)$ is bounded on L^2 and that, by

Proposition 5.1, the operator $m^{\infty}_{\alpha,\frac{\alpha n}{2}}(L)$ is bounded on L^∞. Therefore,

$$\begin{aligned}
&\left|(m^0_{\alpha,\frac{\alpha n}{2}}(L)f_2)_B - (m_{\alpha,\frac{\alpha n}{2}}(L)f)_B\right| \\
&= |B|^{-1}\left|\int_B m^0_{\alpha,\frac{\alpha n}{2}}(L)f_2(x)dx - \int_B m_{\alpha,\frac{\alpha n}{2}}(L)f(x)dx\right| \\
&= |B|^{-1}\left|\int_B m^0_{\alpha,\frac{\alpha n}{2}}(L)f_1(x)dx + \int_B m^{\infty}_{\alpha,\frac{\alpha n}{2}}(L)f(x)dx\right| \\
&\leq |B|^{-1}\left\|m^0_{\alpha,\frac{\alpha n}{2}}(L)f_1\right\|_2 |B|^{\frac{1}{2}} + |B|^{-1}\left\|m^{\infty}_{\alpha,\frac{\alpha n}{2}}(L)f\right\|_\infty |B| \\
&\leq c\,|B|^{-1}\|f\|_\infty |B| + c\,\|f\|_\infty = c\,\|f\|_\infty .
\end{aligned} \tag{6.8}$$

It follows from (6.7), (6.8) and the L^∞ boundedness of $m^{\infty}_{\alpha,\frac{\alpha n}{2}}(L)$ that to prove (6.6), it is enough to show that

$$\int_B \left|m^0_{\alpha,\frac{\alpha n}{2}}(L)f_2(x) - (m^0_{\alpha,\frac{\alpha n}{2}}(L)f_2)_B\right| dx \leq c\,\|f\|_\infty |B| . \tag{6.9}$$

Let us set

$$c_B = \int_{B^c_\alpha} m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1)f_2(x)dx.$$

If $y \in B(y_1,r)$, then

$$m^0_{\alpha,\frac{\alpha n}{2}}(L)f_2(y) - c_B = \int_{B^c_\alpha}\left\{m^0_{\alpha,\frac{\alpha n}{2}}(x,y) - m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1)\right\} f_2(x)dx.$$

Also, if $x \in B(y_1, 2r^{1-\alpha})^c$ and $y \in B(y_1,r)$, then

$$d(x,y_1) > 2r^{1-\alpha} \geq 2d(y,y_1)^{1-\alpha}.$$

Therefore, by Lemma 6.2

$$\begin{aligned}
&\left|m^0_{\alpha,\frac{\alpha n}{2}}(L)f_2(y) - c_B\right| \\
&\leq \int_{B^c_\alpha}\left|m^0_{\alpha,\frac{\alpha n}{2}}(x,y) - m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1)\right| |f_2(x)|\, dx \\
&\leq \|f\|_\infty \int_{d(x,y_1)>2d(y,y_1)^{1-\alpha}}\left|m^0_{\alpha,\frac{\alpha n}{2}}(x,y) - m^0_{\alpha,\frac{\alpha n}{2}}(x,y_1)\right| dx \\
&\leq A\,\|f\|_\infty .
\end{aligned}$$

This implies that

$$\int_B \left| m^0_{\alpha,\frac{\alpha n}{2}}(L) f_2(y) - c_B \right| dy \leq A\,|B|\,\|f\|_\infty\,. \tag{6.10}$$

By (6.10) we have

$$\begin{aligned}
&\int_B \left| m^0_{\alpha,\frac{\alpha n}{2}}(L) f_2(y) - (m^0_{\alpha,\frac{\alpha n}{2}}(L) f_2)_B \right| dy \\
&\leq \int_B \left| m^0_{\alpha,\frac{\alpha n}{2}}(L) f_2(y) - c_B \right| dy + \int_B \left| c_B - (m^0_{\alpha,\frac{\alpha n}{2}}(L) f_2)_B \right| dy \\
&\leq A\,\|f\|_\infty\,|B| + |B| \left| c_B - (m^0_{\alpha,\frac{\alpha n}{2}}(L) f_2)_B \right|.
\end{aligned} \tag{6.11}$$

Finally, by using once more (6.10) we get

$$\begin{aligned}
\left| (m^0_{\alpha,\frac{\alpha n}{2}}(L) f_2)_B - c_B \right| &= |B|^{-1} \left| \int_{B(y_1,r)} m^0_{\alpha,\frac{\alpha n}{2}}(L) f_2(y) dy - \int_B c_B dy \right| \\
&\leq |B|^{-1} \int_B \left| m^0_{\alpha,\frac{\alpha n}{2}}(L) f(y) - c_B \right| dy \leq A\,\|f\|_\infty\,.
\end{aligned} \tag{6.12}$$

7 Proof of the results

In this Section we shall finish the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1: The proof of claims (i) and (ii) of Theorem 1.1 are given in Sections 6 and 5 respectively. It remains to prove claim (ii). This will be done by complex interpolation as in Theorem 6 of [11]. Let us consider the analytic family of operators

$$T_z(L) = e^{z^2} L^{\frac{n\alpha}{4} z} m_{\alpha,\frac{\alpha n}{2}}(L), \quad Rez \in [0,1]\,.$$

If $t \in \mathbb{R}$, then

$$T_{it}(L) = e^{-t^2} L^{i\frac{n\alpha t}{4}} m_{\alpha,\frac{\alpha n}{2}}(L).$$

But the imaginary powers of the Laplacian are bounded on H^1 and

$$\left\| L^{i\gamma} \right\|_{H^1 \to H^1} \leq c \left(1 + \sqrt{|\gamma|} e^{\pi|\gamma|/2} \right), \quad \gamma \in \mathbb{R},$$

(cf. [19]). So, if we combine with Theorem 1.1(i), we get that $T_{it}(L)$ is bounded from $H^1(M)$ to $L^1(M)$ and

$$\|T_{it}(L)\|_{H^1\to L^1} \leq ce^{-t^2}\left(c\sqrt{\pi}+\sqrt{\alpha n\,|t|}e^{\pi\alpha n|t|/8}\right),$$

for all $t \in \mathbb{R}$.

Also, the operators $T_{1+it}(L)$ are bounded on $L^2(M)$ and

$$\|T_{1+it}(L)\|_2 \leq ce^{-t^2}.$$

By complex interpolation between $Rez = 0$ and $Rez = 1$, we obtain that for $\theta \in (0,1)$ and $p \in (1,2)$, the operator $T_\theta(L)$ is bounded on L^p for $\frac{1}{p} = 1 - \frac{\theta}{2}$. If we choose $\theta = 1 - \frac{2\beta}{\alpha n}$, then

$$T_\theta(L) = e^{\theta^2} L^{\frac{n\alpha}{4}} L^{-\frac{n\alpha}{4}\frac{2\beta}{\alpha n}} m_{\alpha,\frac{\alpha n}{2}}(L) = e^{\theta^2} m_{\alpha,\beta}(L)$$

and $\frac{1}{p} - \frac{1}{2} = \frac{\beta}{\alpha n}$. This is the desired result for $p \in (1,2)$. The case $p \in (2,\infty)$ is just the dual result.

Proof of Theorem 1.2: As in [1], by replacing the operator L by $L_1 = t^{2/\alpha}L$, the operators

$$I_{k,\alpha}(L) = kt^{-k}\int_0^t (t-s)^{k-1} e^{isL^{\alpha/2}}ds, \quad 0<\alpha<1,\ k>0,$$

can be written in the form

$$I_{k,\alpha}(L) = M_k(L_1^{\alpha/2}),$$

with

$$M_k(\lambda) = k\int_0^1 (1-s)^{k-1} e^{is|\lambda|}ds.$$

Further, the multiplier $M_k(\lambda)$ can be written as

$$M_k(\lambda) = C_k\psi(\lambda)\lambda^{-k}e^{i\lambda} + \Omega(\lambda),$$

where ψ is as in (1.2) and $\Omega(\lambda)$ satisfies

$$\partial_\lambda^N \Omega(\lambda) = O(\lambda^{-N-1}), \quad as \quad \lambda \to \infty,$$

for all $N \in \mathbb{N}$, (cf. [1], [27], p. 336).

This implies that

$$\left|\hat{\Omega}(t)\right| \leq \frac{c(N,R)}{|t|^{N+1}}, \quad for \quad |t| \geq R.$$

Making use of this and by arguing in exactly the same way as in Proposition 5.1 we can prove that the operator $\Omega(L)$ is bounded on L^p, $p \geq 1$. Furthermore, by Theorem 1.1(ii), $C_k\psi(L_1)L_1^{-\alpha k/2}e^{iL_1^{\alpha/2}}$ is bounded on L^p for $\alpha k \geq \alpha n\left|\frac{1}{p}-\frac{1}{2}\right|$ i.e. for $k \geq n\left|\frac{1}{p}-\frac{1}{2}\right|$, $1 < p < \infty$. This proves the claim (ii) of Theorem 1.2. The claims (i) and (iii) can be deduced in a similar way from Theorem 1.1(i) and (iii).

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