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Silvermann-Toeplitz theorem for double sequences and series and its application to Nörlund means in non-archimedean fields

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Abstract

In this paper, $K$ denotes a complete, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in $K$. In the present paper, we prove the Silvermann-Toeplitz theorem for double sequences and series in $K$ and apply it to Nörlund means for double sequences and series in $K$.

Throughout the present paper, $K$ denotes a complete, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in $K$. In this paper, we prove the Silvermann-Toeplitz theorem for double sequences and series in $K$ (see Theorem 2, proved in the sequel). We then introduce Nörlund means for double sequences and series in $K$ and apply Silvermann-Toeplitz theorem for these means.

For analysis in the classical case a general reference is [2] while for analysis in non-archimedean fields a general reference is [1].

For a given infinite matrix $A = (a_{n,k})$ and a sequence $\{x_k\}$, the sequence $\{y_n\}$ is defined as follows:

$$y_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \quad n = 1, 2, \ldots,$$

it being assumed that the series on the right converge. If $\lim_{n \to \infty} y_n = s$ whenever $\lim_{k \to \infty} x_k = s$, we say that $A$ is regular. The criterion for $A$ to be regular in terms of the entries of the matrix $A$ are well-known (see [4], [6]).

**Theorem 1.** $A = (a_{n,k})$ is regular if and only if

(i) $\sup_{n,k} |a_{n,k}| < \infty$;
In the sequel, the following definitions are needed.

**Definition 1.** Let \( \{x_{m,n}\} \) be a double sequence in \( K \) and \( x \in K \). We say that
\[
\lim_{n \to \infty} x_{m,n} = x
\]
if for each \( \epsilon > 0 \), the set \( \{(m, n) \in \mathbb{N}^2 : |x - x_{m,n}| \geq \epsilon\} \) is finite. In such a case we say that \( x \) is the limit of \( \{x_{m,n}\} \).

**Definition 2.** Let \( \{x_{m,n}\} \) be a double sequence in \( K \) and \( s \in K \). We say that
\[
s = \sum_{m=1, n=1}^{\infty, \infty} x_{m,n},
\]
if
\[
s = \lim_{m+n \to \infty} s_{m,n},
\]
where
\[
s_{m,n} = \sum_{k=1, l=1}^{m, n} x_{k,l}, \ m, n = 1, 2, \ldots
\]

**Remark.** If \( \lim_{m+n \to \infty} x_{m,n} = x \), then the sequence \( \{x_{m,n}\} \) is automatically bounded.

It is easy to prove the following results.

**Lemma 1.** \( \lim_{m+n \to \infty} x_{m,n} = x \) if and only if

(i) \( \lim_{n \to \infty} x_{m,n} = x, \ m = 1, 2, \ldots \),

(ii) \( \lim_{m \to \infty} x_{m,n} = x, \ n = 1, 2, \ldots \),

and

(iii) for each \( \epsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that \( |x - x_{m,n}| < \epsilon \), for all \( m, n \geq N \), which we write as \( \lim_{m,n \to \infty} x_{m,n} = x \).
Lemma 2. \( \lim_{m+n \to \infty} s_{m,n} \) exists if and only if
\[
\lim_{m+n \to \infty} x_{m,n} = 0. \tag{1}
\]

Given the matrix \( A = (a_{m,n,k,l}) \), we define
\[
y_{m,n} = \sum_{k=1, l=1}^{\infty, \infty} a_{m,n,k,l} x_{k,l}, \quad m, n = 1, 2, \ldots, \tag{2}
\]
assuming that the series on the right converge.

Necessary and sufficient conditions for \( A = (a_{m,n,k,l}) \) to be regular for the class of all double sequences and series in the classical case have been found by Kojima [3]. It has been found that convergence and boundedness play a vital role for double sequences and series, a role analogous to that of convergence for simple sequences and series. Robison [8] proved Silvermann-Toeplitz theorem for such a class of bounded and convergent double sequences in the classical case. We prove here its analogue in a complete, non-trivially valued, non-archimedean field.

In this context, the following definition is needed.

Definition 3. If whenever \( \{x_{m,n}\} \) is a convergent sequence, \( \{y_{m,n}\} \) converges to the same value, then the matrix \( A = (a_{m,n,k,l}) \) is said to be regular.

Theorem 2. In order that whenever a sequence \( \{x_{m,n}\} \) has a limit \( x \),

\[
\sum_{k=1, l=1}^{\infty, \infty} a_{m,n,k,l} x_{k,l} \text{ shall converge and } \lim_{m+n \to \infty} \sum_{k=1, l=1}^{\infty, \infty} a_{m,n,k,l} x_{k,l} = x, \text{ i.e., for } A = (a_{m,n,k,l}) \text{ to be regular it is necessary and sufficient that }
\]

(a) \( \lim_{m+n \to \infty} a_{m,n,k,l} = 0, \quad k, l = 1, 2, \ldots; \)

(b) \( \lim_{m+n \to \infty} \sum_{k=1, l=1}^{\infty, \infty} a_{m,n,k,l} = 1; \)

(c) \( \lim_{m+n \to \infty} \sup_{k \geq 1} |a_{m,n,k,l}| = 0, \quad l = 1, 2, \ldots; \)

(d) \( \lim_{m+n \to \infty} \sup_{l \geq 1} |a_{m,n,k,l}| = 0, \quad k = 1, 2, \ldots; \)

and
Proof. Proof of necessity.

Define the sequence \( \{x_{k,l}\} \) as follows: For any fixed \( p, q \), let
\[
x_{k,l} = \begin{cases} 
1, & \text{when } k = p, \ l = q; \\
0, & \text{otherwise}.
\end{cases}
\]  

Then
\[
y_{m,n} = a_{m,n,p,q}.
\]

Since \( \{x_{k,l}\} \) has limit 0, it follows that (a) is necessary.

Define the sequence \( \{x_{k,l}\} \) where \( x_{k,l} = 1, \ k, l = 1, 2, \ldots \).

Now,
\[
y_{m,n} = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}, \ m, n = 1, 2, \ldots.
\]

This shows that \( \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} \) converges for \( m, n = 1, 2, \ldots \).  

Since \( \{x_{k,l}\} \) has limit 1, it follows that
\[
\lim_{m+n \to \infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} = 1,
\]
so that (b) is necessary.

We now show that \( \lim_{m+n \to \infty} \sup_{k \geq 1} |a_{m,n,k,l}| = 0 \) for all \( l \in \mathbb{N} \). Suppose not.

Then there exists \( l_0 \in \mathbb{N} \) such that \( \lim_{m+n \to \infty} \sup_{k \geq 1} |a_{m,n,k,l_0}| = 0 \) does not hold.

So, there exists an \( \epsilon > 0 \), such that
\[
\left\{ (m,n) : \sup_{k \geq 1} |a_{m,n,k,l_0}| > \epsilon \right\} \text{ is infinite.} 
\]

Let us choose \( m_1 = n_1 = r_1 = 1 \). Choose \( m_2, n_2 \in \mathbb{N} \) such that \( m_2 + n_2 > m_1 + n_1 \) and
\[
\sup_{1 \leq k \leq r_1} |a_{m_2,n_2,k,l_0}| < \frac{\epsilon}{8}, \ \text{using (a)};
\]
and
\[
\sup_{k \geq 1} |a_{m_2,n_2,k,l_0}| > \epsilon, \ \text{using (5)}.
\]
Then choose $r_2 \in \mathbb{N}$ such that $r_2 > r_1$ and

$$\sup_{k > r_2} |a_{m_2,n_2,k,l_0}| < \frac{\epsilon}{8};$$

using (b).

Inductively choose $m_p + n_p > m_{p-1} + n_{p-1}$ such that

$$\sup_{1 \leq k \leq r_{p-1}} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8};$$

and then choose $r_p > r_{p-1}$ such that

$$\sup_{k > r_p} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8}.$$

In view of (6), (7), (8), we have

$$\sup_{r_{p-1} < k \leq r_p} |a_{m_p,n_p,k,l_0}| > \epsilon - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \frac{3\epsilon}{4}.$$

We can now find $k_p \in \mathbb{N}$, $r_{p-1} < k_p \leq r_p$ such that

$$|a_{m_p,n_p,k_p,l_0}| > \frac{3\epsilon}{4}.$$

Define the sequence $\{x_{k,l}\}$ as follows:

$$x_{k,l} = \begin{cases} 0, & l \neq l_0; \\ 1, & l = l_0, k = k_p, p = 1, 2, \ldots. \end{cases}$$

We note that $\lim_{k+l \to \infty} x_{k,l} = 0$. Now, in view of (6),

$$\left| \sum_{k=1}^{r_{p-1}} a_{m_p,n_p,k_0,l_0} x_{k,l_0} \right| \leq \sup_{1 \leq k \leq r_{p-1}} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8};$$

Using (8), we have,

$$\left| \sum_{k=r_p+1}^{\infty} a_{m_p,n_p,k_0,l_0} x_{k,l_0} \right| \leq \sup_{k > r_p} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8};$$
and using (9), we get,

\[
\left| \sum_{k=r_p-1+1}^{r_p} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| = \left| a_{m_p,n_p,k,l_0} \right| > \frac{3\varepsilon}{4}.
\]  

(12)

Thus

\[
|y_{m_p,n_p}| = \left| \sum_{k=1}^{\infty} a_{m_p,n_p,k,l_0} x_{k,l_0} \right|
\geq \left| \sum_{k=r_p-1+1}^{r_p} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| - \left| \sum_{k=1}^{r_p-1} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| - \left| \sum_{k=r_p+1}^{\infty} a_{m_p,n_p,k,l_0} x_{k,l_0} \right|
\geq |a_{m_p,n_p,k,l_0}| - \sup_{1 \leq k \leq r_p-1} |a_{m_p,n_p,k,l_0}| - \sup_{k > r_p} |a_{m_p,n_p,k,l_0}|
\geq \frac{3\varepsilon}{4} - \varepsilon - \frac{\varepsilon}{8}, \text{ using (10), (11) and (12)}
= \frac{\varepsilon}{2}, \quad p = 1, 2, \ldots.
\]

Consequently \( \lim_{m+n \to \infty} y_{m,n} = 0 \) does not hold, which is a contradiction. Thus (c) is necessary. The necessity of (d) follows in a similar fashion.

To establish (e), we shall suppose that (e) does not hold and arrive at a contradiction. Since \( K \) is non-trivially valued, there exists \( \pi \in K \) such that \( 0 < \rho = |\pi| < 1 \). Choose \( m_1 = n_1 = 1 \). Using (a), (b), choose \( m_2 + n_2 > m_1 + n_1 \) such that

\[
\sup_{1 \leq k+l \leq m_1+n_1} |a_{m_2,n_2,k,l}| < 2, \text{ using (a)},
\]

\[
\sup_{k+l \geq 1} |a_{m_2,n_2,k,l}| > \left( \frac{2}{\rho} \right)^6;
\]

and

\[
\sup_{k+l > m_1+n_1} |a_{m_2,n_2,k,l}| < 2^2, \text{ using (b) and Lemma 1, Lemma 2}.
\]

It now follows that

\[
\sup_{k+l > m_2+n_2} |a_{m_2,n_2,k,l}| < 2^2.
\]
Choose $m_3 + n_3 > m_2 + n_2$ such that

$$\sup_{1 \leq k+l \leq m_2+n_2} |a_{m_3,n_3,k,l}| < 2^2;$$

$$\sup_{k+l \geq 1} |a_{m_3,n_3,k,l}| > \left(\frac{2}{\rho}\right)^{8};$$

and

$$\sup_{k+l > m_3+n_3} |a_{m_3,n_3,k,l}| < 2^4.$$

Inductively, choose $m_p + n_p > m_{p-1} + n_{p-1}$, such that

$$\sup_{1 \leq k+l \leq m_{p-1}+n_{p-1}} |a_{m_p,n_p,k,l}| < 2^{p-1}; \tag{13}$$

$$\sup_{k+l \geq 1} |a_{m_p,n_p,k,l}| > \left(\frac{2}{\rho}\right)^{2p+2}; \tag{14}$$

and

$$\sup_{k+l > m_p+n_p} |a_{m_p,n_p,k,l}| < 2^{2p-2}. \tag{15}$$

Using (13), (14), (15), we have,

$$\sup_{m_{p-1}+n_{p-1} < k+l \leq m_p+n_p} |a_{m_p,n_p,k,l}| > \left(\frac{2}{\rho}\right)^{2p+2} - 2^{2p-2} - 2^{p-1}$$

$$\geq \left(\frac{2}{\rho}\right)^{p+1} - \left(\frac{2}{\rho}\right)^{p-1}, \text{ since } \frac{1}{\rho} > 1$$

$$= \left(\frac{2}{\rho}\right)^{p-1} \left[ \left(\frac{2}{\rho}\right)^{p+3} - \left(\frac{2}{\rho}\right)^{p-1} - 1 \right], \text{ since } \left(\frac{2}{\rho}\right)^{p-1} \geq 1$$

$$= \left(\frac{2}{\rho}\right)^{p-1} \left[ \left(\frac{2}{\rho}\right)^4 - \left(\frac{2}{\rho}\right)^{p-1} - 2 \left(\frac{2}{\rho}\right)^{p-1} \right],$$

$$> \left(\frac{2}{\rho}\right)^{p-1} \left[ \left(\frac{2}{\rho}\right)^4 - \left(\frac{2}{\rho}\right)^{p-1} \right], \text{ since } \frac{2}{\rho} > 2$$

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\[
\begin{align*}
&= \left( \frac{2}{\rho} \right)^{2p-1} \left[ \left( \frac{2}{\rho} \right)^3 - 1 \right] \\
&> \left( \frac{2}{\rho} \right)^{2p-1} [2^3 - 1], \text{ since } \frac{2}{\rho} > 2 \\
&= 7 \left( \frac{2}{\rho} \right)^{2p-1} \\
&> 4 \left( \frac{2}{\rho} \right)^{2p-1} \\
&= \frac{2^{2p+1}}{\rho^{2p-1}} \\
&> \frac{2^{2p+1}}{\rho^p}, \text{ since } \frac{1}{\rho} > 1. \quad (16)
\end{align*}
\]

Thus there exist \( k_p \) and \( l_p \), \( m_{p-1} + n_{p-1} < k_p + l_p \leq m_p + n_p \) such that
\[
|a_{m_p,n_p,k_p,l_p}| > \frac{2^{2p+1}}{\rho^p}. \quad (17)
\]

Now, define the sequence \( \{x_{k,l}\} \) as follows:
\[
x_{k,l} = \begin{cases} 
\pi^p, & \text{if } k = k_p, l = l_p, p = 1, 2, \ldots; \\
0, & \text{otherwise.}
\end{cases}
\]

We note that \( \lim_{k+l \to \infty} x_{k,l} = 0 \). Now,
\[
|y_{m_p,n_p}| = \left| \sum_{k=1,l=1}^{\infty,\infty} a_{m_p,n_p,k,l} x_{k,l} \right|
\geq \left| \sum_{k+l=(m_{p-1}+n_{p-1})+1}^{m_p+n_p} a_{m_p,n_p,k,l} x_{k,l} \right|
- \left| \sum_{k+l=1}^{m_{p-1}+n_{p-1}} a_{m_p,n_p,k,l} x_{k,l} \right|
- \left| \sum_{k+l=(m_p+n_p)+1}^{\infty} a_{m_p,n_p,k,l} x_{k,l} \right|
\]
Silvermann-Toeplitz Theorem for Double Sequences and Series

\[ |a_{m,n,k,l}| \times |x_{k,l}| - \sup_{1 \leq k+l \leq m,n-1} |a_{m,n,k,l}| - \sup_{m,n<k+l<\infty} |a_{m,n,k,l}| \]

\[ > -p + \rho^p - 2^{p-2} - 2^{p-1}, \text{ using (13), (15) and (17)} \]

\[ = 2^{p+1} - 2^{p-2} - 2^{p-1} \]

\[ = 2^{p-2}(2^3 - 1) - 2^{p-1} \]

\[ = 2^{p-2}(7) - 2^{p-1} \]

\[ = 2^{p-1}[7 \cdot 2^{p-1} - 1] \]

\[ > 2^{p-1}[7 \cdot 2^{p-1} - 2^{p-2}] \]

\[ = 2^{p-1}[2^{p-2}(14 - 1)] \]

\[ = 2^{p-1}[13 \cdot 2^{p-2}] \]

\[ = 13 \cdot 2^{p-3} \]

i.e., \[ |y_{m,n,p}| > 13 \cdot 2^{p-3}, \quad p = 1, 2, \ldots, \]

i.e., \[ \lim_{m+n \to \infty} y_{m,n} = 0 \] does not hold, which is a contradiction. Thus (e) is necessary.

Proof of Sufficiency.
Let \[ \lim_{m+n \to \infty} x_{m,n} = x. \] Then

\[ y_{m,n} - x = x_{k,l} - x. \]

\[ \sum_{k=1}^{\infty} a_{m,n,k,l} x_{k,l} - x. \]

From (b) we have

\[ \sum_{k=1}^{\infty} a_{m,n,k,l} + r_{m,n} = 1, \]

where

\[ \lim_{m+n \to \infty} r_{m,n} = 0. \] (18)

Hence,

\[ y_{m,n} - x = \sum_{k=1}^{\infty} a_{m,n,k,l}(x_{k,l} - x) + r_{m,n}x. \]

Given \( \epsilon > 0 \), we can choose sufficiently large \( p \) and \( q \) such that

\[ \sup_{k+l>p+q} |x_{k,l} - x| < \frac{\epsilon}{5H}. \] (19)
where \( H = \sup_{m,n,k,l \geq 1} |a_{m,n,k,l}| \). Observe that \( H > 0 \) (from (b)).

Let \( L = \sup_{k+l \geq 1} |x_{k,l} - x| \). We now choose \( N \in \mathbb{N} \) such that whenever \( m + n \geq N \), the following are satisfied:

(i) \( \sup_{1 \leq k+l \leq p+q} |a_{m,n,k,l}| < \frac{\epsilon}{5pqL} \), using (a); \hspace{1cm} (20)

(ii) \( \sup_{k \geq 1} |a_{m,n,k,l}| < \frac{\epsilon}{5qL} \), \( l = 1, 2, \ldots, q \), using (c); \hspace{1cm} (21)

(iii) \( \sup_{l \geq 1} |a_{m,n,k,l}| < \frac{\epsilon}{5pL} \), \( k = 1, 2, \ldots, p \), using (d); \hspace{1cm} (22)

and

(iv) \( |r_{m,n}| < \frac{\epsilon}{5|x|} \), from the equation (18). \hspace{1cm} (23)

Whenever \( m + n \geq N \), we thus have,

\[
|y_{m,n} - x| = \left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{m,n,k,l}(x_{k,l} - x) + r_{m,n}x \right| \\
\leq \left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{m,n,k,l}(x_{k,l} - x) \right| + \left| \sum_{k=1}^{\infty} \sum_{l=q+1}^{\infty} a_{m,n,k,l}(x_{k,l} - x) \right| \\
+ \left| \sum_{k=p+1}^{\infty} \sum_{l=1}^{\infty} a_{m,n,k,l}(x_{k,l} - x) \right| + \left| \sum_{k=p+1}^{\infty} \sum_{l=q+1}^{\infty} a_{m,n,k,l}(x_{k,l} - x) \right| \\
+ |r_{m,n}| |x| \\
< \frac{\epsilon}{5pqL} Lpq + \frac{\epsilon}{5pL} Lp + \frac{\epsilon}{5qL} Lq + \frac{\epsilon}{5H} H + \frac{\epsilon}{5|x|} |x| \\
= \epsilon, \hspace{1cm} \text{using (19), (20), (21), (22) and (23).}
\]

Thus

\[ \lim_{m+n \to \infty} y_{m,n} = x, \]

which completes the proof of the theorem.

Nörlund means for simple sequences and series in complete, non-trivially valued, non-archimedean fields were introduced by Srinivasan [9] and studied
later in detail by Natarajan (for instance, see [7]). Nörlund means for double sequences and series in classical analysis were introduced by Moore [5]. We now define Nörlund means for double sequences and series in complete, non-trivially valued, non-archimedean fields and apply Theorem 2 for these means.

**Definition 4.** Given a doubly infinite set of elements \( p_{m,n} \in K, \ m, n = 0, 1, 2, \ldots \), where \( p_{0,0} \neq 0, |p_{i,j}| < |p_{0,0}|, (i, j) \neq (0, 0), i, j = 0, 1, 2, \ldots \), let

\[
P_{m,n} = \sum_{i,j=0}^{m,n} p_{i,j}, \quad m, n = 0, 1, 2, \ldots
\]

Given any double sequence \( \{s_{m,n}\} \) we define

\[
\sigma_{m,n} = (N,p_{m,n})(s_{m,n}) = \frac{S_{m,n}}{P_{m,n}} = \frac{\sum_{i,j=0}^{m,n} p_{m-i,n-j}s_{i,j}}{P_{m,n}}, \quad m, n = 0, 1, 2, \ldots
\]

If \( \lim_{m+n \to \infty} \sigma_{m,n} = \sigma \), we say that the double sequence \( \{s_{m,n}\} \) is summable \( (N,p_{m,n}) \) to the value \( \sigma \), written as

\[s_{m,n} \to \sigma(N,p_{m,n}).\]

Any double series \( \sum_{m,n} u_{m,n} \) is said to be summable \( (N,p_{m,n}) \) to the value \( \sigma \) if the double sequence \( \{s_{m,n}\} \), where

\[s_{m,n} = \sum_{i,j=0}^{m,n} u_{i,j}, \quad m, n = 0, 1, 2, \ldots \]

is summable \( (N,p_{m,n}) \) to the value \( \sigma \).

**Definition 5.** Given the Nörlund means \( (N,p_{m,n}), (N,q_{m,n}) \), we say that they are consistent if

\[s_{m,n} \to \sigma(N,p_{m,n}) \text{ and } s_{m,n} \to \sigma'(N,q_{m,n}) \Rightarrow \sigma = \sigma'.\]

We say that \( (N,p_{m,n}) \) is included in \( (N,q_{m,n}) \), written as

\[(N,p_{m,n}) \subseteq (N,q_{m,n}),\]
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if
\[ s_{m,n} \rightarrow \sigma(N, p_{m,n}) \Rightarrow s_{m,n} \rightarrow \sigma(N, q_{m,n}). \]
The two methods \((N, p_{m,n}), (N, q_{m,n})\) are said to be equivalent if
\[ (N, p_{m,n}) \subseteq (N, q_{m,n}) \text{ and } (N, q_{m,n}) \subseteq (N, p_{m,n}). \]

In view of Theorem 2, it is easy to prove the following result.

**Theorem 3.** The necessary and sufficient conditions for the regularity of the Nörlund means \((N, p_{m,n})\) are:
\[
\lim_{m+n \to \infty} \sup_{0 \leq j \leq n} |p_{m-i,n-j}| = 0, \quad 0 \leq i \leq m; \tag{24}
\]
\[
\lim_{m+n \to \infty} \sup_{0 \leq i \leq m} |p_{m-i,n-j}| = 0, \quad 0 \leq j \leq n. \tag{25}
\]

In the sequel let \((N, p_{m,n}), (N, q_{m,n})\) be two regular Nörlund methods such that each row and each column of the infinite matrices \((p_{m,n}), (q_{m,n})\) is a regular Nörlund mean for simple sequences.

**Theorem 4.** Any two such regular Nörlund methods are consistent.

**Proof.** Given two Nörlund methods \((N, p_{m,n}), (N, q_{m,n})\), where each row and each column of the infinite matrices \((p_{m,n}), (q_{m,n})\) is a regular Nörlund mean for simple sequences, we define a third method \((N, r_{m,n})\) by the equation
\[
r_{m,n} = \sum_{i,j=0}^{m,n} p_{i,j} q_{m-i,n-j}, \quad m, n = 0, 1, 2, \ldots.
\]

We then readily obtain, for \(s = \{s_{m,n}\},\)
\[
(N, r_{m,n})(s) = \sum_{i,j=0}^{m,n} \gamma_{m,n,i,j}(N, q_{i,j})(s),
\]
where
\[
\gamma_{m,n,i,j} = p_{m-i,n-j} Q_{i,j} / \sum_{\mu,\nu=0}^{m,n} p_{m-\mu,n-\nu} Q_{\mu,\nu}.
\]
Since \((N, p_{m,n})\) and \((N, q_{m,n})\) are regular, we have,
\[
\lim_{m+n \to \infty} \sup_{0 \leq j \leq n} |p_{m-i,n-j}| = 0 = \lim_{m+n \to \infty} \sup_{0 \leq i \leq m} |p_{m-i,n-j}|.
\]
Silvermann-Toeplitz Theorem for Double Sequences and Series

It now follows that
\[
\lim_{m+n \to \infty} \sup_{0 \leq j \leq n} \gamma_{m,n,i,j} = 0 = \lim_{m+n \to \infty} \sup_{0 \leq i \leq m} \gamma_{m,n,i,j}.
\]

Consequently \((N, r_{m,n})\) is regular. The regularity of this transformation enables us to infer that
\[
s_{m,n} \to \sigma'(N, q_{m,n}) \Rightarrow s_{m,n} \to \sigma'(N, r_{m,n}).
\]

Similarly we can show that
\[
s_{m,n} \to \sigma(N, p_{m,n}) \Rightarrow s_{m,n} \to \sigma(N, r_{m,n}).
\]

These imply that the two Nörlund methods \((N, p_{m,n})\) and \((N, q_{m,n})\) are consistent, completing the proof of the theorem.

If \((N, p_{m,n}), (N, q_{m,n})\) are regular, in view of conditions (24), (25), we have,

\[
\begin{align*}
P(x, y) &= \sum P_{m,n} x^m y^n, \\
Q(x, y) &= \sum Q_{m,n} x^m y^n, \\
p(x, y) &= \sum p_{m,n} x^m y^n, \\
q(x, y) &= \sum q_{m,n} x^m y^n,
\end{align*}
\]

all converge for \(|x|, |y| < 1\). The series
\[
\begin{align*}
k(x, y) &= \sum k_{m,n} x^m y^n = \frac{q(x, y)}{p(x, y)} = \frac{Q(x, y)}{P(x, y)}, \\
l(x, y) &= \sum l_{m,n} x^m y^n = \frac{p(x, y)}{q(x, y)} = \frac{P(x, y)}{Q(x, y)}
\end{align*}
\]

are convergent for \(|x|, |y| < 1\) and further
\[
\begin{align*}
\sum_{i,j=0}^{m,n} k_{i,j} p_{m-i,n-j} &= q_{m,n}; & \sum_{i,j=0}^{m,n} k_{i,j} P_{m-i,n-j} &= Q_{m,n}, \\
\sum_{i,j=0}^{m,n} l_{i,j} q_{m-i,n-j} &= p_{m,n}; & \sum_{i,j=0}^{m,n} l_{i,j} Q_{m-i,n-j} &= P_{m,n}.
\end{align*}
\]
Theorem 5. If \((N, p_{m,n}), (N, q_{m,n})\) are regular, then \((N, p_{m,n}) \subseteq (N, q_{m,n})\) if and only if \(\lim_{m+n \to \infty} k_{m,n} = 0\).

Proof. Let \(s(x, y) = \sum s_{m,n} x^m y^n\). Then for \(|x|, |y| < 1\), we have,

\[
\sum Q_{m,n}(N, q_{m,n})(s)x^m y^n = \sum \left(\sum_{i,j=0}^{m,n} q_{m-i,n-j}s_{i,j}\right)x^m y^n
\]

Similarly,

\[
\sum P_{m,n}(N, p_{m,n})(s)x^m y^n = p(x, y)s(x, y);
\]

Thus

\[
\sum Q_{m,n}(N, q_{m,n})(s)x^m y^n = \sum k_{m,n} x^m y^n \sum P_{m,n}(N, p_{m,n})(s)x^m y^n
\]

which implies that

\[
Q_{m,n}(N, q_{m,n})(s) = \sum_{i,j=0}^{m,n} k_{m-i,n-j}P_{i,j}(N, p_{i,j})(s).
\]

Hence,

\[
(N, q_{m,n})(s) = \sum_{i,j=0}^{m,n} c_{m,n,i,j}(N, p_{i,j})(s), \tag{28}
\]

where

\[
c_{m,n,i,j} = k_{m-i,n-j}P_{i,j}/Q_{m,n}.
\]

If \((N, p_{m,n}) \subseteq (N, q_{m,n}), (c_{m,n,i,j})\) is regular and so, by Theorem 2 (a),

\[
\lim_{m+n \to \infty} c_{m,n,0,0} = 0,
\]

i.e.,

\[
\lim_{m+n \to \infty} \frac{|k_{m,n}|}{|q_{0,0}|} = 0,
\]

which implies that \(\lim_{m+n \to \infty} k_{m,n} = 0\).

Conversely, if \(\lim_{m+n \to \infty} k_{m,n} = 0\), we can easily verify that \((c_{m,n,i,j})\) is regular. Consequently, using (28), it follows that \((N, p_{m,n}) \subseteq (N, q_{m,n})\). This completes the proof of the theorem.
Theorem 6, stated below, is an immediate consequence of Theorem 5.

**Theorem 6.** If \((N, p_{m,n})\) and \((N, q_{m,n})\) are regular Nörlund methods, then they are equivalent if and only if \(\lim_{m+n \to \infty} k_{m,n} = 0\) and \(\lim_{m+n \to \infty} l_{m,n} = 0\).

**Remark.** For the analogue of Theorem 6 in the classical case, see [5], Theorem III. Theorem 5, Theorem 6, in the case of regular Nörlund means for simple sequences, were established earlier by Natarajan (see [7], Theorem 3, Theorem 4).

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**References**


