Annales mathématiques Blaise Pascal

P.N. Natarajan

Some properties of the Y-method of summability in complete ultrametric fields

Annales mathématiques Blaise Pascal, tome 9, nº 1 (2002), p. 79-84
<http://www.numdam.org/item?id=AMBP_2002__9_1_79_0>

ANNALES MATHEMATIQUES BLAISE PASCAL **9**, 79-84 (2002)

Some properties of the Y- method of summability in complete ultrametric fields

P. N. Natarajan

Abstract

In this paper, a few results regarding the Y-method of summability in complete ultrametric fields are proved.

Let K be a complete ultrametric field. Throughout the present paper, infinite matrices, sequences and series have entries in K. Given an infinite matrix $A = (\alpha_i^j), i, j = 0, 1, 2, \cdots$ and a sequence $\{u_j\}, j = 0, 1, 2, \cdots$, by the A-transform of $\{u_j\}$, we mean the sequence $\{v_i\}$,

$$v_i = \sum_{j=0}^{\infty} \alpha_i^j u_j, i = 0, 1, 2, \cdots,$$

where it is assumed that the series on the right converge. If $\lim_{i \to \infty} v_i = s$, we say that the sequence $\{u_j\}$ is A-summable to s.

The Y-method of summability in K is defined as follows: the Y-method is given by the infinite matrix $Y = (\alpha_i^j)$, where

$$\alpha_i^j = \lambda_{i-j},$$

$\{\lambda_n\}$ being a bounded sequence in K. Srinivasan's method [4] is a particular case with $K = \mathbb{Q}_p$, the p-adic field for a prime p, $\lambda_0 = \lambda_1 = \frac{1}{2}, \lambda_n = 0, n > 1$.

We shall prove a few results about the Y-method using properties of analytic functions (a general reference in this direction is [2]).

Let U be the closed unit disk in K and let $H(U)$ be the set of all power series converging in U, with coefficients in K. Let $h(x) = \sum_{n=0}^{\infty} u_n x^n$ and

AMS subject classification: 40

Keywords: complete ultrametric field, Y- method of summability, p-adic field, power series, analytic functions, summability factors.

$l(x) = \sum_{n=0}^{\infty} v_n x^n$. The following result is easily proved.

Lemma 1. *The sequence $\{u_n\}$ is Y- summable to s if and only if the function l is of the form*

$$l(x) = \frac{s}{1-x} + \psi(x),$$

where $\psi \in H(U)$. We now have

Lemma 2. *Let $\phi(x) = \sum_{n=0}^{\infty} \lambda_n x^n$. The Y-transform $\{v_n\}$ of the sequence $\{u_n\}$ satisfies*

$$l(x) = \phi(x)h(x),$$

i.e., The Y-transform $\{v_n\}$ of $\{u_n\}$ is the convolution product of $\{u_n\}$ and $\{\lambda_n\}$.

Most of the theorems that are proved in the sequel use the following basic Lemma which is true in any complete ultrametric field and which follows as a corollary of the Hensel Lemma.

Lemma 3. *Let $h \in H(U)$ and $a \in U$ such that $h(a) = 0$. Then there exists $t \in H(U)$ such that*

$$h(x) = (x-a)t(x).$$

We now prove the main results of the paper.

Theorem 1. *If $\{a_n\}$ is Y-summable to $0, \{b_n\}$ is Y-summable to B, then $\{c_n\}$ is Y-summable to $B\left(\sum_{n=0}^{\infty} a_n\right)$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}, n = 0, 1, 2, \cdots$, i.e., $\{c_n\}$ is the convolution product of $\{a_n\}$ and $\{b_n\}$.*

Proof. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$. Then $\phi(x)f(x) \in H(U)$ and $\phi(x)g(x) = \frac{B}{1-x} + \theta(x)$, where $\theta \in H(U)$. Consequently the convolution product $\{c_n\}$ of the sequences $\{a_n\}$ and $\{b_n\}$ satisfies:

$$\begin{aligned}
\phi(x)\sum_{n=0}^{\infty} c_n x^n &= (\phi(x)g(x))f(x) \\
&= \left(\frac{B}{1-x} + \theta(x)\right) f(x) \\
&= \left(\frac{B}{1-x} + \theta(x)\right) \{f(1) + (f(x) - f(1))\}
\end{aligned}$$

$$= \frac{Bf(1)}{1-x} + \frac{B(f(x)-f(1))}{1-x} + \theta(x)f(x).$$

In view of Lemma 3, $\frac{f(x)-f(1)}{1-x} \in H(U)$. So

$$\phi(x)\sum_{n=0}^{\infty} c_n x^n = \frac{Bf(1)}{1-x} + \gamma(x),$$

where $\gamma \in H(U)$. Using Lemma 1, the result follows.

Theorem 2. *Let K be a complete ultrametric field of characteristic $\neq 2$. Let $\lambda_0 = \lambda_1 = \frac{1}{2}, \lambda_n = 0, n > 1$. If $\{a_n\}$ is Y-summable to $A, \{b_n\}$ is Y-summable to B, then*

$$\lim_{n\to\infty} (\gamma_{n+2} - \gamma_n) = 2AB,$$

where $\{\gamma_n\}$ is the Y-transform of $\{c_n\}$.

Proof. Let us retain the same notations regarding f, g. Let $F(x) = \phi(x)g(x)f(x)$. Again $\phi(x)g(x) = \frac{B}{1-x} + \theta(x), \phi(x)f(x) = \frac{A}{1-x} + \xi(x)$, where $\theta, \xi \in H(U)$. Hence

$$\phi^2(x)f(x)g(x) = \frac{AB}{(1-x)^2} + \frac{A\theta(x) + B\xi(x)}{1-x} + \xi(x)\theta(x).$$

On the other hand, let $h(x) = \sum_{n=0}^{\infty} \gamma_n x^n$. Then $h(x) = \phi(x)f(x)g(x)$ so that $\phi^2(x)f(x)g(x) = \phi(x)h(x)$ and consequently

$$\phi(x)h(x) = \frac{AB}{(1-x)^2} + \frac{\omega(x)}{1-x},$$

where $\omega \in H(U)$. Now,

$$(1-x)\phi(x)h(x) = \frac{AB}{1-x} + \omega(x).$$

Since $\lambda_0 = \lambda_1 = \frac{1}{2}$, $\lambda_n = 0, n > 1, \phi(x) = \frac{1+x}{2}$ and so

$$(1-x)\left(\frac{1+x}{2}\right)h(x) = \frac{AB}{1-x} + \omega(x)$$

i.e.,
$$\left(\frac{1-x^2}{2}\right)h(x) = \frac{AB}{1-x} + \omega(x)$$

i.e.,
$$\sum_{n=0}^{\infty}\left(\frac{\gamma_n - \gamma_{n-2}}{2}\right)x^n = \frac{AB}{1-x} + \omega(x).$$

Now the result follows using Lemma 1.

We now return back to the general case when $\{\lambda_n\}$ is a bounded sequence in any complete ultrametric field K and $\alpha_i^j = \lambda_{i-j}, i, j = 0, 1, 2, \cdots$.

Definition. The series $\sum_{k=0}^{\infty} a_k$ is said to be Y-summable to l if $\{s_n\}$ is Y-summable to l, where $s_n = \sum_{k=0}^{n} a_k, n = 0, 1, 2, \cdots$.

We now have

Theorem 3. *Suppose* $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^{\infty} a_n = l$. *Let* $\sum_{n=0}^{\infty} b_n$ *be* Y*-summable to* m. *Then* $\sum_{n=0}^{\infty} c_n$ *is* Y*-summable to* lm.

Proof. Let $t_n = \sum_{k=0}^{n} b_k, w_n = \sum_{k=0}^{n} c_k, n = 0, 1, 2, \cdots$. Let f, g have the same meaning as in the preceding theorems. We notice that

$$\sum_{n=0}^{\infty} t_n x^n = g(x)\left(\sum_{n=0}^{\infty} x^n\right) = \frac{g(x)}{1-x}.$$

Since $\{t_n\}$ is Y-summable to m, we have,

$$\frac{\phi(x)g(x)}{1-x} = \frac{m}{1-x} + \psi(x),$$

where $\psi \in H(U)$. Hence

$$\begin{aligned}\frac{\phi(x)f(x)g(x)}{1-x} &= m\frac{f(x)-f(1)}{1-x} + \frac{mf(1)}{1-x} + \psi(x)\\ &= \frac{mf(1)}{1-x} + \theta(x),\end{aligned}$$

where $\theta \in H(U)$ (this is so because $\frac{f(x)-f(1)}{1-x} \in H(U)$) and $f(1) = l$. The proof is now complete.

Remark 1. In the classical case, we have the following result: If $\sum_{n=0}^{\infty} |a_n| < \infty$ and $\sum_{n=0}^{\infty} a_n = l, \sum_{n=0}^{\infty} b_n$ is Y-summable to m, then $\sum_{n=0}^{\infty} c_n$ is Y-summable to

lm. Theorem 3 thus gives yet another instance where absolute convergence in classical analysis is effectively replaced by ordinary convergence in non-archimedean analysis.

In the context of summability factors (For the definition of summability factors or convergence factors, see, for instance, [3], pp.38-39), the following result about the $Y-$ method is interesting.

Theorem 4. *Let* $\lim_{n\to\infty} \lambda_n = 0$. *If* $\sum_{n=0}^{\infty} a_n$ *is* Y*- summable and* $\{b_n\}$ *converges, then* $\sum_{n=0}^{\infty} a_n b_n$ *is* Y*-summable.*

Proof. Let $s_n = \sum_{k=0}^{n} a_k, n = 0, 1, 2, \cdots, \{s_n\}$ be Y- summable to s, $\lim_{n\to\infty} b_n = m$. Let $b_n = m + \epsilon_n$ so that $\lim_{n\to\infty} \epsilon_n = 0$. Since $\lim_{n\to\infty} \lambda_n = 0, \phi \in H(U)$. Since $\{s_n\}$ is Y- summable to s, we have,

$$\frac{\phi(x)f(x)}{1-x} = \frac{s}{1-x} + \psi(x),$$

where $\psi \in H(U)$. Now,

$$\frac{\phi(x)\sum_{n=0}^{\infty} a_n b_n x^n}{1-x} = \frac{m\phi(x)f(x)}{1-x} + \frac{\phi(x)\theta(x)}{1-x},$$

where $\theta(x) = \sum_{n=0}^{\infty} \epsilon_n x^n$ and $\theta \in H(U)$. Consequently

$$\begin{aligned} \phi(x)\sum_{n=0}^{\infty} a_n b_n x^n &= \frac{ms}{1-x} + \psi(x) + \frac{\phi(x)\theta(x)}{1-x} \\ &= \frac{ms + \phi(1)\theta(1)}{1-x} + \omega(x), \end{aligned}$$

where $\omega \in H(U)$ so that $\sum_{n=0}^{\infty} a_n b_n$ is Y-summable to $ms+\phi(1)\theta(1)$, completing the proof of the theorem.

Acknowledgement. The author thanks the referee very much for pointing out the remarkable connection between the Y-method and the properties of analytic functions.

References

[1] G. Bachman, Introduction to $p - adic$ numbers and valuation theory. Academic Press, 1964.

[2] A. Escassut, Analytic elements in p-adic Analysis. World Scientific Publishing Co., 1995.

[3] Peyerimhoff, Lectures on summability, Springer,1969.

[4] V.K. Srinivasan, On certain summation processes in the p-adic field. *Indag. Math.* **27**(1965), 319-325.

P. N. NATARAJAN
DEPARTMENT OF MATHEMATICS
RAMAKRISHNA MISSION
VIVEKANANDA COLLEGE
CHENNAI 600 004
INDIA