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Abstract

In this paper, a few results regarding the $Y$-method of summability in complete ultrametric fields are proved.

Let $K$ be a complete ultrametric field. Throughout the present paper, infinite matrices, sequences and series have entries in $K$. Given an infinite matrix $A = (\alpha^i_j), i, j = 0, 1, 2, \cdots$ and a sequence $\{u_j\}, j = 0, 1, 2, \cdots$, by the $A$-transform of $\{u_j\}$, we mean the sequence $\{v_i\}$,

$$v_i = \sum_{j=0}^{\infty} \alpha^i_j u_j, \; i = 0, 1, 2, \cdots,$$

where it is assumed that the series on the right converge. If $\lim_{i \to \infty} v_i = s$, we say that the sequence $\{u_j\}$ is $A$-summable to $s$.

The $Y$-method of summability in $K$ is defined as follows: the $Y$-method is given by the infinite matrix $Y = (\alpha^i_j)$, where

$$\alpha^i_j = \lambda_{i-j},$$

$\{\lambda_n\}$ being a bounded sequence in $K$. Srinivasan’s method [4] is a particular case with $K = \mathbb{Q}_p$, the $p$-adic field for a prime $p$, $\lambda_0 = \lambda_1 = \frac{1}{2}, \lambda_n = 0, n > 1$.

We shall prove a few results about the $Y$-method using properties of analytic functions (a general reference in this direction is [2]).

Let $U$ be the closed unit disk in $K$ and let $H(U)$ be the set of all power series converging in $U$, with coefficients in $K$. Let $h(x) = \sum_{n=0}^{\infty} u_n x^n$ and

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\[ l(x) = \sum_{n=0}^{\infty} v_n x^n. \] The following result is easily proved.

**Lemma 1.** The sequence \( \{u_n\} \) is \( Y \)-summable to \( s \) if and only if the function \( l \) is of the form

\[ l(x) = \frac{s}{1 - x} + \psi(x), \]

where \( \psi \in H(U) \). We now have

**Lemma 2.** Let \( \phi(x) = \sum_{n=0}^{\infty} \lambda_n x^n \). The \( Y \)-transform \( \{v_n\} \) of the sequence \( \{u_n\} \) satisfies

\[ l(x) = \phi(x)h(x), \]

i.e., The \( Y \)-transform \( \{v_n\} \) of \( \{u_n\} \) is the convolution product of \( \{u_n\} \) and \( \{\lambda_n\} \).

Most of the theorems that are proved in the sequel use the following basic Lemma which is true in any complete ultrametric field and which follows as a corollary of the Hensel Lemma.

**Lemma 3.** Let \( h \in H(U) \) and \( a \in U \) such that \( h(a) = 0 \). Then there exists \( t \in H(U) \) such that

\[ h(x) = (x - a)t(x). \]

We now prove the main results of the paper.

**Theorem 1.** If \( \{a_n\} \) is \( Y \)-summable to 0, \( \{b_n\} \) is \( Y \)-summable to \( B \), then

\( \{c_n\} \) is \( Y \)-summable to \( B \left( \sum_{n=0}^{\infty} a_n \right) \), where \( c_n = \sum_{k=0}^{n} a_k b_{n-k}, n = 0, 1, 2, \ldots \),

i.e., \( \{c_n\} \) is the convolution product of \( \{a_n\} \) and \( \{b_n\} \).

**Proof.** Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( g(x) = \sum_{n=0}^{\infty} b_n x^n \). Then \( \phi(x)f(x) \in H(U) \) and \( \phi(x)g(x) = \frac{B}{1 - x} + \theta(x) \), where \( \theta \in H(U) \). Consequently the convolution product \( \{c_n\} \) of the sequences \( \{a_n\} \) and \( \{b_n\} \) satisfies:

\[
\sum_{n=0}^{\infty} c_n x^n = (\phi(x)g(x))f(x)
\]

\[
= \left( \frac{B}{1 - x} + \theta(x) \right) f(x)
\]

\[
= \left( \frac{B}{1 - x} + \theta(x) \right) \{f(1) + (f(x) - f(1))\}
\]
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\[ \frac{Bf(1)}{1-x} + \frac{B(f(x) - f(1))}{1-x} + \theta(x)f(x). \]

In view of Lemma 3, \( f(x) - \frac{f(1)}{1-x} \in H(U) \). So

\[ \phi(x) \sum_{n=0}^{\infty} c_n x^n = \frac{Bf(1)}{1-x} + \gamma(x), \]

where \( \gamma \in H(U) \). Using Lemma 1, the result follows.

**Theorem 2.** Let \( K \) be a complete ultrametric field of characteristic \( \neq 2 \). Let \( \lambda_0 = \lambda_1 = \frac{1}{2}, \lambda_n = 0, n > 1 \). If \( \{a_n\} \) is \( Y \)-summable to \( A \), \( \{b_n\} \) is \( Y \)-summable to \( B \), then

\[ \lim_{n \to \infty} (\gamma_{n+2} - \gamma_n) = 2AB, \]

where \( \{\gamma_n\} \) is the \( Y \)-transform of \( \{c_n\} \).

**Proof.** Let us retain the same notations regarding \( f, g \). Let \( F(x) = \phi(x)g(x)f(x) \).

Again \( \phi(x)g(x) = \frac{B}{1-x} + \theta(x), \phi(x)f(x) = \frac{A}{1-x} + \xi(x) \), where \( \theta, \xi \in H(U) \).

Hence

\[ \phi^2(x)f(x)g(x) = \frac{AB}{(1-x)^2} + \frac{A\theta(x) + B\xi(x)}{1-x} + \xi(x)\theta(x). \]

On the other hand, let \( h(x) = \sum_{n=0}^{\infty} \gamma_n x^n \). Then \( h(x) = \phi(x)f(x)g(x) \) so that

\[ \phi^2(x)f(x)g(x) = \phi(x)h(x) \]

and consequently

\[ \phi(x)h(x) = \frac{AB}{(1-x)^2} + \frac{\omega(x)}{1-x}, \]

where \( \omega \in H(U) \). Now,

\[ (1-x)\phi(x)h(x) = \frac{AB}{1-x} + \omega(x). \]

Since \( \lambda_0 = \lambda_1 = \frac{1}{2}, \lambda_n = 0, n > 1, \phi(x) = \frac{1+x}{2} \) and so

\[ (1-x)\left( \frac{1+x}{2} \right) h(x) = \frac{AB}{1-x} + \omega(x) \]

i.e.,

\[ \left( \frac{1-x^2}{2} \right) h(x) = \frac{AB}{1-x} + \omega(x) \]

i.e.,

\[ \sum_{n=0}^{\infty} \left( \frac{\gamma_n - \gamma_{n-2}}{2} \right) x^n = \frac{AB}{1-x} + \omega(x). \]
Now the result follows using Lemma 1.

We now return back to the general case when \( \{ \lambda_n \} \) is a bounded sequence in any complete ultrametric field \( K \) and \( \alpha^j_i = \lambda_{i-j}, i, j = 0, 1, 2, \ldots \).

**Definition.** The series \( \sum_{k=0}^{\infty} a_k \) is said to be \( Y \)-summable to \( l \) if \( \{ s_n \} \) is \( Y \)-summable to \( l \), where \( s_n = \sum_{k=0}^{n} a_k, n = 0, 1, 2, \ldots \).

We now have

**Theorem 3.** Suppose \( \lim_{n \to \infty} a_n = 0 \) and \( \sum_{n=0}^{\infty} a_n = l \). Let \( \sum_{n=0}^{\infty} b_n \) be \( Y \)-summable to \( m \). Then \( \sum_{n=0}^{\infty} c_n \) is \( Y \)-summable to \( lm \).

**Proof.** Let \( t_n = \sum_{k=0}^{n} b_k, w_n = \sum_{k=0}^{n} c_k, n = 0, 1, 2, \ldots \). Let \( f, g \) have the same meaning as in the preceding theorems. We notice that

\[
\sum_{n=0}^{\infty} t_n x^n = g(x) \left( \sum_{n=0}^{\infty} x^n \right) = \frac{g(x)}{1 - x}.
\]

Since \( \{ t_n \} \) is \( Y \)-summable to \( m \), we have,

\[
\frac{\phi(x)g(x)}{1 - x} = \frac{m}{1 - x} + \psi(x),
\]

where \( \psi \in H(U) \). Hence

\[
\frac{\phi(x)f(x)g(x)}{1 - x} = m \frac{f(x) - f(1)}{1 - x} + \frac{mf(1)}{1 - x} + \psi(x) = \frac{mf(1)}{1 - x} + \theta(x),
\]

where \( \theta \in H(U) \) (this is so because \( \frac{f(x) - f(1)}{1 - x} \in H(U) \)) and \( f(1) = l \). The proof is now complete.

**Remark 1.** In the classical case, we have the following result: If \( \sum_{n=0}^{\infty} |a_n| < \infty \) and \( \sum_{n=0}^{\infty} a_n = l, \sum_{n=0}^{\infty} b_n \) is \( Y \)-summable to \( m \), then \( \sum_{n=0}^{\infty} c_n \) is \( Y \)-summable to
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Theorem 3 thus gives yet another instance where absolute convergence in classical analysis is effectively replaced by ordinary convergence in non-archimedean analysis.

In the context of summability factors (For the definition of summability factors or convergence factors, see, for instance, [3], pp.38-39), the following result about the $Y-$ method is interesting.

**Theorem 4.** Let $\lim_{n \to \infty} \lambda_n = 0$. If $\sum_{n=0}^{\infty} a_n$ is $Y$- summable and $\{b_n\}$ converges, then $\sum_{n=0}^{\infty} a_nb_n$ is $Y$-summable.

**Proof.** Let $s_n = \sum_{k=0}^{n} a_k, n = 0, 1, 2, \ldots, \{s_n\}$ be $Y$- summable to $s$, $\lim_{n \to \infty} b_n = m$.

Let $b_n = m + \epsilon_n$ so that $\lim_{n \to \infty} \epsilon_n = 0$. Since $\lim_{n \to \infty} \lambda_n = 0, \phi \in H(U)$. Since $\{s_n\}$ is $Y$- summable to $s$, we have,

$$\frac{\phi(x)f(x)}{1-x} = \frac{s}{1-x} + \psi(x),$$

where $\psi \in H(U)$. Now,

$$\frac{\phi(x)\sum_{n=0}^{\infty} a_nb_nx^n}{1-x} = \frac{m\phi(x)f(x)}{1-x} + \frac{\phi(x)\theta(x)}{1-x},$$

where $\theta(x) = \sum_{n=0}^{\infty} \epsilon_nx^n$ and $\theta \in H(U)$. Consequently

$$\phi(x)\sum_{n=0}^{\infty} a_nb_nx^n = \frac{ms}{1-x} + \psi(x) + \frac{\phi(x)\theta(x)}{1-x}$$

$$= \frac{ms + \phi(1)\theta(1)}{1-x} + \omega(x),$$

where $\omega \in H(U)$ so that $\sum_{n=0}^{\infty} a_nb_n$ is $Y$-summable to $ms + \phi(1)\theta(1)$, completing the proof of the theorem.

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References


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