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THE PERIOD FUNCTION NEAR A POLYCYCLE
WITH TWO SEMI-HYPERBOLIC VERTICES

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ABSTRACT. Let $P$ be a polycycle of an analytic vector field on an open subset of the plane $\mathbb{R}^2$. Suppose that $P$ is the union of two semi-hyperbolic singular points (vertices of $P$) connected by two trajectories (sides of $P$). Assume that one side is part of the center manifold of each vertex. Denote by $L$ the other side. Assume also that $P$ is a boundary component of an annulus of periodic orbits. Let $\Sigma$ be a Poincaré section at the polycycle that intersects $L$. We show that the period function defined on $\Sigma$ has a principal part of the form $kx^{-n}$, $k > 0$, $n \in \mathbb{N}$.

1. INTRODUCTION

Consider an analytic ordinary differential equation $E$ on an open subset of the plane $\mathbb{R}^2$. Suppose that $E$ has an annulus of periodic orbits, not necessarily bounded. It is known that a boundary component union of such an annulus is a polycycle $P$ (cf. [P]); that is a finite connected union of singularities (vertices of $P$) and integral curves (sides of $P$) of $E$. A unique singular point may be considered as a polycycle. Let $\Sigma$ be a small Poincaré section at the polycycle with a local coordinate $s$ whose origin lies at the polycycle. The integral curve of $E$ that passes through a point of $\Sigma$ is a periodic orbit. The period function assigns to $s$ the (minimum) period $T(s)$ of the corresponding periodic solution.

We are interested in the qualitative behavior of $T$, mainly in the asymptotic expansion of $T$ and of its derivative, for small argument. The fact that a polycycle has a period function with an asymptotic expansion, and also the expansion itself, are problems of general interest. Also, the oscillatory character of $T$ is of interest to us. We say that a function is oscillatory if the set of its critical points has accumulation points. The derivative of an oscillatory function either does not have an asymptotic expansion or it has an asymptotic expansion identically zero.

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The behavior of $T$ and of its derivative, as its argument approach zero, depends on whether the polycycle is bounded or not, and also on the type of its vertices. We would like to know how the analytic local invariants of these singular points intervene in the behavior of such a function.

The period function $T$ is analytic at every strictly positive coordinate $s$ and it is analytic at the origin only if the polycycle is a nondegenerate center. Generally, both $T$ and its derivative grow without bound when their arguments approach zero. Bounded polycycles have nonoscillatory period functions, [C-D]. On the other hand, in [S, Sa] it is proved that if the vertices of a polycycle (bounded or not) are formally linearizable after desingularization, then $T$ and its derivative have asymptotic expansions in $\{s^\mu\}$ and $\{s^\delta \log s\}$, $\mu, \delta \in \mathbb{R}$. A consequence if such a polycycle is unbounded and has a finite vertex then it has a nonoscillatory period function.

In this work we are interested in a class of polycycles with semi-hyperbolic vertices. We determine the principal part of the period function for bounded polycycles in this class.

More precisely, we consider polycycles with two semi-hyperbolic singular points as vertices with a side that lies in the center manifold of both vertices. We call such a side the center side of the polycycle. The other side is called hyperbolic side. We prove that the period function defined on a transversal section through the hyperbolic side is of the form

$$T(s) = k^{-n}(1 + o(s)), \quad k > 0, \quad n \in \mathbb{N}.$$ 

To prove this we decompose the period function in local time functions through the polycycle sides and the saddle sectors. The period function is the sum of each of these local time functions composed on the right with an appropriate transition map.

2. PRINCIPAL PART OF $T$

Consider the analytic differential equation $E : \frac{dx}{dt} = A(x, y), \frac{dy}{dt} = B(x, y)$ on an open subset of the plane. A polycycle $P$ of $E$ is a connected union consisting of a finite number of singularities of $E$ (vertices of $P$) and the integral curves of $E$ (sides of $P$) such that a unilateral return map $R$ exists, that is, there is an analytic curve

$$\gamma : [0, 1] \to \Sigma \subset \mathbb{R}^2, \quad \gamma(0) \in P$$

transverse to $P$ such that the integral curve through $\gamma(s)$ intersects $\Sigma$ again for the first time at $\gamma(R(s))$ for each sufficiently small $s$. 

The period function $T : ]0, \epsilon[ \rightarrow \mathbb{R}^+$, is defined when $R$ is the identity map, given by $s \mapsto T(s)$ as the period of the periodic orbit through $\gamma(s)$.

The function $T$ is analytic on $]0, \epsilon[$. But $T$ is not necessarily defined or analytic at $s = 0$, and $T(s)$ may converge to infinity as $s \rightarrow 0^+$.

Next, consider a polycycle $P$ with two vertices. Suppose that such vertices are semi-hyperbolic singular points of $E$ (that is, if $a$ is a vertex then the Jacobian matrix of $(A, B)$ at $a$ has one eigenvalue equal to zero and the other one different to zero). We suppose, moreover, that one side of $P$ (center side) is contained in the center manifold of each vertex (see Fig. 1). Therefore

Fig. 1

**Theorem.** Let $E$ be a Poincaré section at $P$ such that $E$ intersects the hyperbolic side. The period function $T$ defined on $\Sigma$ satisfies

$$T(s) = k^{-n}(1 + \epsilon(s)),$$

where $k > 0$, $n \in \mathbb{N}$ with $\epsilon(s) \rightarrow 0$ if $s \rightarrow 0^+$.

**Proof.** The orientation of the trajectories of $E$ defines a sense of direction of $P$. Let $a_1, a_2$ be the vertices of $P$ and let $L_1$ and $L_2$ be the hyperbolic side and the center side respectively, such that the end of $L_1$ and the start of $L_2$ is $a_1$ (see Fig. 2). We assume that $a_1$ is crossed by running first through $L_1$ and then through $L_2$. On a neighborhood of each vertex $a_i$ ($i = 1, 2$) of $P$, we choose two analytic semi-transversals

$$\gamma_i : [0, 1] \rightarrow \Sigma_i \subset \mathbb{R}^2, \quad \gamma_i(0) = p_i \in P,$$

and

$$\pi_i : [0, 1] \rightarrow \Pi_i \subset \mathbb{R}^2, \quad \pi_i(0) = q_i \in P,$$
where $p_1, p_2 \in L_1$ and $q_1, q_2 \in L_2$. We choose $\Sigma = \Sigma_1$.

Define the functions

$$g_1 : [0, \delta_1] \rightarrow \mathbb{R}^+ \text{ given by } s \mapsto g_1(s) \text{ and}$$

$$S : [0, \delta_1] \rightarrow \mathbb{R}^+ \text{ given by } s \mapsto S(s),$$

such that the positive semi-orbit of $E$ through $\gamma_1(s) (s \neq 0)$ intersects $\Pi_1 - \{q_1\}$ at $\pi_1(g_1(s))$ and $\Sigma_2 - \{p_2\}$ at $\gamma_2(S(s))$.

Define also the functions

$$\sigma_1 : [0, \delta_1] \rightarrow \mathbb{R}^+ \text{ given by } s \mapsto \sigma_1(s),$$

$$\tau_1 : [0, \epsilon_1] \rightarrow \mathbb{R}^+ \text{ given by } \kappa \mapsto \tau_1(\kappa)$$

and

$$\tau_2 : [0, \epsilon_2] \rightarrow \mathbb{R}^+ \text{ given by } \xi \mapsto \tau_2(\xi),$$

where, $\sigma_1(s)$ is the time required for the integral curve starting at $\gamma_1(s)$ to intersect the transversal $\Pi_1$ for the first time at $\pi_1(g_1(s))$, the number $\tau_1(\kappa)$ is the time required for the integral curve starting at $\pi_1(\kappa) \in \Pi_1$ to intersect, for the first time the transversal $\Pi_2$, and $\tau_2(\xi)$ is the time required for the integral curve starting at $\gamma_2(\xi) \in \Sigma_2$ to intersect, for the first time, the transversal $\Sigma_1$ at $\gamma_1(s)$.

Next, consider the equation

$$E^* : \frac{dx}{dt} = -A(x, y); \frac{dy}{dt} = -B(x, y)$$

and define the function $\sigma_2 : [0, \epsilon_2] \rightarrow \mathbb{R}^+ \xi \mapsto \sigma_2(\xi)$, where $\sigma_2(\xi)$ is the time required for the integral curve of $E^*$ starting at $\gamma_2(\xi) \in \Sigma_2$ to intersect the transversal $\Pi_2$ for the first time at $\pi_2(g_2(\xi))$. The functions $\sigma_i \ (i = 1, 2)$ are called the corner passage time functions relative to $\Sigma_i$ and $\Pi_i$. Therefore, the function $T$ is given by

$$T(s) = \sigma_1(s) + \tau_1(g_1(s)) + \sigma_2(S(s)) + \tau_2(S(s)) \quad (*)$$
The functions $\tau_1$ and $\tau_2$ are analytic at zero with $\tau_1(0) \neq 0$ and $\tau_2(0) \neq 0$. In effect, consider the function $\tau_1$. From the Flow Box Theorem, analytic coordinates $(u, v)$ exist on a neighborhood of the side $L_2$ of $P$ such that the axis $v = 0$ is the side $L_2$, while the semitransversals $\Pi_1$ and $\Pi_2$ are graphs of the analytic functions, $k$ and $l$, on $v \geq 0$ (the functions $k$ and $l$ are defined and analytic on a neighborhood of zero, and the points $(k(0), 0)$ and $(l(0), 0)$ correspond to $q_1$ and $q_2$ respectively). In these coordinates, $X$ (the vector field associated with the differential equation $E$) becomes

$$X = A_1(u, v) \frac{\partial}{\partial u},$$

where $A_1$ is a strictly positive analytic function. We obtain that the time required for the integral curve starting at $(k(v), v)$ to intersect $\Pi_2$ at $(l(v), v)$ is the integral

$$\tau(v) = \int_{k(v)}^{l(v)} \frac{1}{A_1(u, v)} \, du.$$

Thus $\tau$ is an analytic function with $\tau(0) \neq 0$. Since the coordinate change $\kappa \mapsto v$ is analytic and fixes zero, we obtain that $\tau_1$ is an analytic function on a neighborhood of zero with $\tau_1(0) \neq 0$. In the same way $\tau_2$ is an analytic function with $\tau_2(0) \neq 0$.

Since $g_1(0) = 0$ and $S(0) = 0$ (with $g_1$ and $S$ defined, at zero, by continuity), the principal parts of $\tau_1(g_1(s))$ and $\tau_2(S(s))$ are $\tau_1(0)$ and $\tau_2(0)$, respectively.

Now, from (*), it remains to evaluate the principal part of $\sigma_1(s) + \sigma_2(S(s))$. For that purpose we shall need the following propositions (to be proved in the appendix).

**Proposition 1.** There exist a positive integer $n$, such that
\[ \sigma_1(s) = k_1 s^{-n} + o(s^{-n}) \]
and
\[ \sigma_2(\xi) = k_2 \xi^{-n} + o(\xi^{-n}), \]
where \( k_1 \) and \( k_2 \) are strictly positive numbers.

**Proposition 2.** The function \( S \) has an asymptotic expansion in \( s^\delta \) and \( s^\mu (\log s)^m \), where \( \delta \) and \( \mu \) are strictly positive rational numbers and \( m \) is a strictly positive integer. More precisely, there exists a \( C^{\infty} \) function \( S_1 \) on a neighborhood of \( (0,0) \) such that \( S_1(0,0) = 0 \) and
\[ S(s) = c \cdot s \left( 1 + S_1(s, s^n \log s) \right), \]
where \( c > 0 \) with \( n \) is given by Proposition 1.

Hence, from (*) and the Propositions 1 and 2, it follows that
\[ T(s) = k_1 s^{-n} + \tau_1(0) + k_2 c^{-n} s^{-n} + \tau_2(0) + o(s^{-n}). \]
Since \( n > 0 \) we have
\[ T(s) = ks^{-n} + o(s^{-n}), \]
with \( k > 0 \). The last equality constitutes the main result of the present work. \( \blacksquare \)

**Remark.** If the transversal section \( \Sigma \) intersects the center side, the calculations are more complicated; in this case, we do not know how to calculate the function \( S \) defined on \( \Sigma \).
where \( f_1 \) is a strictly positive \( C^\infty \) function, with \( n_1 \in \mathbb{N} \) and \( \lambda_1 \in \mathbb{R} \) (cf. [M]). We can consider that \( \Sigma_1 \) and \( \Pi_1 \) are the segments \( 0 \leq x_1 < 1, \ y_1 = 1 \) and \( x_1 = 1, \ 0 \leq y_1 < 1 \), respectively.

In the coordinates \((x_2, y_2)\) the equation \( E^* \) (recall that \( E^* = -E \)) is given by

\[
E^* : \quad \frac{dx_2}{dt} = x_2^{n_2+1} f_2(x_2, y_2) ; \quad \frac{dy_2}{dt} = -y_2(n_2 + \lambda_2 x_2^n) f_2(x_2, y_2)
\]

where \( f_2 \) is a strictly positive \( C^\infty \) function, with \( n_2 \in \mathbb{N} \) and \( \lambda_2 \in \mathbb{R} \). We suppose again that \( \Sigma_2 \) and \( \Pi_2 \) are the segments \( 0 \leq x_2 < 1, \ y_2 = 1 \) and \( x_2 = 1, \ 0 \leq y_2 < 1 \), respectively.

1 Proof of Proposition 1. The definitions of \( \sigma_1 \) and \( \sigma_2 \) are similar, therefore we shall only consider the equation \( E \) and omit the subindex 1 of the coordinates \((x_1, y_1)\), \( n_1 \) and \( \lambda_1 \).

From the equation (**), we deduce that the corner passage time function, defined on the transversal \( 0 < x < 1, \ y = 1 \), is the line integral

\[
\sigma(x_1) = \int_{\gamma_{x_1}} \frac{1}{x^{n+1} f(x, y)} \, dx
\]

where \( \gamma_{x_1} \) is the orbit arc of \( E \) that joins \((x_1, 1)\) to the point \((1, y_1(x_1)) \in \Pi_1 \).

A first integral of equation (***) is

\[
I(x, y) = y \cdot x^n \exp(-\frac{1}{x^n}),
\]

hence \( \gamma_{x_1} \) is done by the equation \( y = \left( \frac{x_1}{x} \right)^\lambda \exp(\frac{1}{x^n} - \frac{1}{x_1^n}) \), \( x_1 \leq x \leq 1 \). We obtain the equality

\[
\sigma(x_1) = \int_{x_1}^1 x^{-n-1} F(x, \left( \frac{x_1}{x} \right)^\lambda \exp(\frac{1}{x^n} - \frac{1}{x_1^n})) \, dx , \ 0 < x_1 < 1,
\]

where \( F = \frac{1}{x} \).

We know that two \( C^\infty \) functions \( F_1 \) and \( F_2 \) exist on \( \mathbb{R}^2 \) such that
F(x, y) = F(0, 0) + xF_1(x, y) + yF_2(x, y).

Therefore

\[ \sigma(x_1) = \frac{F(0, 0)}{n}x_1^{-n} - \frac{F(0, 0)}{n} + H_1(x_1) + H_2(x_1), \]

where

\[ H_1(x_1) = \int_{x_1}^{1} x^{-n}F_1(x, \frac{x_1}{x})\exp\left(\frac{1}{x^n} - \frac{1}{x_1^n}\right)dx \]

and

\[ H_2(x_1) = x_1^\lambda \exp\left(-\frac{1}{x_1^n}\right) \int_{x_1}^{1} x^{-n-\lambda-1} \exp\left(\frac{1}{x^n}\right)F_2(x, \frac{x_1}{x})\exp\left(\frac{1}{x^n} - \frac{1}{x_1^n}\right)dx. \]

Since the set

\[ \{(x, (\frac{x_1}{x})^\lambda \exp\left(\frac{1}{x^n} - \frac{1}{x_1^n}\right)) \mid x_1 \leq x \leq 1, x_1 \in ]0, 1[ \} \]

is contained in \([0, 1] \times [0, 1]\), a constant \(K > 0\) exists such that

\[ |F_1(x, (\frac{x_1}{x})^\lambda \exp\left(\frac{1}{x^n} - \frac{1}{x_1^n}\right))| \leq K \]

for all \(x_1 \in ]0, 1[\) and \(x \in [x_1, 1]\). Consequently

\[ |x_1^nH_1(x_1)| = |x_1^n\int_{x_1}^{1} x^{-n}F_1(x, (\frac{x_1}{x})^\lambda \exp\left(\frac{1}{x^n} - \frac{1}{x_1^n}\right))dx| \leq Kx_1^n\int_{x_1}^{1} x^{-n}dx. \]

We conclude that \(H_1(x_1) = o(x_1^{-n})\). By a similar calculation, we have \(H_2(x_1) = o(x_1^{-n})\).

Thus,

\[ \sigma(x_1) = \frac{F(0, 0)}{n}x_1^{-n} + o(x_1^{-n}). \]

Since the \(C^\infty\) coordinate change \(s \mapsto x_1\) fixes zero, we obtain that \(\sigma_1(s)\) satisfies the equality

\[ \sigma_1(s) = k_1s^{-n} + o(s^{-n}), \]
with \( k_1 > 0 \).
In the same way,
\[
\sigma_2(\xi) = k_2 \xi^{-n_2} + o(\xi^{-n_2}),
\]
with \( k_2 > 0 \). This proves Proposition 1. ■

2 Proof of Proposition 2. Let \( h(\kappa) \) be the number such that the integral curve through \( \pi_1(\kappa) \in \Pi_1 \) intersects the transversal \( \Pi_2 \) for the first time at \( \pi_2(h(\kappa)) \). The function \( h \) is a strictly increasing analytic function with \( h(0) = 0 \). Recall that the integral curve of \( E^* \) through \( \gamma_2(\xi) \in \Sigma_2 \) intersects the transversal \( \Pi_2 \) at \( \pi_2(g_2(\xi)) \). This defines the function \( \xi \mapsto g_2(\xi) \). Thus, the function \( S \) satisfies the equality

\[
S = g_2^{-1} \circ h \circ g_1.
\]

Next, consider the expression of the functions \( g_1, g_2 \) and \( h \) in the coordinates \( x_1, x_2, y_1 \) and \( y_2 \). That is,

\[
g_1 : x_1 \mapsto y_1, \quad g_2 : x_2 \mapsto y_2 \quad \text{and} \quad h : y_1 \mapsto y_2.
\]

Hence, the function \( S \) is given by

\[
S = g_2^{-1} \circ h \circ g_1 : x_1 \mapsto x_2.
\]

To find the expression \( x_2 = S(x_1) \), consider the equality

\[
g_2(x_2) = (h \circ g_1)(x_1).
\]

Recall that, on a neighborhood of \( a_1 \), a first integral of equation \( E \) is

\[
I(x_1, y_1) = y_1 \cdot x_1^{\lambda_1} \exp(-\frac{1}{x_1^{n_1}}).
\]

So that \( y_1 = g_1(x_1) \) is the solution of the equation \( I(1, y_1) = I(x_1, 1) \).

Thus, \( g_1(x_1) \) is given by

\[
g_1(x_1) = x_1^{\lambda_1} \exp(1 - \frac{1}{x_1^{n_1}}).
\]

Similarly, for \( g_2(x_2) \):

\[
g_2(x_2) = x_2^{\lambda_2} \exp(1 - \frac{1}{x_2^{n_2}}).
\]
Moreover, since $h : y_1 \mapsto y_2$ is a strictly increasing $C^\infty$ function with $h(0) = 0$ ($y_1$ and $y_2$ are $C^\infty$ coordinates of $\Pi_1$ and $\Pi_2$ respectively, and the coordinates change are strictly increasing), there exists a $C^\infty$ function $h_1$ such that $h_1(0) = 0$ and

$$h(y_1) = \beta y_1(1 + h_1(y_1)),$$

with $\beta > 0$.

Hence, the equation $g_2(x_2) = (h \circ g_1)(x_1)$ is equivalent to

$$x_2^\lambda \exp(1 - \frac{1}{x_2^\beta}) = \beta x_1^\lambda \exp(1 - \frac{1}{x_1^\beta})(1 + h_1(x_1^\lambda \exp(1 - \frac{1}{x_1^\beta})).$$

Applying the logarithmic function, we obtain

$$\lambda x_2 \log x_2 - \frac{1}{x_2^\beta} = \log \beta + \lambda x_1 \log x_1 - \frac{1}{x_1^\beta} + \epsilon(x_1)$$

where $\epsilon(x_1) = \log(1 + h_1(x_1^\lambda \exp(1 - \frac{1}{x_1^\beta})))$ is a $C^\infty$ function, which is flat at zero (that is, its Taylor series at zero is equal to zero).

Consider the variable $z$ through

$$x_2^\beta = \frac{x_1^\beta}{1 + z}.$$

From (1), we obtain

$$\lambda x_1 x_2^\beta \log x_1 - \lambda x_2^\beta \log(1 + z) - n_2z = n_2 x_1^\beta \log \beta + \lambda x_2^\beta \log x_1 + n_2 x_1^\beta \epsilon(x_1).$$

Next, put $w = x_1^\beta \log x_1$ and consider the function $G$ defined as

$$G(x, w, z) = \lambda x_1 w - \lambda x_2^\beta \log(1 + z) - n_2z - n_2 x_1^\beta \log \beta - \lambda x_2^\beta w - n_2 x_1^\beta \epsilon(x_1).$$

The function $G$ is $C^\infty$ on a neighborhood of the origin $(x_1, w, z) = (0, 0, 0)$, with $G(0, 0, 0) = 0$ and $\frac{\partial G}{\partial z}(0, 0, 0) = -n_2 \neq 0$. Thus, from the Implicit Function Theorem, there exists a $C^\infty$ function $z = z(x_1, w)$, defined on a neighborhood of $(x_1, w) = (0, 0)$ such that $z(0, 0) = 0$ and $G(x_1, w, z(x_1, w)) = 0$. Therefore,
From this, it follows that the function \( S \) is given by

\[
S(x_1) = x_1^{n_2} (1 + v_1(x_1, x_1^{n_1} \log x_1))
\]  

where \( v_1 \) is a \( C^\infty \) function on a neighborhood of \((0,0)\) such that \( v_1(0, 0) = 0 \).

Now, the coordinate change \( s \mapsto x_1 \) is a \( C^\infty \) diffeomorphism on a neighborhood of zero and fixes zero, that is, \( x_1 = c_1 s(1 + b(s)) \), where \( c_1 > 0 \) and \( b \) is a \( C^\infty \) function with \( b(0) = 0 \). Hence, substituting this expression in (3), we obtain that the function \( S \) satisfies the equality

\[
S(s) = c \cdot s^{n_2} (1 + v(s, s^{n_1} \log s)),
\]

where \( c > 0 \) and \( v \) is a \( C^\infty \) function with \( v(0, 0) = 0 \).

Note that the return function \( R \) is the composition of the function \( S \) plus a \( C^\infty \) diffeomorphism from \( \Sigma_2 \) to \( \Sigma_1 \). Since \( R \) is the identity map it follows that \( n_1 = n_2 \) and therefore

\[
S(s) = c \cdot s (1 + v(s, s^n \log s)),
\]

where \( c > 0 \) and \( v \) is as above. This proves the Proposition 2 and the proof of the main result is now complete. \( \blacksquare \)

**Remark.** If the polycycle have an arbitrary number of vertices, it seems possible to find a principal part of the period function. The composition of the corner passage functions (\( g_1 \) and \( g_2 \) in the present work), involves the composition of exponential functions. Therefore, the asymptotic expansion of the return function (function \( S \) here) is not anymore in \( \{s^n\} \) and \( \{s^\delta \} \log s \). This case will be the object of another paper.

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