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# Some boundary optimal control problems related to a singular cost functional

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**Résumé :** Dans ce travail, nous étudions un problème de contrôle optimal dépendant d'un petit paramètre $\epsilon > 0$, où l'état est solution d'une équation de Laplace sur un ouvert borné régulier $\Omega$, de l'espace Euclidien à n dimensions, avec des conditions du type Robin sur la frontière de cet ouvert. Les ensembles de contrôles admissibles sont des sous-ensembles convexes fermés et bornés de l'espace de Hilbert des fonctions de carré sommable sur la frontière vérifiant certaines conditions. La fonction coût utilisée ici est singulière. Nous prouvons l'existence et unicité des contrôles optimaux et nous étudions leur convergence en moyenne quadratique ainsi que celle des états correspondants dans l'espace de Sobolev $H^1(\Omega)$ quand $\epsilon \longrightarrow 0$. Nous traitons aussi un autre problème de contrôle optimal dont la fonction coût est obtenue par perturbation de notre fonction coût précédente pour pouvoir étendre nos méthodes aux cas des ensembles de contrôles admissibles non bornés.

**Abstract :** We are concerned by a class of linear boundary optimal control systems associated to Laplace operator on a regular bounded domain in the $n$ dimensional Euclidean space obtained by perturbing a singular system. The sets of admissible controls are bounded and closed convex subsets of the Hilbert space of all square integrable functions on the boundary. The cost functional, used here, is singular. For these systems, we prove the existence of the (perturbed) states and optimal controls, and study their convergence under some natural conditions upon the sets of admissible controls and the decision function. We treat also another related optimal control problem associated to a modified cost functional in which we consider the case of unbounded sets of admissible controls.

**AMS (MOS) subject classification :** 49J20, 49J40, 93C10, and 93C20.

**Keywords** : Perturbed states. Linear boundary optimal control systems. Singular cost functionals.

## 1. Introduction

**1.1.** Let $\Omega$ be a connected and simply connected regular and bounded open subset of the Euclidean space $\mathbb{R}^n$ with a smooth boundary $\Gamma := \partial\Omega$. Let $H^1(\Omega)$ be the classical (real)

Sobolev space equipped with its usual inner product and associated norm $\|.\|_{H^1(\Omega)}$, and let $L^2(\Gamma)$ be the usual Lebesgue space of all (real) square integrable functions on $\Gamma$. For every $v \in L^2(\Gamma)$, we consider the following linear boundary system :

$$\begin{cases} \Delta y(v) = 0, \quad \text{on } \Omega, \\ \dfrac{\partial}{\partial \nu} y(v) = v, \quad \text{at } \Gamma = \partial\Omega, \\ y(v) \in H^1(\Omega), \end{cases} \qquad (P)(v)$$

where $\frac{\partial}{\partial \nu} y(v)$ is the normal derivative of $y(v)$. One can easily see that $(P)(v)$ has a solution if and only if $v \in L_0^2(\Gamma) := \{u \in L^2(\Gamma) : \int_\Gamma u \, d\gamma = 0\}$. In this case, each solution $y(v)$ of this system is given by $y(v) = y_0(v) + c$ where $c$ is a constant and $y_0(v)$ is the unique element of the Sobolev space $V := \{y \in H^1(\Omega) : \int_\Gamma y \, d\gamma = 0\}$ verifying $(P)(v)$. For this reason, we can say that $(P)(v)$ is a singular sytem.

**1.2.** Let $\mathcal{U}_{ad}$ be a closed convex subset of $L^2(\Gamma)$, we are concerned for each positive number $\epsilon > 0$ by finding $u_\epsilon \in \mathcal{U}_{ad}$ such that

$$J_\epsilon(u_\epsilon) = \min\{J_\epsilon(v); v \in \mathcal{U}_{ad}\}, \qquad (Q_\epsilon)$$

where

$$J_\epsilon(v) = \int_\Gamma (y_\epsilon(v) - h)^2 \, d\gamma$$

$h$ is a fixed (decision) function in $L^2(\Omega)$, and $y_\epsilon(v)$ is a solution of the following problem :

$$\begin{cases} \Delta y_\epsilon(v) = 0, \quad \text{on } \Omega, \\ \dfrac{\partial}{\partial \nu} y_\epsilon(v) + \epsilon y_\epsilon(v) = v, \quad \text{at } \Gamma = \partial\Omega, \\ y_\epsilon(v) \in H^1(\Omega). \end{cases} \qquad (P_\epsilon)(v)$$

There are two questions. The first one is when does such solution exist ?. The second question is how do behave the net of optimal controls $u_\epsilon$ and the corresponding net of states $y_\epsilon$ ?. The purpose of this work is to bring some responses to these questions.

**1.3.** This paper is organized as follows. In the next section, we establish some preliminary results. In the third section, we discuss the existence and uniqueness of optimal controls. The fourth section contains our main result (see theorem 4.3) where the problems of convergence are solved under some natural conditions upon the the decision function and the set $\mathcal{U}_{ad}$ of bounded admissible controls. In the fifth (and last) section we treat the case of unbounded sets of admissible controls for a cost functional $\mathcal{J}_\epsilon$ obtained from our singular cost functional $J_\epsilon$ by adding the term $\epsilon\|v\|^2$.

## 2. Some preliminary results

**2.1.** In all this paper we suppose that the boundary is smooth enough, and that at least $\Gamma = \partial\Omega \in C^2$. By using Lax-Milgram theorem, we see that for each $\epsilon > 0$ and for each $v \in L^2(\Gamma)$, there exists a unique solution $y_\epsilon(v) \in H^1(\Omega)$ to the system $(P_\epsilon)(v)$. The aim of the next proposition is to give some properties of the map $T_\epsilon : L^2(\Gamma) \longrightarrow L^2(\Gamma)$ which associates to every $v \in L^2(\Gamma)$ the trace (on $\Gamma$) of the solution $y_\epsilon(v)$.

**2.2 Proposition :** *For each $\epsilon > 0$, we have*

(2.2.1) *The mapping $T_\epsilon : L^2(\Gamma) \longrightarrow L^2(\Gamma)$, $\quad v \longrightarrow y_\epsilon(v)$ is linear and injective.*

(2.2.2) *The mapping $T_\epsilon : L^2(\Gamma) \longrightarrow L^2(\Gamma)$ is compact and has a dense range.*

**Proof :** a) The linearity of the map $T_\epsilon$ is evident. Let $v \in L^2(\Gamma)$ such that $T_\epsilon(v) = y_\epsilon(v) = 0$ on $\Gamma$. Then $y_\epsilon(v) \in H_0^1(\Omega)$, and by the variational formulation of the problem $(P_\epsilon(v))$, it verifies

$$\int_\Omega |\nabla y_\epsilon(v)|^2 \, d\omega + \epsilon \int_\Omega y_\epsilon(v)^2 \, d\omega = 0,$$

which gives $y_\epsilon(v) = 0$. Thus $T_\epsilon$ is injective.

b) Let us prove the density of the range $T_\epsilon(L^2(\Gamma))$ in $L^2(\Gamma)$. We recall that $C^2(\Gamma)$ (the set of functions such that their first and second derivatives are continuous on $\Gamma$) is dense in $L^2(\Gamma)$. So, for every $w \in L^2(\Gamma)$, we may find a sequence $(f_n)$ of elements in $C^2(\Gamma)$ converging to $w$ in the Hilbert space $L^2(\Gamma)$. Now, by a well known result (see [8], p. 223) the following Dirichlet problem :

$$\begin{cases} \Delta\psi_n = 0, & \text{on } \Omega, \\ \psi_n|_\Gamma = f_n, & \text{at } \Gamma = \partial\Omega, \end{cases} \qquad (D_n)$$

has a unique solution $\psi_n \in H^2(\Omega)$. For every integer $n$, we put $u_{n,\epsilon} := \epsilon\psi_n|_\Gamma + \frac{\partial}{\partial\nu}\psi_n|_\Gamma$. Then $u_{n,\epsilon}$ belongs to $L^2(\Gamma)$, and clearly, $\psi_n$ is the solution to the system $(P_\epsilon)(u_{n,\epsilon})$. Therefore the sequence $(f_n = T_\epsilon(\psi_n))_n$ converges to $w$ in $L^2(\Gamma)$. Thus, the map $T_\epsilon$ has a dense range. It remains to show the compacity of $T_\epsilon$. Let $B$ be a bounded subset of $L^2(\Gamma)$. Then $\{y_\epsilon(v) : v \in B\}$ is bounded in $H^1(\Omega)$. Therefore (see [8], Theorem 4, p. 143) the set of its traces on $\Gamma$ is conditionally compact in the Hilbert space $L^2(\Gamma)$. This proves our proposition. $\square$

Concerning the cost functional $J_\epsilon$, one has the following lemma.

**2.3 Lemma :** *For each $\epsilon > 0$, the map $J_\epsilon : L^2(\Gamma) \longrightarrow [0, +\infty[$, $\quad v \longrightarrow J_\epsilon(v)$ is strictly convex and weakly l.s.c. (i.e., lower semicontinuous) on $L^2(\Gamma)$.*

**Proof :** Since $T_\epsilon$ is injective, then $J_\epsilon$ is strictly convex. To show that $J_\epsilon$ is weakly l.s.c. on $L^2(\Gamma)$, one can use the compacity of $T_\epsilon$ and the fact that the norm in any Hilbert space is weakly l.s.c.. $\square$

## 3. On the existence of optimal controls

It is not true that optimal controls always exist. Indeed, there are some particular cases where no solution exists for the problem $(Q_\epsilon)$. Let us give an example of such cases.

**3.1 Example :** Take $\mathcal{U}_{ad} = L^2(\Gamma)$, and let $h$ (the decision function) be such that $h \in L^2(\Gamma) \setminus H^1(\Gamma)$. Then, according to proposition 2.1, we have

$$\min\{J_\epsilon(v); v \in L^2(\Gamma)\} = 0.$$

If an optimal control $u_\epsilon$ exists then we must have $h = y_\epsilon(u_\epsilon) \in H^1(\Gamma)$, a contradiction.

Next, we give some sufficient conditions where the optimal control exists and is unique. It is an interesting problem of finding necessary and sufficient conditions (holding on $\mathcal{U}_{ad}$ and the decision function $h$) ensuring the existence/uniqueness of the optimal control $u_\epsilon$.

**3.2 Proposition :** *We suppose that $\mathcal{U}_{ad}$ is a closed convex subset of $L^2(\Gamma)$, verifying one of the following assumptions :*

$(A_1)$ *$\mathcal{U}_{ad}$ is bounded (i.e. there exists a positive constant $M > 0$, such that $\|u\|_{L^2(\Gamma)} \leq M$, $\forall u \in \mathcal{U}_{ad}$.)*

$(A_2)$ *There exists a finite dimensional subspace $\mathcal{U}$ of $L^2(\Gamma)$, containing $\mathcal{U}_{ad}$.*

*Then, there exists a unique optimal control $u_\epsilon \in \mathcal{U}_{ad}$.*

**Proof :** The unicity of $u_\epsilon$ results from the fact that $J_\epsilon$ is strictly convex. Since the map $v \to J_\epsilon(v)$ is weakly lower semicontinuous on the space $\mathcal{U}_{ad}$, then the existence of $u_\epsilon$ is clear when $\mathcal{U}_{ad}$ is bounded. If $\mathcal{U}_{ad}$ is not bounded in $L^2(\Gamma)$ but verifying $(A_2)$, then by a classical result of J.L. Lions (see [6]) in order to prove the existence of optimal controls, it suffices to verify the following condition :

($\bullet$) For every sequence $(v_n)$ in $\mathcal{U}_{ad}$, such that $\|v_n\|_{L^2(\Gamma)} \longrightarrow +\infty$, then $J_\epsilon(v_n) \longrightarrow +\infty$, when $n \longrightarrow +\infty$.
Since $\mathcal{U}$ has finite dimension and $T_\epsilon$ is injective, then according to the closed graph theorem, we can find a positive constant $\delta_\epsilon > 0$, such that the following inequality holds true :

$$\delta_\epsilon \|v\|_{L^2(\Gamma)} \leq \|y_\epsilon(v)\|_{L^2(\Gamma)}, \ \forall v \in \mathcal{U}.$$

This completes the proof of our proposition. □

## 4. Convergence of $u_\epsilon$ and $y_\epsilon$ when $\mathcal{U}_{ad}$ is bounded

We start by establishing the following lemma.

**4.1 Lemma :** *Let $v \in L^2(\Gamma)$ and let $y_\epsilon(v)$ designate the unique solution to the system $(P_\epsilon)(v)$. Then the net $(y_\epsilon(v))_\epsilon$ is bounded in $H^1(\Omega)$ if and only if $v$ blongs to $L^2_0(\Gamma)$. In this case, $y_\epsilon(v) \in V = \{y \in H^1(\Omega) : \int_\Gamma y\, d\gamma = 0\}$ for all $\epsilon > 0$, and $(y_\epsilon(v))_\epsilon$ converges to $y_0(v)$ strongly in the Sobolev space $H^1(\Omega)$, where $y_0(v)$ is the unique element in $V$ satisfying the system $(P)(v)$.*

**Proof :** a) Let $v \in L^2(\Gamma)$. Then $y_\epsilon(v)$ is the unique element in $H^1(\Omega)$ verifying

$$\int_\Omega \nabla y_\epsilon(v) \nabla z\, d\omega + \epsilon \int_\Gamma y_\epsilon(v) z\, d\gamma = \int_\Gamma v z\, d\gamma, \quad \forall z \in H^1(\Omega). \tag{1}$$

We set $z = 1$ in (1) and obtain $\int_\Gamma v\, d\gamma = \epsilon \int_\Gamma y_\epsilon(v)\, d\gamma$. If $y_\epsilon(v)$ is bounded in $H^1(\Omega)$ then necessarily $\int_\Gamma v\, d\gamma = 0$.

b) Conversely, suppose that $v \in L^2_0(\Gamma)$. Then we obtain from (1) that $\int_\Gamma y_\epsilon(v)\, d\gamma = 0$. (i.e. that $y_\epsilon(v) \in V$.) By using (1) and the variational formulation of the system $(P)(v)$, we get

$$\int_\Omega |\nabla y_\epsilon(v) - y_0(v)|^2\, d\omega = -\epsilon \int_\Gamma y_\epsilon(v)[y_\epsilon(v) - y_0(v)]\, d\gamma, \tag{2}$$

from which we obtain

$$\||\nabla y_\epsilon(v) - y_0(v)|\|^2_{L^2(\Omega)} \leq \epsilon \|y_\epsilon(v)\|_{L^2(\Gamma)} \|y_\epsilon(v) - y_0(v)\|_{L^2(\Gamma)}. \tag{3}$$

We recall that $\|z\|_V := \||\nabla z|\|_{L^2(\Omega)}$ is a norm in $V$ equivalent to the restricted norm $\|.\|_{H^1(\Omega)}$ of $H^1(\Omega)$ to $V$. Now, according to the trace theorem, we can find a positive constant $\lambda > 0$ such that

$$\|y\|_{L^2(\Gamma)} \leq \lambda \|y\|_V, \quad \forall y \in V. \tag{4}$$

Therefore, we get

$$\|y_\epsilon(v) - y_0(v)\|_V \leq \epsilon \lambda^2 \|y_\epsilon(v)\|_V. \tag{5}$$

Using one more time the variational formulation of the system $(P_\epsilon)(v)$, we obtain

$$\|y_\epsilon(v)\|^2_V \leq \int_\Gamma v y_\epsilon(v)\, d\gamma \leq \|v\|_{L^2(\Gamma)} \|y_\epsilon(v)\|_{L^2(\Gamma)} \leq \lambda \|v\|_{L^2(\Gamma)} \|y_\epsilon(v)\|_V. \tag{6}$$

We conclude that we have proved the following inequality

$$\|y_\epsilon(v) - y_0(v)\|_V \leq \epsilon \lambda^3 \|v\|_{L^2(\Gamma)}. \quad \forall \epsilon > 0. \tag{7}$$

Wich implies that the net $(y_\epsilon(v))_\epsilon$ is bounded and converges strongly in $H^1(\Omega)$ to $y_0(v)$ when $\epsilon \longrightarrow 0$. □

**4.2 Notations and assumptions :** Before stating our main result, we need to introduce some notations and precise our assumptions. $\mathcal{U}_{ad}$ will be a bounded, closed and convex subset of $L^2(\Gamma)$ such that $\mathcal{K}_{ad} := \mathcal{U}_{ad} \cap L^2_0(\Gamma) \neq \emptyset$. For every $w \in \mathcal{K}_{ad}$, let $y_0(w)$ be the unique solution (belonging to $V$) to the system $(P)(w)$, and set

$$J_0(w) := \|y_0(w) - h\|^2_{L^2(\Gamma)}. \tag{8}$$

We denote by $u_*$ the unique element in $\mathcal{K}_{ad}$ such that

$$J_0(u_*) = \min \{J_0(w) : w \in \mathcal{K}_{ad}\}, \tag{9}$$

We set $\sigma := \frac{1}{|\Gamma|} \int_\Gamma h\, d\gamma$, where $|\Gamma|$ is the Lebesgue measure of the set $\Gamma$. With these considerations we have the following theorem.

**4.3 Theorem :** *Under the assumptions made in 4.2, we have*

(4.3.1) *If $\sigma = 0$ then the net $(u_\epsilon)_\epsilon$ of optimal controls converges weakly in $L^2(\Gamma)$ to $u_*$, and the net $(y_\epsilon(u_\epsilon))_\epsilon$ of states converges strongly in the space $H^1(\Omega)$ to $z = y_0(u_*)$.*

(4.3.2) *If $\sigma \in \mathcal{U}_{ad}$, then the net $(u_\epsilon)_\epsilon$ of optimal controls converges weakly in $L^2(\Gamma)$ to $u_*$, and the net $(y_\epsilon(u_\epsilon))_\epsilon$ of states converges strongly in the space $H^1(\Omega)$ to $z = y_0(u_*) + \sigma$.*

(4.3.3) *In the general case, there exists a subsequence (named $(u_\epsilon)_\epsilon$) of $(u_\epsilon)_\epsilon$ converging weakly in $L^2(\Gamma)$ to an element $u_* \in \mathcal{K}_{ad}$, such that the net $(y_\epsilon(u_\epsilon))_\epsilon$ of states converges* *strongly in the space $H^1(\Omega)$ to $z = y_0(u_\bullet) + \beta$, where $\beta \in [\sigma - |\sigma|, \sigma + |\sigma|]$. Furtheremore, one has the follwing inequalities :*

$$\|y_0(u_\bullet) + \beta - h\|^2_{L^2(\Gamma)} \leq J_0(u_*) \leq J_0(u_\bullet). \tag{10}$$

**Proof :** a) Since $\mathcal{U}_{ad}$ is bounded, then there exists a subsequence (called again $(u_\epsilon)_\epsilon$) converging weakly in $L^2(\Gamma)$ to a unique element $u_\bullet \in \mathcal{U}_{ad}$. For every $\epsilon > 0$, we have

$$0 \leq J_\epsilon(u_\epsilon) \leq J_\epsilon(u_*) = \int_\Gamma (y_\epsilon(u_*) - h)^2 \, d\gamma. \tag{11}$$

By Lemma 4.1, we know that $y_\epsilon(u_*)$ converges in $H^1(\Omega)$ to $y_0(u_*)$, then it is bounded in $L^2(\Gamma)$. Using (11), we deduce that the net $(y_\epsilon)_\epsilon$ of traces is bounded in $L^2(\Gamma)$.

b) To simplify the notations, we set $y_\epsilon(u_\epsilon) = y_\epsilon$. Then by using the variational formulation equivalent to the system $(P_\epsilon)(u_\epsilon)$, we get

$$\||\nabla y_\epsilon\||^2_{L^2(\Omega)} \leq \|u_\epsilon\|_{L^2(\Gamma)} \|y_\epsilon\|_{L^2(\Gamma)} \cdot \tag{12}$$

From (a) and (12), we conclude that the net $(y_\epsilon)_\epsilon$ of states is bounded in $H^1(\Omega)$. Therefore, it is converging weakly to an element say $z \in H^1(\Omega)$. It is easy to see that $z$ satisfies the following

$$\int_\Omega \nabla z \nabla q \, d\omega = \int_\Gamma u_\bullet q \, d\gamma, \quad \forall q \in H^1(\Omega). \tag{13}$$

which implies that $u_\bullet \in \mathcal{K}_{ad}$ and that $z$ is a solution of the system $(P)(u_\bullet)$. Hence, there exists a real constant $\beta$ such that $z = y_0(u_\bullet) + \beta$.

c) By using the compacity of the embedding of $H^1(\Omega)$ into $L^2(\Gamma)$ (see for example [8], p. 143), we can find a subsequence (called again $(y_\epsilon)_\epsilon$) which is converging strongly to $z$ in the space $L^2(\Gamma)$. According to the lower semicontinuity of the norm in $L^2(\Gamma)$, we obtain

$$\|y_0(u_\bullet) + \beta - h\|^2_{L^2(\Gamma)} \leq J_0(u_*) \leq J_0(u_\bullet), \tag{14}$$

from which, we obtain after some easy calculations,

$$|\Gamma| \beta[\beta - 2\sigma] \leq J_0(u_*) - J_0(u_\bullet) \leq 0. \tag{15}$$

The last inequalities imply that $\beta \in [\sigma - |\sigma|, \sigma + |\sigma|]$. Next we show the strong convergence of (the subsequence) $y_\epsilon$ to $z$ in $H^1(\Omega)$.

d) Indeed, by using the variational formulations for the systems $(P_\epsilon)(u_\epsilon)$ and $(P)(u_\bullet)$, we obtain the following equality

$$\int_\Omega [\nabla y_\epsilon - \nabla z] \nabla q \, d\omega = -\epsilon \int_\Gamma y_\epsilon q \, d\gamma + \int_\Gamma [u_\epsilon - u_\bullet] q \, d\gamma, \quad \forall q \in H^1(\Omega). \tag{16}$$

By setting $q = y_\epsilon - z$ in (16), and using Cauchy-Schwarz inequality, we get

$$|||\nabla y_\epsilon - \nabla z|||^2_{L^2(\Omega)} \leq \left[\epsilon \, \|u_\epsilon\|_{L^2(\Gamma)} + 2M\right] \|y_\epsilon - z\|_{L^2(\Gamma)}, \tag{17}$$

where $M > 0$ is a positive constant such that $\|u\|_{L^2(\Gamma)} \leq M$ for every $u \in \mathcal{U}_{ad}$. Therefore, (the subsequence) $y_\epsilon$ converges strongly to $z$ in $H^1(\Omega)$. This achieves the proof of (4.3.3).

e) If $\sigma = 0$, then we get from the inequalities (14) and (15), that $J_0(u_\bullet) \leq J_0(u_*) \leq J_0(u_\bullet)$, which implies that $u_* = u_\bullet$, and from the steps above, the assertion 4.3.1 is proved.

f) To complete the proof of our theorem, we suppose that $\sigma \in \mathcal{U}_{ad}$, and consider subsequences such that all the conclusions of the step d) hold true. For each $\epsilon \in ]0,1]$ we set $v_\epsilon := (1-\epsilon)u_\bullet + \epsilon\sigma$. Then $v_\epsilon \in \mathcal{U}_{ad}$ for all $\epsilon \in ]0,1]$ and satisfies $J_\epsilon(u_\epsilon) \leq J_\epsilon(v_\epsilon)$. It is easy to see that

$$J_\epsilon(v_\epsilon) = \|(1-\epsilon)y_0(u_\bullet) + \sigma - h\|^2_{L^2(\Gamma)}, \tag{18}$$

which converges to $\|y_0(u_\bullet) + \sigma - h\|^2_{L^2(\Gamma)}$, when $\epsilon \to 0$. On the other hand, we already know that $J_\epsilon(u_\epsilon)$ conveges to $\|y_0(u_\bullet) + \beta - h\|^2_{L^2(\Gamma)}$, when $\epsilon \to 0$. Thus we obtain

$$\|y_0(u_\bullet) + \beta - h\|^2_{L^2(\Gamma)} \leq \|y_0(u_\bullet) + \sigma - h\|^2_{L^2(\Gamma)}. \tag{19}$$

Since the map $t \to \|y_0(u_\bullet) + t - h\|^2_{L^2(\Gamma)}$ reaches its minimal value in the set $\mathbb{R}$ only for $t = \sigma$, therefore we must have $\beta = \sigma$. To achieve the proof of the assertion (4.3.2), let us set $w_\epsilon := (1-\epsilon)u_* + \epsilon\sigma$. Then $w_\epsilon \in \mathcal{U}_{ad}$ for all $\epsilon \in ]0,1]$ and satisfies $J_\epsilon(u_\epsilon) \leq J_\epsilon(w_\epsilon)$. Letting $\epsilon \longrightarrow 0$ in both members of this inequality, we get

$$\|y_0(u_\bullet) + \sigma - h\|^2_{L^2(\Gamma)} \leq \|y_0(u_*) + \sigma - h\|^2_{L^2(\Gamma)}. \tag{20}$$

From which we obtain after some computations that $J_0(u_\bullet) \leq J_0(u_*)$. Therefore, we must have $u_\bullet = u_*$. At the end, we use the previous steps to conclude that the assertion (4.3.2) holds true. Therefore, our theorem is completely proved. □

**4.4 Remark :** Let $\mathcal{U}_{ad}$ be a closed and convex subset of $L^2(\Gamma)$ such that $\mathcal{U}_{ad} \cap L^2_0(\Gamma) \neq \emptyset$, and consider the problem of finding

$$\min \left\{ J(u,y) : u \in \mathcal{K}_{ad} = \mathcal{U}_{ad} \cap L^2_0(\Gamma) \; ; \; y \in S(u) \right\},$$

where $S(u)$ is the set of solutions of $(P)(u)$ and $J(u,y) := \|y - h\|^2_{L^2(\Gamma)}$. Then, it is easy to see that this problem has a unique solution $(u_\infty, y_\infty) \in \mathcal{K}_{ad} \times S(u_\infty)$ given by $u_\infty = u_*$ and $y_\infty = y(u_*) + \sigma$. According to (4.3.2), if $\mathcal{U}_{ad}$ is bounded and if $\sigma \in \mathcal{U}_{ad}$ then $u_\epsilon$ converges weakly in $L^2(\Gamma)$ to $u_\infty$ and that $y_\epsilon$ converges strongly in $H^1(\Omega)$ to $y_\infty$. An interesting problem arises. It consists of determining the sets $\mathcal{U}_{ad}$ of admissible controls verifying the assumptions of 4.2, and satisfying (i) for each $\epsilon > 0$ there exists a unique optimal control $u_\epsilon$, and (ii) a convergence (in some sense) of $(u_\epsilon, y_\epsilon)$ to $(u_\infty, y_\infty)$ happens when $\epsilon \longrightarrow 0$.

## 5. A related perturbed problem

**5.1** In order to be able to consider the case of unbounded sets of admissible controls, we consider a new cost functional named $\mathcal{J}_\epsilon$ given for all $v \in L^2(\Gamma)$ by

$$\mathcal{J}_\epsilon(v) := J_\epsilon(v) + \epsilon\|v\|^2_{L^2(\Gamma)}.$$

Let $\mathcal{U}_{ad}$ be any arbitrary closed and convex subset of $L^2(\Gamma)$. Then, by a classical result of J.L. Lions (see [6]), for all $\epsilon > 0$ there exists an optimal problem $u_\epsilon$. Since $\mathcal{J}_\epsilon$ is strictly convex, this optimal control is unique. In general, the estimates are not easy to obtain, but if $\mathcal{U}_{ad}$ verifies some supplementary conditions, one could get some results. In this respect, we have the following theorem which generalizes our previous Theorem 4.3.

**5.2 Theorem :** *Let $\mathcal{U}_{ad}$ be a (possibly unbounded) closed convex subset of $L^2(\Gamma)$ satisfying,*
*(1) $\mathcal{K}_{ad} := \mathcal{U}_{ad} \cap L^2_0(\Gamma) \neq \emptyset$.*
*(2) $\pi_0(\mathcal{U}_{ad})$ is bounded in $L^2_0(\Gamma)$, where $\pi_0$ is the orthogonal projection on the hyperplane $L^2_0(\Gamma)$.*
*Then we have*
(5.2.1) *If $\sigma = 0$ then the net $(u_\epsilon)_\epsilon$ of optimal controls converges weakly in $L^2(\Gamma)$ to $u_*$, and the net $(y_\epsilon(u_\epsilon))_\epsilon$ of states converges strongly in the space $H^1(\Omega)$ to $z = y_0(u_*)$.*
(5.2.2) *If $\sigma \in \mathcal{U}_{ad}$, then the net $(u_\epsilon)_\epsilon$ of optimal controls converges weakly in $L^2(\Gamma)$ to $u_*$, and the net $(y_\epsilon(u_\epsilon))_\epsilon$ of states converges strongly in the space $H^1(\Omega)$ to $z = y_0(u_*) + \sigma$.*
(5.2.3) *In the general case, there exists a subsequence (named $(u_\epsilon)_\epsilon$) of $(u_\epsilon)_\epsilon$ converging weakly in $L^2(\Gamma)$ to an element $u_\bullet \in \mathcal{K}_{ad}$, such that the net $(y_\epsilon(u_\epsilon))_\epsilon$ of states converges strongly in the space $H^1(\Omega)$ to $z = y_0(u_\bullet) + \beta$, where $\beta \in [\sigma - |\sigma|, \sigma + |\sigma|]$. Furtheremore, one has the follwing inequalities :*

$$\|y_0(u_\bullet) + \beta - h\|^2_{L^2(\Gamma)} \leq J_0(u_*) \leq J_0(u_\bullet). \tag{10}$$

**Proof :** All we need to prove is the boundedness of the set of optimal controls. For every $\epsilon > 0$ we write $u_\epsilon = u^0_\epsilon + u^1_\epsilon$, where $u^1_\epsilon := \frac{1}{|\Gamma|}\int_\Gamma u_\epsilon\, d\gamma$, and $u^0_\epsilon = \pi_0(u_\epsilon) = u_\epsilon - u^1_\epsilon \in L^2_0(\Gamma)$. Then we get

$$y_\epsilon = y_\epsilon(u^0_\epsilon) + \frac{1}{\epsilon|\Gamma|}\int_\Gamma u_\epsilon\, d\gamma,$$

and

$$\mathcal{J}_\epsilon(u_\epsilon) = \left\|y_\epsilon(u^0_\epsilon) - h^0\right\|^2 + \left\|\frac{1}{\epsilon|\Gamma|}\int_\Gamma u_\epsilon\, d\gamma - h^1\right\|^2 + \epsilon\|u_\epsilon\|^2_{L^2(\Gamma)}.$$

Take and fix an element $w \in \mathcal{K}_{ad}$. By Lemma 4.1, $y_\epsilon(w)$ is a bounded net in $H^1(\Omega)$. Therefore, there exists a positive constant named $C$ such that for all $\epsilon \in ]0, 1]$, we have

$$\mathcal{J}_\epsilon(u_\epsilon) \leq J_\epsilon(w) + \epsilon\|w\|^2_{L^2(\Gamma)} \leq C.$$

From which we deduce that there exists another positive constant called again $C$ such that for all $\epsilon \in ]0,1]$, we have

$$|u_\epsilon^1| \leq \epsilon |\Gamma| C.$$

Since, by assumption, the set $\{u_\epsilon^0 : \epsilon > 0\}$ is bounded in $L_0^2(\Gamma)$, we conclude that the set $\{u_\epsilon : 0 < \epsilon \leq 1\}$ is bounded in $L^2(\Gamma)$. We end the proof of this theorem by following the steps of the proof of Theorem 4.3. □

**5.3 Remark :** The idea of introducing the cost functional $\mathcal{J}_\epsilon$ was suggested to us by the referee. We think that it is a nice problem to investigate and seek other classes of unbounded sets of admissible controls where one could apply the methods developped here for the bounded case. We think that the estimates will not be easy to obtain for the unbounded cases.

It is worthy to notice that our cost functional used here may be viewed as the limit case ($N = 0$) for the functional cost

$$J_N(v) := \int_\Gamma (y_\epsilon(v) - h)^2 \, d\gamma + N\|v\|^2_{L^2(\Gamma)},$$

where $N > 0$, considered in the work [3] for the same problem $(P_\epsilon)(v)$. We emphasize the fact that our methods are completely different from those used in [3] where the case $N = 0$ is not treated. This motivates the choice we have made for the singular cost functional $J_\epsilon$.

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