ANNALES MATHÉMATIQUES BLAISE PASCAL

PEI-CHU HU

CHUNG-CHUN YANG

Notes on a generalized *abc*-conjecture over function fields

Annales mathématiques Blaise Pascal, tome 8, nº 1 (2001), p. 61-71

<http://www.numdam.org/item?id=AMBP_2001__8_1_61_0>

Ann. Math. Blaise Pascal, Vol. 8, N° 1, 2001, pp.61-71

Notes on a generalized *abc*-conjecture over function fields

Pei-Chu Hu & Chung-Chun Yang*

Abstract

In this note, we weaken a condition in the generalized *abc*-conjecture proposed by us in a previous paper, and prove its analogue for non-Archimedean entire functions, as well as a generalized Mason's theorem for polynomials.

1 Introduction

In all the paper, κ will denote an algebraically closed field of characteristic zero.

Let a be a non-zero integer. Then

$$|a| = p_1^{i_1} \cdots p_n^{i_n}$$

holds for distinct primes $p_1, ..., p_n$. For a positive integer k, define

$$r_k(a) = \prod_{\nu=1}^{n} p_\nu^{\min\{i_\nu, k\}}.$$

Mathematics Subject Classification 2000 (MSC2000). Primary 11D88, 11E95. Secondary 30D35.

*The work of first author was partially supported by NSFC of China and second author was partially supported by the UGC grant of Hong Kong; proj. no. HKUST 6180/99p.

Key words and phrases: generalized *abc*-conjecture, non-Archimedean, entire functions, meromorphic functions, characteristic functions, linear independent

Conjecture 1.1 ([6],[10],[11]) *Let $a_j (j = 0, \cdots, k)$ be nonzero integers such that the greatest common factor of $a_0, ..., a_k$ is 1,*

$$a_1 + \cdots + a_k = a_0, \tag{1}$$

and no proper subsum of (1) is equal to 0. Then for $\varepsilon > 0$, there exists a number $C(k, \varepsilon)$ such that $\max_{0 \le j \le k}\{|a_j|\} \le C(k, \varepsilon) \left(\prod_{i=0}^{k} r_{k-1}(a_i)\right)^{1+\varepsilon}$,
$\max_{0 \le j \le k}\{|a_j|\} \le C(k, \varepsilon) r_{\frac{k(k-1)}{2}} (a_0 \cdots a_k)^{1+\varepsilon}$.

If $k = 2$, this corresponds to the well known *abc*-conjecture which also is a consequence of the Vojta's Conjecture (see Vojta [19]). Some special cases of Conjecture 1.1 were given in [8] and [9]. In this note, we will prove the analogue of Conjecture 1.1 for entire functions defined over non-Archimedean fields:

Theorem 1.1 *Let κ be complete for a non-trivial non-Archimedean absolute value $|\cdot|$. Let $f_j (j = 0, \cdots, k)$ be entire functions on κ such that $f_0, ..., f_k$ have no common zeros, f_j $(j = 1, \cdots, k)$ be linearly independent on κ and*

$$f_1 + \cdots + f_k = f_0. \tag{2}$$

Then the Nevanlinna functions $T(r, f_j)$ and $N_h(r, f_j)$ satisfy
$$\max_{0 \le j \le k}\{T(r, f_j)\} \le \sum_{i=0}^{k} N_{k-1}\left(r, \frac{1}{f_i}\right) - \frac{k(k-1)}{2}\log r + O(1)$$
$$\max_{0 \le j \le k}\{T(r, f_j)\} \le N_{\frac{k(k-1)}{2}}\left(r, \frac{1}{f_0 \cdots f_k}\right) - \frac{k(k-1)}{2}\log r + O(1).$$

For the meaning of the above notations, we refer the reader to § 2. Under a stronger condition that f_0, f_j have no common zeros for $j = 1, ..., k$, some special cases of Theorem 1.1 were given in [7], [9]. If f is a polynomial, it is easy to show

$$\deg(f) = \lim_{r \to \infty} \frac{T(r, f)}{\log r}, \quad r_k(f) := \lim_{r \to \infty} n_k\left(r, \frac{1}{f}\right) = \lim_{r \to \infty} \frac{N_k\left(r, \frac{1}{f}\right)}{\log r}.$$

As a direct consequence of Theorem 1.1, we obtain

Theorem 1.2 *Let $f_j (j = 0, \cdots, k)$ be polynomials on κ such that $f_0, ..., f_k$ have no common zeros, f_j $(j = 1, \cdots, k)$ be linearly independent on κ and*

$$f_1 + \cdots + f_k = f_0. \tag{3}$$

Then $\max_{0 \le j \le k}\{\deg(f_j)\} \le \sum_{i=0}^{k} r_{k-1}(f_i) - \frac{k(k-1)}{2}$,
$\max_{0 \le j \le k}\{\deg(f_j)\} \le r_{\frac{k(k-1)}{2}}(f_0 \cdots f_k) - \frac{k(k-1)}{2}$.

When $k = 2$, it reduces to a Mason's theorem (see [12], [13], [14], [15] and [18]) which has been generalized recently to fields of any characteristic p by Boutabaa and Escassut [2]. If $k \geq 2$, the following example

$$f_0(z) = (z+1)^{k-1}, \quad f_{i+1}(z) = k - 1iz^i \ (i = 0, ..., k-1),$$

which obviously satisfy the conditions in Theorem 1.2, shows that the inequalities in the theorem, in fact, become equality for this example. Under the stronger assumption that f_j, f_0 have no common zeros for $j = 1, ..., k$, Theorem 1.2 was obtained by Hu-Yang [9]. For any positive integer k and any polynomial f on κ, note that

$$r_k(f) \leq kr_1(f).$$

Theorem 1.2 yields immediately the following:

Theorem 1.3 *Let $f_1, f_2, \cdots, f_k$ ($k \geq 2$) be linearly independent polynomials in κ. Put $f_0 = f_1 + f_2 + \cdots + f_k$ and assume that $f_0, ..., f_k$ have no common zeros. Then the following inequalities* $\max_{0 \leq j \leq k}\{\deg(f_j)\} \leq (k-1)\sum_{i=0}^{k} r_1(f_i) - \frac{k(k-1)}{2}$,
$\max_{0 \leq j \leq k}\{\deg(f_j)\} \leq \frac{k(k-1)}{2}(r_1(f_0 \cdots f_k) - 1)$, *hold.*

The inequality (1.3) was obtained independently by J. F. Voloch [20], W. D. Brownawell and D. Masser [4]. Earlier R. C. Mason [16] derived this estimate with $\frac{1}{2}k(k-1)$ replaced by 4^{k-1}. J. Browkin and J. Brzeziński [3] conjectured that the value $\frac{1}{2}k(k-1)$ in (1.3) would be replaced by $2k-3$. If the restriction on the linear independence of polynomials $f_1, ..., f_k$ is removed, we have

Theorem 1.4 *For fixed integer $k \geq 1$, let f_j ($j = 0, ..., k$) be non-zero polynomials on κ such that $f_1 + \cdots + f_k = f_0$. Assume also that not all the f_j are constants, and the f_j are pairwise relatively prime. Then*

$$\max_{0 \leq j \leq k}\{\deg(f_j)\} \leq (d-1)(r_1(f_0 \cdots f_k) - 1), \tag{4}$$

where d is the dimension of the vector space spanned by the f_i over κ.

As an application of Theorem 1.2, we can derive the following:

Theorem 1.5 *Given polynomials $f_1, f_2, \cdots, f_k$ ($k \geq 2$) in κ and positive integers $l_j (1 \leq j \leq k)$ such that*

(a) $f_1^{l_1}, f_2^{l_2}, \cdots, f_k^{l_k}$ *are linearly independent over* κ;

(b) $f_0, f_1^{l_1}, f_2^{l_2}, \cdots, f_k^{l_k}$ *have no common zeros, where*

$$f_0 = \sum_{j=1}^{k} f_j^{l_j}.$$

Then the following inequality

$$\left\{1 - \sum_{j=1}^{k} \frac{k-1}{l_j}\right\} \max_{1 \le j \le k} \deg\left(f_j^{l_j}\right) \le r_{k-1}(f_0) - \frac{k(k-1)}{2} \tag{5}$$

holds.

Obviously, the inequality (5) implies

$$\left\{1 - \sum_{j=1}^{k} \frac{k-1}{l_j}\right\} \max_{1 \le j \le k} \deg\left(f_j^{l_j}\right) \le \deg\left(\sum_{j=1}^{k} f_j^{l_j}\right) - \frac{k(k-1)}{2}. \tag{6}$$

For the case

$$k = 2, \quad l_1 = 2, \quad l_2 = 3, \tag{7}$$

and

$$f_1 = f, \quad f_2 = -g, \tag{8}$$

the inequality (6) yields

$$\frac{1}{2} \deg(g) \le \deg\left(f^2 - g^3\right) - 1, \tag{9}$$

which was proved for complex case by Davenport [5]. In fact, Davenport proved that (9) is true as long as $f^2 - g^3 \neq 0$ (also see [1],[18]).

2 Basic facts

Let κ be an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedean absolute value $|\cdot|$. Define

$$\kappa[0; r] = \{z \in \kappa \mid |z| \le r\}.$$

Let $\mathcal{A}(\kappa)$ be the set of entire functions on κ. Then each $f \in \mathcal{A}(\kappa)$ can be given by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (a_n \in \kappa), \tag{10}$$

such that for any $z \in \kappa$, one has $|a_n z^n| \to 0$ as $n \to \infty$. Define the *maximum term*:

$$\mu(r, f) = \max_{n \geq 0} |a_n| r^n$$

with the associated the *central index*:

$$n\left(r, \frac{1}{f}\right) = \max_{n \geq 0}\{n \mid |a_n| r^n = \mu(r, f)\}.$$

Then $n\left(r, \frac{1}{f}\right)$ just is the *counting function* of zeros of f, which denotes the number of zeros (counting multiplicity) of f with absolute value $\leq r$. Fix a real ρ_0 with $\rho_0 > 0$. Define the *valence function* of zeros of f by

$$N\left(r, \frac{1}{f}\right) = \int_{\rho_0}^{r} \frac{n(t, \frac{1}{f})}{t} dt \quad (r > \rho_0). \tag{11}$$

The field of fractions of $\mathcal{A}(\kappa)$ will be denoted by $\mathcal{M}(\kappa)$. An element f in the set $\mathcal{M}(\kappa)$ will be called a *meromorphic function* on κ. Take $f \in \mathcal{M}(\kappa)$. Since greatest common divisors of any two elements in $\mathcal{A}(\kappa)$ exist, then there are $g, h \in \mathcal{A}(\kappa)$ with $f = \frac{g}{h}$ such that g and h have no any common zeros in the ring $\mathcal{A}(\kappa)$. We can uniquely extend μ to a meromorphic function $f = \frac{g}{h}$ by defining

$$\mu(r, f) = \frac{\mu(r, g)}{\mu(r, h)} \; (0 \leq r < \infty).$$

Then the following *Jensen formula*

$$N\left(r, \frac{1}{f}\right) - N(r, f) = \log \mu(r, f) - \log \mu(\rho_0, f) \tag{12}$$

holds, where

$$N(r, f) = N\left(r, \frac{1}{h}\right).$$

Note that

$$\mu(r, f_1 f_2) = \mu(r, f_1)\mu(r, f_2), \quad f_1, f_2 \in \mathcal{M}(\kappa).$$

Thus the Jensen formula implies

$$N\left(r,\frac{1}{f_1f_2}\right)-N(r,f_1f_2)=N\left(r,\frac{1}{f_1}\right)+N\left(r,\frac{1}{f_2}\right)-N(r,f_1)-N(r,f_2). \tag{13}$$

Define the *compensation function* by

$$m(r,f)=\max\{0,\log\mu(r,f)\}.$$

As usual, we define the *characteristic function*:

$$T(r,f)=m(r,f)+N(r,f)\ (\rho_0<r<\infty).$$

Then the following formula (see [8])

$$T(r,f)=\max\left\{N\left(r,\frac{1}{f-a}\right),N\left(r,\frac{1}{f-b}\right)\right\}+O(1) \tag{14}$$

holds for any two distinct elements $a,b\in\kappa\cup\{\infty\}$. In particular, if f is a non-constant entire function in κ, then

$$N\left(r,\frac{1}{f-a}\right)=T(r,f)+O(1) \tag{15}$$

for all $a\in\kappa$.

We also denote the number of distinct zeros of $f-a$ on $\kappa[0;r]$ by $\overline{n}(r,\frac{1}{f-a})$ and define

$$\overline{N}\left(r,\frac{1}{f-a}\right)=\int_{\rho_0}^{r}\frac{\overline{n}(t,\frac{1}{f-a})}{t}dt\quad(r>\rho_0).$$

Let $n_k(r,\frac{1}{f-a})$ denote the the number of zeros of $f-a$ on $\kappa[0;r]$, where a zero of $f-a$ with multiplicity m will be counted as $\min\{m,k\}$ in $n_k(r,\frac{1}{f-a})$. Write

$$N_k\left(r,\frac{1}{f-a}\right)=\int_{\rho_0}^{r}\frac{n_k(t,\frac{1}{f-a})}{t}dt\quad(r>\rho_0).$$

The following result is a non-Archimedean analogue of a result of Nevanlinna[17]:

Lemma 2.1 ([8], [9]) *Let $f_j (j=1,\cdots,k)$ be linearly independent meromorphic functions on κ such that*

$$f_1+\cdots+f_k=1. \tag{16}$$

Then

$$T(r,f_j) < \sum_{i=1}^{k} N\left(r,\frac{1}{f_i}\right) - \sum_{i\neq j} N(r,f_i) + N(r,\mathbf{W})$$
$$-N\left(r,\frac{1}{\mathbf{W}}\right) - \frac{k(k-1)}{2}\log r + O(1), \quad 1\le j\le k \quad (17)$$

where $\mathbf{W}$ *is the Wronskian of* $f_1,...,f_k$.

3 Proof of the main theorems

Proof of Theorem 1.1. Applying Lemma 2.1 to $f_1/f_0,...,f_k/f_0$, we obtain

$$T\left(r,\frac{f_j}{f_0}\right) < \sum_{i=1}^{k} N\left(r,\frac{f_0}{f_i}\right) - \sum_{i\neq j} N\left(r,\frac{f_i}{f_0}\right) + N(r,\mathbf{W})$$
$$-N\left(r,\frac{1}{\mathbf{W}}\right) - \frac{k(k-1)}{2}\log r + O(1), \quad 1\le j\le k, \quad (18)$$

where $\mathbf{W} = \mathbf{W}(f_1/f_0,...,f_k/f_0)$ is the Wronskian of $f_1/f_0,...,f_k/f_0$. Note that

$$\mathbf{W} = \mathbf{W}\left(\frac{f_1}{f_0},...,\frac{f_k}{f_0}\right) = \frac{\mathbf{W}_1}{f_0^k},$$

where $\mathbf{W}_1 = \mathbf{W}(f_1,...,f_k)$ is the Wronskian of $f_1,...,f_k$. By the formula (13), we obtain easily $N\left(r,\frac{f_0}{f_i}\right) - N\left(r,\frac{f_i}{f_0}\right) = N\left(r,\frac{1}{f_i}\right) - N\left(r,\frac{1}{f_0}\right)$, $N(r,\mathbf{W}) - N\left(r,\frac{1}{\mathbf{W}}\right) = kN\left(r,\frac{1}{f_0}\right) - N\left(r,\frac{1}{\mathbf{W}_1}\right)$, and hence, for $1\le j\le n$, we obtain

$$T\left(r,\frac{f_j}{f_0}\right) \le \sum_{i=0}^{n} N\left(r,\frac{1}{f_i}\right) - N\left(r,\frac{1}{\mathbf{W}_1}\right) + N\left(r,\frac{f_0}{f_j}\right)$$
$$-N\left(r,\frac{1}{f_j}\right) - \frac{k(k-1)}{2}\log r + O(1). \quad (19)$$

Obviously, for each $j = 1,...,k$, we can choose entire functions h_j, $\tilde{f}_{0j}$ and $\tilde{f}_j$ such that $\tilde{f}_{0j}$ and $\tilde{f}_j$ have no common zeros, and

$$f_0 = h_j\tilde{f}_{0j}, \quad f_j = h_j\tilde{f}_j.$$

By simple observation, we find

$$N\left(r,\frac{1}{f_j}\right)-N\left(r,\frac{f_0}{f_j}\right)=N\left(r,\frac{1}{h_j}\right). \tag{20}$$

Noting that, by (14) and (15),

$$\begin{aligned} T\left(r,\frac{f_j}{f_0}\right)+N\left(r,\frac{1}{h_j}\right) &= \max\left\{N\left(r,\frac{1}{f_0}\right),N\left(r,\frac{1}{f_j}\right)\right\}+O(1) \\ &= \max\{T(r,f_0),T(r,f_j)\}+O(1). \end{aligned} \tag{21}$$

Thus Theorem 1.1 follows from (19), (21) and the following estimates

$$\sum_{i=0}^{k}\mu^0_{f_i}-\mu^0_{\mathbf{W}_1}\leq\sum_{i=0}^{k}\mu^0_{f_i,k-1}, \tag{22}$$

$$\sum_{i=0}^{k}\mu^0_{f_i}-\mu^0_{\mathbf{W}_1}\leq\mu^0_{f_0\cdots f_k,\frac{k(k-1)}{2}}, \tag{23}$$

where μ^a_f is the a-valued multiplicity of an element $f\in\mathcal{M}(\kappa)$, and

$$\mu^a_{f,k}(z)=\min\{\mu^a_f(z),k\}.$$

Take $z_0\in\kappa$. Then $\mu^0_{f_s}(z_0)=0$ for some $s\in\{0,...,n\}$ since $f_0,...,f_k$ have no common zeros. Note that, by the identity (2),

$$\mathbf{W}_1=\mathbf{W}(f_1,...,f_{s-1},f_0,f_{s+1},...,f_n).$$

Obviously we have

$$\mu^0_{f_i^{(j)}}(z_0)\geq\mu^0_{f_i}(z_0)-\mu^0_{f_i,j}(z_0)\geq\mu^0_{f_i}(z_0)-\mu^0_{f_i,k-1}(z_0),\quad i\neq s,\ 1\leq j\leq k-1,$$

and, hence,

$$\mu^0_{\mathbf{W}_1}(z_0)\geq\sum_{i\neq s}\{\mu^0_{f_i}(z_0)-\mu^0_{f_i,k-1}(z_0)\},$$

that is, $\sum_{i=0}^{k}\mu^0_{f_i}(z_0)-\mu^0_{\mathbf{W}_1}(z_0)=\sum_{i\neq s}\mu^0_{f_i}(z_0)-\mu^0_{\mathbf{W}_1}(z_0)$ $\leq\sum_{i\neq s}\mu^0_{f_i,k-1}(z_0)=\sum_{i=0}^{k}\mu^0_{f_i,k-1}(z_0)$. The inequality (23) can be obtained similarly by comparing the multiplicities of zeros of $f_0\cdots f_k$ and $\mathbf{W}_1$. Then Theorem 1.1 follows from (19), (21), (22) and (23).

Proof of Theorem 1.4. In the sequel, we will use the notation

$$f_i \equiv 0\{f_{i_1}, ..., f_{i_{s_i}}\} \tag{24}$$

to denote that $\{i_1, ..., i_{s_i}\} \subset \{0, 1, ..., k\} - \{i\}$ are distinct, $f_{i_1}, ..., f_{i_{s_i}}$ linearly independent, and

$$f_i = \sum_{\alpha=1}^{s_i} c_\alpha f_{i_\alpha}, \quad c_\alpha \in \kappa - \{0\} \ (1 \le \alpha \le s_i).$$

We proceed the proof of Theorem 1.4 by induction on k. For $k = 1$ it is obviously true since if $f_0 = f_1$, f_0 and f_1 relatively prime, then they both are constants. Assume the theorem is true for all cases k' with $2 \le k' < k$, and consider that of $k+1$ polynomials. By the assumptions in Theorem 1.4, at least two of the f_i are non-constant. Note that if two of the f_i are constants, then we may either eliminate them if their sum is zero or replace them by their sum when it is not zero. Then the inductive hypothesis could be applied to yield the desired result. Thus we may assume that at most one of the f_i is a constant. For each $i \in \{0, 1, ..., k\}$, it is easy to show that

$$f_i \equiv 0\{f_{i_1}, ..., f_{i_{s_i}}\}$$

for some $i_1, ..., i_{s_i}$. Obviously, $d \ge s_i \ge 2$ and $f_{i_1}, ..., f_{i_{s_i}}$ have no common zeros since the f_j are pairwise relatively prime. So by Theorem 1.3, we have

$$\max_{0 \le \alpha \le s_i} \{\deg(f_{i_\alpha})\} \le (s_i - 1) \sum_{\alpha=0}^{s_i} r_1(f_{i_\alpha}) - \frac{s_i(s_i - 1)}{2}, \tag{25}$$

where $i_0 = i$. Therefore, we obtain
$\max_{0 \le \alpha \le s_i} \{\deg(f_{i_\alpha})\} \le (s_i - 1)\left(\sum_{\alpha=0}^{s_i} r_1(f_{i_\alpha}) - 1\right)$
$\le (d-1)\left(\sum_{i=0}^{k} r_1(f_i) - 1\right) = (d-1)\left(r_1(f_0 \cdots f_k) - 1\right)$, that is, for each $i \in \{0, 1, ..., k\}$, $\deg(f_i) \le (d-1)\left(r_1(f_0 \cdots f_k) - 1\right)$. Hence Theorem 1.4 is proved.

Proof of Theorem 1.5. Theorem 1.2 implies

$$\max_{1 \le j \le k} \deg\left(f_j^{l_j}\right) \le \max_{0 \le j \le k} \deg\left(f_j^{l_j}\right) \le \sum_{j=0}^{k} r_{k-1}\left(f_j^{l_j}\right) - \frac{k(k-1)}{2}, \tag{26}$$

where $l_0 = 1$. Note that

$$r_{k-1}\left(f_j^{l_j}\right) \le (k-1) r_1(f_j) \le (k-1)\deg(f_j) = \frac{k-1}{l_j}\deg\left(f_j^{l_j}\right), \quad j \ne 0. \tag{27}$$

Hence (5) follows from (26) and (27).

Acknowledgement: The authors are indebt to the anonymous referee for his/her careful reading of the manuscript, with helpful comments and suggestions.

References

[1] Birch, B. J., Chowla, S., M. Hall Jnr. and Schinzel, A., On the difference $x^3 - y^2$, Norske Vid. Selsk. Forh. (Trondheim) 38 (1965), 65-69.

[2] Boutabaa, A. and Escassut, A., Nevanlinna theory in characteristic p, and applications, preprint.

[3] Browkin, J. and Brzezinski, J., Some remarks on the *abc*-conjecture, Mathematics of Computation 62 (1994), 931-939.

[4] Brownawell, W. D. and Masser, D., Vanishing sums in function fields, Math. Proc. Cambridge Philos. Soc. 100(1986), 427-434.

[5] Davenport, H., On $f^3(t) - g^2(t)$, Norske Vid. Selsk. Forh. (Trondheim) 38 (1965), 86-87.

[6] Hu, P. C., Li, P. and Yang, C. C., Unicity of meromorphic mappings, manuscript.

[7] Hu, P. C. and Yang, C. C., The "abc" conjecture over function fields, Proc. Japan Acad. 76, Ser. A(2000), 118-120.

[8] Hu, P. C. and Yang, C. C., Meromorphic functions over non-Archimedean fields, Mathematics and Its Applications 522, Kluwer Academic Publishers, 2000.

[9] Hu, P. C. and Yang, C. C., A generalized *abc*-conjecture over function fields, to appear in Journal of Number Theory.

[10] Hu, P. C. and Yang, C. C., Some progresses in non-Archimedean analysis, preprint.

[11] Hu, P. C. and Yang, C. C., A note on the *abc*-conjecture, preprint.

[12] Lang, S., Old and new conjectured Diophantine inequalities, Bull. Amer. Math. Soc. 23 (1990), 37-75.

[13] Mason, R. C., The hyperelliptic equation over function fields, Math. Proc. Cambridge Philos. Soc. 93(1983), 219-230.

[14] Mason, R. C., Equations over function fields, Lecture Notes in Math. 1068 (1984), 149-157, Springer.

[15] Mason, R. C., Diophantine equations over function fields, London Math. Soc. Lecture Note Series, Vol. 96, Cambridge University Press, United Kingdom, 1984.

[16] Mason, R. C., Norm form equations I, J. Number Theory 22 (1986), 190-207.

[17] Nevanlinna, R., Le théorème de Picard-Borel et la théorie des fonctions meromorphes, Gauthier Villars, Paris, 1929; Reprint, Chelsea, New York, 1974.

[18] Stothers, W. W., Polynomial identities and Hauptmoduln, Quart. J. Math. Oxford Ser. (2) 32 (1981), no. 127, 349-370.

[19] Vojta, P., Diophantine approximation and value distribution theory, Lecture Notes in Math. 1239, Springer, 1987.

[20] Voloch, J. F., Diagonal equations over function fields, Bol. Soc. Brasil. Mat. 16 (1985), 29-39.

Department of Mathematics
Shandong University
Jinan 250100, Shandong
China

Department of Mathematics
The Hong Kong University of
Science and Technology
Clear Water Bay, Kowloon
Hong Kong