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Abstract

In this paper, we consider the second-order delay differential inclusion $x"(t) \in Ax(t) + F(t, x_t)$ in a Banach space and we study some properties of its solution set. We prove a relaxation theorem which reveals the connection between the solution sets of a second-order delay differential inclusion and its convexified version, under some weak conditions.

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1 Introduction

Many problems in applied mathematics, such as those in control theory, lead to the study of second-order delay differential inclusions

$$x"(t) \in Ax(t) + F(t, x_t), \qquad (1)$$

where A is the infinitesimal generator of a C_0-propagator of linear operators $(C(t))_{t\in\mathbb{R}}$ on a Banach space $(E, |.|_E)$ and F is a nonlinear multimapping, satisfying assumptions to be specified in the third section.
As particular cases of relations of the form (1) we have:
i) The second-order delay differential equation

$$x"(t) = Ax(t) + f(t, x_t)$$

where $F(t, x_t) = f(t, x_t)$.
ii) The differential inequalities

$$|x"(t) - Ax(t) - f(t, x_t)|_E \leq g(t, x_t)$$

where $F(t, x_t)$ is the ball of radius $g(t, x_t)$ centered at $Ax(t) + f(t, x_t)$.
iii) Control problems where the control $u(t)$ and the trajectory $x(t)$ are related by the second-order delay differential equation

$$x"(t) = Ax(t) + f(t, x_t, u(t)),\ u(t) \in U(t).$$

Here, the control function $u(t)$ is a measurable function and $F(t,x_t) = f(t,x_t,U(t))$. This paper is concerned with the second-order delay differential inclusion (1) and its mild trajectories. We show that many results which allow us to apply differential inclusions, see for example [1,3,8,10,13] and references therein, are valid as well for (1). In our relaxation theorem, the assumption of integrale boundedness (condition (H_4)) will be replaced by an integrability condition (condition (H_3')). We also give some properties of the solution set of the inclusion (1).

2 Preliminaries

For a real Banach space $(E,|.|_E)$ and $J := [-r,0]$ $(r>0)$, let $\mathcal{C} := C([-r,0];E)$ be the Banach space of continuous functions from J to E with the usual supremum norm $\|.\|$. For any continuous function $x \in C([-r,\omega];E)$ $(\omega > 0)$ and any $t \in I := [0,\omega]$ we denote by x_t the element of $\mathcal{C}$ defined by $x_t(\theta) = x(t+\theta)$, $\theta \in J$.
For a subset $A \subset E$, coA, $\overline{co}A$ and clA are respectively the convex hull, the closed convex hull and the closure. We denote by $\mathcal{F}(E)$ (resp. $\mathcal{F}_c(E)$) the family of all nonempty closed (resp. closed convex) subsets of E, and by δ the Hausdorff distance in $\mathcal{F}(E)$, i.e. for $A,B \in \mathcal{F}(E)$

$$\delta(A,B) = \max[\sup_{a\in A}(d(a,B), \sup_{b\in B} d(b,A)]$$

where $d(a,B) = \inf_{b\in B} d(a,b)$.
Next we present some basic concepts concerning multimappings.
Let X be another Banach space, for a multimapping $G : X \to \mathcal{P}(E)$ (the family of all nonempty subsets of E), we define its lim sup and lim inf at $x \in X$ in the Kuratowski sense by

$$\limsup_{y\to x} G(y) = \{z \in E : \liminf_{y\to x} d(z,G(y)) = 0\}$$

and

$$\liminf_{y\to x} G(y) = \{z \in E : \lim_{y\to x} d(z,G(y)) = 0\}.$$

We say that the limit of $G(y)$ as y tends to x exists in the Kuratowski sense if

$$\limsup_{y\to x} G(y) = \liminf_{y\to x} G(y).$$

We denote this limit by $\lim_{y\to x} G(y) = G(x)$. We say that G is upper (resp. lower) semicontinuous at x if

$$\limsup_{y\to x} G(y) \subseteq G(x) \text{ (resp. } G(x) \subseteq \liminf_{y\to x} G(y)).$$

If G is both upper and lower semicontinuous at x then we say that G is continuous at x. If G is continuous or semicontinuous for all $x \in X$, we say that G is continuous or semicontinuous on X.

Let $G : I \to \mathcal{P}(E)$ be a multimapping. A function $g : I \to E$ such that $g(t) \in G(t)$ for every $t \in I$ is called a selection of G.
G is called measurable if, for almost all $t \in I$

$$G(t) \subseteq cl\{g_n(t) : n \geq 1\}$$

where g_n are measurable selections of G. This definition of the mesurability is given by Zhu [13], when E is separable and $G(t) \in \mathcal{F}(E)$ for every $t \in I$ this definition is the same as the classic one (see for example [3]).
By the symbol of I_G^1 we will denote the set of all Bochner integrable selections of the multimapping G, i.e.

$$I_G^1 = \{g \in L^1(I; E) : g(t) \in G(t) \text{ a.e.}\}.$$

If $I_G^1 \neq \emptyset$, then the measurable multimapping G is called integrable and

$$\int_I G(t)dt = \{\int_I g(t)dt : g \in I_G^1\}.$$

Clearly if G is measurable and integrably bounded, i.e. there exists $\nu \in L_+^1(I)$ such that

$$\|G(t)\| := \sup\{|e|_E : e \in G(t)\} \leq \nu(t) \text{ a.e.}$$

then G is integrable. But the converse is not true.
We will also need the following properties (see [13]) which will be used later.
Lemma 2.1 Let $G : I \to \mathcal{P}(E)$ be a measurable multimapping. Then so is $\overline{co}G$.
Lemma 2.2 Let $G : I \to \mathcal{P}(E)$ be an integrable multimapping. Then $cl \int_I G(t)dt$ is a convex set and

$$cl \int_I G(t)dt = cl \int_I coG(t)dt = cl \int_I \overline{co}G(t)dt.$$

Remark If $G : I \to P(E)$ is an integrable multimapping, then so is $\overline{G}$ where $\overline{G}(t) = clG(t)$ and

$$cl \int_I G(t)dt = cl \int_I \overline{G}(t)dt$$

(indeed $cl \int_I G(t)dt \subset cl \int_I \overline{G}(t)dt \subset cl \int_I \overline{co}G(t)dt = cl \int_I G(t)dt$).
Lemma 2.3 Let $G : I \to \mathcal{P}(E)$ be a measurable multimapping and $u : I \to E$ a measurable function. Then for any measurable function $v : I \to \mathbb{R}^+$, there exists a measurable selection g of G such that

$$|g(t) - u(t)|_E \leq d(u(t), G(t)) + v(t) \text{ a.e.}$$

At last, we give some important properties of a C_0-propagator and its infinitesimal generator (see [7]).
A strongly continuous propagator $(C(t))_{t\in\mathbb{R}}$ of continuous operators on E is a family of continuous linear mappings $C(t) : E \to E$, $t \in \mathbb{R}$, satisfying
i) $C(0) = I$;
ii) $C(t + s) + C(t - s) = 2C(t)C(s)$;

iii) for $x \in E$, $C(.)x : \mathbb{R} \to E$ is continuous.
A strongly continuous propagator of continuous linear mappings is also called a C_0-propagator. A linear operator A is associated with a propagator, it plays the role of the infinitesimal generator for C_0-semigroups:

$$D(A) = \{x \in E : \lim_{h \searrow 0} \frac{2}{h^2}[C(h) - I]x \text{ exists}\}$$

and

$$Ax = \lim_{h \searrow 0} \frac{2}{h^2}[C(h) - I]x \text{ for } x \in D(A)$$

is the infinitesimal generator of the C_0-propagator $(C(t))_{t \in \mathbb{R}}$, $D(A)$ is the domain of A. We have:
- There exist constants $\alpha \geq 0$ and $\eta \geq 1$ such that

$$\|C(t)\| \leq \eta e^{\alpha|t|} \text{ for } t \in \mathbb{R}.$$

- $D(A)$ is dense in E and A is a closed linear operator.
- For every $x \in D(A)$ and $t \in \mathbb{R}$, then $C(t)x \in D(A)$ and

$$\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax.$$

- Let $a, b \in E$ and $f \in L^1(I; E)$, the function $u \in C(I; E)$ given by

$$u(t) = C(t)a + S(t)b + \int_0^t S(t-s)(f(s))ds, \ t \in I$$

is the mild solution on I of the initial value problem

$$\begin{cases} u''(t) = Au(t) + f(t), \ t \in I \\ u(0) = a, \ u'(0) = b \end{cases}$$

where $S(t) = \int_0^t C(s)ds$. Moreover

$$|u(t)|_E \leq \eta e^{\alpha t} |a|_E + \eta \alpha^{-1}(e^{\alpha t} - 1) |b|_E + \eta \alpha^{-1}(e^{\alpha \omega} - 1) \|f\|_1, \ t \in I$$

($\alpha^{-1}(e^{\alpha t} - 1)$ is replaced by t when $\alpha = 0$). If $a = 0$ then u is continuously differentiable and

$$|u'(t)|_E \leq \eta e^{\alpha t} |b|_E + \eta e^{\alpha \omega} \|f\|_1, \ t \in I.$$

3 The solution set of a second-order delay differential inclusion and a relaxation theorem

Consider the functional differential inclusion

$$x''(t) \in Ax(t) + F(t, x_t) \text{ a.e. in } I \tag{3.1}$$

Definition 3.1 A function $x \in \mathcal{C}_\omega := C([-r,\omega];E)$ is called a mild trajectory of (3.1), if there exist $\varphi \in \mathcal{B} := \{\varphi \in \mathcal{C} : \varphi'(0) \text{ exists}\}$ and a Bochner integrable function $f \in L^1(I;E)$ such that

$$f(t) \in F(t, x_t) \text{ a.e. in } I \tag{2}$$

and

$$x(t) = \begin{cases} \varphi(t), & t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds, & t \in I \end{cases} \tag{3}$$

i.e., f is a Bochner integrable selection of the multimapping $t \longmapsto F(t,x_t)$ and x is a mild solution of the initial value problem

$$(4) \begin{cases} x"(t) = Ax(t) + f(t), & t \in I \\ x_0 = \varphi, \ \varphi \in \mathcal{B}. \end{cases}$$

For $\varphi \in \mathcal{B}$, we define $S_F(\varphi) = \{x \in \mathcal{C}_\omega : x \text{ is a mild trajectory of (3.1) with } x_0 = \varphi\}$ to be the solution set of (3.1) from the point φ.
Let $\psi \in \mathcal{B}$, $g \in L^1(I,E)$ and $y \in \mathcal{C}_\omega$ be a mild solution of the problem

$$(C) \begin{cases} y"(t) = Ay(t) + g(t), & t \in I \\ y_0 = \psi. \end{cases}$$

Suppose that the multimapping $F : I \times \mathcal{C} \to \mathcal{F}(E)$ satisfies the following conditions:
H_1) For every $\phi \in \mathcal{C}$, the multimapping $F(.,\phi)$ is measurable on I.
H_2) There is an integrable function $k : I \to \mathbb{R}^+$ such that for every $\phi, \xi \in \mathcal{C}$,

$$\delta(F(t,\phi), F(t,\xi)) \leq k(t)\|\phi - \xi\| \text{ a.e. in } I.$$

H_3) The function $q : t \longmapsto d(g(t), F(t,y_t))$ is integrable on I.
H_3') For any function $x \in \mathcal{C}_\omega$, the multimapping $t \longmapsto F(t,x_t)$ is integrable on I.
H_4) There is an integrable function $\nu \in L^1_+(I)$ such that

$$\|F(t,\phi)\| := \sup\{|y|_E : y \in F(t,\phi)\} \leq \nu(t)$$

for all $\phi \in \mathcal{C}$ and almost all $t \in I$.

Remarks

- When F satisfies (H_1) and (H_2), then $t \to F(t,y_t)$ and q are measurable on I.
- If q is measurable, then the condition (H_3') gives (H_3).
- When F satisfies (H_1) and (H_2) it satisfies (H_3') if and only if it satisfies: there is $z \in \mathcal{C}_\omega$ such that the multimapping $t \to F(t,z_t)$ is integrable (see [13]).
- When F satisfies (H_2), then for every integrable function $k' > k$ and $\phi, \xi \in \mathcal{C}$,

$$F(t,\phi) \subset F(t,\xi) + k'(t)\|\phi - \xi\|B \text{ a.e. in } I$$

where B denotes the closed unit ball in E.

Next we present a useful result on the relationships between the trajectories of (3.1) and the solutions of problem (C).

Theorem 3.1 Let $\psi \in \mathcal{B}$, $g \in L^1(I;E)$ and $y \in \mathcal{C}_\omega$ be a mild solution of problem (C). Assume that $(H_1) - (H_3)$ hold true and let $\mu \geq 0$. Then for all $\varphi \in \mathcal{B}$ with $\|\varphi - \psi\| \leq \mu$, $|\varphi'(0) - \psi'(0)|_E \leq \mu$ and for all integrable function $v : I \to \mathbb{R}^+$, there exist $x \in \mathcal{C}_\omega$ and $f \in L^1(I;E)$ satisfying (2), (3) and

$$\|x - y\|_w \leq K(\omega)m(\omega), \quad \|f - g\|_1 \leq K(\omega)m(\omega)$$

where $M = \eta(e^{\alpha\omega} + \frac{e^{\alpha\omega}-1}{\alpha})$, ($\frac{e^{\alpha\omega}-1}{\alpha}$ is replaced by ω when $\alpha = 0$),

$$K(t) = MexpM \int_0^t 2k(s)ds, \; m(t) = \mu + \int_0^t (q(s) + v(s))ds.$$

Proof. By lemma 2.3, there is a measurable selection f_1 of the multimapping $t \longmapsto F(t, y_t)$ such that, for almost all $t \in I$,

$$\begin{aligned} |f_1(t) - g(t)|_E &\leq d(g(t), F(t, y_t)) + v(t) \\ &\leq q(t) + v(t) \end{aligned}$$

and then $f_1 \in L^1(I;E)$. Set

$$x^1(t) = \begin{cases} \varphi(t) & \text{if } \; t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_1(s))ds & \text{if } \; t \in I \end{cases}$$

we have $x^1 \in \mathcal{C}_\omega$ and for all $t \in I$,

$$\begin{aligned} \|x_t^1 - y_t\| &= \sup_{\theta \in J} |x^1(t+\theta) - y(t+\theta)|_E \\ &\leq M(\mu + \int_0^t |f_1(s) - g(s)|_E ds) \\ &\leq M(\mu + \int_0^t (q(s) + v(s))ds). \end{aligned}$$

By using lemma 2.3, there is a measurable selection f_2 of the multimapping $t \longmapsto F(t, x_t^1)$ such that, for almost all $t \in I$,

$$\begin{aligned} |f_2(t) - f_1(t)|_E &\leq 2d(f_1(t), F(t, x_t^1)) \\ &\leq 2\delta(F(t, y_t), F(t, x_t^1)) \\ &\leq 2k(t)\|x_t^1 - y_t\| \end{aligned}$$

and then $f_2 \in L^1(I;E)$. Set

$$x^2(t) = \begin{cases} \varphi(t) & \text{if } \; t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_2(s))ds & \text{if } \; t \in I. \end{cases}$$

Thus, we can define by induction two sequences (x^n) and (f_n) with $x^n \in \mathcal{C}_\omega$ and $f_n \in L^1(I;E)$ such that:
i) $x^0 = y$ and for all $n \geq 1$,

$$x^n(t) = \begin{cases} \varphi(t) & \text{if } \; t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_n(s))ds & \text{if } \; t \in I; \end{cases}$$

ii) $f_0 = g$ and for all $n \geq 1$

$$f_n(t) \in F(t, x_t^{n-1}) \text{ a.e. in } I;$$

iii) for almost all $t \in I$ and $n \geq 1$,

$$|f_{n+1}(t) - f_n(t)|_E \leq 2k(t)\|x_t^n - x_t^{n-1}\|.$$

It follows then from (iii) that
iv) for all $t \in I$ and $n \geq 1$,

$$\begin{aligned}
\|x_t^{n+1} - x_t^n\| &\leq M\int_0^t |f_{n+1}(t_1) - f_n(t_1)|_E dt_1 \\
&\leq M\int_0^t 2k(t_1)\|x_{t_1}^n - x_{t_1}^{n-1}\|dt_1 \\
&\leq M\int_0^t 2k(t_1)[M\int_0^{t_1} 2k(t_2)\|x_{t_2}^{n-1} - x_{t_2}^{n-2}\|dt_2]dt_1 \\
&\vdots \\
&\leq M^n\int_0^t 2k(t_1)\int_0^{t_1} 2k(t_2)\cdots\int_0^{t_{n-1}} 2k(t_n)\|x_{t_n}^1 - y_{t_n}\|dt_n\cdots dt_1 \\
&\leq M[\eta + \int_0^t (q(s)+v(s))ds].\frac{[M\int_0^t 2k(s)ds]^n}{n!}.
\end{aligned}$$

Then, for all $n \geq 1$

$$\begin{aligned}
\|x^{n+1} - x^n\|_\omega \quad &: = \max(\|x^{n+1} - x^n\|\ , \sup_{t\in I}|x^{n+1}(t) - x^n(t)|_E) \\
&= \sup_{t\in I}|x^{n+1}(t) - x^n(t)|_E \\
&\leq \sup_{t\in I}\|x_t^{n+1} - x_t^n\| \\
&\leq Mm(\omega)\frac{[M\int_0^w 2k(t)dt]^n}{n!}
\end{aligned}$$

By (iv) we obtain for all $t \in I$ and $n \geq 1$,

$$\begin{aligned}
\|x_t^{n+1} - y_t\| &\leq \|x_t^1 - y_t\| + \sum_{i=1}^n \|x_t^{i+1} - x_t^i\| \\
&\leq Mm(t)[1 + \sum_{i=1}^n \frac{[M\int_0^t 2k(s)ds]^i}{i!}] \\
&\leq K(t)m(t).
\end{aligned}$$

We deduce that (x^n) is a Cauchy sequence of a continuous functions, converging uniformly to a function $x \in \mathcal{C}_\omega$ and for almost all $t \in I$, $(f_n(t))$ is a Cauchy sequence in

E, hence $(f_n(.))$ converges pointwise almost everywhere to a measurable function $f(.)$ in E. But for almost all $t \in I$ and $n \in \mathbb{N}$

$$\begin{aligned} |f_{n+1}(t) - g(t)|_E &\leq \sum_{i=1}^{n} |f_{i+1}(t) - f_i(t)|_E + |f_1(t) - g(t)|_E \\ &\leq 2k(t) \sum_{i=1}^{n} \|x_t^i - x_t^{i-1}\| + q(t) + v(t) \\ &\leq 2k(t)K(\omega)m(\omega) + q(t) + v(t) \end{aligned}$$

hence, $|f_{n+1}(t)|_E \leq |g(t)|_E + 2k(t)K(\omega)m(\omega) + q(t) + v(t)$, thus (f_n) converges to f in $L^1(I;E)$ and then $(x^n(t))$ $(t \in [-r,\omega])$ converges in E to

$$\begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I, \end{cases}$$

we obtain

$$x(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I. \end{cases}$$

Furthermore, for almost all $t \in I$

$$\begin{aligned} d(f(t), F(t, x_t)) &\leq |f(t) - f_n(t)|_E + d(f_n(t), F(t, x_t)) \\ &\leq |f(t) - f_n(t)|_E + \delta(F(t, x_t^{n-1}), F(t, x_t)) \\ &\leq |f(t) - f_n(t)|_E + k(t)\|x_t^{n-1} - x_t\|. \end{aligned}$$

The right hand side tends to zero almost everywhere on I as $n \to +\infty$. Thus, for almost all $t \in I$, $f(t) \in F(t, x_t)$.
Consequently $x \in S_F(\varphi)$, moreover, for all $n \in \mathbb{N}$

$$\begin{aligned} \|x^{n+1} - y\|_\omega &\leq \sup_{t \in I} \|x_t^{n+1} - y_t\| \\ &\leq K(\omega)m(\omega). \end{aligned}$$

Taking limits in the precedent inequality, we have $\|x - y\|_\omega \leq K(\omega)m(\omega)$.
We now show $\|f - g\|_1 \leq K(\omega)m(\omega)$.
For almost all $t \in I$ and $n \in \mathbb{N}$, we have

$$|f_{n+1}(t) - g(t)|_E \leq q(t) + v(t) + 2k(t)Mm(\omega) \sum_{i=1}^{n} \frac{[M \int_0^t 2k(s)ds]^{i-1}}{(i-1)!}$$

thus,

$$\begin{aligned} \|f_{n+1} - g)\|_1 &\leq m(\omega)[1 + \sum_{i=1}^{n} \frac{[M \int_0^\omega 2k(t))dt]^i}{i!}] \\ &\leq m(\omega)K(\omega). \end{aligned}$$

Taking the limit in the above inequality, we obtain $\|f - g\|_1 \leq m(\omega)K(\omega)$. ■

In the next theorem we compare trajectories of (3.1) and of the convexified (relaxed) second-order delay differential inclusion $x"(t) \in Ax(t) + \overline{co}F(t, x_t)$ (3.2).
For $\varphi \in \mathcal{B}$, we put

$$S_{\overline{co}F}(\varphi) = \{x \in \mathcal{C}_\omega : \ x \text{ is a trajectory of (3.2) with } x_0 = \varphi\}.$$

Theorem 3.2 Assume that F satisfies conditions $(H_1), (H_2)$ and (H_3'). Then, for all $\varphi \in \mathcal{B}$,

$$clS_F(\varphi) = clS_{\overline{co}F}(\varphi).$$

Proof. It is easy to see that $clS_F(\varphi) \subset clS_{\overline{co}F}(\varphi)$. Conversly, we shall show that $S_{\overline{co}F}(\varphi) \subset clS_F(\varphi)$. Let $y \in S_{\overline{co}F}(\varphi)$, then there exists $g \in L^1(I; E)$ such that

$$y(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(g(s))ds & \text{if } t \in I \end{cases}$$

where $g(s) \in \overline{co}F(s, y_s)$ a.e. in I.
The following result follows immediately from [3 p. 85].
Lemma 3.1
Let $G : I \rightarrow P(E)$ be a measurable multimapping, then so is $s \rightarrow S(t-s)G(s)$. Moreover if $f(s) \in S(t-s)G(s)$ then, there exists a measurable selection $g(s) \in G(s)$ such that $f(s) = S(t-s)g(s)$ a.e. in I.
By (H_3') for all fixed t in I, the multimapping $s \longmapsto S(t-s)F(s, y_s)$ is integrable on I and by lemma 2.2 and its remark we obtain

$$s \longmapsto clS(t-s)F(s, y_s) \text{ and } s \longmapsto \overline{co}S(t-s)F(s, y_s)$$

are also integrable on I and

$$\begin{aligned} cl\int_I S(t-s)F(s, y_s)ds &= cl\int_I clS(t-s)F(s, y_s)ds \\ &= cl\int_I \overline{co}S(t-s)F(s, y_s)ds \end{aligned}$$

but, $\overline{co}S(t-s)F(s, y_s) = clS(t-s)\overline{co}F(s, y_s)$, indeed

$$S(t-s)F(s, y_s) \subset clS(t-s)\overline{co}F(s, y_s)$$

which is a closed convex set and then

$$\overline{co}S(t-s)F(s, y_s) \subset clS(t-s)\overline{co}F(s, y_s),$$

conversly, it suffice to see that

$$S(t-s)\overline{co}F(s, y_s) \subset \overline{co}S(t-s)F(s, y_s)$$

let $f(s) \in S(t-s)\overline{co}F(s,y_s)$, then there exists $g(s) \in \overline{co}F(s,y_s)$ such that $f(s) = S(t-s)g(s)$ hence, there exists a sequence $(g_n(s))$ such that $g_n(s) \in coF(s,y_s)$ and $\lim_{n\to+\infty} g_n(s) = g(s)$, we put

$$f_n(s) = S(t-s)g_n(s) \in S(t-s)coF(s,y_s) = coS(t-s)F(s,y_s)$$

and taking the limit as $n \to +\infty$, we obtain

$$f(s) = S(t-s)g(s) \in cl\ coS(t-s)F(s,y_s)$$

thus,

$$\begin{aligned} cl\int_I S(t-s)F(s,y_s)ds &= cl\int_I clS(t-s)\overline{co}F(s,y_s)ds \\ &= cl\int_I S(t-s)\overline{co}F(s,y_s)ds \end{aligned}$$

(see remark of lemma 2.2).
By lemma 3.1, we obtain for all $\varepsilon > 0$ an integrable selection $h(s) \in F(s,y_s)$ a.e. such that

$$|\int_I S(t-s)(g(s))ds - \int_I S(t-s)(h(s))ds|_E < \frac{\varepsilon}{K(\omega)(\|k\|_1+\omega)+1},$$

set

$$z(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(h(s))ds & \text{if } t \in I \end{cases}$$

then z is a mild solution of problem

$$\begin{cases} z''(t) = Az(t) + h(t) \\ z_0 = \varphi. \end{cases}$$

Moreover by assumption (H_3'), the function $t \longmapsto q(t) = d(h(t), F(t,z_t))$ is integrable on I. It follows from theorem 3.1 for $\mu = 0$ and $v(t) = \frac{\varepsilon}{K(\omega)(\|k\|_1+\omega)+1}$ there exists $x \in S_F(\varphi)$ such that

$$\begin{aligned} \|x-z\|_\omega &\le K(\omega)[\int_0^\omega q(t)dt + \int_0^\omega v(t)dt] \\ &\le \frac{\varepsilon K(\omega)(\|k\|_1+\omega)}{K(\omega)(\|k\|_1+\omega)+1} \end{aligned}$$

thus,

$$\begin{aligned} \|x-y\|_\omega &\le \|x-z\|_\omega + \|z-y\|_\omega \\ &\le \frac{\varepsilon K(\omega)(\|k\|_1+\omega)}{K(\omega)(\|k\|_1+\omega)+1} + \frac{\varepsilon}{K(\omega)(\|k\|_1+\omega)+1} \\ &\le \varepsilon. \ \blacksquare \end{aligned}$$

4 Some properties of the solution set

In this section, we discuss the continuous dependence of the solution set on parameters and initial value. We suppose that E is a reflexive Banach space.

Theorem 4.1. Let (Λ, d_Λ) be a metric space, $F_\lambda : I \times \mathcal{C} \to \mathcal{F}_c(E)$ a family of multimappings satisfying conditions $(H_1), (H_2)$ with the same function k and (H_4) for the same function ν. If for any $(t, \phi) \in I \times \mathcal{C}$, $\lim\limits_{\lambda \to \lambda_0} \delta(F_\lambda(t,\phi), F_{\lambda_0}(t,\phi)) = 0$, then for all $\varphi \in \mathcal{B}$, $\lambda \longmapsto S_{F_\lambda}(\varphi)$ is upper semicontinuous at λ_0.

Proof. Let $x \in \limsup\limits_{\lambda \to \lambda_0} S_{F_\lambda}(\varphi)$, there exists a sequence (λ_n) such that $\lim\limits_{n \to +\infty} \lambda_n = \lambda_0$ and $x^{\lambda_n} \in S_{F_{\lambda_n}}(\varphi)$ such that $\lim\limits_{n \to +\infty} x^{\lambda_n} = x$ in $\mathcal{C}_\omega$, hence

$$x^{\lambda_n}(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_{\lambda_n}(s))ds & \text{if } t \in I \end{cases}$$

where $f_{\lambda_n}(s) \in F_{\lambda_n}(s, x_s^{\lambda_n})$ a.e. in I.

The sequence (f_{λ_n}) is integrably bounded and E is reflexive, then by the Dunford-Pettis theorem [12], taking a subsequence and keeping the same notation, we may assume that it converges weakly in $L^1(I;E)$ to some function $f \in L^1(I;E)$. For each $t \in I$, the mapping

$$g \in L^1(I;E) \to \int_0^t S(t-s)(g(s))ds$$

is a continuous linear operator from $L^1(I;E)$ into E. It remains continuous if these spaces are endowed with the weak topologies [2]. Therefore for each $t \in I$, the sequence $(x^{\lambda_n}(t))$ converges weakly to $C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds$. Since by assumption $(x^{\lambda_n}(t))$ converges to $x(t)$ in E, we have

$$x(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds.$$

We claim that $f(s) \in F_{\lambda_0}(s, x_s)$ a.e. According to Mazur's theorem [6], the weak convergence implies the existence of the double sequence of nonnegative numbers $(\alpha_{m,n})$ such that

i) $\alpha_{m,n} = 0$ for $n \geq n_0(m)$;

ii) $\sum\limits_{n=m}^{n_0(m)} \alpha_{m,n} = 1$ for $m \in \mathbb{N}$;

iii) the sequence $(\tilde{f}_m)$, where $\tilde{f}_m(t) = \sum\limits_{n=m}^{n_0(m)} \alpha_{m,n} f_{\lambda_n}(t)$, converges to f with respect to the norm of the space $L^1(I,E)$. Passing if necessary to a subsequence we can assume that $(\tilde{f}_{m_j})$ converges to f almost everywhere on I. Moreover for almost everywhere $s \in I$

$$\begin{aligned} d(f_{\lambda_n}(s), F_{\lambda_0}(s,x_s) &\leq \delta(F_{\lambda_n}(s, x_s^{\lambda_n}), F_{\lambda_0}(s,x_s)) \\ &\leq \delta(F_{\lambda_n}(s, x_s^{\lambda_n}), F_{\lambda_n}(s,x_s)) + \delta(F_{\lambda_n}(s,x_s), F_{\lambda_0}(s,x_s)) \\ &\leq k(s) \left\| x_s^{\lambda_n} - x_s \right\| + \delta(F_{\lambda_n}(s,x_s), F_{\lambda_0}(s,x_s)) \end{aligned}$$

and since $\lim_{\lambda \to \lambda_0} \delta(F_\lambda(t,\phi), F_{\lambda_0}(t,\phi)) = 0$, then

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N,\ f_{\lambda_n}(s) \in F_{\lambda_0}(s, x_s) + 2\varepsilon B \ \text{ a.e. in } I$$

where B is the closed unit ball in E, and then, for all $n > N$

$$\tilde{f}_{m_j}(s) \in \sum_{n=m_j}^{n_0(m_j)} \alpha_{m_j,n}(F_{\lambda_0}(s, x_s) + 2\varepsilon B) = F_{\lambda_0}(s, x_s) + 2\varepsilon B$$

taking the limit in the above formula, we deduce that for all $\varepsilon > 0$,
$f(s) \in F_{\lambda_0}(s, x_s) + 2\varepsilon B$ a.e. in I, and then

$$f(s) \in F_{\lambda_0}(s, x_s) \text{ a.e. in } I.$$

Remark Since, in the theorem 4.1, the assumption E is reflexive is used only for deducing the sequence (f_{λ_n}) converges weakly in $L^1(I; E)$, it may be replaced by the following assumption: there exists a $k \geq 0$ such that for all bounded subset $\Omega \subset \mathcal{C}$

$$\chi(F(t, \Omega)) \leq k\chi_0(\Omega) \ \text{ for all } t \in I$$

where χ (resp. χ_0) is the measure of noncompactness in E (resp. $\mathcal{C}$) (see for example [4, 11]). In this case, we obtain

$$\chi(\{f_{\lambda_n}(t) : n \in \mathbb{N}\}) \leq k\chi_0(\{x_t^{\lambda_n} : n \in \mathbb{N}\}) = 0$$

for almost all $t \in I$, i.e. the set $\{f_{\lambda_n}(t) : n \in \mathbb{N}\}$ is relatively compact in E a.e. in I and since $\sup_{n\in\mathbb{N}} \|f_{\lambda_n}\|_1 < +\infty$, then from Diestel'theorem [4] it follows that the sequence (f_{λ_n}) is relatively weak compact in the space $L^1(I; E)$.

Theorem 4.2 (E is not reflexive). Let (Λ, d_Λ) be a metric space, $F_\lambda : I \times \mathcal{C} \to \mathcal{F}(E)$ a family of multimappings satisfying the conditions $(H_1), (H_2)$ with the same function k. If for any $(t, \phi) \in I \times \mathcal{C}$ the multimapping $\lambda \longmapsto F_\lambda(t, \phi)$ is lower semicontinuous at $\lambda_0 \in \Lambda$, then for all $\varphi \in \mathcal{B}$, $\lambda \longmapsto S_{F_\lambda}(\varphi)$ is lower semicontinuous at λ_0.

Proof. Since the case $S_{F_{\lambda_0}}(\varphi) = \emptyset$ is trivial, we assume that $S_{F_{\lambda_0}}(\varphi) \neq \emptyset$. Let $x \in S_{F_{\lambda_0}}(\varphi)$ then,

$$x(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I \end{cases}$$

where $f(s) \in F_{\lambda_0}(s, x_s) \subset \liminf_{\lambda \to \lambda_0} F_\lambda(s, x_s)$ a.e. in I, thus
$\lim_{\lambda \to \lambda_0} d(f(s), F_\lambda(s, x_s)) = 0$ a.e., and then for $\varepsilon > 0$, there exists $\rho > 0$ such that $d_\Lambda(\lambda, \lambda_0) < \rho$ implies $d(f(s), F_\lambda(s, x_s)) < \frac{\varepsilon}{2\omega K(\omega)}$. Thus for $\lambda \in \Lambda$ such that

$$d_\Lambda(\lambda, \lambda_0) < \rho,\ t \longmapsto d(f(t), F_\lambda(t, x_t)) = q(t)$$

is integrable and x is a mild solution of

$$\begin{cases} x''(t) = Ax(t) + f(t) \\ x_0 = \varphi \end{cases}$$

and by theorem 3.1 with $\mu = 0$ and $v(t) = \frac{\varepsilon}{2\omega K(\omega)}$ there exists a function $x^\lambda \in S_{F_\lambda}(\varphi)$ (for $d_\Lambda(\lambda, \lambda_0) < \rho$) such that

$$\|x^\lambda - x\|_\omega \leq K(\omega)m(\omega) = K(\omega)[\int_0^\omega (q(t) + v(t))dt] = \varepsilon,$$

hence $x \in \liminf_{\lambda \to \lambda_0} S_{F_\lambda}(\varphi)$. ■

Combining theorems 4.1 and 4.2, we obtain.

Corollary Let (Λ, d_Λ) be a metric space, $F_\lambda : I \times \mathcal{C} \to \mathcal{F}_c(E)$ a family of multimappings satisfying the conditions (H_1), (H_2) with the same function k and (H_4) with the same function ν. If for any $(t, \phi) \in I \times \mathcal{C}$, $\lim_{\lambda \to \lambda_0} \delta(F_\lambda(t, \phi), F_{\lambda_0}(t, \phi)) = 0$, then for all $\varphi \in \mathcal{B}$, $\lambda \longmapsto S_{F_\lambda}(\varphi)$ is continuous at λ_0.

Theorem 4.3 Assume that $F : I \times \mathcal{C} \to \mathcal{F}_c(E)$ satisfying the conditions $(H_1), (H_2)$ and (H_4). Then $S_F : \mathcal{C}^1 \to \mathcal{P}(\mathcal{C}_\omega)$ is continuous on $\mathcal{C}^1$, where $\mathcal{C}^1 := C^1(J; E)$ denote the Banach space of continuously differentiable E-valued functions on J with the norm $\|\varphi\|_{\mathcal{C}^1} = \|\varphi\| + \|\varphi'\|$.

Proof. For any $\varphi_1, \varphi_2 \in \mathcal{C}^1$, let $F_{\varphi_2}(t, \phi) = F(t, \phi + (\widetilde{\varphi_2})_t - (\widetilde{\varphi_1})_t)$ for all $(t, \phi) \in I \times \mathcal{C}$ then $S_F(\varphi_2) = S_{F_{\varphi_2}}(\varphi_1) + \tilde{\varphi}_2 - \tilde{\varphi}_1$ where

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) & \text{if } t \in I \end{cases}$$

indeed,

$$\begin{aligned} x \in S_{F_{\varphi_2}}(\varphi_1) &\Leftrightarrow x(t) = \begin{cases} \varphi_1(t) & \text{if } t \in J \\ C(t)\varphi_1(0) + S(t)\varphi_1'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I \end{cases} \\ \text{where } f(s) &\in F_{\varphi_2}(s, x_s) \text{ a.e.} \\ &\Leftrightarrow x(t) + \tilde{\varphi}_2(t) - \tilde{\varphi}_1(t) = \begin{cases} \varphi_2(t) \\ C(t)\varphi_2(0) + S(t)\varphi_2'(0) + \int_0^t S(t-s)(f(s))ds \end{cases} \\ \text{where } f(s) &\in F(s, x_s + (\widetilde{\varphi_2})_s - (\widetilde{\varphi_1})_s) = F(s, (x + \tilde{\varphi}_2 - \tilde{\varphi}_1)_s) \text{ a.e.} \\ &\Leftrightarrow x + \tilde{\varphi}_2 - \tilde{\varphi}_1 \in S_F(\varphi_2). \end{aligned}$$

Furthermore, it is clear that $\varphi_2 \longmapsto F_{\varphi_2}(t, \phi)$ (for all $(t, \phi) \in I \times \mathcal{C}$) is continuous at φ_1 and the family $(F_{\varphi_2})_{\varphi_2 \in \mathcal{C}^1}$ satisfy the assumptions of precedent corollary, therefore for all $\varphi \in \mathcal{C}^1, \varphi_2 \longmapsto S_{F_{\varphi_2}}(\varphi)$ is continuous at φ_1 and then

$$\begin{aligned} \lim_{\varphi_2 \to \varphi_1} S_F(\varphi_2) &= \lim_{\varphi_2 \to \varphi_1} (S_{F_{\varphi_2}}(\varphi_1) + \tilde{\varphi}_2 - \tilde{\varphi}_1) \\ &= S_{F_{\varphi_1}}(\varphi_1) \\ &= S_F(\varphi_1). \ ■ \end{aligned}$$

Theorem 4.4 (E is not reflexive) Assume that $F : I \times \mathcal{C} \to \mathcal{F}_c(E)$ satisfying the conditions $(H_1), (H_2)$ and (H_4') i.e. there exists a compact $K \subset E$ such that for every $(t, \phi) \in I \times \mathcal{C}$, $F(t, \phi) \subset K$. Then for all $\varphi \in \mathcal{B}$, $S_F(\varphi)$ is compact.

Proof. We prove first that $S_F(\varphi)$ is relatively compact. Let (x^n) be a sequence of $S_F(\varphi)$, then for all $n \in \mathbb{N}$

$$x^n(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_n(s))ds & \text{if } t \in I \end{cases}$$

where $f_n(s) \in F(s, x_s^n)$ a.e. in I.
We shall show that $\mathcal{A} := \{x^n_{|_I} : n \in \mathbb{N}\}$ is equicontinuous. For each $0 \leq t_0 < t \leq \omega$ and $n \in \mathbb{N}$

$$\begin{aligned} |x^n(t) - x^n(t_0)|_E \leq\ & |C(t)\varphi(0) - C(t_0)\varphi(0)|_E + |S(t)\varphi'(0) - S(t_0)\varphi'(0)|_E + \\ & \int_0^{t_0} \|S(t-s) - S(t_0-s)\| \ |f_n(s)|_E ds + \int_{t_0}^t \|S(t-s)\| \ |f_n(s)|_E ds \end{aligned}$$

but,

$$\begin{aligned} \|S(t-s) - S(t_0-s)\| &= \left\| \int_0^{t-s} C(\tau)d\tau - \int_0^{t_0-s} C(\tau)d\tau \right\| \\ &\leq \int_{t_0-s}^{t-s} \|C(\tau)\| \, d\tau \\ &\leq \int_{t_0-s}^{t-s} \eta e^{\alpha\tau} d\tau \\ &\leq \eta\alpha^{-1}[e^{\alpha(t-s)} - e^{\alpha(t_0-s)}] \\ &\leq \eta(t-t_0)e^{\alpha\omega} \end{aligned}$$

($\alpha^{-1}[e^{\alpha(t-s)} - e^{\alpha(t_0-s)}]$ is replaced by $t - t_0$ when $\alpha = 0$), then

$$\int_0^{t_0} \|S(t-s) - S(t_0-s)\| \ |f_n(s)|_E ds \leq \eta(t-t_0)e^{\alpha\omega} \int_0^{t_0} |f_n(s)|_E \, ds.$$

Also,

$$\int_{t_0}^t \|S(t-s)\| \ |f_n(s)|_E ds \leq \eta(t-t_0)e^{\alpha\omega} \int_{t_0}^t |f_n(s)|_E \, ds.$$

Since f_n are integrably bounded and the maps $t \to C(t)\varphi(0), t \to S(t)\varphi'(0)$ are uniformly continuous on I, we obtain that $\mathcal{A}$ is equicontinuous, clearly it is also bounded. Now, we prove that $\mathcal{A}(t) = \{x^n(t) : n \in \mathbb{N}\}$ is relatively compact. For all $s \in I$, $S(t-s) : E \to E$ is continuous, then by assumption (H_4') we have that
$K_1 = \{S(t-s)f_n(s) : s \in [0,t] \text{ and } n \in \mathbb{N}\}$ is relatively compact, thus $K_2 = \overline{co}K_1$ is compact and $K_3 = \{tx : (t,x) \in I \times K_2\}$ is compact. Consequently
$\mathcal{A}(t) \subset C(t)\varphi(0) + S(t)\varphi'(0) + K_3$ is relatively compact. From the Ascoli theorem [4, 11] we may assume that the sequence (x^n) converges to some $x \in \mathcal{C}_\omega$. We prove next that $x \in S_F(\varphi)$. By condition (H_4'), the set $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact in E and since $\sup_{n \in IN} \|f_n\|_1 < +\infty$, then from Diestel's theorem [4] it follows that the sequence (f_n) is relatively weak compact in the space $L^1(I; E)$ and by using exactly the same method as in the proof of theorem 4.1 we obtain $x \in S_F(\varphi)$. ∎