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Abstract

In this paper, we consider the second-order delay differential inclusion \( x''(t) \in Ax(t) + F(t, x_t) \) in a Banach space and we study some properties of its solution set. We prove a relaxation theorem which reveals the connection between the solution sets of a second-order delay differential inclusion and its convexified version, under some weak conditions.

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1 Introduction

Many problems in applied mathematics, such as those in control theory, lead to the study of second-order delay differential inclusions

\[ x''(t) \in Ax(t) + F(t, x_t), \]

where \( A \) is the infinitesimal generator of a \( C_0 \)-propagator of linear operators \( (C(t))_{t \in \mathbb{R}} \) on a Banach space \( (E, |.|_E) \) and \( F \) is a nonlinear multimapping, satisfying assumptions to be specified in the third section.

As particular cases of relations of the form (1) we have:

i) The second-order delay differential equation

\[ x''(t) = Ax(t) + f(t, x_t), \]

where \( F(t, x_t) = f(t, x_t) \).

ii) The differential inequalities

\[ |x''(t) - Ax(t) - f(t, x_t)|_E \leq g(t, x_t) \]

where \( F(t, x_t) \) is the ball of radius \( g(t, x_t) \) centered at \( Ax(t) + f(t, x_t) \).

iii) Control problems where the control \( u(t) \) and the trajectory \( x(t) \) are related by the second-order delay differential equation

\[ x''(t) = Ax(t) + f(t, x_t, u(t)), \quad u(t) \in U(t). \]
Here, the control function \( u(t) \) is a measurable function and \( F(t, x_t) = f(t, x_t, U(t)) \).
This paper is concerned with the second-order delay differential inclusion (1) and its mild trajectories. We show that many results which allow us to apply differential inclusions, see for example [1, 3, 8, 10, 13] and references therein, are valid as well for (1). In our relaxation theorem, the assumption of integrable boundedness (condition \((H_4)\)) will be replaced by an integrability condition (condition \((H'_4)\)). We also give some properties of the solution set of the inclusion (1).

2 Preliminaries

For a real Banach space \((E, ||.||_E)\) and \( J := [-r, 0) \) \((r > 0)\), let \( C := C([-r, 0]; E) \) be the Banach space of continuous functions from \( J \) to \( E \) with the usual supremum norm \(||.||\). For any continuous function \( x \in C([-r, \omega]; E) \) \((\omega > 0)\) and any \( t \in I := [0, \omega] \) we denote by \( x_t \) the element of \( C \) defined by \( x_t(\theta) = x(t + \theta), \theta \in J \).
For a subset \( A \subset E \), \( \text{co} A \), \( \overline{\text{co}} A \) and \( \text{cl} A \) are respectively the convex hull, the closed convex hull and the closure. We denote by \( \mathcal{F}(E) \) (resp. \( \mathcal{F}_c(E) \)) the family of all nonempty closed (resp. closed convex) subsets of \( E \), and by \( \delta \) the Hausdorff distance in \( \mathcal{F}(E) \), i.e. for \( A, B \in \mathcal{F}(E) \)

\[
\delta(A, B) = \max \left[ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right]
\]

where \( d(a, B) = \inf_{b \in B} d(a, b) \).
Next we present some basic concepts concerning multimappings.
Let \( X \) be another Banach space, for a multimapping \( G : X \rightarrow \mathcal{P}(E) \) (the family of all nonempty subsets of \( E \)), we define its lim sup and lim inf at \( x \in X \) in the Kuratowski sense by

\[
\limsup_{y \to x} G(y) = \{ z \in E : \liminf_{y \to x} d(z, G(y)) = 0 \}
\]

and

\[
\liminf_{y \to x} G(y) = \{ z \in E : \lim_{y \to x} d(z, G(y)) = 0 \}.
\]

We say that the limit of \( G(y) \) as \( y \) tends to \( x \) exists in the Kuratowski sense if

\[
\limsup_{y \to x} G(y) = \liminf_{y \to x} G(y).
\]
We denote this limit by \( \lim_{y \to x} G(y) = G(x) \). We say that \( G \) is upper (resp. lower) semicontinuous at \( x \) if

\[
\limsup_{y \to x} G(y) \subseteq G(x) \quad (\text{resp. } G(x) \subseteq \liminf_{y \to x} G(y)).
\]
If \( G \) is both upper and lower semicontinuous at \( x \) then we say that \( G \) is continuous at \( x \). If \( G \) is continuous or semicontinuous for all \( x \in X \), we say that \( G \) is continuous or semicontinuous on \( X \).
Let $G : I \to \mathcal{P}(E)$ be a multimapping. A function $g : I \to E$ such that $g(t) \in G(t)$ for every $t \in I$ is called a selection of $G$.

$G$ is called measurable if, for almost all $t \in I$

$$G(t) \subseteq \text{cl}\{g_n(t) : n \geq 1\}$$

where $g_n$ are measurable selections of $G$. This definition of the measurability is given by Zhu [13], when $E$ is separable and $G(t) \in \mathcal{F}(E)$ for every $t \in I$ this definition is the same as the classic one (see for example [3]).

By the symbol of $I^1_G$ we will denote the set of all Bochner integrable selections of the multimapping $G$, i.e.

$$I^1_G = \{g \in L^1(I; E) : g(t) \in G(t) \text{ a.e.}\}.$$ 

If $I^1_G \neq \emptyset$, then the measurable multimapping $G$ is called integrable and

$$\int_I G(t) dt = \{\int_I g(t) dt : g \in I^1_G\}.$$ 

Clearly if $G$ is measurable and integrably bounded, i.e. there exists $\nu \in L^1(I)$ such that

$$\|G(t)\| = \sup\{|e|_E : e \in G(t)\} \leq \nu(t) \text{ a.e.}$$

then $G$ is integrable. But the converse is not true.

We will also need the following properties (see [13]) which will be used later.

**Lemma 2.1** Let $G : I \to \mathcal{P}(E)$ be a measurable multimapping. Then so is $\text{co}G$.

**Lemma 2.2** Let $G : I \to \mathcal{P}(E)$ be an integrable multimapping. Then $\text{cl} \int_I G(t) dt$ is a convex set and

$$\text{cl} \int_I G(t) dt = \text{cl} \int_I \text{co}G(t) dt = \text{cl} \int_I \overline{\text{co}}G(t) dt.$$ 

**Remark** If $G : I \to \mathcal{P}(E)$ is an integrable multimapping, then so is $\overline{G}$ where $\overline{G}(t) = \text{cl}G(t)$ and

$$\text{cl} \int_I G(t) dt = \text{cl} \int_I \overline{G}(t) dt$$

(Indeed $\text{cl} \int_I G(t) dt \subseteq \text{cl} \int_I \overline{G}(t) dt \subseteq \text{cl} \int_I \overline{\text{co}}G(t) dt = \text{cl} \int_I G(t) dt$).

**Lemma 2.3** Let $G : I \to \mathcal{P}(E)$ be a measurable multimapping and $u : I \to E$ a measurable function. Then for any measurable function $v : I \to \mathbb{R}^+$, there exists a measurable selection $g$ of $G$ such that

$$|g(t) - u(t)|_E \leq d(u(t), G(t)) + v(t) \text{ a.e.}$$

At last, we give some important properties of a $C_0$-propagator and its infinitesimal generator (see [7]).

A strongly continuous propagator $(C(t))_{t \in \mathbb{R}}$ of continuous operators on $E$ is a family of continuous linear mappings $C(t) : E \to E$, $t \in \mathbb{R}$, satisfying

i) $C(0) = I$;

ii) $C(t + s) + C(t - s) = 2C(t)C(s)$;
iii) for $x \in E$, $C(.)x : \mathbb{R} \rightarrow E$ is continuous.

A strongly continuous propagator of continuous linear mappings is also called a $C_0$-propagator. A linear operator $A$ is associated with a propagator, it plays the role of the infinitesimal generator for $C_0$-semigroups:

$$D(A) = \{ x \in E : \lim_{h \searrow 0} \frac{2}{h^2} [C(h) - I]x \text{ exists} \}$$

and

$$Ax = \lim_{h \searrow 0} \frac{2}{h^2} [C(h) - I]x \text{ for } x \in D(A)$$

is the infinitesimal generator of the $C_0$-propagator $(C(t))_{t \in \mathbb{R}}$, $D(A)$ is the domain of $A$. We have:

- There exist constants $\alpha \geq 0$ and $\eta \geq 1$ such that
  $$\|C(t)\| \leq \eta e^{\alpha |t|} \text{ for } t \in \mathbb{R}.$$  

- $D(A)$ is dense in $E$ and $A$ is a closed linear operator.

- For every $x \in D(A)$ and $t \in \mathbb{R}$, then $C(t)x \in D(A)$ and
  $$\frac{d^2}{dt^2} C(t)x = AC(t)x = C(t)Ax.$$  

- Let $a, b \in E$ and $f \in L^1(I; E)$, the function $u \in C(I; E)$ given by
  $$u(t) = C(t)a + S(t)b + \int_0^t S(t-s)(f(s))ds, \; t \in I$$

is the mild solution on $I$ of the initial value problem

$$\begin{cases} 
  u''(t) = Au(t) + f(t), \; t \in I \\
  u(0) = a, \; u'(0) = b 
\end{cases}$$

where $S(t) = \int_0^t C(s)ds$. Moreover

$$|u(t)|_E \leq \eta e^{\alpha t} |a|_E + \eta \alpha^{-1}(e^{\alpha t} - 1) |b|_E + \eta \alpha^{-1}(e^{\omega \alpha} - 1) \|f\|_1, \; t \in I$$

($\alpha^{-1}(e^{\alpha t} - 1)$ is replaced by $t$ when $\alpha = 0$). If $a = 0$ then $u$ is continuously differentiable and

$$|u'(t)|_E \leq \eta e^{\alpha t} |b|_E + \eta e^{\omega \alpha} \|f\|_1, \; t \in I.$$ 

3 The solution set of a second-order delay differential inclusion and a relaxation theorem

Consider the functional differential inclusion

$$x''(t) \in Ax(t) + F(t, x_t) \text{ a.e. in } I$$

(3.1)
**Definition 3.1** A function \( x \in C_\omega := C([-r, \omega]; E) \) is called a mild trajectory of (3.1) if there exist \( \varphi \in B := \{ \varphi \in C : \varphi'(0) \) exists\} and a Bochner integrable function \( f \in L^1(I; E) \) such that

\[
f(t) \in F(t, x_t) \text{ a.e. in } I
\]  

and

\[
x(t) = \begin{cases} 
\varphi(t), & t \in J \\
C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds, & t \in I
\end{cases}
\]  

i.e., \( f \) is a Bochner integrable selection of the multimapping \( t \mapsto F(t, x_t) \) and \( x \) is a mild solution of the initial value problem

\[
\begin{cases}
x''(t) = Ax(t) + f(t), & t \in I \\
x_0 = \varphi, & \varphi \in B.
\end{cases}
\]  

For \( \varphi \in B \), we define \( S_F(\varphi) = \{ x \in C_\omega : x \text{ is a mild trajectory of } (3.1) \text{ with } x_0 = \varphi \} \) to be the solution set of (3.1) from the point \( \varphi \).

Let \( \psi \in B, g \in L^1(I, E) \) and \( y \in C_\omega \) be a mild solution of the problem

\[
(C) \begin{cases}
y''(t) = Ay(t) + g(t), & t \in I \\
y_0 = \psi.
\end{cases}
\]

Suppose that the multimapping \( F : I \times C \to \mathcal{F}(E) \) satisfies the following conditions:

\( H_1 \) For every \( \varphi \in C \), the multimapping \( F(., \phi) \) is measurable on \( I \).

\( H_2 \) There is an integrable function \( k : I \to \mathbb{R}^+ \) such that for every \( \phi, \xi \in C \),

\[
d(F(t, \phi), F(t, \xi)) \leq k(t)\|\phi - \xi\| \text{ a.e. in } I.
\]

\( H_3 \) The function \( q : t \mapsto d(g(t, F(t, y_t))) \) is integrable on \( I \).

\( H'_3 \) For any function \( x \in C_\omega \), the multimapping \( t \mapsto F(t, x_t) \) is integrable on \( I \).

\( H_4 \) There is an integrable function \( \nu \in L^1(I) \) such that

\[
\|F(t, \phi)\| := \sup\{|y|_E : y \in F(t, \phi)\} \leq \nu(t)
\]

for all \( \phi \in C \) and almost all \( t \in I \).

**Remarks**

- When \( F \) satisfies \((H_1)\) and \((H_2)\), then \( t \to F(t, y_t) \) and \( q \) are measurable on \( I \).
- If \( q \) is measurable, then the condition \((H'_3)\) gives \((H_3)\).
- When \( F \) satisfies \((H_1)\) and \((H_2)\) it satisfies \((H'_3)\) if and only if it satisfies: there is \( z \in C_\omega \) such that the multimapping \( t \to F(t, z_t) \) is integrable (see [13]).
- When \( F \) satisfies \((H_2)\), then for every integrable function \( k' > k \) and \( \phi, \xi \in C \),

\[
F(t, \phi) \subset F(t, \xi) + k'(t)\|\phi - \xi\|B \text{ a.e. in } I
\]

where \( B \) denotes the closed unit ball in \( E \).

Next we present a useful result on the relationships between the trajectories of (3.1) and the solutions of problem \((C)\).
Theorem 3.1 Let \( \psi \in B, g \in L^1(I; E) \) and \( y \in C_0 \) be a mild solution of problem (C). Assume that \((H_1) - (H_3)\) hold true and let \( \mu \geq 0 \). Then for all \( \varphi \in B \) with \( \| \varphi - \psi \| \leq \mu, \| \varphi(0) - \psi(0) \|_E \leq \mu \) and for all integrable function \( v : I \to \mathbb{R}^+ \), there exist \( x \in C_\omega \) and \( f \in L^1(I; E) \) satisfying (2), (3) and

\[
\| x - y \|_\omega \leq K(\omega)m(\omega), ~ \| f - g \|_1 \leq K(\omega)m(\omega)
\]

where \( M = \eta(e^{\alpha \omega} + e^{\alpha^2 \omega - 1}) \), \( (e^{\alpha^2 \omega - 1}) \) is replaced by \( \omega \) when \( \alpha = 0 \),

\[
K(t) = M\exp M \int_0^t 2k(s)ds, ~ m(t) = \mu + \int_0^t (q(s) + v(s))ds.
\]

Proof. By lemma 2.3, there is a measurable selection \( f_1 \) of the multimapping \( t \mapsto F(t, y_t) \) such that, for almost all \( t \in I \),

\[
|f_1(t) - g(t)|_E \leq d(g(t), F(t, y_t)) + v(t) \leq q(t) + v(t)
\]

and then \( f_1 \in L^1(I; E) \). Set

\[
x^1(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_1(s))ds & \text{if } t \in I \end{cases}
\]

we have \( x^1 \in C_\omega \) and for all \( t \in I \),

\[
\| x^1_t - y_t \| = \sup_{\theta \in J} |x^1(t + \theta) - y(t + \theta)|_E \\
\leq M(\mu + \int_0^t |f_1(s) - g(s)|_E ds) \\
\leq M(\mu + \int_0^t (q(s) + v(s))ds).
\]

By using lemma 2.3, there is a measurable selection \( f_2 \) of the multimapping \( t \mapsto F(t, x^1_t) \) such that, for almost all \( t \in I \),

\[
|f_2(t) - f_1(t)|_E \leq 2d(f_1(t), F(t, x^1_t)) \\
\leq 2\delta(F(t, y_t), F(t, x^1_t)) \\
\leq 2k(t)\| x^1_t - y_t \|
\]

and then \( f_2 \in L^1(I; E) \). Set

\[
x^2(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_2(s))ds & \text{if } t \in I \end{cases}
\]

Thus, we can define by induction two sequences \( (x^n) \) and \( (f_n) \) with \( x^n \in C_\omega \) and \( f_n \in L^1(I; E) \) such that:

i) \( x^0 = y \) and for all \( n \geq 1 \),

\[
x^n(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_n(s))ds & \text{if } t \in I \end{cases}
\]
ii) \( f_0 = g \) and for all \( n \geq 1 \)
\[
f_n(t) \in F(t, x_t^{n-1}) \text{ a.e. in } I;
\]

iii) for almost all \( t \in I \) and \( n \geq 1 \),
\[
|f_{n+1}(t) - f_n(t)|_E \leq 2k(t)\|x_t^n - x_t^{n-1}\|.
\]

It follows then from (iii) that

iv) for all \( t \in I \) and \( n \geq 1 \),
\[
\|x_t^{n+1} - x_t^n\| \leq M \int_0^t |f_{n+1}(t_1) - f_n(t_1)|_E dt_1 \\
\leq M \int_0^t 2k(t_1)\|x_{t_1}^n - x_{t_1}^{n-1}\| dt_1 \\
\leq M \int_0^t 2k(t_1)[M \int_0^{t_1} 2k(t_2)\|x_{t_2}^{n-1} - x_{t_2}^{n-2}\| dt_2] dt_1 \\
\vdots \\
\leq M^n \int_0^t 2k(t_1) \int_0^{t_1} 2k(t_2) \cdots \int_0^{t_{n-1}} 2k(t_n)\|x_{t_n}^1 - y_{t_n}\| dt_n \cdots dt_1 \\
\leq M[\eta + \int_0^t (g(s) + v(s))ds].\left(\frac{M \int_0^t 2k(s)ds}{n!}\right)^n.
\]

Then, for all \( n \geq 1 \)
\[
\|x_t^{n+1} - x_t^n\|_\omega := \max(\|x_t^{n+1} - x_t^n\|, \sup_{t \in I} |x_t^{n+1}(t) - x_t^n(t)|_E) \\
= \sup_{t \in I} |x_t^{n+1}(t) - x_t^n(t)|_E \\
\leq \sup_{t \in I} \|x_t^{n+1} - x_t^n\| \\
\leq Mm(\omega)\left(\frac{M \int_0^t 2k(t)dt}{n!}\right)^n.
\]

By (iv) we obtain for all \( t \in I \) and \( n \geq 1 \),
\[
\|x_t^{n+1} - y_t\| \leq \|x_t^1 - y_t\| + \sum_{i=1}^n \|x_t^{i+1} - x_t^i\| \\
\leq Mm(t)[1 + \sum_{i=1}^n \left(\frac{M \int_0^t 2k(s)ds}{i!}\right)^i] \\
\leq K(t)m(t).
\]

We deduce that \((x^n)\) is a Cauchy sequence of a continuous functions, converging uniformly to a function \( x \in C_\omega \) and for almost all \( t \in I \), \((f_n(t))\) is a Cauchy sequence in
hence \( (f_n(.)) \) converges pointwise almost everywhere to a measurable function \( f(.) \) in \( E \). But for almost all \( t \in I \) and \( n \in \mathbb{N} \)

\[
|f_{n+1}(t) - g(t)|_E \leq \sum_{i=1}^{n} |f_{i+1}(t) - f_i(t)|_E + |f_1(t) - g(t)|_E \\
\leq 2k(t) \sum_{i=1}^{n} \|x^i_t - x^{i-1}_t\| + q(t) + v(t) \\
\leq 2k(t)K(\omega)m(\omega) + q(t) + v(t)
\]
hence, \( |f_{n+1}(t)|_E \leq |g(t)|_E + 2k(t)K(\omega)m(\omega) + q(t) + v(t) \), thus \( (f_n) \) converges to \( f \) in \( L^1(I; E) \) and then \( (x^n(t)) \) \( (t \in [-r, \omega]) \) converges in \( E \) to

\[
\begin{cases}
\varphi(t) & \text{if } t \in J \\
C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t - s)(f(s))ds & \text{if } t \in I,
\end{cases}
\]
we obtain

\[
x(t) = \begin{cases}
\varphi(t) & \text{if } t \in J \\
C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t - s)(f(s))ds & \text{if } t \in I.
\end{cases}
\]
Furthermore, for almost all \( t \in I \)

\[
d(f(t), F(t, x_t)) \leq |f(t) - f_n(t)|_E + d(f_n(t), F(t, x_t)) \\
\leq |f(t) - f_n(t)|_E + \delta(F(t, x_t^{n-1}), F(t, x_t)) \\
\leq |f(t) - f_n(t)|_E + k(t)\|x^i_t - x_t\|.
\]
The right hand side tends to zero almost everywhere on \( I \) as \( n \to +\infty \). Thus, for almost all \( t \in I \), \( f(t) \in F(t, x_t) \).
Consequently \( x \in S_F(\varphi) \), moreover, for all \( n \in \mathbb{N} \)

\[
\|x^{n+1} - y\|_\omega \leq \sup_{t \in I} \|x^{n+1}_t - y_t\| \\
\leq K(\omega)m(\omega).
\]
Taking limits in the precedent inequality, we have \( \|x - y\|_\omega \leq K(\omega)m(\omega) \).
We now show \( \|f - g\|_1 \leq K(\omega)m(\omega) \).
For almost all \( t \in I \) and \( n \in \mathbb{N} \), we have

\[
|f_{n+1}(t) - g(t)|_E \leq q(t) + v(t) + 2k(t)Mm(\omega)\sum_{i=1}^{n} \left[ M \int_0^t 2k(s)ds \right]^{i-1} \\
\frac{(i-1)!}{(i-1)!}
\]
thus,

\[
\|f_{n+1} - g\|_1 \leq m(\omega)[1 + \sum_{i=1}^{n} \left[ M \int_0^{\omega} 2k(t)dt \right]^{i}] \\
\leq m(\omega)K(\omega).
\]
Taking the limit in the above inequality, we obtain \( \|f - g\|_1 \leq m(\omega)K(\omega). \)

In the next theorem we compare trajectories of (3.1) and of the convexified (relaxed) second-order delay differential inclusion \( x''(t) \in Ax(t) + \overline{co}F(t, x_t) \) (3.2).

For \( \varphi \in B \), we put

\[
S_{\overline{co}F}(\varphi) = \{ x \in C_\omega : x \text{ is a trajectory of } (3.2) \text{ with } x_0 = \varphi \}.
\]

**Theorem 3.2** Assume that \( F \) satisfies conditions \((H_1), (H_2) \) and \((H'_2)\). Then, for all \( \varphi \in B \),

\[
clS_F(\varphi) = clS_{\overline{co}F}(\varphi).
\]

**Proof.** It is easy to see that \( clS_F(\varphi) \subseteq clS_{\overline{co}F}(\varphi) \). Conversely, we shall show that \( S_{\overline{co}F}(\varphi) \subseteq clS_F(\varphi) \). Let \( y \in S_{\overline{co}F}(\varphi) \), then there exists \( g \in L^1(I; E) \) such that

\[
y(t) = \begin{cases} 
\varphi(t) & \text{if } t \in J \\
C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t - s)(g(s))ds & \text{if } t \in I
\end{cases}
\]

where \( g(s) \in \overline{co}F(s, y_s) \) a.e. in \( I \).

The following result follows immediately from [3 p. 85].

**Lemma 3.1**

Let \( G : I \rightarrow P(E) \) be a measurable multimapping, then so is \( s \rightarrow S(t - s)G(s) \). Moreover if \( f(s) \in S(t - s)G(s) \) then, there exists a measurable selection \( g(s) \in G(s) \) such that \( f(s) = S(t - s)g(s) \) a.e. in \( I \).

By \((H'_2)\) for all fixed \( t \) in \( I \), the multimapping \( s \mapsto S(t - s)F(s, y_s) \) is integrable on \( I \) and by lemma 2.2 and its remark we obtain

\[
s \mapsto clS(t - s)F(s, y_s) \text{ and } s \mapsto \overline{co}S(t - s)F(s, y_s)
\]

are also integrable on \( I \) and

\[
cl \int_I S(t - s)F(s, y_s)ds = cl \int_I clS(t - s)F(s, y_s)ds = cl \int_I \overline{co}S(t - s)F(s, y_s)ds
\]

but, \( \overline{co}S(t - s)F(s, y_s) = clS(t - s)\overline{co}F(s, y_s) \), indeed

\[
S(t - s)F(s, y_s) \subset clS(t - s)\overline{co}F(s, y_s)
\]

which is a closed convex set and then

\[
\overline{co}S(t - s)F(s, y_s) \subset clS(t - s)\overline{co}F(s, y_s),
\]

conversely, it suffice to see that

\[
S(t - s)\overline{co}F(s, y_s) \subset \overline{co}S(t - s)F(s, y_s)
\]
let $f(s) \in S(t - s)\overline{co}F(s, y_s)$, then there exists $g(s) \in \overline{co}F(s, y_s)$ such that $f(s) = S(t - s)g(s)$ hence, there exists a sequence $(g_n(s))$ such that $g_n(s) \in coF(s, y_s)$ and $\lim_{n \to +\infty} g_n(s) = g(s)$, we put

$$f_n(s) = S(t - s)g_n(s) \in S(t - s)coF(s, y_s) = coS(t - s)F(s, y_s)$$

and taking the limit as $n \to +\infty$, we obtain

$$f(s) = S(t - s)g(s) \in cl \, coS(t - s)F(s, y_s)$$

thus,

$$cl \int_I S(t - s)F(s, y_s)ds = cl \int_I cl \, S(t - s)\overline{co}F(s, y_s)ds$$

$$= cl \int_I S(t - s)\overline{co}F(s, y_s)ds$$

(see remark of lemma 2.2).

By lemma 3.1, we obtain for all $\varepsilon > 0$ an integrable selection $h(s) \in F(s, y_s)$ a.e. such that

$$|\int_I S(t - s)(g(s))ds - \int_I S(t - s)(h(s))ds|_E < \frac{\varepsilon}{K(\omega)(\|k\|_1 + \omega) + 1},$$

set

$$z(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t - s)(h(s))ds & \text{if } t \in I \end{cases}$$

then $z$ is a mild solution of problem

$$\begin{cases} z''(t) = Az(t) + h(t) \\ z_0 = \varphi. \end{cases}$$

Moreover by assumption $(H'3)$, the function $t \mapsto q(t) = d(h(t), F(t, z_t))$ is integrable on $I$. It follows from theorem 3.1 for $\mu = 0$ and $v(t) = \frac{\frac{\varepsilon}{K(\omega)(\|k\|_1 + \omega)} + 1}{\int_0^\omega q(t)dt + \int_0^\omega v(t)dt}$ there exists $x \in S_F(\varphi)$ such that

$$\|x - z\|_\omega \leq K(\omega)\left[\int_0^\omega q(t)dt + \int_0^\omega v(t)dt\right]$$

$$\leq \frac{\varepsilon K(\omega)(\|k\|_1 + \omega)}{K(\omega)(\|k\|_1 + \omega) + 1}$$

thus,

$$\|x - y\|_\omega \leq \|x - z\|_\omega + \|z - y\|_\omega$$

$$\leq \frac{\varepsilon K(\omega)(\|k\|_1 + \omega)}{K(\omega)(\|k\|_1 + \omega) + 1} + \frac{\varepsilon}{K(\omega)(\|k\|_1 + \omega) + 1}$$

$$\leq \varepsilon.$$
4 Some properties of the solution set

In this section, we discuss the continuous dependence of the solution set on parameters and initial value. We suppose that $E$ is a reflexive Banach space.

**Theorem 4.1.** Let $(A, d_A)$ be a metric space, $F_\lambda : I \times C \to \mathcal{F}_c(E)$ a family of multimappings satisfying conditions $(H_1), (H_2)$ with the same function $k$ and $(H_4)$ for the same function $\nu$. If for any $(t, \phi) \in I \times C$, $\lim_{\lambda \to \lambda_0} \delta(F_\lambda(t, \phi), F_{\lambda_0}(t, \phi)) = 0$, then for all $\varphi \in B$, $\lambda \mapsto S_{F_\lambda}(\varphi)$ is upper semicontinuous at $\lambda_0$.

**Proof.** Let $x \in \limsup_{\lambda \to \lambda_0} S_{F_\lambda}(\varphi)$, there exists a sequence $(\lambda_n)$ such that $\lim_{n \to +\infty} \lambda_n = \lambda_0$ and $x^{\lambda_n} \in S_{F_{\lambda_n}}(\varphi)$ such that $\lim_{n \to +\infty} x^{\lambda_n} = x$ in $C_w$, hence

$$x^{\lambda_n}(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_{\lambda_n}(s))\,ds & \text{if } t \in I \end{cases}$$

where $f_{\lambda_n}(s) \in F_{\lambda_n}(s, x^{\lambda_n})$ a.e. in $I$.

The sequence $(f_{\lambda_n})$ is integrably bounded and $E$ is reflexive, then by the Dunford-Pettis theorem [12], taking a subsequence and keeping the same notation, we may assume that it converges weakly in $L^1(I; E)$ to some function $f \in L^1(I; E)$. For each $t \in I$, the mapping

$$g \in L^1(I; E) \rightarrow \int_0^t S(t-s)(g(s))\,ds$$

is a continuous linear operator from $L^1(I; E)$ into $E$. It remains continuous if these spaces are endowed with the weak topologies [2]. Therefore for each $t \in I$, the sequence $(x^{\lambda_n}(t))$ converges weakly to $C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))\,ds$. Since by assumption $(x^{\lambda_n}(t))$ converges to $x(t)$ in $E$, we have

$$x(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))\,ds.$$

We claim that $f(s) \in F_{\lambda_0}(s, x_s)$ a.e. According to Mazur’s theorem [6], the weak convergence implies the existence of the double sequence of nonnegative numbers $(\alpha_{m,n})$ such that

i) $\alpha_{m,n} = 0$ for $n \geq n_0(m)$;

ii) $\sum_{n=m}^{n_0(m)} \alpha_{m,n} = 1$ for $m \in \mathbb{N}$;

iii) the sequence $(\tilde{f}_m)$, where $\tilde{f}_m(t) = \sum_{n=m}^{n_0(m)} \alpha_{m,n} f_{\lambda_n}(t)$, converges to $f$ with respect to the norm of the space $L^1(I, E)$. Passing if necessary to a subsequence we can assume that $(\tilde{f}_m)$ converges to $f$ almost everywhere on $I$. Moreover for almost everywhere $s \in I$

$$d(f_{\lambda_n}(s), F_{\lambda_0}(s, x_s)) \leq \delta(F_{\lambda_n}(s, x^{\lambda_n}_s), F_{\lambda_0}(s, x_s))$$

$$\leq \delta(F_{\lambda_n}(s, x^{\lambda_n}_s), F_{\lambda_n}(s, x_s)) + \delta(F_{\lambda_n}(s, x_s), F_{\lambda_0}(s, x_s))$$

$$\leq k(s) \|x^{\lambda_n}_s - x_s\| + \delta(F_{\lambda_n}(s, x_s), F_{\lambda_0}(s, x_s))$$
and since \( \lim_{\lambda \to \lambda_0} \delta(F_\lambda(t, \phi), F_{\lambda_0}(t, \phi)) = 0 \), then

\[
\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, f_{\lambda_n}(s) \in F_{\lambda_0}(s, x_s) + 2\varepsilon B \text{ a.e. in } I
\]

where \( B \) is the closed unit ball in \( E \), and then, for all \( n > N \)

\[
\tilde{f}_{m_j}(s) = \sum_{n=m_j}^{n_0(m_j)} \alpha_{m_j, n}(F_{\lambda_0}(s, x_s) + 2\varepsilon B) = F_{\lambda_0}(s, x_s) + 2\varepsilon B
\]

taking the limit in the above formula, we deduce that for all \( \varepsilon > 0 \),

\[
f(s) \in F_{\lambda_0}(s, x_s) + 2\varepsilon B \text{ a.e. in } I, \text{ and then then}
\]

\[
f(s) \in F_{\lambda_0}(s, x_s) \text{ a.e. in } I.
\]

**Remark** Since, in the theorem 4.1, the assumption \( E \) is reflexive is used only for deducing the sequence \( (f_{\lambda_n}) \) converges weakly in \( L^1(I; E) \), it may be replaced by the following assumption: there exists a \( k \geq 0 \) such that for all bounded subset \( \Omega \subset C \)

\[
\chi(F(t, \Omega)) \leq k\chi_0(\Omega) \text{ for all } t \in I
\]

where \( \chi \) (resp. \( \chi_0 \)) is the measure of noncompactness in \( E \) (resp. \( C \)) (see for example [4, 11]). In this case, we obtain

\[
\chi(\{f_{\lambda_n}(t) : n \in \mathbb{N}\}) \leq k\chi_0(\{x_t^{\lambda_n} : n \in \mathbb{N}\}) = 0
\]

for almost all \( t \in I \), i.e. the set \( \{f_{\lambda_n}(t) : n \in \mathbb{N}\} \) is relatively compact in \( E \) a.e. in \( I \) and since \( \sup \|f_{\lambda_n}\|_1 < +\infty \), then from Diestel’s theorem [4] it follows that the sequence \( \{f_{\lambda_n}\} \) is relatively weak compact in the space \( L^1(I; E) \).

**Theorem 4.2** \((E \text{ is not reflexive})\). Let \((\Lambda, d_\Lambda)\) be a metric space, \( F_\lambda : I \times C \to \mathcal{F}(E) \) a family of multimappings satisfying the conditions \((H_1), (H_2)\) with the same function \( k \). If for any \((t, \phi) \in I \times C\) the multimapping \( \lambda \mapsto F_\lambda(t, \phi) \) is lower semicontinuous at \( \lambda_0 \in \Lambda \), then for all \( \phi \in B, \lambda \mapsto S_{F_{\lambda_0}}(\phi) \) is lower semicontinuous at \( \lambda_0 \).

**Proof.** Since the case \( S_{F_{\lambda_0}}(\phi) = \emptyset \) is trivial, we assume that \( S_{F_{\lambda_0}}(\phi) \neq \emptyset \). Let \( x \in S_{F_{\lambda_0}}(\phi) \) then,

\[
x(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I \end{cases}
\]

where \( f(s) \in F_{\lambda_0}(s, x_s) \subset \liminf_{\lambda \to \lambda_0} \lambda F_{\lambda}(s, x_s) \text{ a.e. in } I \) thus

\[
\lim_{\lambda \to \lambda_0} d(f(s), F_\lambda(s, x_s)) = 0 \text{ a.e., and then for } \varepsilon > 0, \text{ there exists } \rho > 0 \text{ such that }
\]

\[
d_\Lambda(\lambda, \lambda_0) < \rho \text{ implies } d(f(s), F_\lambda(s, x_s)) < \frac{\varepsilon}{\lambda_0 K(\omega)}. \text{ Thus for } \lambda \in \Lambda \text{ such that }
\]

\[
d_\Lambda(\lambda, \lambda_0) < \rho, \quad t \mapsto d(f(t), F_\lambda(t, x_t)) = q(t)
\]

is integrable and \( x \) is a mild solution of

\[
\begin{cases} x''(t) = Ax(t) + f(t) \\ x_0 = \varphi
\end{cases}
\]
and by theorem 3.1 with \( \mu = 0 \) and \( v(t) = \frac{\epsilon}{2\omega k(\omega)} \) there exists a function \( x^\lambda \in S_{F_\lambda}(\varphi) \) (for \( d_\lambda(\lambda, \lambda_0) < \rho \)) such that

\[
\|x^\lambda - x\|_\omega \leq K(\omega)m(\omega) = K(\omega)[\int_0^\omega (q(t) + v(t))dt] = \epsilon,
\]

hence \( x \in \liminf_{\lambda \to \lambda_0} S_{F_\lambda}(\varphi) \).

Combining theorems 4.1 and 4.2, we obtain.

**Corollary** Let \((\Lambda, d_\lambda)\) be a metric space, \(F_\lambda : I \times C \to \mathcal{F}_c(E)\) a family of multimappings satisfying the conditions \((H_1), (H_2)\) with the same function \(k\) and \((H_4)\) with the same function \(\nu\). If for any \((t, \phi) \in I \times C\), \(\lim_{\lambda \to \lambda_0} \delta(F_\lambda(t, \phi), F_{\lambda_0}(t, \phi)) = 0\), then for all \(\varphi \in B\), \(\lambda \mapsto S_{F_\lambda}(\varphi)\) is continuous at \(\lambda_0\).

**Theorem 4.3** Assume that \(F : I \times C \to \mathcal{F}_c(E)\) satisfying the conditions \((T_i), (H_2)\) and \((H_4)\). Then \(S_F : C^1 \to \mathcal{P}(C_\omega)\) is continuous on \(C^1\), where \(C^1 := C^1(J; E)\) denote the Banach space of continuously differentiable \(E\)-valued functions on \(J\) with the norm \(\|\varphi\|_{C^1} = \|\varphi\| + \|\varphi'|\|\).

**Proof.** For any \(\varphi_1, \varphi_2 \in C^1\), let \(F_{\varphi_2}(t, \phi) = F(t, \phi + (\overline{\varphi_2}_t - (\overline{\varphi_1}_t)_t)\) for all \((t, \phi) \in I \times C\) then \(S_F(\varphi_2) = S_{F_{\varphi_2}}(\varphi_1) + \hat{\varphi}_2 - \hat{\varphi}_1\) where

\[
\hat{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) & \text{if } t \in I \end{cases}
\]

indeed,

\[
x \in S_{F_{\varphi_2}}(\varphi_1) \iff x(t) = \begin{cases} \varphi_1(t) & \text{if } t \in J \\ C(t)\varphi_1(0) + S(t)\varphi_1'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I \end{cases}
\]

where \(f(s) \in F_{\varphi_2}(s, x_s)\) a.e.

\[
\iff x(t) + \hat{\varphi}_2(t) - \hat{\varphi}_1(t) = \begin{cases} \varphi_2(t) & \text{if } t \in J \\ C(t)\varphi_2(0) + S(t)\varphi_2'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I \end{cases}
\]

where \(f(s) \in F(s, x_s + (\overline{\varphi_2}_s - (\overline{\varphi_1}_s)_s)) = F(s, (x + \hat{\varphi}_2 - \hat{\varphi}_1)_s)\) a.e.

\[
\iff x + \hat{\varphi}_2 - \hat{\varphi}_1 \in S_F(\varphi_2).
\]

Furthermore, it is clear that \(\varphi_2 \mapsto F_{\varphi_2}(t, \phi)\) (for all \((t, \phi) \in I \times C\)) is continuous at \(\varphi_1\) and the family \((F_{\varphi_2})_{\varphi_2 \in C^1}\) satisfy the assumptions of preceedent corollary, therefore for all \(\varphi \in C^1, \varphi_2 \mapsto S_{F_{\varphi_2}}(\varphi)\) is continuous at \(\varphi_1\) and then

\[
\lim_{\varphi_2 \to \varphi_1} S_F(\varphi_2) = \lim_{\varphi_2 \to \varphi_1} \left( S_{F_{\varphi_2}}(\varphi_1) + \hat{\varphi}_2 - \hat{\varphi}_1 \right) = S_{F_{\varphi_1}}(\varphi_1) = S_F(\varphi_1) \text{.}
\]

**Theorem 4.4** \((E\text{ is not reflexive})\) Assume that \(F : I \times C \to \mathcal{F}_c(E)\) satisfying the conditions \((H_1), (H_2)\) and \((H_4)\) i.e. there exists a compact \(K \subset E\) such that for every \((t, \phi) \in I \times C\), \(F(t, \phi) \subset K\). Then for all \(\varphi \in B\), \(S_F(\varphi)\) is compact.
Proof. We prove first that $S_F(\varphi)$ is relatively compact. Let $(x^n)$ be a sequence of $S_F(\varphi)$, then for all $n \in \mathbb{N}$

$$x^n(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_n(s))ds & \text{if } t \in I \end{cases}$$

where $f_n(s) \in F(s,x^n_s)$ a.e. in $I$.

We shall show that $A := \{x^n_t : n \in \mathbb{N}\}$ is equicontinuous. For each $0 \leq t_0 < t \leq \omega$ and $n \in \mathbb{N}$

$$|x^n(t) - x^n(t_0)|_E \leq |C(t)\varphi(0) - C(t_0)\varphi(0)|_E + |S(t)\varphi'(0) - S(t_0)\varphi'(0)|_E +$$

$$\int_0^{t_0} \|S(t-s) - S(t_0-s)\| |f_n(s)|_E ds + \int_{t_0}^t \|S(t-s)\| |f_n(s)|_E ds$$

but,

$$\|S(t-s) - S(t_0-s)\| = \left\| \int_0^{t-s} C(\tau)d\tau - \int_0^{t_0-s} C(\tau)d\tau \right\|$$

$$\leq \int_0^{t-s} |C(\tau)|d\tau$$

$$\leq \int_0^{t-s} e^{\alpha\tau}d\tau$$

$$\leq \eta \alpha^{-1}[e^{\alpha(t-s)} - e^{\alpha(t_0-s)}]$$

$$\leq \eta (t-t_0)e^{\alpha \omega}$$

$(\alpha^{-1}[e^{\alpha(t-s)} - e^{\alpha(t_0-s)}]$ is replaced by $t-t_0$ when $\alpha = 0$), then

$$\int_0^{t_0} \|S(t-s) - S(t_0-s)\| |f_n(s)|_E ds \leq \eta(t-t_0)e^{\alpha \omega} \int_0^{t_0} |f_n(s)|_E ds.$$

Also,

$$\int_0^t \|S(t-s)\| |f_n(s)|_E ds \leq \eta(t-t_0)e^{\alpha \omega} \int_0^t |f_n(s)|_E ds.$$

Since $f_n$ are integrably bounded and the maps $t \to C(t)\varphi(0), t \to S(t)\varphi'(0)$ are uniformly continuous on $I$, we obtain that $A$ is equicontinuous, clearly it is also bounded.

Now, we prove that $A(t) = \{x^n(t) : n \in \mathbb{N}\}$ is relatively compact. For all $s \in I$, $S(t-s) : E \to E$ is continuous, then by assumption $(H_4')$ we have that $K_1 = \{S(t-s)f_n(s) : s \in [0,t] \text{ and } n \in \mathbb{N}\}$ is relatively compact, thus $K_2 = \overline{\partial K_1}$ is compact and $K_3 = \{tx : (t,x) \in I \times K_2\}$ is compact. Consequently $A(t) \subset C(t)\varphi(0) + S(t)\varphi'(0) + K_3$ is relatively compact. From the Ascoli theorem [4,11] we may assume that the sequence $(x^n)$ converges to some $x \in C_\omega$. We prove next that $x \in S_F(\varphi)$. By condition $(H_4')$, the set $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact in $E$ and since $\sup_{n \in \mathbb{N}} \|f_n\|_1 < +\infty$, then from Diestel’s theorem [4] it follows that the sequence $(f_n)$ is relatively weak compact in the space $L^1(I;E)$ and by using exactly the same method as in the proof of theorem 4.1 we obtain $x \in S_F(\varphi)$. 

\[ \square \]
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