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# The measure-theoretical approach to $p$ -adic probability theory

Andrei Khrennikov, Shinichi Yamada, Arnoud van Rooij

## 1 Introduction

The development of a non-Archimedean (especially,  $p$ -adic) mathematical physics [20]-[22], [1]-[4], [6], [8]-[13] induced some new mathematical structures over non-Archimedean fields. In particular, probability theory with  $p$ -adic valued probabilities was developed in [11], [8], [4]<sup>1</sup>.

The first theory with  $p$ -adic probabilities was the frequency theory in which probabilities were defined as limits of relative frequencies  $\nu_N = n/N$  in the  $p$ -adic topology<sup>2</sup>. This frequency probability theory was a natural extension of the frequency probability theory of R. von Mises [15], [16].

The next step was the creation of  $p$ -adic probability formalism on the basis of a theory of  $p$ -adic valued probability measures. It was natural to do this by following the fundamental work of A.N. Kolmogorov [14] in which he had proposed the measure-theoretical axiomatics of probability theory. Kolmogorov used properties of the frequency probability (non-negativity, normalization by 1 and additivity) as the basis of his axiomatics. Then he added the technical condition of  $\sigma$ -additivity for using Lebesgue's integration theory. In works [11],[8] we tried to follow A.N. Kolmogorov.  $p$ -adic frequency probability has also the properties of additivity: it is normalized by 1 and the set of possible values of this probability is the whole field of  $p$ -adic numbers  $\mathbb{Q}_p$ . Thus it was natural to define  $p$ -adic probability as a  $\mathbb{Q}_p$ -valued measure normalized by 1.

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<sup>1</sup> $p$ -adic probability theory appeared in connection with a model of quantum mechanics with  $p$ -adic valued wave functions [12]. The main task of this probability formalism was to present the probability interpretation for  $p$ -adic valued wave functions.

<sup>2</sup>The following trivial fact is the cornerstone of this theory: the relative frequencies belong to the field of rational numbers  $\mathbb{Q}$ ; we can study their behaviour not only in the real topology on  $\mathbb{Q}$ , but also in the  $p$ -adic topologies on  $\mathbb{Q}$ .

[18], [19]. Therefore the creators of non-Archimedean integration theory (A. Monna and T. Springer [17]) did not try to develop abstract measure theory, but they proposed an integration formalism via Bourbaki based on integrals of continuous functions. This integration theory has been used for creating  $p$ -adic probability theory in the measure-theoretical framework [8]. The main disadvantage of this probability model is the strong connection with the topological structure of a sample space<sup>3</sup>.

An abstract theory of non-Archimedean measures has been developed in [19]. The basic idea of this approach is to study measures defined on *rings* which in principle cannot be extended to measures on  $\sigma$ -rings. This gives the possibility for constructing non-discrete valued measures with values in non-Archimedean fields (and, in particular, in fields of  $p$ -adic numbers). On the other hand, the condition of continuity for measures in [19] implies the  $\sigma$ -additivity in all natural cases.

In this paper we develop a  $p$ -adic probability formalism based on measure theory of [19]. By probabilistic reasons we use the special case of this measure theory: (1) measures are defined on *algebras* (such measures have some special properties); (2) measures take values in fields of  $p$ -adic numbers (here values of probabilities can be approximated by rational relative frequencies).

However, probabilistic applications stimulate also the development of the general theory of non-Archimedean measures defined on rings. We prove the formula of the change of variables for these measures and use this formula for developing the formalism of conditional expectations for  $p$ -adic valued random variables.

## 2 Measures

Everywhere below  $K$  denotes a complete non-Archimedean field,  $\mathbf{R}$  denotes the field of real numbers. The valuations on these fields are denoted by the same symbol  $|\cdot|$ . We set  $U_R(a) = \{x \in K : |x-a| \leq R\}$ ,  $a \in K$ ,  $R \in \mathbf{R}$ ,  $R > 0$ . By definition these are balls in  $K$ .

Let  $X$  be an arbitrary set and let  $\mathcal{R}$  be a ring of subsets of  $X$ . The pair  $(X, \mathcal{R})$  is called a *measurable space*. The ring  $\mathcal{R}$  is said to be *separating* if for every two distinct elements,  $x$  and  $y$ , of  $X$  there exists an  $A \in \mathcal{R}$  such

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<sup>3</sup>This is quite similar to the old probability formalisms of Frechet [6] and Cramer [5] in which the topological structure of the sample space played the important role.

that  $x \in A, y \notin A$ . We shall consider measurable spaces only over separating rings which cover the set  $X$ .

Every ring  $\mathcal{R}$  can be used as a base for the zero-dimensional topology which we shall call the  $\mathcal{R}$ -topology. This topology is Hausdorff iff  $\mathcal{R}$  is separating.

Throughout this section,  $\mathcal{R}$  is a separating ring of a set  $X$ .

A subcollection  $\mathcal{S}$  of  $\mathcal{R}$  is said to be *shrinking* if the intersection of any two elements of  $\mathcal{S}$  contains an element of  $\mathcal{S}$ . If  $\mathcal{S}$  is shrinking, and if  $f$  is a map  $\mathcal{R} \rightarrow K$  or  $\mathcal{R} \rightarrow \mathbf{R}$ , we say that  $\lim_{A \in \mathcal{S}} f(A) = 0$  if for every  $\epsilon > 0$ , there exists an  $A_0 \in \mathcal{S}$  such that  $|f(A)| \leq \epsilon$  for all  $A \in \mathcal{S}, A \subset A_0$ .

A *measure* on  $\mathcal{R}$  is a map  $\mu : \mathcal{R} \rightarrow K$  with the properties: (i)  $\mu$  is additive; (ii) for all  $A \in \mathcal{R}, \|A\|_\mu = \sup\{|\mu(B)| : B \in \mathcal{R}, B \subset A\} < \infty$ ; (iii) if  $\mathcal{S} \subset \mathcal{R}$  is shrinking and has empty intersection, then  $\lim_{A \in \mathcal{S}} \mu(A) = 0$ .

We call these conditions respectively *additivity, boundedness, continuity*. The latter condition is equivalent to the following:  $\lim_{A \in \mathcal{S}} \|A\|_\mu = 0$  for every shrinking collection  $\mathcal{S}$  with empty intersection. Further, we shall briefly discuss the main properties of measures, see [19] for the details.

For any set  $D$ , we denote its characteristic function (the indicator) by the symbol  $i_D$ . For  $f : X \rightarrow K$  and  $\phi : X \rightarrow [0, \infty)$ , put  $\|f\|_\phi = \sup_{x \in X} |f(x)|\phi(x)$ . We set  $N_\mu(x) = \inf_{U \in \mathcal{R}, x \in U} \|U\|_\mu$  for  $x \in X$ . Then  $\|A\|_\mu = \|i_A\|_{N_\mu}$  for any  $A \in \mathcal{R}$ . We set  $\|f\|_\mu = \|f\|_{N_\mu}$ .

A *step function* (or  $\mathcal{R}$ -step function) is a function  $f : X \rightarrow K$  of the form  $f(x) = \sum_{k=1}^N c_k i_{A_k}(x)$  where  $c_k \in K$  and  $A_k \in \mathcal{R}, A_k \cap A_l = \emptyset, k \neq l$ . We set for such a function  $\int_X f(x)\mu(dx) = \sum_{k=1}^N c_k \mu(A_k)$ . Denote the space of all step functions by the symbol  $S(X)$ . The integral  $f \rightarrow \int_X f(x)\mu(dx)$  is the linear functional on  $S(X)$  which satisfies the inequality

$$\left| \int_X f(x)\mu(dx) \right| \leq \|f\|_\mu. \tag{1}$$

A function  $f : X \rightarrow K$  is called  $\mu$ -*integrable* if there exists a sequence of step functions  $\{f_n\}$  such that  $\lim_{n \rightarrow \infty} \|f - f_n\|_\mu = 0$ . The  $\mu$ -integrable functions form a vector space  $L_1(X, \mu)$  (and  $S(X) \subset L_1(X, \mu)$ ). The integral is extended from  $S(X)$  on  $L_1(X, \mu)$  by continuity. The inequality (1) holds for  $f \in L_1(X, \mu)$ .

Let  $\mathcal{R}_\mu = \{A : A \subset X, i_A \in L_1(X, \mu)\}$ . This is a ring. Elements of this ring are called  $\mu$ -measurable sets. By setting  $\mu(A) = \int_X i_A(x)\mu(dx)$  the measure  $\mu$  is extended to a measure on  $\mathcal{R}_\mu$ . This is the *maximal extension* of  $\mu$ , i.e., if we repeat the previous procedure starting with the ring  $\mathcal{R}_\mu$ , we will obtain this ring again.

Set  $X_\epsilon = \{x \in X : N_\mu(x) \geq \epsilon\}$ ,  $X_0 = \{x \in X : N_\mu(x) = 0\}$ ,  $X_+ = X \setminus X_0$ . Every  $A \subset X_0$  belongs to  $\mathcal{R}_\mu$ . We call such sets  $\mu$ -negligible.

Now we construct product measures. Let  $\mu_j, j = 1, 2, \dots, n$ , be measures on (separating) rings  $\mathcal{R}_j$  of subsets of sets  $X_j$ . The finite unions of the sets  $A_1 \times \dots \times A_n, A_j \in \mathcal{R}_j$ , form a (separating) ring  $\mathcal{R}_1 \times \dots \times \mathcal{R}_n$  of  $X_1 \times \dots \times X_n$ . Then there exists a unique measure  $\mu_1 \times \dots \times \mu_n$  on  $\mathcal{R}_1 \times \dots \times \mathcal{R}_n$  such that  $\mu_1 \times \dots \times \mu_n(A_1 \times \dots \times A_n) = \mu_1(A_1) \times \dots \times \mu_n(A_n)$ . We have

$$N_{\mu_1 \times \dots \times \mu_n}(x_1, \dots, x_n) = N_{\mu_1}(x_1) \times \dots \times N_{\mu_n}(x_n). \quad (2)$$

Let  $X$  be a zero-dimensional topological space<sup>4</sup>. We denote the ring of *clopen* (i.e., at the same time open and closed) subsets of  $X$  by the symbol  $B(X)$  (in fact, this is an algebra). We denote the space of continuous bounded functions  $f : X \rightarrow K$  by the symbol  $C_b(X)$ . We use the norm  $\|f\|_\infty = \sup_{x \in X} |f(x)|$  on this space.

First we remark that if  $X$  is compact and  $\mathcal{R} = B(X)$  then the condition (iii) in the definition of a measure is redundant. If  $X$  is not compact then there exist bounded additive set functions which are not continuous.

Let  $X$  be zero-dimensional  $\mathbf{N}$ -compact topological space, i.e., there exists a set  $S$  such that  $X$  is homeomorphic to a closed subset of  $\mathbf{N}^S$ . We remark that every product of  $\mathbf{N}$ -compact spaces is  $\mathbf{N}$ -compact; every closed subspace of an  $\mathbf{N}$ -compact space is  $\mathbf{N}$ -compact. Then every bounded  $\sigma$ -additive function  $\mu : B(X) \rightarrow K$  is a measure. On the other hand, if  $X$  is a zero-dimensional space such that every bounded  $\sigma$ -additive function  $B(X) \rightarrow K$  is a measure, then  $X$  is  $\mathbf{N}$ -compact.

In the theory of integration a crucial role is played by the  $\mathcal{R}_\mu$ -topology, i.e., the (zero-dimensional) topology that has  $\mathcal{R}_\mu$  as a base. Of course,  $\mathcal{R}_\mu$ -topology is stronger than  $\mathcal{R}$ -topology. Every  $\mu$ -negligible set is  $\mathcal{R}_\mu$ -clopen. The following two theorems [19] will be important for our considerations.

**Theorem 2.1.** (i) *If  $\mu$  is a measure on  $\mathcal{R}$ , then  $N_\mu$  is  $\mathcal{R}$ -upper semicontinuous, (hence,  $\mathcal{R}_\mu$ -upper semicontinuous) and for every  $A \in \mathcal{R}_\mu$  and  $\epsilon > 0$  the set  $A_\epsilon = A \cap X_\epsilon$  is  $\mathcal{R}_\mu$ -compact.*

(ii) *Conversely, let  $\mu : \mathcal{R} \rightarrow K$  be additive. Assume that there exists an  $\mathcal{R}$ -upper semicontinuous  $\phi : X \rightarrow [0, \infty)$  such that  $|\mu(A)| \leq \sup_{x \in A} \phi(x), A \in \mathcal{R}$ , and  $\{x \in A : \phi(x) \geq \epsilon\}$  is  $\mathcal{R}$ -compact ( $A \in \mathcal{R}, \epsilon > 0$ ). Then  $\mu$  is a measure and  $N_\mu \leq \phi$ .*

**Theorem 2.2.** *Let  $\mu : \mathcal{R} \rightarrow K$  be a measure. A function  $f : X \rightarrow K$  is  $\mu$ -integrable iff it has the following two properties: (1)  $f$  is  $\mathcal{R}_\mu$ -continuous;*

<sup>4</sup>We consider only Hausdorff spaces.

(2) for every  $\epsilon > 0$ , the set  $\{x : |f(x)|N_\mu(x) \geq \epsilon\}$  is  $\mathcal{R}_\mu$ -compact.

We shall also use the following fact.

**Theorem 2.3.** *Let  $f \in L_1(X, \mu)$  and let*

$$\int_A f(x)\mu(dx) = 0 \text{ for every } A \in \mathcal{R}. \quad (3)$$

Then  $\text{supp } f \subset X_0$ .

**Proof.** Let us assume that  $f$  satisfies (3) and there exists  $x_0 \in X_+$  (hence  $N_\mu(x_0) = \alpha > 0$ ) such that  $|f(x_0)| = c > 0$ . Let  $\{f_k\}$  be a sequence of  $\mathcal{R}$ -step functions which approximates  $f$ . For every  $\epsilon > 0$  there exist  $N_\epsilon$  such that  $\|f - f_k\|_\mu < \alpha\epsilon$  for all  $k \geq N_\epsilon$ . In particular, this implies that  $|f_k(x_0)| \geq c - \epsilon$ ,  $k \geq N_\epsilon$ . Then we have

$$\Delta_{B,k} = \left| \int_B f_k(x)\mu(dx) \right| = \left| \int_B f_k(x)\mu(dx) - \int_B f(x)\mu(dx) \right| < \alpha\epsilon, \quad B \in \mathcal{R}.$$

Let

$$f_k(x) = \sum_j c_{kj} i_{B_{kj}}(x), \quad c_{kj} \in K, B_{kj} \in \mathcal{R}, B_{kj} \cap B_{ki} = \emptyset, i \neq j,$$

and let  $x_0 \in B_{kj_0}$ . If  $B \subset B_{kj_0}$ ,  $B \in \mathcal{R}$ , then  $\Delta_{B,k} = |c_{kj_0}| \mu(B) = |f_k(x_0)| \mu(B) < \alpha\epsilon$ . On the other hand, as  $\|B_{kj_0}\|_\mu \geq \alpha$ , then for every  $\delta > 0$ , there exists  $B \subset B_{kj_0}$ ,  $B \in \mathcal{R}$ , such that  $|\mu(B)| \geq (\alpha - \delta)$ . Thus we obtain for this  $B$ :  $\Delta_{B,k} \geq (\alpha - \delta)(c - \epsilon)$ . By choosing  $\epsilon > 0$ ,  $\delta > 0$ , such that  $(\alpha - \delta)(c - \epsilon) > \alpha\epsilon$ , we arrive to a contradiction.

Let  $(X_j, \mathcal{R}_j)$ ,  $j = 1, 2$ , be two measurable spaces. A function  $f : X_1 \rightarrow X_2$  such that  $f^{-1}(\mathcal{R}_2) \subset \mathcal{R}_1$  is said to be measurable ( $(\mathcal{R}_1, \mathcal{R}_2)$ -measurable). We shall use the following simple fact.

**Lemma 2.1.** *Let  $(X_j, \mathcal{R}_j)$ ,  $j = 1, 2$ , be measurable spaces and let  $f : X_1 \rightarrow X_2$  be measurable. If  $\mathcal{S}$  is shrinking in  $\mathcal{R}_2$  then  $f^{-1}(\mathcal{S})$  is shrinking in  $\mathcal{R}_1$ . If  $\mathcal{S}$  has empty intersection, then  $f^{-1}(\mathcal{S})$  has also empty intersection.*

**Lemma 2.2.** *Let  $(X_j, \mathcal{R}_j)$ ,  $j = 1, 2$ , be measurable spaces and let  $\eta : X_1 \rightarrow X_2$  be a measurable function. Then, for every measure  $\mu : \mathcal{R}_1 \rightarrow K$ , the function  $\mu_\eta : \mathcal{R}_2 \rightarrow K$  defined by the equality  $\mu_\eta(A) = \mu(\eta^{-1}(A))$  is a measure on  $\mathcal{R}_2$  and, for every  $\mathcal{R}_2$ -continuous function,  $h : X_2 \rightarrow K$  the following inequality holds:*

$$\|h\|_{\mu_\eta} \leq \|h \circ \eta\|_\mu. \quad (4)$$

**Proof.** We have for every  $A \in \mathcal{R}_2$ ,

$$\|A\|_{\mu_\eta} = \sup\{|\mu(\eta^{-1}(B))| : B \in \mathcal{R}_2, B \subset A\} \leq \|\eta^{-1}(A)\|_\mu < \infty. \quad (5)$$

Thus  $\mu_\eta$  is bounded. We now prove that  $\mu_\eta$  is continuous on  $\mathcal{R}_2$ . Let  $\mathcal{S}$  be shrinking in  $\mathcal{R}_2$  which has the empty intersection. By Lemma 2.1  $\eta^{-1}(\mathcal{S})$  is shrinking in  $\mathcal{R}_1$  which has also the empty intersection. By (5) we obtain that  $\lim_{A \in \mathcal{S}} \|A\|_{\mu_\eta} = 0$ .

We prove inequality (4). Let  $h : X_2 \rightarrow K$  be  $\mathcal{R}_2$ -continuous. We wish to prove that  $|h(b)|N_{\mu_\eta}(b) \leq \|h \circ \eta\|_\mu$  for all  $b \in X_2$ . So we choose  $b \in X_2$  with  $h(b) \neq 0$ . Then the set  $C_b = \{y \in X_2 : |h(y)| = |h(b)|\}$  is  $\mathcal{R}_2$ -open. Hence there is a  $B \in \mathcal{R}_2$  with  $b \in B \subset C_b$ . Then

$$\begin{aligned} |h(b)|N_{\mu_\eta}(b) &\leq |h(b)|\|B\|_{\mu_\eta} \leq |h(b)|\|\eta^{-1}(B)\|_\mu = \\ \sup_{x \in \eta^{-1}(B)} |h(b)|N_\mu(x) &\leq \sup_{x \in \eta^{-1}(B)} |(h \circ \eta)(x)|N_\mu(x) \leq \|h \circ \eta\|_\mu. \end{aligned}$$

**Theorem 2.4.** (Change of variables) *Let  $(X_j, \mathcal{R}_j)$ ,  $j = 1, 2$ , be measurable spaces and let  $\eta : X_1 \rightarrow X_2$  be a measurable function, and let  $\mu : \mathcal{R}_1 \rightarrow K$  be a measure. If  $f : X_2 \rightarrow K$  is an  $\mathcal{R}_2$ -continuous function such that the function  $f \circ \eta$  belongs to  $L_1(X_1, \mu)$ , then  $f \in L_1(X_2, \mu_\eta)$  and*

$$\int_{X_1} f(\eta(x))\mu(dx) = \int_{X_2} f(y)\mu_\eta(dy). \quad (6)$$

**Proof.** It suffices to prove that for every  $\epsilon > 0$  there exists a  $\mathcal{R}_2$ -step function  $g$  such that  $\|f - g\|_{\mu_\eta} \leq \epsilon$  and  $\|f \circ \eta - g \circ \eta\|_\mu \leq \epsilon$ . By (4) the first follows from the second. So we fix  $\epsilon > 0$ .

By Theorem 2.2 the set

$$A = \{x \in X_1 : |(f \circ \eta)(x)|N_\mu(x) \geq \epsilon\}$$

is  $\mathcal{R}_1$ -compact and therefore contained in an element of  $\mathcal{R}_1$ . But  $N_\mu$  is bounded on every element of  $\mathcal{R}_1$ , so  $N_\mu$  is bounded on  $A$ . We choose  $\delta > 0$  so that

$$\delta N_\mu(x) \leq \epsilon \text{ for all } x \in A.$$

As  $A$  is compact,  $f(\eta(A))$  is also compact. We can cover  $f(\eta(A))$  by disjoint closed balls of radius  $\delta$ :  $f(\eta(A)) \subset U_\delta(\alpha_0) \cup \dots \cup U_\delta(\alpha_N)$ , where  $\alpha_0$  is chosen to be 0 in order to obtain:

$$|\alpha_n| \leq |t| \text{ for } t \in U_\delta(\alpha_n), n = 0, 1, \dots, N. \quad (7)$$

For each  $n$ ,  $C_n = \{C \in \mathcal{R}_2 : C \subset f^{-1}(U_\delta(\alpha_n))\}$  is a collection of open sets covering the compact set  $\eta(A) \cap f^{-1}(U_\delta(\alpha_n))$ . Thus, for each  $n$  there is a  $C_n \in \mathcal{C}_n$  such that  $\eta(A) \cap f^{-1}(U_\delta(\alpha_n)) \subset C_n$ . We now have

$$C_0, \dots, C_N \in \mathcal{R}_2, \quad (8)$$

$$C_n \subset f^{-1}(U_\delta(\alpha_n)), n = 0, 1, \dots, N, \quad (9)$$

$$\eta(A) \subset C_0 \cup \dots \cup C_N. \quad (10)$$

Put  $g(x) = \sum_{n=0}^N \alpha_n i_{C_n}(x)$ . Then  $g$  is a  $\mathcal{R}_2$ -step function. We wish to show that, for all  $a \in X$ ,

$$\Delta(a) = |(f \circ \eta)(a) - (g \circ \eta)(a)|N_\mu(a) \leq \epsilon.$$

Thus, take  $a \in X$  :

(1) If  $a \in A$ , then there is a unique  $n$  with  $\eta(a) \in C_n$ . Then  $\Delta(a) = |(f \circ \eta)(a) - \alpha_n|N_\mu(a) \leq \delta N_\mu(a) \leq \epsilon$ .

(2) If  $a \notin A$ , but  $\eta(a) \in C_n$  for some  $n$ , then by (7) we obtain that  $\Delta(a) = |(f \circ \eta)(a) - \alpha_n|N_\mu(a) \leq |(f \circ \eta)(a)|N_\mu(a) \leq \epsilon$ .

(3) If  $a \notin C_0 \cup \dots \cup C_N$ , then  $g(\eta(a)) = 0$ . Thus  $\Delta(a) = |(f \circ \eta)(a)|N_\mu(a) \leq \epsilon$  (as  $a \notin A$ ).

**Open problem.** To find a condition for functions  $f$  which is weaker than continuity, but implies the formula of the change of variables.

Further we shall obtain some properties of measures which are specific for measures defined on algebras<sup>5</sup>.

Throughout this paper,  $\mathcal{A}$  is a separating algebra of a set  $X$ . First we remark that if we start with a measure  $\mu$  defined on the algebra  $\mathcal{A}$  then the system  $\mathcal{A}_\mu$  of  $\mu$ -integrable sets is again an algebra.

**Proposition 2.1.** *Let  $\mu : \mathcal{A} \rightarrow K$  be a measure. Then for each  $\epsilon > 0$ , the set  $X_\epsilon$  is  $\mathcal{A}_\mu$ -compact.*

This fact is a consequence of Theorem 2.1.

**Proposition 2.2.** *Let  $\mu : \mathcal{A} \rightarrow K$  be a measure. Then the algebra  $B(X)$  of  $\mathcal{A}_\mu$ -clopen sets coincides with the algebra  $\mathcal{A}_\mu$ .*

**Proof.** We use Theorem 2.2 and the previous proposition. Let  $B \in B(X)$ . Then  $i_B$  is  $\mathcal{A}_\mu$ -continuous and  $\{x : |i_B(x)|N_\mu(x) \geq \epsilon\} = B \cap X_\epsilon$ . As  $B$  is closed and  $X_\epsilon$  is compact,  $B \cap X_\epsilon$  is compact. Thus  $B(X) \subset \mathcal{A}_\mu$ .

As a consequence of Proposition 2.2, we obtain that  $C_b(X) \subset L_1(X, \mu)$  (for the space  $X$  endowed with  $\mathcal{A}_\mu$ -topology) and the following inequality holds:

$$\left| \int_X f(x)\mu(dx) \right| \leq \|f\|_\infty \|X\|_\mu, \quad f \in C_b(X). \quad (11)$$

Let  $X$  be zero dimensional topological space. A measure  $\mu$  defined on the algebra  $B(X)$  of the clopen sets is called a *tight* measure. Thus by

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<sup>5</sup>An algebra of  $X$  is a ring of subsets of  $X$  containing  $X$ .



Proposition 2.2 every measure  $\mu : \mathcal{A} \rightarrow K$  is extended to a tight measure on the space  $X$  endowed with the  $\mathcal{A}_\mu$ -topology.

**Proposition 2.3.** *Let  $\mu : \mathcal{A} \rightarrow K$  be a measure and let  $f \in L_1(X, \mu)$ . Then  $f$  is  $(\mathcal{A}_\mu, B(K))$ -measurable.*

**Proof.** By Theorem 2.2  $f$  is  $\mathcal{A}_\mu$ -continuous. Thus  $f^{-1}(B(K)) \subset B(X)$ . But by Proposition 2.2 we have that  $\mathcal{A}_\mu = B(X)$ .

### 3 $p$ -adic probability space

Let  $\mu : \mathcal{A} \rightarrow \mathbf{Q}_p$  be a measure defined on a separating algebra  $\mathcal{A}$  of subsets of the set  $\Omega$  which satisfies the normalization condition  $\mu(\Omega) = 1$ . We set  $\mathcal{F} = \mathcal{A}_\mu$  and denote the extension of  $\mu$  on  $\mathcal{F}$  by the symbol  $\mathbf{P}$ . A triple  $(\Omega, \mathcal{F}, \mathbf{P})$  is said to be a  $p$ -adic probability space ( $\Omega$  is a sample space,  $\mathcal{F}$  is an algebra of events,  $\mathbf{P}$  is a probability).

As in general measure theory we set  $\Omega_\alpha = \{\omega \in \Omega : N_{\mathbf{P}}(\omega) \geq \alpha\}$ ,  $\alpha > 0$ ,  $\Omega_+ = \cup_{\alpha > 0} \Omega_\alpha$ ,  $\Omega_0 = \Omega \setminus \Omega_+$ . Everywhere below, if a property  $\Xi$  is valid on the subset  $\Omega_+$  we say that  $\Xi$  is valid a.e. (mod  $\mathbf{P}$ ).

Everywhere below  $(G, \Gamma)$  denotes a measurable space over the algebra  $\Gamma$ . Functions  $\xi : \Omega \rightarrow G$  which are  $(\mathcal{F}, \Gamma)$ -measurable are said to be random variables.

Everywhere below  $Y$  is a zero dimensional topological space. We consider  $Y$  as the measurable space over the algebra  $B(Y)$ . Every random variable  $\xi : \Omega \rightarrow Y$  is continuous in the  $\mathcal{F}$ -topology. In particular,  $\mathbf{Q}_p$ -valued random variables are  $(\mathcal{F}, B(\mathbf{Q}_p))$ -measurable functions. If  $\xi \in L_1(\Omega, \mathbf{P})$ , we introduce an expectation of this random variable by setting  $E\xi = \int_\Omega \xi(\omega) \mathbf{P}(d\omega)$ . We note that every bounded random variable  $\xi : \Omega \rightarrow \mathbf{Q}_p$  belongs to  $L_1(\Omega, \mathbf{P})$ .

Let  $\eta : \Omega \rightarrow G$  be a random variable. The measure  $\mathbf{P}_\eta$  is said to be a distribution of the random variable. By Theorem 2.4 we have that

$$E f(\eta) = \int_{\mathbf{Q}_p} f(y) \mathbf{P}_\eta(dy) \quad (12)$$

for every  $\Gamma$ -continuous function  $f : G \rightarrow \mathbf{Q}_p$  such that  $f \circ \eta \in L_1(\Omega, \mathbf{P})$ . In particular, we have the following result.

**Proposition 3.1.** *Let  $\eta : \Omega \rightarrow Y$  be a random variable and let  $f \in C_b(Y)$ . Then the formula (12) holds.*

We shall also use the following technical result.

**Proposition 3.2.** *Let  $\eta : \Omega \rightarrow Y$  be a random variable and let  $\zeta \in L_1(\Omega, \mathbf{P})$ , and let  $f \in C_b(Y)$ . Then  $\xi(\omega) = \zeta(\omega)f(\eta(\omega))$  belongs  $L_1(\Omega, \mathbf{P})$  and*

$$\mathbf{E}\xi = \int_{\mathbf{Q}_p \times Y} xf(y)\mathbf{P}_z(dx dy), \quad z(\omega) = (\zeta(\omega), \eta(\omega)).$$

**Proof.** We have only to show that  $\xi \in L_1(\Omega, \mathbf{P})$ . This fact is a consequence of Theorem 2.2.

The random variables  $\xi, \eta : \Omega \rightarrow G$  are called independent if

$$\mathbf{P}(\xi \in A, \eta \in B) = \mathbf{P}(\xi \in A)\mathbf{P}(\eta \in B) \text{ for all } A, B \in \Gamma. \quad (13)$$

**Proposition 3.3.** *Let  $\xi, \eta : \Omega \rightarrow Y$  be independent random variables and functions  $f, g \in C_b(Y)$ . Then we have:*

$$\mathbf{E}f(\xi)g(\eta) = \mathbf{E}f(\xi)\mathbf{E}g(\eta). \quad (14)$$

**Proof.** If  $f$  and  $g$  are locally constant functions then (14) is a consequence of (13). Arbitrary functions  $f, g \in C_b(Y)$  can be approximated by locally constant functions (with the convergence of corresponding integrals) by using the technique developed in the proof of Theorem 2.4.

**Remark 3.1.** In fact, the formula (14) is valid for the continuous  $f, g$  such that the random variables  $f(\xi), g(\eta)$  and  $f(\xi)g(\eta)$  belong to  $L_1(\Omega, \mathbf{P})$ .

**Proposition 3.4.** *Let  $\xi$  and  $\eta$  be independent random variables. Then the random vector  $z = (\xi, \eta)$  has the probability distribution  $\mathbf{P}_z = \mathbf{P}_\eta \times \mathbf{P}_\xi$ .*

This fact is a direct consequence of (13).

Let  $\xi$  and  $\eta$  be respectively  $\mathbf{Q}_p$  and  $G$  valued random variables and  $\xi \in L_1(\Omega, \mathbf{P})$ . A conditional expectation  $\mathbf{E}[\xi|\eta = y]$  is defined as a function  $m \in L_1(G, \mathbf{P}_\eta)$  such that

$$\int_{\{\omega \in \Omega: \eta(\omega) \in B\}} \xi(\omega)\mathbf{P}(d\omega) = \int_B m(y)\mathbf{P}_\eta(dy) \text{ for every } B \in \Gamma.$$

**Proposition 3.5.** *The conditional expectation if it exists, is defined uniquely a.e. mod  $\mathbf{P}_\eta$ .*

**Proof.** We assume that there exist two conditional expectations  $m_j \in L_1(G, \mathbf{P}_\eta)$  and  $m_1(x_0) \neq m_2(x_0)$  at some point  $x_0$  and  $N_{\mathbf{P}_\eta}(x_0) > 0$ . Set  $m(x) = m_1(x) - m_2(x)$ . We have :  $\int_B m(x)\mathbf{P}_\eta(dx) = 0$  for every  $B \in \Gamma$ . To obtain the contradiction, it is sufficient to use Theorem 2.3.

As there is no analogue of the Radon-Nikodym theorem in the non-Archimedean case [17], [18], [19], it may happens that a conditional expectation does not exist. Everywhere below we assume that  $m(y) = \mathbf{E}[\xi|\eta = y]$  is well defined and moreover, that it belongs to the class  $C_b(Y)$ .

**Proposition 3.6.** *Let  $\xi : \Omega \rightarrow \mathbf{Q}_p, \eta : \Omega \rightarrow Y$  be random variables, and  $\xi \in L_1(\Omega, \mathbf{P})$ . The equality*

$$\mathbf{E}f(\eta)\xi = \mathbf{E}f(\eta(\omega))\mathbf{E}[\xi(\omega)|\eta = \eta(\omega)] \quad (15)$$

holds for every function  $f \in C_b(Y)$ .

**Proof.** By Proposition 3.2 we obtain  $\mathbf{E}\xi f(\eta) = \int_{\mathbf{Q}_p \times Y} xf(y)\mathbf{P}_z(dx dy)$ , where  $z(\omega) = (\xi(\omega), \eta(\omega))$ . Set for  $A \in B(Y)$ ,

$$\lambda(A) = \int_{\mathbf{Q}_p \times Y} xi_A(y)\mathbf{P}_z(dx dy).$$

As  $\lambda(A) = \int_{\eta^{-1}(A)} \xi(\omega)\mathbf{P}(d\omega) = \int_Y m(y)\mathbf{P}_\eta(dy)$ , it is a tight measure on  $Y$ . Then

$$\int_{\mathbf{Q}_p \times Y} xf(y)\mathbf{P}_z(dx dy) = \int_Y f(y)\lambda(dy) = \int_Y f(y)m(y)\mathbf{P}_\eta(dy) = \mathbf{E}f(\eta)m(\eta).$$

The authors plan to apply the measure-theoretical framework developed in this paper for studying of the limits theorems, random walks for  $p$ -adic probabilities (compare with the paper [3] in that  $p$ -adic random walk was studied on the basis of conventional probability theory).

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