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Weakly Compact Operators and the Dunford-Pettis Property on Uniform Spaces

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ABSTRACT. Let $(X, \mathcal{U})$ be a Hausdorff uniform space and $C_b(X)$ the space of all bounded continuous real-valued functions on X. The subspace of $C_b(X)$, consisting of the all uniformly continuous functions with respect to $\mathcal{U}$, is denote by $C_{ub}(X)$.In this paper we give a characterization of weakly compact operators and β_u – *continuous* defined from $C_{ub}(X)$ into a Banach space E,where β_u is the finest locally convex topology agreeing with the pointwise topology on each uniformly equicontinuous and bounded subsets of $C_{ub}(X)$. We also show that $(C_{ub}(X), \beta_u)$ has the Strict Dunford-Pettis Property and the Dunford-Pettis Property, both under special conditions.

1. INTRODUCTION AND NOTATIONS

All uniform spaces $(X, \mathcal{U})$ are assumed to be Hausdorff uniform spaces. Basic references for the measures theory on topological spaces are in Varadarajan [7]. We will denote by $C_b(X)$ the space of all real-valued bounded continuous functions defined on X, and $C_{ub}(X)$, the subspace of $C_b(X)$, consists of those functions which are uniformly continuous. $\mathcal{H}$ will denote the collection of all uniformly equicontinuous and bounded (U.E.B.) subsets of $C_{ub}(X)$. β_u will denote the finest locally convex topology agreeing with the pointwise topology on each $H \in \mathcal{H}$. A uniform measure on X is defined to be a bounded linear functional on $C_{ub}(X)$ which is pointwise continuous on each $H \in \mathcal{H}$ (see [1], [2], [4]) and the space of all uniform measures will be denoted by $M_u(X)$. It is well known that the dual of $(C_{ub}(X), \beta_u)$ is $M_u(X)$.

Let $M(X)$ be the dual of $(C_{ub}(X), \|.\|)$, where the $\|.\|$ denotes the supremum norm. We denote by $\mathcal{H}$-top the locally convex topology on $M(X)$ of uniform convergence on the U.E.B. sets. Let $M_d(X)$ be the subspace of $M(X)$ generated by the Dirac measures. It was proved in [4] that the $\mathcal{H}$-top closure of $M_d(X)$ is the space $M_u(X)$.

We denote by β_t the locally convex topology on $C_b(X)$ agreeing with the compact-open topology on each norm-bounded subset of $C_b(X)$. Sentilles [6] proved that the

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dual of $(C_b(X), \beta_t)$ is the space $M_t(X)$ of all tight measures on X. One of the Sentilles's results that we will use here is the following: "a subset A of $M_t(X)$ is uniformly tight if, and only if, A is β_t–equicontinuous. It is also known that $C_{ub}(X)$ is β_t–dense on $C_b(X)$ (see [1]).

The proof of the following technical lemma is a simple verification and it will be omitted..

Lemma 1. *The canonical mapping* $\Phi : (C_{ub}(X), \beta_u) \to \left(C_{ub}(\widehat{X}), \beta_u\right)$ *defined by* $f \to \widehat{f}$ *is an isomorphism, where* $\widehat{X}$ *denotes the completion of* $(X, \mathcal{U})$ *and* $\widehat{f}$ *is the unique uniform extension of* f *to* $\widehat{X}$.

2. Weakly compact operators

Let E be a Banach space. In this section we will study E–valued linear norm-continuous operators on $C_{ub}(X)$, in particular, weakly compact operators.

Definition 1. *Let* T *be a* E*-valued linear continuous operator on* $C_{ub}(X)$. *We shall say that* T *is a tight additive operator if its restriction to the unit ball of* $C_{ub}(X)$ *is continuous for the compact-open convergence topology.*

Note that T is tight additive if, and only if, T is β_t-continuous and so, by the density of $C_{ub}(X)$ in $C_b(X)$ via the topology β_t, T has a unique continuous extension to $C_b(X)$.

From now on we will assume that $T : C_{ub}(X) \to E$ is a weakly compact operator, that is, T transforms the unit ball of $C_{ub}(X)$ into a relatively weakly compact subset of E. By the denseness, $(C_{ub}(X), \beta_t)' = (C_b(X), \beta_t)' = M_t(X)$, the space of all tight measures (see [6]), and by the weakly compactness of T, $T''(C_{ub}(X)'') \subset E$. On the other hand, T has a unique β_t-continuous extension $\widetilde{T}$ to $C_b(X)$ and, by the latter,

$$\widetilde{T}'' (C_b(X)'') = \widetilde{T}'' (M_t(X)') = T''(C_{ub}(X)'') \subset E.$$

Then, $\widetilde{T}$ is also a weakly compact operator.

The following theorem will give a characterization of tight operators and the proof is based on the well known result which says that β_u and the norm topology have the same bounded sets (see [1]).

Theorem 2. *Suppose that* $\mathcal{U}$ *is metrizable and* $(X, \mathcal{U})$ *is complete. Then, the following statements are equivalent:*

(a) T *is* β_u*-continuous*

(b) T *is tight additive*

Proof. Since $(X,\mathcal{U})$ is metrizable and complete, we have that the pointwise topology and the compact-open topology coincide on each U.E.B. subset of $C_{ub}(X)$.
$(a) \Rightarrow (b)$Since T is β_u-continuous, we have that $\{x' \circ T : \|x'\| \leq 1\} \subset M_u(X)$.
On the other hand, since T is also weakly compact operator,

$$T'(\{x' : \|x'\| \leq 1\}) = \{x' \circ T : \|x'\| \leq 1\}$$

is relatively $\sigma(M(X), M(X)')$-compact. By this fact and since $M(X)$ is an AL-space, the closure of solid hull of $\{x' \circ T : \|x'\| \leq 1\}$ is $\sigma(M(X), M(X)')$-compact (see Cor. 8.8, p. 119, [5]). Since $\sigma(M(X), M(X)')$ is finer than $\sigma(M(X), C_{ub}(X))$, we have that $\{|x' \circ T| : \|x'\| \leq 1\}$ is relatively $\sigma(M(X), C_{ub}(X))$-compact. Then, by [1], p. 239, $\{|x' \circ T| : \|x'\| \leq 1\}$ is a tight set.
Now, take a net $(f_\alpha)_{\alpha \in I}$ in the unit ball of $C_{ub}(X)$ converging to 0 in the topology of compact-open convergence; hence by the above and the tightness of

$$\{|x' \circ T|(f_\alpha) : \|x'\| \leq 1\},$$

we have $|x' \circ T(f_\alpha)| \leq |x' \circ T|(f_\alpha) \to 0$ uniformly for $\|x'\| \leq 1$. This argument shows that $\|Tf_\alpha\| \to 0$.
$(b) \Rightarrow (a)$ By the tightness of T, its restriction to each U.E.B. set is continuous for the topology of compact-open convergence. But, this topology coincides with the topology of pointwise convergence on each U.E.B. set; therefore T is $\beta_u - continuous$.

In the next theorem we shall use the following notations: If $(X,\mathcal{U})$ is a uniform space and d is uniformly continuous pseudometrics (*u.c.p.*) on X, then $\hat{X}_d$ denotes the completion of the metric space which comes from X, d and the corresponding projection, π_d.

Theorem 3. *Let T be a weakly compact $E-$valued operator defined on $C_{ub}(X)$. Then, the following statements are equivalent:*

1. *T is $\beta_u - continuous$*
2. *$T_{|H}$ is pointwise continuous for each U.E.B. set H.*
3. *$\{x' \circ T : x' \in E'; \|x'\| \leq 1\}$ is $\beta_u - equicontinuous$.*
4. *$\{|x' \circ T| : x' \in E'; \|x'\| \leq 1\}$ is relatively $\sigma(M_u(X), C_{ub}(X)) - compact$*
5. *For each u.c.p. d on X, $\pi_d \circ T$ (natural definition) is a tight additive operator.*

Proof. The equivalences (3) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2) are clear. The equivalence (3) $\Leftrightarrow$ (4) follows from [1, *p.* 228 and 241].
(1) $\Rightarrow$ (5) Let d be a *u.c.p.* on X. Since π_d is uniformly continuous, we have that $\pi_d \circ T : C_{ub}(X) \to E$, defined by $(\pi_d \circ T)(\hat{f}) = T\left(\hat{f} \circ \pi_d\right)$, is $\beta_u -$ *continuous* and a weakly compact operator. Therefore, by Th. 2.2, $\pi_d \circ T$ is a tight additive operator.
(5) $\Rightarrow$ (1) Let $H \in \mathcal{H}$; hence $d_H(x,y) = \sup\{|f(x) - f(y)| : f \in H\}$ is a *u.c.p.* on X. Denote by π_H the corresponding projection of d_H and defined by $\hat{f}\,(\pi_H(x)) = f(x)$, for any $f \in H$. It is not difficult to see that the function $\hat{f}$ is well defined, it belongs to $C_{ub}(\hat{X}_{d_H})$ and the $\hat{H} = \left\{\hat{f} : f \in H\right\}$ is a U.E.B. subset of $C_{ub}(\hat{X}_{d_H})$.

Take a net $(f_\alpha)_{\alpha \in I}$ in H such that $f_\alpha \to 0$ pointwise. It easily follows that $\hat{f}_\alpha \to 0$ pointwise and, from the hypothesis, $\left\|(\pi_d \circ T)(\hat{f}_\alpha)\right\| \to 0$. Therefore, since $\|Tf\| = \left\|(\pi_d \circ T)(\hat{f})\right\|$, for any $f \in H$, we get that T is $\beta_u -$ *continuous.*

3. Dunford-Pettis and Strict Dunford-Pettis Property

In this section we will analyze the Strict Dunford-Pettis and the Dunford-Pettis Property of the locally convex space $(C_{ub}(X), \beta_u)$. We begin with the definition of these properties which were given by Grothendieck in his well known paper "Sur les applications linéares faiblement compact d'espace du type $C(K)$",Canad. J. Math. 5(1974), 183-201.

Definition 2. *We shall say that a Hausdorff locally convex space E has the Dunford-Pettis Property (resp. Strict Dunford-Pettis Property) if for any Banach space F and every linear continuous and weakly compact operator $T : E \to F$, $T(C)$ is relatively compact (resp. $\{T(x_n)\}$ is Cauchy) in F for any absolutely convex weakly compact subset C (resp. weak Cauchy sequence $\{x_n\}$) in E.*

Theorem 4. *If $\mathcal{U}$ is metrizable, then $(C_{ub}(X), \beta_u)$ has the Dunford-Pettis and Strict Dunford-Pettis Properties.*

Proof. First we shall assume that $(X, \mathcal{U})$ is a complete metrizable uniform space. Let T be a weakly compact and $\beta_u -$ *continuous* linear operator defined from $C_{ub}(X)$ into an arbitrary Banach space F. By Th. 2.2, T is a tight operator and then T admits a unique extension $\tilde{T}$ to $C_b(X)$ which is $\beta_t -$ *continuous.*
We shall first prove that $(C_{ub}(X), \beta_u)$ has the Strict Dunford-Pettis Property. Let $\{f_n\}_{n \in \mathbb{N}}$ be a $\sigma(C_{ub}(X), M_u(X)) -$ *Cauchy* sequence in $C_{ub}(X)$. Since $M_u(X) =$

$M_t(X)$ and $\{f_n\}_{n\in\mathbb{N}}$ is in $C_b(X)$, we have that this sequence is $\sigma(C_b(X), M_t(X))-$ *Cauchy*. Now, by [3], we know that $(C_b(X), \beta_t))$ has the Strict Dunford-Pettis Property, therefore $\{Tf_n\}_{n\in\mathbb{N}} = \left\{\tilde{T} f_n\right\}_{n\in\mathbb{N}}$ is convergent in F.
If $\mathcal{U}$ is metrizable, then the conclusion follows from Lemma 1.1, since

$$T \circ \Phi : C_{ub}(\hat{X}) \to F$$

is β_u – *continuous* and weakly compact operator.
Now, to prove that $(C_{ub}(X), \beta_u)$ has the Dunford-Pettis Property, we again assume that $(X, \mathcal{U})$ is complete metrizable uniform space and follows the similar argument given above.

One of the open problems that we still face, is whether or not $(C_{ub}(X), \beta_u)$ has the Strict Dunford-Pettis Property. We already proved that the answer is yes if $\mathcal{U}$ is metrizable. In the next theorem we will assume that $(C_{ub}(X), \beta_u)$ has the Strict Dunford-Pettis Property and we will prove that it has the Dunford-Pettis Property under the condition that X is $\sigma-$compact.

Theorem 5. *If $(C_{ub}(X), \beta_u)$ possesses the Strict Dunford-Pettis Property and X is $\sigma-$compact, then $(C_{ub}(X), \beta_u)$ has the Dunford-Pettis Property.*

Proof. Let $\{K_n\}_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets of X such that $\bigcup_{n=1}^{\infty} K_n$ is dense on X. We will denote by L_n the closed absolutely convex hull of K_n in $M_u(X)$ (X is a uniform subspace of $M_u(X)$). Since K_n is a compact subset of $M_u(X)$ in the $\mathcal{H}$-top and $(M_u(X), \mathcal{H}-\text{top})$ is complete, we have that L_n is a compact subset of $M_u(X)$. Moreover, $\{L_n\}_{n\in\mathbb{N}}$ is an increasing sequence.
We claim that $\bigcup_{n=1}^{\infty} L_n$ is $\mathcal{H}$-top dense in $M_u(X)$. In fact, take $\mu \in M_u(X)$ and a balanced neighborhood V of μ. Since $M_d(X)$ is $\mathcal{H}$-top dense in $M_u(X)$, V contains some element $\nu = \sum_{i=1}^{p} \alpha_i \delta_{x_i}$ of $M_d(X)$, with $x_1, x_2, ..., x_p \in X$. Suppose that $0 < \alpha = \sum_{i=1}^{p} |\alpha_i| \leq 1$ (if $\alpha = 0$, $V \cap \left(\bigcup_{n=1}^{\infty} L_n\right) \neq \emptyset$ and we are done) and take neighborhoods W_i of δ_{x_i}, $i = 1, 2, .., p$, such that $\sum_{i=1}^{p} \alpha_i W_i \subset V$. Since $W_i \cap X$ is a neighborhood of δ_{x_i} in X, we get $\delta_{y_i} \in K_{n_i}$ such that $\delta_{y_i} \in W_i \cap X$. Thus, $\sum_{i=1}^{p} \alpha_i \delta_{y_i} \in \sum_{i=1}^{p} \alpha_i W_i \subset V$ and $\sum_{i=1}^{p} \alpha_i \delta_{y_i} \in L_N$, where $N = \max\{n_i : i = 1, 2, ..., p\}$. Therefore, $V \cap \left(\bigcup_{n=1}^{\infty} L_n\right) \neq \emptyset$.
Suppose now that $\alpha > 1$; hence $\alpha \sum_{i=1}^{p} \frac{\alpha_i}{\alpha} \delta_{x_i} \in V$ and so $\sum_{i=1}^{p} \frac{\alpha_i}{\alpha} \delta_{x_i} \in \frac{1}{\alpha} V \subset V$. Applying

the above argument to $\frac{1}{\alpha}V$, we get $V \cap \left(\bigcup_{n=1}^{\infty} L_n\right) \neq \emptyset$.
From this, we have that $(M_u(X), \mathcal{H} - \text{top})$ is a $\sigma-$compact space, which implies that $(M_u(X), \sigma(M_u(X), C_{ub}(X)))$ is also a $\sigma-$compact space. The conclusion of the theorem follows from [3], Th. 1

References

[1] J. B. Cooper and W. Schachermayer, Uniform measures and cosaks spaces, Springer Verlarg, Lectures Notes 843 (1981), 217-246.

[2] M. Z. Frolik, Measures uniformes, C. R. Acad. Sci., Paris, 277 (1973),105-108.

[3] S. S. Khurana, Dunford-Pettis Property, J. Math. Anal. Appl., 65 (1978), 361-364.

[4] J. Pachl, Measures as functional of uniformly continuous functions, Pacific J. Math., 82 (1979), 515-521.

[5] H. H. Schaefer, Banach Lattices and Positives Operators, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[6] F, Sentilles, Bounded Continuous Functions on a Completely Regular Spaces, Trans. Amer. Math. Soc., V. 168(1972).

[7] V. S. Varadarajan, Measures on topological spaces, Amer. Math. Soc. Transl., (2) 48(1965), 161-220