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Approximation Results in the Strict Topology

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Abstract: In this paper we prove results of the Weierstrass-Stone type for subsets $W$ of the vector space $V$ of all continuous and bounded functions from a topological space $X$ into a real normed space $E$, when $V$ is equipped with the strict topology $\beta$. Our main results characterize the $\beta$-closure of $W$ when (1) $W$ is $\beta$-truncation stable; (2) $E = \mathbb{R}$ and $W$ is a subalgebra; (3) $E = \mathbb{R}$ and $W$ is the convex cone of all positive elements of some algebra; (4) $W$ is uniformly bounded; (5) $X$ is a completely regular Hausdorff space and $W$ is convex.

§1. Introduction and definitions

Let $X$ be a topological space and let $E$ be a real normed space. We denote by $B(X; E)$ the normed space of all bounded $E$-valued functions on $X$, equipped with the supremum norm

$$||f||_X = \sup\{||f(x)||; x \in X\}$$

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for each \( f \in B(X; E) \). We denote by \( B_0(X; E) \) the subset of all \( f \in B(X; E) \) that vanish at infinity, i.e., those \( f \) such that for every \( \varepsilon > 0 \), the set \( K = \{ t \in X; ||f(t)|| \geq \varepsilon \} \) is compact (or empty). And we denote by \( B_{00}(X; E) \) the subset of all \( f \in B(X; E) \) which have compact support. We denote by \( C(X; E) \) the vector space of all continuous \( E \)-valued functions on \( X \), and set

\[
C_b(X; E) = C(X; E) \cap B(X; E), \\
C_0(X; E) = C(X; E) \cap B_0(X; E), \\
C_{00}(X; E) = C(X; E) \cap B_{00}(X; E)
\]

We denote by \( I(X) \) the set of all \( \varphi \in B(X; \mathbb{R}) \) such that \( 0 \leq \varphi(x) \leq 1 \), for all \( x \in X \). We then define

\[
D(X) = C_b(X; \mathbb{R}) \cap I(X), \\
D_0(X) = B_0(X; \mathbb{R}) \cap I(X).
\]

The strict topology \( \beta \) on \( C_b(X; E) \) is the locally convex topology determined by the family of seminorms

\[
p_\varphi(f) = \sup\{ \varphi(x)||f(x)||; x \in X \}
\]

for \( f \in C_b(X; E) \), when \( \varphi \) ranges over \( D_0(X) \). Clearly, given \( \varphi \in D_0(X) \) there is a compact subset \( K \) such that \( \varphi(x) < \varepsilon \) for all \( x \not\in K \). Therefore, our strict topology is coarser than the strict topology introduced by R. Giles [3]. To see that they actually coincide, let \( \psi \in B(X; \mathbb{R}) \) be such that, for each \( \varepsilon > 0 \) there is a compact subset \( K \) such that \( \psi(x) < \varepsilon \) for all \( x \not\in K \). We may assume \( ||\psi||_X < 1 \). Choose compact sets \( K_n \) with \( \phi = K_0 \subset K_1 \subset K_2 \subset \ldots \) such that \( |\psi(x)| < 2^{-n} \), for all \( x \not\in K_n \).

Let \( \psi_n \in B_0(X; \mathbb{R}) \) be the characteristic function of \( K_n \) multiplied by \( 2^{-n} \), i.e., \( \psi_n(x) = 2^{-n} \), if \( x \in K_n \); and \( \psi_n(x) = 0 \) if \( x \not\in K_n \). Let \( \varphi = \sum_{n=1}^{\infty} \psi_n \). For each \( \varepsilon > 0 \), we claim that the set \( K = \{ x \in X; \varphi(x) \geq \varepsilon \} \) is compact (or empty). If \( \varepsilon > 1 \), then \( K = \phi \). If \( \varepsilon = 1 \), then \( K = K_1 \), because \( \varphi(t) = 1 \) precisely for \( t \in K_1 \). If \( \varepsilon < 1 \),
let $n \geq 0$ be such that $2^{-(n+1)} \leq \varepsilon < 2^{-n}$. Then $K = K_{n+1}$. Hence $\varphi \in D_0(X)$. We claim now that $\psi(x) \leq \varphi(x)$ for all $x \in X$. We first notice that $\varphi(x) = 0$ if, and only if $x \not\in \bigcup_{n=1}^{\infty} K_n$. Indeed, if the point $x \not\in \bigcup_{n=1}^{\infty} K_n$, then $\psi_k(x) = 0$ for all $n = 1, 2, 3, \ldots$, and so $\varphi(x) = 0$. Conversely, if $\varphi(x) = 0$, then $\psi_n(x) = 0$ for all $n = 1, 2, 3, \ldots$ and therefore $x \not\in K_n$ for all $n = 1, 2, 3, \ldots$. Hence $x \not\in \bigcup_{n=1}^{\infty} K_n$. Let now $x \in X$. If $\varphi(x) = 0$, then $x \not\in K_n$ for all $n = 1, 2, 3, \ldots$ and so $|\psi(x)| < 2^{-n}$ for all $n = 1, 2, 3, \ldots$. Hence $\psi(x) = 0$ and so $\psi(x) = \varphi(x)$. Suppose now $\varphi(x) > 0$. Then $x \in \bigcup_{n=1}^{\infty} K_n$. Let $N$ be the smallest positive integer $n \geq 1$ such that $x \in K_n$. If $N = 1$, then $x \in K_1$ and so $\varphi(x) = 1 > \psi(x)$. If $N > 1$, then $x \in K_N$ and $x \not\in K_{N-1}$. Hence

$$\varphi(x) = \sum_{n=N}^{\infty} 2^{-n} = 2^{-(N-1)}$$

and $\psi(x) < 2^{-(N-1)}$, since $x \not\in K_{N-1}$. Therefore $\psi(x) < \varphi(x)$, whenever $\varphi(x) > 0$.

Given any non-empty subset $S \subset C(X; E)$ we denote by $x \equiv y \pmod{S}$ the equivalence relation defined by $f(x) = f(y)$ for all $f \in S$. For each $x \in X$, the equivalence class of $x \pmod{S}$ is denoted by $[x]_S$, i.e.,

$$[x]_S = \{ t \in X \ ; \ x \equiv t \pmod{S} \}$$

For any non-empty subset $K \subset X$ and any $f : X \to E$, we denote by $f_K$ its restriction to $K$. If $S \subset C(X; E)$ and $K \subset X$, then for each $x \in K$ one has

$$[x]_{S_K} = K \cap [x]_S.$$ 

If $S \subset C_b(X; \mathbb{R})$, we define $S^+$ by

$$S^+ = \{ f \in S \ ; \ f \geq 0 \}.$$ 

If $S = C_b(X; \mathbb{R})$, we write $S^+ = C_b^+(X; \mathbb{R})$. 

Approxi...
Definition 1. Let $S \subset C_b(X; \mathbb{R})$ and let $W \subset C_b(X; E)$ be given. We say that $W$ is $\beta$-localizable under $S$ if, for every $f \in C_b(X; E)$, the following are equivalent:

1. $f$ belongs to the $\beta$-closure of $W$;
2. for every $\varphi \in D_0(X)$, every $\varepsilon > 0$ and every $x \in X$, there is some $g_x \in W$ such that $\varphi(t)||f(t) - g_x(t)|| < \varepsilon$ for all $t \in [x]_S$.

Remark. Clearly, (1) $\Rightarrow$ (2) in any case. Hence a set $W$ is $\beta$-localizable under $S$ if, and only if, (2) $\Rightarrow$ (1). Notice also that if $W$ is $\beta$-localizable under $S$ and $T \subset S$, then $W$ is $\beta$-localizable under $T$. Indeed, $T \subset S$ implies $[x]_S \subset [x]_T$.

Definition 2. We say that a set $W \subset C_b(X, E)$ is $\beta$-truncation stable if, for every $f \in H_\beta$ and every $M > 0$, the function $T_M \circ f$ belongs to the $\beta$-closure of $W$, where $T_M : E \to E$ is the mapping defined by

$$
T_M(v) = \begin{cases} v, & \text{if } ||v|| < 2M; \\ \frac{v}{||v||}2M, & \text{if } ||v|| \geq 2M. \end{cases}
$$

Notice that, when $E = \mathbb{R}$, the mapping $T_M : \mathbb{R} \to \mathbb{R}$ is given by

$$
T_M(r) = \begin{cases} r, & \text{if } ||r|| < 2M; \\ 2M, & \text{if } r > 2M; \\ -2M, & \text{if } r > -2M. \end{cases}
$$

Remark that, for every $f \in C_b(X; E)$, one has $||T_M \circ f||_X \leq 2M$.

Notice that when $W \subset C^+_b(X; \mathbb{R})$, then $W$ is $\beta$-truncation stable if, for every $f \in W$ and every constant $M > 0$, the function $P_M \circ f$ belongs to the $\beta$-closure of $W$, where $P_M : \mathbb{R} \to \mathbb{R}_+$ is the mapping defined by $P_M = \max(0, T_M)$, i.e.,

$$
P_M(r) = \begin{cases} 0, & \text{if } r < 0; \\ r, & \text{if } 0 \leq r \leq 2M; \\ 2M, & \text{if } r > 2M. \end{cases}
$$
Definition 3. Let $W \subset C_b(X; E)$ be a non-empty subset. A function $\psi \in D(X)$ is called a multiplier of $W$ if $\psi f + (1 - \psi)g$ belongs to $W$, for each pair, $f$ and $g$, of elements of $W$.

Definition 4. A subset $S \subset D(X)$ is said to have property $V$ if

(a) $\psi \in S$ implies $(1 - \psi) \in S$;
(b) the product $\varphi \psi$ belongs to $S$, for any pair, $\varphi$ and $\psi$, of elements of $S$.

Notice that the set of all multipliers of a subset $W \subset C_b(X; E)$ has property $V$. Indeed, condition (a) is clear and the equation

$$(\varphi \psi)f + (1 - \varphi \psi)g = \varphi[\psi f + (1 - \psi)g] + (1 - \varphi)g$$

show that (b) holds as well.

When $X$ is locally compact, R.C. Buck [1] proved a Weierstrass-Stone Theorem for subalgebras of $C_b(X; \mathbb{R})$ equipped with the strict topology. This result was extended and generalized by Glicksberg [4], Todd [7], Wells [8] and Giles [3]. See also Buck [2], where modules are dealt with, and Prolla [5], where the strict topology is considered as an example of weighted spaces.

Our versions of the Weierstrass-Stone Theorem are analogues of Chapter 4 of Prolla [6] for arbitrary subsets of $C(X; E)$ equipped with the uniform convergence topology, $X$ compact. Whereas the previous results dealt only with algebras or vector spaces which are modules over an algebra, our results now go much further: we are able to cover the case of convex sets (when $X$ is completely regular) or $\beta$-truncation stable sets (when $X$ is just a topological space). The latter case cover both algebras and the convex cones obtained by taking the set of positive elements.
of an algebra.

§ 2. $\beta$-truncation stable subsets

Theorem 1. Let $W \subset C_b(X; E)$ be a $\beta$-truncation stable non-empty subset, and let $A$ be the set of all multipliers of $W$. Then $W$ is $\beta$-localizable under $A$.

Proof. Let $f \in C_b(X; E)$ be given and assume condition (2) of Definition 1, with $S = A$. Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. Without loss of generality we may assume that $\varphi$ is not identically zero. Choose $M > 0$ so big that $M > \|f\|_X, M > \varepsilon$ and the compact set $K = \{ t \in X ; \varphi(t) \geq \varepsilon/(6M) \}$ is non-empty. Consider the non-empty subset $W_K \subset C(K; E)$. Clearly, the set $A_K \subset D(K)$ is a set of multipliers of $W_K$. Take a point $x \in K$. By condition (2) applied to $\varphi/(12M)$, there exists $g_x \in W$ such that $\varphi/(12M)$ for all $t \in [x]_A$. Let $M \subset D(K)$ be the set of all multipliers of $W_K \subset C(K; E)$. Then $M$ has property $V$. Now $A_K \subset M$ implies

$$[x]_M \subset [x]_{A_K} = [x]_A \cap K.$$

Hence $\varphi(t)||f(t) - g_x(t)|| < \varepsilon^2/(12M)$ holds for all $t \in K$ such that $t \in [x]_M$. Now $\varphi(t) \geq \varepsilon/(6M)$ for all $t \in K$ and therefore

$$||f(t) - g_x(t)|| < \varepsilon/2$$

for all $t \in [x]_M$. By Theorem 1, Chapter 4, of Prolla [6] applied to $W_K \subset C(K; E)$ and to the set $M \subset D(K)$, there is $g_1 \in W$ such that

$$||f(t) - g_1(t)|| < \varepsilon/2$$

for all $t \in K$. Let $h = T_M \circ g_1$. By hypothesis, $h$ belongs to the $\beta$-closure of $W$, and there is $g \in W$ such that $p_\varphi(h - g) < \varepsilon/2$. We claim that $p_\varphi(f - h) < \varepsilon/2$. Let

...
$t \in K$. Then

$$||g_1(t)|| \leq ||f(t) - g_1(t)|| + ||f(t)|| < \varepsilon/2 + M < 2M$$

and so $h(t) = T_M(g_1(t)) = g_1(t)$. Hence

$$\varphi(t)||f(t) - h(t)|| = \varphi(t)||f(t) - g_1(t)|| \leq ||f(t) - g_1(t)|| < \varepsilon/2.$$  

Suppose now $t \notin K$. Then

$$\varphi(t)||f(t) - h(t)|| < \frac{\varepsilon}{6M} ||f(t) - h(t)||$$

$$\leq \frac{\varepsilon}{6M}(||f||_X + ||h||_X) < \frac{\varepsilon}{6M}3M = \frac{\varepsilon}{2},$$

because $||h||_X \leq 2M$, and $||f||_X < M$.

This establishes our claim that $p_\varphi(f - h) < \frac{\varepsilon}{2}$. Hence $p_\varphi(f - g) < \varepsilon$, and $f$ belongs to the $\beta$-closure of $W$.  

$\square$

**Theorem 2.** Let $W \subset C_b(X; E)$ be a $\beta$-truncation stable non-empty subset, and let $B$ be any non-empty set of multipliers of $W$. Then $W$ is $\beta$-localizable under $B$.

**Proof.** Let $A$ be the set of all multipliers of $W$. By Theorem 1 the set $W$ is $\beta$-localizable under $A$. Now $B \subset A$, so $W$ is also $\beta$-localizable under $B$.  

$\square$

§3. The case of subalgebras

**Lemma 1.** If $B \subset C_b(X; \mathbb{R})$ is a uniformly closed subalgebra, and $T : \mathbb{R} \to \mathbb{R}$ is a continuous mapping, with $T(0) = 0$, then $T \circ f$ belongs to $B$, for every $f \in B$.  

Proof. Let \( f \in B \) and \( \varepsilon > 0 \) be given. Choose \( k \geq ||f||_X \). By Weierstrass' Theorem, there exists an algebraic polynomial \( p \) such that \( |T(t) - p(t)| < \varepsilon \) for all \( t \in \mathbb{R} \) with \( |t| \leq k \), and we may assume \( p(0) = T(0) = 0 \). Hence, for every \( x \in X \), we have \( |T(f(x)) - p(f(x))| < \varepsilon \), because \( |f(x)| \leq k \). Now \( p \circ f \) belongs to \( B \), and therefore \( T \circ f \) belongs to the uniform closure of \( B \), that is \( B \) itself.

Corollary 1. Every subalgebra \( W \subset C_b(X; \mathbb{R}) \) is \( \beta \)-truncation stable.

Proof. Let \( f \in W \) and \( M > 0 \) be given. Let \( B \) be the \( \beta \)-closure of \( W \) in \( C_b(X; \mathbb{R}) \). We know that \( B \) is then a uniformly closed subalgebra. By Lemma 1 applied to \( T = T_M \), we see that \( T_M \circ f \) belongs to the \( \beta \)-closure of \( W \) as claimed.

Corollary 2. Every uniformly closed subalgebra of \( C_b(X; \mathbb{R}) \) is a lattice.

Proof. Since
\[
\max(f, g) = \frac{1}{2} \left( f + g + |f - g| \right)
\]
\[
\min(f, g) = \frac{1}{2} \left( f + g - |f - g| \right)
\]
it suffices to show that \( |f| \in B \), for every \( f \in B \). This follows from Lemma 1, by taking \( T : \mathbb{R} \to \mathbb{R} \) to be the mapping \( T(t) = |t| \), for \( t \in \mathbb{R} \).

Theorem 3. Every subalgebra \( W \subset C_b(X; \mathbb{R}) \) is \( \beta \)-localizable under itself.

Proof. Let \( f \in C_b(X; \mathbb{R}) \) and assume that condition (2) of Definition 1 holds with \( S = W \). Notice that for every \( x \in X \) one has
\[
[x]_W = [x]_B
\]
where \( B \) is the \( \beta \)-closure of \( W \). Let now
\[
V = \{ \psi \in B; ||\psi||_X \leq 1 \} \quad \text{and} \quad A = \{ \psi \in B; 0 \leq \psi \leq 1 \}.
\]
It is easy to see that

\[ [x]_B = [x]_V \subset [x]_A , \]

for each \( x \in X \). Notice that, by Corollary 2, every \( \psi \in V \) can be written in the form \( \psi = \psi^+ - \psi^- \), with \( \psi^+ \) and \( \psi^- \) in \( A \). Hence \([x]_A \subset [x]_V \) is also true. Hence \( f \) satisfies condition (2) of Definition 1 with respect to \( S = A \). Now \( A \) is a set of multipliers of \( B \), and the algebra \( B \), by Corollary 1, is \( \beta \)-truncation stable. Hence, by Theorem 3, the function \( f \) belongs to the \( \beta \)-closure of \( B \), that is \( B \) itself. We have proved that \( f \) belongs to the \( \beta \)-closure of \( W \). Hence \( W \) is \( \beta \)-localizable under \( S = W \).

**Corollary 3.** Let \( W \subset C_b(X; \mathbb{R}) \) be a subalgebra, and let \( f \in C_b(X; \mathbb{R}) \) be given. Then \( f \) belongs to the \( \beta \)-closure of \( W \) if, and only if, the following conditions are satisfied:

1. for each pair, \( x \) and \( y \), of elements of \( X \) such that \( f(x) \neq f(y) \), there is some \( g \in W \) such that \( g(x) \neq g(y) \);
2. for each \( x \in X \) such that \( f(x) \neq 0 \) there is some \( g \in W \) such that \( g(x) \neq 0 \).

**Proof.** Clearly, if \( f \in \overline{W}^{\beta} \), then (1) and (2) are satisfied. Conversely, assume that conditions (1) and (2) are verified.

Let \( x \in X \) be given. By condition (1) the function \( f \) is constant on \([x]_W \). Let \( f(x) \) be its value. If \( f(x) = 0 \), then \( g_x = 0 \) belongs to \( W \) and \( f(t) = f(x) = 0 = g_x(t) \) for all \( t \in [x]_W \). If \( f(x) \neq 0 \), by condition (2) there is \( g \in W \) such that \( g(x) \neq 0 \). Define \( g_x = [f(x)/g(x)]g \). Then \( g_x \in W \) and \( g_x(t) = f(x) = f(t) \) for all \( t \in [x]_W \). Hence \( f \) satisfies condition (2) of Definition 1 with respect to \( S = W \). By Theorem 3, we conclude that \( f \) belongs to the \( \beta \)-closure of \( W \).

Corollary 3 implies the following results.
Corollary 4. Let $A$ be a subalgebra of $C_b(X; \mathbb{R})$ which for each $x \in X$ contains a function $g$ with $g(x) \neq 0$, and let $f \in C_b(X; \mathbb{R})$ be given. Then $f$ belongs to the $\beta$-closure of $A$ if, and only if, for each pair, $x$ and $y$, of elements of $X$ such that $f(x) \neq f(y)$, there is some $g \in A$ such that $g(x) \neq g(y)$.

Corollary 5. Let $A$ be a subalgebra of $C_b(X; \mathbb{R})$ which separates the points of $X$ and for each $x \in X$ contains a function $g$ with $g(x) \neq 0$. Then $A$ is $\beta$-dense in $C_b(X; \mathbb{R})$.

Corollary 6. If $X$ is a locally compact Hausdorff space, then $C_{00}(X; \mathbb{R})$ is $\beta$-dense in $C_b(X; \mathbb{R})$.

Lemma 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(t) \geq 0$ for all $t \in \mathbb{R}$ and $f(0) = 0$. If $k > 0$ and $\varepsilon > 0$ are given, there is a real algebraic polynomial $p$ such that $p(t) \geq 0$ for all $0 \leq t \leq k$, $p(0) = 0$ and $|p(t) - f(t)| \leq \varepsilon$ for all $0 \leq t \leq k$.

Proof. Define $g : [0, 1] \to \mathbb{R}$ by setting $g(u) = f(ku)$, for each $u \in [0, 1]$. Clearly, $g(u) \geq 0$, for all $0 \leq u \leq 1$ and $g(0) = 0$. Now, given $\varepsilon > 0$, choose $n$ so that the $n$-th Bernstein polynomial of $g$, written $B_n g$, is such that

$$|(B_n g)(u) - g(u)| < \varepsilon$$

for all $0 \leq u \leq 1$. For $t \in \mathbb{R}$, define $p(t) = (B_n g)(t/k)$. Since $B_n g \geq 0$ in $[0, 1]$, it follows that $p(t) \geq 0$, for $t \in [0, k]$. Since $(B_n g)(0) = g(0) = f(0) = 0$, we see that $p(0) = 0$. It remains to notice that, for any $0 \leq t \leq k$ we have $0 \leq t/k \leq 1$ and

$$|p(t) - f(t)| = |(B_n g)(t/k) - g(t/k)| < \varepsilon$$

Lemma 3. If $A \subset C_b(X; \mathbb{R})$ is a subalgebra, then $A^+$ is $\beta$-truncation stable.
Proof. Let $f \in A^+$ and $M > 0$ be given. We claim that $P_M \circ f$ belongs to the $eta$-closure of $A^+$. Let $k > 0$ be such that $0 \leq f(x) \leq k$ for all $x \in X$. Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. By Lemma 2 above there exists a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $p(t) \geq 0$ for all $0 \leq t \leq k$, $p(0) = 0$ and $|p(t) - P_M(t)| < \varepsilon$ for all $0 \leq t \leq k$. Let $x \in X$. Then $\varphi(x) \leq 1$ and so $\varphi(x)|p(f(x)) - P_M(f(x))| < \varepsilon$. Now $p \circ f$ belongs to $A$ (since $p(0) = 0$) and $p(f(x)) \geq 0$ for all $x \in X$, since $0 \leq f(x) \leq k$. Hence $p \circ f \in A^+$. This ends the proof that $P_M \circ f$ belongs to the $eta$-closure of $A^+$ as claimed.

Theorem 4. If $A \subset C_b(X; \mathbb{R})$ is a subalgebra, then $A^+$ is localizable under itself.

Proof. Let $f \in C_b(X; \mathbb{R})$ be given satisfying condition (2) of Definition 1 with respect to $S = A^+$. Define $B = \{f \in A; 0 \leq f \leq 1\}$. It is easy to see that $[x]_S = [x]_B$, for every $x \in X$. Hence $f$ satisfies condition (2) of Definition 1 with respect to $B$, which is a set of multipliers of $A^+$. By Lemma 3, the set $A^+$ is $\beta$-truncation stable. Therefore $A^+$ is $\beta$-localizable under $B$, by Theorem 2. Hence $f$ belongs to the $\beta$-closure of $A^+$.

Theorem 4. Let $A \subset C_b(X; \mathbb{R})$ be a subalgebra and let $f \in C_b^+(X; \mathbb{R})$ be given. Then $f$ belongs to the $\beta$-closure of $A^+$ if, and only if, the following two conditions hold:

1. for each pair, $x$ and $y$, of elements of $X$ such that $f(x) \neq f(y)$, there is some $g \in A^+$ such that $g(x) \neq g(y)$;
2. for each $x \in X$ such that $f(x) > 0$ there is some $g \in A^+$ such that $g(x) > 0$.

Proof. If $f$ belongs to the $\beta$-closure of $A^+$ the two conditions (1) and (2) above are easily seen to hold. Conversely, assume that conditions (1) and (2) above hold. Let $x \in X$ be given. By condition (1), the function $f$ is constant on $[x]_S$ where $S = A^+$. Let $f(x) \geq 0$ be its constant value. If $f(x) = 0$, then $g_x = 0$ belongs to
$A^+$ and $f(t) = f(x) = 0 = g_x(t)$ for all $t \in [x]_S$. If $f(x) > 0$, then by condition (2) there is $g_x \in A^+$ such that $g(x) > 0$. Let $g_x = [f(x)/g(x)]g$. Then $g_x \in A^+$ and $g_x(t) = f(x) = f(t)$ for all $t \in [x]_S$. Hence $f$ satisfies condition (2) of Definition 1 with respect to $W = A^+$ and $S = A^+$. By Theorem 4, we conclude that $f$ belongs to the $\beta$-closure of $A^+$. 

§4. The case of uniformly bounded subsets

Theorem 5. Let $W$ be a uniformly bounded subset of $C_b(X; E)$ and let $A$ be the set of all multipliers of $W$. Then $W$ is $\beta$-localizable under $A$.

Proof. Let $f \in C_b(X; E)$ be given and assume that condition (2) of Definition 1 holds with $S = A$. Let $\varepsilon > 0$ and $\varphi \in D_0(X)$ be given. Choose $M > 0$ so big that $M > \|f\|_X$ and $M > k = \sup\{\|g\|_X; g \in W\}$, and the compact set $K = \{t \in X; \varphi(t) \geq \varepsilon/(2M)\}$ is non-empty. (Without loss of generality we may assume that $\varphi$ is not identically zero). Consider the non-empty set $W_K \subset C(K; E)$. Clearly, the set $A_K$ is a set of multipliers of $W_K$. Take a point $x \in K$. By condition (2) applied to $\varepsilon^2/(2M)$, there exists some $g_x \in W$ such that

$$\varphi(t)\|f(t) - g_x(t)\| < \varepsilon^2/(2M)$$

for all $t \in [x]_A$. Hence $\|f(t) - g_x(t)\| < \varepsilon$ for all $t \in [x]_{A_K}$ since $\varphi(t) \geq \varepsilon/(2M)$ for all $t \in K$. Let now $M$ be the set of all multipliers of $W_K \subset C(K; E)$. Since $A_K \subset M$, it follows that $[x]_M \subset [x]_{A_K}$ and so $\|f(t) - g_x(t)\| < \varepsilon$ for all $t \in [x]_M$. By Theorem 1, Chapter 4 of Prolla [6] there is $g \in W$ such that $\|f(t) - g(t)\| < \varepsilon$ for all $t \in K$. We claim that $p_\varphi(t - g) < \varepsilon$. Let $x \in X$. If $x \in K$, then $\varphi(x) \leq 1$ and

$$\varphi(x)\|f(x) - g(x)\| \leq \|f(x) - g(x)\| < \varepsilon.$$

If $x \notin K$, then

$$\varphi(x)\|f(x) - g(x)\| \leq \frac{\|f\|_X}{2M}[\|f\|_X + \|g\|_X] < \varepsilon.$$
Hence $f$ belongs to the $\beta$-closure of $W$ and so $W$ is $\beta$-localizable under $A$.

Theorem 6. Let $W$ be a uniformly bounded subset of $C_b(X; E)$ and let $B$ be any non-empty set of multipliers of $W$. Then $W$ is $\beta$-localizable under $B$.

Proof. Let $A$ be the set of all multipliers of $W$. Since $B \subseteq A$ and by Theorem 5 the set $W$ is $\beta$-localizable under $A$, it follows that $W$ is also $\beta$-localizable under $B$.

Theorem 7. Let $A$ be a non-empty subset of $D(X)$ with property $V$ and let $f \in D(X)$. Then $f$ belongs to the $\beta$-closure of $A$ if, and only if, the following two conditions hold:

1. For every pair of points, $x$ and $y$, of $X$ such that $f(x) \neq f(y)$, there exists $g \in A$ such that $g(x) \neq g(y)$;
2. For every $x \in X$ such that $0 < f(x) < 1$, there exists $g \in A$ such that $0 < g(x) < 1$.

Proof. It is easy to see that conditions (1) and (2) are necessary for $f$ to belong to the $\beta$-closure of $A$. Conversely, assume that $f$ satisfies conditions (1) and (2).

Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. Without loss of generality we may assume that $\varphi$ is not identically zero. Choose $\delta > 0$ so small that $2\delta < \varepsilon$ and the compact set $K = \{t \in X; \varphi(t) \geq \delta\}$ is non-empty. Clearly, $A_K$ has property $V$. Since conditions (1) and (2) hold, we may apply Theorem 1, Chapter 8, Prolla [6] to conclude that $f_K$ belongs to the uniform closure of $A_K$. Hence there is some $g \in A$ such that $|f(t) - g(t)| < \varepsilon$ for all $t \in K$. We claim that $p_\varphi(f - g) < \varepsilon$. Let $x \in X$. If $x \in K$, then $\varphi(x) \leq 1$ and $\varphi(x)|f(x) - g(x)| \leq |f(x) - g(x)| < \varepsilon$.

If $x \notin K$, then $\varphi(x) < \delta$ and

$$\varphi(x)|f(x) - g(x)| \leq \delta[|f||x| + |g||x|] \leq 2\delta < \varepsilon.$$

Hence $f$ belongs to the $\beta$-closure of $A$.

Remark. We say that a subset $A \subseteq D(X)$ has property $VN$ if $fg + (1 - f)h \in A$. 

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for all $f, g, h \in A$. Clearly, if $A$ has property $VN$ and contains $0$ and $1$, then $A$ has property $V$.

**Corollary 6.** Let $A$ be a non-empty subset of $D(X)$ with property $V$, and let $W$ be its $\beta$-closure. Then $W$ has property $VN$ and $W$ is a lattice.

**Proof.**

(a) *W has property VN:* Let $f, g, \varphi$ belong to $W$, and let $h = \varphi f + (1 - \varphi)g$. Assume $h(x) \neq h(y)$. Then at least one of the following three equalities is necessarily false: $\varphi(x) = \varphi(y)$, $f(x) = f(y)$ and $g(x) = g(y)$. Since $\varphi, f$ and $g$ belong all three to $W$, there exists $a \in A$ such that $a(x) \neq a(y)$. Hence $h$ satisfies condition (1) of Theorem 7. Suppose now that $0 < h(x) < 1$. If $0 < \varphi(x) < 1$, then $0 < a(x) < 1$ for some $a \in A$, because $\varphi$ belongs to the $\beta$-closure of $A$. Assume that $\varphi(x) = 0$. Then $h(x) = g(x)$ and so $0 < g(x) < 1$. Since $g \in W$, it follows that $0 < a(x) < 1$ for some $a \in A$. Similarly, if $\varphi(x) = 1$ then $h(x) = f(x)$ and so $0 < f(x) < 1$. Since $f \in W$, there is $a \in A$ such that $0 < a(x) < 1$. Hence $h$ satisfies condition (2) of Theorem 7. By Theorem 7 above, the function $h$ belongs to $W$.

(b) *W is lattice:* Let $f$ and $g$ belong to $W$. Let $h = \max(f, g)$. Let $x$ and $y$ be a pair of points of $X$ such that $h(x) \neq h(y)$. Then at least one of the two equalities $f(x) = f(y)$, $g(x) = g(y)$ must be false. Since $f$ and $g$ both belong to the $\beta$-closure of $A$, there exists $a \in A$ such that $a(x) \neq a(y)$. On the other hand, let $x \in X$ be such that $0 < h(x) < 1$. If $f(x) \geq g(x)$, then $h(x) = f(x)$ and so $0 < f(x) < 1$. Since $f \in W$, there exists $a \in A$ such that $0 < a(x) < 1$. Assume now $f(x) < g(x)$. Then $h(x) = g(x)$ and so $0 < g(x) < 1$. Since $g \in W$, there exists $a \in A$ such that $0 < a(x) < 1$. By Theorem 7 above, the function $h$ belongs to $W$. Similarly, one shows that the function $\min(f, g)$ belongs to $W$. \[\Box\]

**Corollary 7.** Let $A$ be a $\beta$-closed non-empty subset of $D(X)$ with property $V$. Then $A$ has property $VN$ and $A$ is a lattice.
Proof. Immediate from Corollary 6.

§5. The case of convex subsets

In this section we suppose that $X$ is a completely regular Hausdorff space. We denote its Stone-Čech compactification by $\beta X$, and by $\beta : C_b(X; \mathbb{R}) \to C(\beta X; \mathbb{R})$ the linear isometry which to each $f \in C_b(X; \mathbb{R})$ assigns its (unique) continuous extension to $\beta X$. Since $\beta$ is an algebra (and lattice) isomorphism, the image $\beta(A)$ of any subset $A \subset C_b(X, \mathbb{R})$ with property $V$ is contained in $D(\beta X)$ and has property $V$. If $B = \beta(A)$, then for each $x \in X$ one has

$$[x]_A = [x]_B \cap X.$$

If $Y$ denotes the quotient space of $\beta X$ by the equivalence relation $x \equiv y$ if and only if $\varphi(x) = \varphi(y)$, for all $\varphi \in B$, then $Y$ is a compact Hausdorff space.

If $x \in X$ and $K_x \subset X$ is a compact subset disjoint from $[x]_A$, then $\pi(K_x)$ is a compact subset in $Y$ which does not contain the point $\pi(x)$. (Here we have denoted by $\pi$ the canonical projection $\pi : \beta X \to Y$. Indeed, if $\pi(x) \in \pi(K_x)$, then $\pi(x) = \pi(y)$ for some $y \in K_x$. Now $y \notin [x]_B$ because that $y \in [x]_A$. But $K_x \cap [x]_A = \emptyset$, and we have reached a contradiction. Hence $\pi(x) \notin \pi(K_x)$. We will apply these remarks in the proof of the following lemma.

Lemma 4. Let $A \subset D(X)$ be a subset with property $V$ and containing some constant $0 < c < 1$. Let $x \in X$ and let $K_x \subset X$ be a compact subset, disjoint from $[x]_A$. Then, there exists an open neighborhood $W(x)$ of $[x]_A$ in $X$, disjoint from $K_x$ and such that given $0 < \delta < 1$ there is $\varphi \in A$ such that

1. $\varphi(t) < \delta$, for all $t \in K_x$;
2. $\varphi(t) > 1 - \delta$, for all $t \in W(x)$.

Proof. Let $N(x)$ be the complement of $K_x$ in $\beta X$. Then $N(x)$ is an open neigh-
borhood of $[x]_A$ in $\beta X$. We know that $s(K_x)$ is a compact subset of $Y$ which does not contain the point $y = \pi(x)$. Let $f \in C(Y; R)$ be a mapping such that $0 \leq f \leq 1$, $f(y) = 0$ and $f(t) = 1$ for all $t \in s(K_x)$. Let $g = f \circ \pi$. By Theorem 1, Chapter 8, Prolla [6], the function $g$ belongs to the uniform closure of $B$ in $D(\beta X)$. Notice that $a(x) = 0$ and $g(u) = 1$, for all $u \in K_x$. Define $N(x) = \{ t \in \beta X; g(t) < 1/4 \}$. Clearly, $[x]_B \subseteq N(x)$, since $g(t) = 0$ for all $t \in [x]_B$. It is also clear that $N(x)$ is disjoint from $K_x$. Let us define $W(x) = N(x) \cap X$. Then $W(x)$ is an open neighborhood of $[x]_A$ in $X$, which is disjoint from $K_x$.

Given $0 < \delta < 1$, let $p$ be a polynomial determined by Lemma 1, Chapter 1, Prolla [6], applied to $a = 1/4$ and $b = 3/4$, and $\varepsilon = \delta/2$. Let $h(t) = p(g(t))$, for all $t \in \beta X$. Since $\overline{B}$ has property $V$, it follows that $h \in \overline{B}$. If $t \in K_x$, then $g(t) = 1$ and so $h(t) < \delta/2$. If $t \in W(x)$, then $g(t) < 1/4$ and so $h(t) > 1 - \delta/2$. Choose now $\psi \in B$ with $||\psi - h||_X < \delta/2$, and let $\varphi \in A$ be such that $\beta(\varphi) = \psi$. Then $\varphi \in A$ satisfies conditions (1) and (2). □

**Theorem 8.** Let $W \subseteq C_b(X; E)$ be a non-empty subset and let $A$ be a set of multipliers of $W$ which has property $V$ and contains some constant $0 < c < 1$. Then $W$ is $\beta$-localizable under $A$.

**Proof.** Assume that condition (2) of Definition 1 is true with $S = A$. For each $x \in X$, there is some $g_x \in W$ such that, for all $t \in [x]_A$, one has $\varphi(t)||f(t) - g_x(t)|| < \varepsilon/2$. Consider the compact subset $K_x$ of $X$ defined by

$$K_x = \{ t \in X; \varphi(t)||f(t) - g_x(t)|| \geq \frac{\varepsilon}{2} \}.$$ 

Clearly, $K_x$ is disjoint from $[x]_A$. Now for each $x \in X$, select an open neighborhood $W(x)$ of $[x]_A$, disjoint from $K_x$, according to Lemma 4.

Select and fix a point $x_1 \in X$. Let $K = K_{x_1}$. By compactness of $K$, there exists a finite set $\{ x_2, \ldots, x_m \} \subseteq K$ such that

$$K \subseteq W(x_2) \cup W(x_3) \cup \ldots \cup W(x_m)$$
Let \( k = \sum_{i=1}^{m} p_i(f - g_{x_i}) \) and let \( 0 < \delta < 1 \) be so small that \( \delta k < \varepsilon/2 \).

By Lemma 4, there are \( \varphi_2, \ldots, \varphi_m \in A \) such that
(a) \( \varphi_i(t) < \delta \), for all \( t \in K_{x_i} \);
(b) \( \varphi_i(t) > 1 - \delta \), for all \( t \in W(x_i) \)
for \( i = 2, \ldots, m \). Define

\[
\begin{align*}
\psi_2 &= \varphi_2 \\
\psi_3 &= (1 - \varphi_2)\varphi_3 \\
&\vdots \\
\psi_m &= (1 - \varphi_2)(1 - \varphi_3)\ldots(1 - \varphi_{m-1})\varphi_m.
\end{align*}
\]

Clearly, \( \psi_i \in A \) for all \( i = 2, \ldots, m \). Now

\[
\psi_2 + \ldots + \psi_j = 1 - (1 - \varphi_2)(1 - \varphi_3)\ldots(1 - \varphi_j)
\]
for all \( j \in \{2, \ldots, m\} \), can be easily seen by induction. Define

\[
\psi_1 = (1 - \varphi_2)(1 - \varphi_3)\ldots(1 - \varphi_m)
\]
then \( \psi_1 \in A \) and \( \psi_1 + \psi_2 + \ldots + \psi_m = 1 \).

Notice that

(c) \( \psi_i(t) < \delta \) for all \( t \in K_{x_i} \),
for each \( i = 1, 2, \ldots, m \). Indeed, if \( i > 2 \) then (c) follows from (a). If \( i = 1 \), then for \( t \in K \), we have \( t \in W(x_j) \) for some \( j = 2, \ldots, m \). By (b), one has \( 1 - \varphi_j(t) < \delta \) and so

\[
\psi_1(t) = (1 - \varphi_j(t)) \prod_{i \neq j}(1 - \varphi_i(t)) < \delta.
\]

Let us write \( g_i = g_{x_i} \) for \( i = 1, 2, \ldots, m \).

Define \( g = \psi_1 g_1 + \psi_2 g_2 + \ldots + \psi_m g_m \).

Notice that

\[
g = \varphi_2 g_2 + (1 - \varphi_2)[\varphi_3 g_3 + (1 - \varphi_3)[\varphi_4 g_4 + \ldots + (1 - \varphi_{m-1})[\varphi_m g_m + (1 - \varphi_m)g_1] \ldots]].
\]
Hence $g \in W$. Let $x \in X$ be given. Then

$$
\varphi(x)\|f(x) - g(x)\| = \varphi(x)\left\| \sum_{i=1}^{m} \psi_i(x)(f(x) - g_i(x)) \right\| 
\leq \varphi(x)\| \sum_{i=1}^{m} \psi_i(x)\|(f(x) - g_i(x))\|
$$

Define $I = \{1 \leq \tau \leq m; x \notin K_{x_i}\}; J = \{1 \leq i \leq m; x \in K_{x_i}\}$.

If $i \in I$, then $x \notin K_{x_i}$ and

$$
\varphi(x)\|f(x) - g_i(x)\| < \frac{\varepsilon}{2}
$$

and therefore

$$
(*) \sum_{i \in I} \varphi(x)\psi_i(x)\|f(x) - g_i(x)\| \leq \frac{\varepsilon}{2} \sum_{i \in I} \psi_i(x) \leq \frac{\varepsilon}{2}
$$

If $i \in J$, then by (c), $\psi_i(x) < \delta$ and so

$$
(**) \sum_{i \in J} \varphi(x)\psi_i(x)\|f(x) - g_i(x)\| \leq \delta k < \frac{\varepsilon}{2}
$$

From (*) and (**) we get $\varphi(x)\|f(x) - q(x)\| < \varepsilon$. \hfill \Box

**Theorem 9.** Let $W \subset C_b(X; E)$ be a non-empty convex subset and let $A$ be the set of all multipliers of $W$. Then $W$ is $\beta$-localizable under $A$.

**Proof.** The set $A$ has property $V$ and, since $W$ is convex, every constant $0 < c < 1$ belongs to $A$. \hfill \Box

**Theorem 10.** Let $W \subset C_b(X; E)$ be a non-empty convex subset and let $B$ be any non-empty set of multipliers of $W$. Then $W$ is $\beta$-localizable under $B$.

**Proof.** Similar to that of Theorem 6, using now Theorem 9 instead of Theorem 5.
Corollary 8. Let $W \subseteq C_b(X; E)$ be a non-empty convex subset such that the set of all multipliers of $W$ separates the points of $X$. Then, for each $f \in C_b(X; \mathbb{R})$ the following are equivalent:

1. $f$ belongs to the $\beta$-closure of $W$;
2. for each $\varepsilon > 0$ and each $x \in X$, there is some $g \in W$ such that $\|f(x) - g(x)\| < \varepsilon$.

Proof. Clearly, (1) $\Rightarrow$ (2). Suppose now that (2) holds. Let $\varphi \in D_0(X), \varepsilon > 0$ and $x \in X$ be given. Notice that $[x]_W = \{x\}$. If $\varphi(x) = 0$, for any $g \in W$ one has $\varphi(x)\|f(x) - g(x)\| = 0 < \varepsilon$. If $\varphi(x) > 0$, by (2) there is $g \in W$ such that $\|f(x) - g(x)\| < \varepsilon/\varphi(x)$. Hence $\varphi(x)\|f(x) - g(x)\| < \varepsilon$, and by Theorem 9, (1) is true. \hfill $\square$

Corollary 9. Let $S \subseteq X$ be a non-empty closed subset and let $V \subseteq E$ be a non-empty convex subset. Let $W = \{g \in C_b(X; E); g(S) \subseteq V\}$. Then, for each $f \in C_b(X; E)$ the following are equivalent:

1. $f$ belongs to the $\beta$-closure of $W$;
2. for each $x \in S$, $f(x)$ belongs to the closure of $V$ in $E$

Hence, $\overline{W}^\beta = \{f \in C_b(X; E); f(S) \subseteq \overline{V}\}$, where $\overline{V}$ is the closure of $V$ in $E$.

Proof. Clearly, (1) $\Rightarrow$ (2). Conversely, assume that (2) holds. Clearly, $W$ is a convex set such that $D(X)$ is the set of all multipliers of $W$. Since $X$ is a completely regular Hausdorff space, $D(X)$ separates the points of $X$. Let $\varepsilon > 0$ and $x \in X$ be given. If $x \in S$ there is $v \in V$ such that $\|f(x) - v\| < \varepsilon$, and the constant mapping on $X$ whose value is $v$ belongs to $W$ and $g(x) = v$. If $x \not\in S$, choose $\varphi \in C_b(X; \mathbb{R}), 0 \leq \varphi \leq 1, \varphi(t) = 1$ for all $t \in S$ and $\varphi(x) = 0$; and let $g \in C_b(X; E)$
be defined by \( g = \varphi \otimes v_0 + (1 - \varphi) \otimes f(x) \), where \( v_0 \in V \) is chosen arbitrarily. Then \( g(t) = v_0 \) for all \( t \in S \), and therefore \( g \in W \), and \( g(x) = f(x) \). Hence (2) of Corollary 8 is verified and so \( f \) belongs to the \( \beta \)-closure of \( W \). \( \Box \)

**Corollary 10.** Let \( W \subset C_b(X; E) \) be a non-empty convex subset such that the set of all multipliers of \( W \) separates the points of \( X \) and, for each \( x \in X \), the set \( W(x) = \{ g(x) ; g \in W \} \) is dense in \( E \). Then \( W \) is \( \beta \)-dense in \( C_b(X; E) \).

**Proof.** Apply Corollary 8. \( \Box \)

**Corollary 11.** The vector subspace \( W = C_b(X; \mathbb{R}) \otimes E \) is \( \beta \)-dense in \( C_b(X; E) \).

**Proof.** The set \( A \) of all multipliers of \( W \) is \( D(X) \), and \( W(x) = E \), for each \( x \in X \). It remains to apply Corollary 10. \( \Box \)

**Corollary 12.** If \( X \) is a locally compact Hausdorff space, then \( C_{00}(X; \mathbb{R}) \otimes E \) is \( \beta \)-dense in \( C_b(X; E) \).

**Proof.** Let \( W = C_{00}(X; \mathbb{R}) \otimes E \). As in the previous corollary, the set \( A \) of all multipliers of \( W \) is \( D(X) \), and for each \( x \in X \), \( W(x) = E \). \( \Box \)

**Theorem 11.** Let \( A \subset C_b(X; \mathbb{R}) \) be a subalgebra and let \( W \subset C_b(X; E) \) be a vector subspace which is an \( A \)-module, i.e., \( AW \subset W \). Then \( W \) is \( \beta \)-localizable under \( A \).

**Proof.** Let \( f \in C_b(X; E) \) be given. Assume that condition (2) of Definition 1 holds with \( S = A \). Without loss of generality we may assume that \( A \) is \( \beta \)-closed and contains the constants. Let \( M \) be the set of all multipliers of \( W \). We claim that, for each \( x \in X \), one has \( [x]_M \subset [x]_A \). Indeed, let \( t \in [x]_M \) and let \( \varphi \in A \). If \( \varphi = 0 \), then \( \varphi \in M \) and \( \varphi(t) = \varphi(x) \). Assume \( \varphi \neq 0 \). Write \( \varphi = \varphi^+ - \varphi^- \),
where $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = \max(-\varphi, 0)$. By Corollary 2, §3, both $\varphi^+$ and $\varphi^-$ belong to $A$. If $\varphi^+ = 0$, then $\varphi^+$ belongs to $M$ and $\varphi^+(t) = \varphi^+(x)$. If $\varphi^+ \neq 0$, let $\psi = \varphi^+ / ||\varphi^+||_X$. Now $\psi$ belongs to $A$ and $0 \leq \psi \leq 1$. Hence $\psi \in M$ and therefore $\psi(t) = \psi(x)$. Consequently, one has $\varphi^+(t) = \varphi^+(x)$. Similarly, one proves that $\varphi^-(t) = \varphi^-(x)$. Hence $\varphi(t) = \varphi(x)$. This ends the proof that $[x]_M \subset [x]_A$ for all $x \in X$. Hence condition (2) of Definition 1 is verified with $S = M$. By Theorem 9, $W$ is $\beta$-localizable under $M$. Hence $f$ belongs to the $\beta$-closure of $W$.

**Corollary 13.** Let $W \subset C_b(X; E)$ be a vector subspace, and let

$$A = \{\psi \in C_b(X; \mathbb{R}); \psi g \in W \text{ for all } g \in W\}.$$  

Then $W$ is $\beta$-localizable under $A$.

**Proof.** Clearly $A$ is a subalgebra of $C_b(X; \mathbb{R})$ and $W$ is an $A$-module. 

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