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Approximation Results in the Strict Topology

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Abstract: In this paper we prove results of the Weierstrass-Stone type for subsets $W$ of the vector space $V$ of all continuous and bounded functions from a topological space $X$ into a real normed space $E$, when $V$ is equipped with the strict topology $\beta$. Our main results characterize the $\beta$-closure of $W$ when (1) $W$ is $\beta$-truncation stable; (2) $E = \mathbb{R}$ and $W$ is a subalgebra; (3) $E = \mathbb{R}$ and $W$ is the convex cone of all positive elements of some algebra; (4) $W$ is uniformly bounded; (5) $X$ is a completely regular Hausdorff space and $W$ is convex.

§1. Introduction and definitions

Let $X$ be a topological space and let $E$ be a real normed space. We denote by $B(X; E)$ the normed space of all bounded $E$-valued functions on $X$, equipped with the supremum norm

$$
\|f\|_X = \sup\{|f(x)|; x \in X\}
$$

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for each \( f \in B(X; E) \). We denote by \( B_0(X; E) \) the subset of all \( f \in B(X; E) \) that vanish at infinity, i.e., those \( f \) such that for every \( \varepsilon > 0 \), the set \( K = \{ t \in X; ||f(t)|| \geq \varepsilon \} \) is compact (or empty). And we denote by \( B_{00}(X; E) \) the subset of all \( f \in B(X; E) \) which have compact support. We denote by \( C(X; E) \) the vector space of all continuous \( E \)-valued functions on \( X \), and set

\[
C_b(X; E) = C(X; E) \cap B(X; E),
C_0(X; E) = C(X; E) \cap B_0(X; E),
C_{00}(X; E) = C(X; E) \cap B_{00}(X; E).
\]

We denote by \( I(X) \) the set of all \( \varphi \in B(X; \mathbb{R}) \) such that \( 0 \leq \varphi(x) \leq 1 \), for all \( x \in X \). We then define

\[
D(X) = C_b(X; \mathbb{R}) \cap I(X),
D_0(X) = B_0(X; \mathbb{R}) \cap I(X).
\]

The strict topology \( \beta \) on \( C_b(X; E) \) is the locally convex topology determined by the family of seminorms

\[
p_\varphi(f) = \sup\{\varphi(x)||f(x)||; x \in X\}
\]

for \( f \in C_b(X; E) \), when \( \varphi \) ranges over \( D_0(X) \). Clearly, given \( \varphi \in D_0(X) \) there is a compact subset \( K \) such that \( \varphi(x) < \varepsilon \) for all \( x \not\in K \). Therefore, our strict topology is coarser than the strict topology introduced by R. Giles [3]. To see that they actually coincide, let \( \psi \in B(X; \mathbb{R}) \) be such that, for each \( \varepsilon > 0 \) there is a compact subset \( K \) such that \( \psi(x) < \varepsilon \) for all \( x \not\in K \). We may assume \( ||\psi||_X < 1 \). Choose compact sets \( K_n \) with \( \phi = K_0 \subset K_1 \subset K_2 \subset \ldots \) such that \( |\psi(x)| < 2^{-n} \), for all \( x \not\in K_n \).

Let \( \psi_n \in B_0(X; \mathbb{R}) \) be the characteristic function of \( K_n \) multiplied by \( 2^{-n} \), i.e., \( \psi_n(x) = 2^{-n} \), if \( x \in K_n \); and \( \psi_n(x) = 0 \) if \( x \not\in K_n \). Let \( \varphi = \sum_{n=1}^{\infty} \psi_n \). For each \( \varepsilon > 0 \), we claim that the set \( K = \{ x \in X; \varphi(x) \geq \varepsilon \} \) is compact (or empty). If \( \varepsilon > 1 \), then \( K = \phi \). If \( \varepsilon = 1 \), then \( K = K_1 \), because \( \varphi(t) = 1 \) precisely for \( t \in K_1 \). If \( \varepsilon < 1 \),
let \( n \geq 0 \) be such that \( 2^{-(n+1)} \leq \varepsilon < 2^{-n} \). Then \( K = K_{n+1} \). Hence \( \varphi \in D_0(X) \).

We claim now that \( \psi(x) \leq \varphi(x) \) for all \( x \in X \). We first notice that \( \varphi(x) = 0 \) if, and only if \( x \not\in \bigcup_{n=1}^{\infty} K_n \). Indeed, if the point \( x \not\in \bigcup_{n=1}^{\infty} K_n \), then \( \psi_k(x) = 0 \) for all \( n = 1, 2, 3, \ldots \), and so \( \varphi(x) = 0 \). Conversely, if \( \varphi(x) = 0 \), then \( \psi_n(x) = 0 \) for all \( n = 1, 2, 3, \ldots \) and therefore \( x \not\in K_n \) for all \( n = 1, 2, 3, \ldots \). Hence \( x \not\in \bigcup_{n=1}^{\infty} K_n \). Let now \( x \in X \). If \( \varphi(x) = 0 \), then \( x \not\in K_n \) for all \( n = 1, 2, 3, \ldots \) and so \( |\psi(x)| < 2^{-n} \) for all \( n = 1, 2, 3, \ldots \). Hence \( \psi(x) = 0 \) and so \( \psi(x) = \varphi(x) \). Suppose now \( \varphi(x) > 0 \).

Then \( x \in \bigcup_{n=1}^{\infty} K_n \). Let \( N \) be the smallest positive integer \( n \geq 1 \) such that \( x \in K_n \). If \( N = 1 \), then \( x \in K_1 \) and so \( \varphi(x) = 1 > \psi(x) \). If \( N > 1 \), then \( x \in K_N \) and \( x \not\in K_{N-1} \).

Hence

\[
\varphi(x) = \sum_{n=N}^{\infty} 2^{-n} = 2^{-(N-1)}
\]

and \( \psi(x) < 2^{-(N-1)} \), since \( x \not\in K_{N-1} \). Therefore \( \psi(x) < \varphi(x) \), whenever \( \varphi(x) > 0 \).

Given any non-empty subset \( S \subset C(X; E) \) we denote by \( x \equiv y \pmod{S} \) the equivalence relation defined by \( f(x) = f(y) \) for all \( f \in S \). For each \( x \in X \), the equivalence class of \( x \pmod{S} \) is denoted by \([x]_S\), i.e.,

\[
[x]_S = \{ t \in X \; ; \; x \equiv t \pmod{S} \}
\]

For any non-empty subset \( K \subset X \) and any \( f : X \to E \), we denote by \( f_K \) its restriction to \( K \). If \( S \subset C(X; E) \) and \( K \subset X \), then for each \( x \in K \) one has

\[
[x]_{S_K} = K \cap [x]_S.
\]

If \( S \subset C_b(X; R) \), we define \( S^+ \) by

\[
S^+ = \{ f \in S \; ; \; f \geq 0 \}.
\]

If \( S = C_b(X; R) \), we write \( S^+ = C^+_b(X; R) \).
Definition 1. Let $S \subseteq C_b(X; \mathbb{R})$ and let $W \subseteq C_b(X; E)$ be given. We say that $W$ is $\beta$-localizable under $S$ if, for every $f \in C_b(X; E)$, the following are equivalent:

1. $f$ belongs to the $\beta$-closure of $W$;
2. for every $\varphi \in D_0(X)$, every $\varepsilon > 0$ and every $x \in X$, there is some $g_x \in W$ such that $\varphi(t)||f(t) - g_x(t)|| < \varepsilon$ for all $t \in [x]_S$.

Remark. Clearly, (1) $\Rightarrow$ (2) in any case. Hence a set $W$ is $\beta$-localizable under $S$ if, and only if, (2) $\Rightarrow$ (1). Notice also that if $W$ is $\beta$-localizable under $S$ and $T \subseteq S$, then $W$ is $\beta$-localizable under $T$. Indeed, $T \subseteq S$ implies $[x]_T \subseteq [x]_S$.

Definition 2. We say that a set $W \subseteq C_b(X, E)$ is $\beta$-truncation stable if, for every $f \in W$ and every $M > 0$, the function $T_M \circ f$ belongs to the $\beta$-closure of $W$, where $T_M : E \to E$ is the mapping defined by

$$
T_M(v) = \begin{cases} 
  v, & \text{if } ||v|| < 2M; \\
  \frac{v}{||v||}2M, & \text{if } ||v|| \geq 2M 
\end{cases}
$$

Notice that, when $E = \mathbb{R}$, the mapping $T_M : \mathbb{R} \to \mathbb{R}$ is given by

$$
T_M(r) = \begin{cases} 
  r, & \text{if } ||r|| < 2M; \\
  2M, & \text{if } r > 2M; \\
  -2M, & \text{if } r > -2M.
\end{cases}
$$

Remark that, for every $f \in C_b(X; E)$, one has $||T_M \circ f||_X \leq 2M$.

Notice that when $W \subseteq C_b^+(X; \mathbb{R})$, then $W$ is $\beta$-truncation stable if, for every $f \in W$ and every constant $M > 0$, the function $P_M \circ f$ belongs to the $\beta$-closure of $W$, where $P_M : \mathbb{R} \to \mathbb{R}_+$ is the mapping defined by $P_M = \max(0, T_M)$, i.e.,

$$
P_M(r) = \begin{cases} 
  0, & \text{if } r < 0; \\
  r, & \text{if } 0 \leq r \leq 2M; \\
  2M, & \text{if } r > 2M.
\end{cases}
$$
Definition 3. Let \( W \subseteq C_b(X; E) \) be a non-empty subset. A function \( \psi \in D(X) \) is called a multiplier of \( W \) if \( \psi f + (1 - \psi)g \) belongs to \( W \), for each pair, \( f \) and \( g \), of elements of \( W \).

Definition 4. A subset \( S \subseteq D(X) \) is said to have property \( V \) if

(a) \( \psi \in S \) implies \( (1 - \psi) \in S \);

(b) the product \( \varphi \psi \) belongs to \( S \), for any pair, \( \varphi \) and \( \psi \), of elements of \( S \).

Notice that the set of all multipliers of a subset \( W \subseteq C_b(X; E) \) has property \( V \). Indeed, condition (a) is clear and the equation

\[
(\varphi \psi) f + (1 - \varphi \psi) g = \varphi [\psi f + (1 - \psi) g] + (1 - \varphi) g
\]

show that (b) holds as well.

When \( X \) is locally compact, R.C. Buck [1] proved a Weierstrass-Stone Theorem for subalgebras of \( C_b(X; \mathbb{R}) \) equipped with the strict topology. This result was extended and generalized by Glicksberg [4], Todd [7], Wells [8] and Giles [3]. See also Buck [2], where modules are dealt with, and Prolla [5], where the strict topology is considered as an example of weighted spaces.

Our versions of the Weierstrass-Stone Theorem are analogues of Chapter 4 of Prolla [6] for arbitrary subsets of \( C(X; E) \) equipped with the uniform convergence topology, \( X \) compact. Whereas the previous results dealt only with algebras or vector spaces which are modules over an algebra, our results now go much further: we are able to cover the case of convex sets (when \( X \) is completely regular) or \( \beta \)-truncation stable sets (when \( X \) is just a topological space). The latter case cover both algebras and the convex cones obtained by taking the set of positive elements.
§2. \(\beta\)-truncation stable subsets

**Theorem 1.** Let \(W \subset C_0(X; E)\) be a \(\beta\)-truncation stable non-empty subset, and let \(A\) be the set of all multipliers of \(W\). Then \(W\) is \(\beta\)-localizable under \(A\).

**Proof.** Let \(f \in C_0(X; E)\) be given and assume condition (2) of Definition 1, with \(S = A\). Let \(\varphi \in D_0(X)\) and \(\varepsilon > 0\) be given. Without loss of generality we may assume that \(\varphi\) is not identically zero. Choose \(M > 0\) so big that \(M > \|f\|_X, M > \varepsilon\) and the compact set \(K = \{t \in X; \varphi(t) \geq \varepsilon/(6M)\}\) is non-empty. Consider the non-empty subset \(W_K \subset C(K; E)\). Clearly, the set \(A_K \subset D(K)\) is a set of multipliers of \(W_K\). Take a point \(x \in K\). By condition (2) applied to \(\varphi^2/(12M)\), there exists \(g_x \in W\) such that \(\varphi(t)^2/(12M)\) for all \(t \in [x]^A\). Let \(M \subset D(K)\) be the set of all multipliers of \(W_K \subset C(K; E)\). Then \(M\) has property \(V\). Now \(A_K \subset M\) implies 

\[
[x]_M \subset [x]_{A_K} = [x]_A \cap K.
\]

Hence \(\varphi(t)^2/(12M)\) holds for all \(t \in K\) such that \(t \in [x]_M\). Now \(\varphi(t) \geq \varepsilon/(6M)\) for all \(t \in K\) and therefore 

\[
\|f(t) - g_x(t)\| < \varepsilon/2
\]

for all \(t \in [x]_M\). By Theorem 1, Chapter 4, of Prolla [6] applied to \(W_K \subset C(K; E)\) and to the set \(M \subset D(K)\), there is \(g_1 \in W\) such that 

\[
\|f(t) - g_1(t)\| < \varepsilon/2
\]

for all \(t \in K\). Let \(h = T_M \circ g_1\). By hypothesis, \(h\) belongs to the \(\beta\)-closure of \(W\), and there is \(g \in W\) such that \(p_\varphi(h - g) < \varepsilon/2\). We claim that \(p_\varphi(f - h) < \varepsilon/2\). Let


$t \in K$. Then

\[ \|g_1(t)\| \leq \|f(t) - g_1(t)\| + \|f(t)\| < \varepsilon/2 + M < 2M \]

and so \( h(t) = T_M(g_1(t)) = g_1(t) \). Hence

\[ \varphi(t)\|f(t) - h(t)\| = \varphi(t)\|f(t) - g_1(t)\| \leq \|f(t) - g_1(t)\| < \varepsilon/2. \]

Suppose now \( t \not\in K \). Then

\[ \varphi(t)\|f(t) - h(t)\| < \frac{\varepsilon}{6M}\|f(t) - h(t)\| \leq \frac{\varepsilon}{6M}(\|f\|_X + \|h\|_X) < \frac{\varepsilon}{6M}3M = \frac{\varepsilon}{2}, \]

because \( \|h\|_X \leq 2M \), and \( \|f\|_X < M \).

This establishes our claim that \( p_\varphi(f - h) < \frac{\varepsilon}{2} \). Hence \( p_\varphi(f - g) < \varepsilon \), and \( f \) belongs to the \( \beta \)-closure of \( W \). \( \Box \)

**Theorem 2.** Let \( W \subset C_0(X; E) \) be a \( \beta \)-truncation stable non-empty subset, and let \( B \) be any non-empty set of multipliers of \( W \). Then \( W \) is \( \beta \)-localizable under \( B \).

**Proof.** Let \( A \) be the set of all multipliers of \( W \). By Theorem 1 the set \( W \) is \( \beta \)-localizable under \( A \). Now \( B \subset A \), so \( W \) is also \( \beta \)-localizable under \( B \). \( \Box \)

§3. The case of subalgebras

**Lemma 1.** If \( B \subset C_0(X; \mathbb{R}) \) is a uniformly closed subalgebra, and \( T : \mathbb{R} \to \mathbb{R} \) is a continuous mapping, with \( T(0) = 0 \), then \( T \circ f \) belongs to \( B \), for every \( f \in B \).
Proof. Let $f \in B$ and $\varepsilon > 0$ be given. Choose $k \geq ||f||_X$. By Weierstrass' Theorem, there exists an algebraic polynomial $p$ such that $|T(t) - p(t)| < \varepsilon$ for all $t \in \mathbb{R}$ with $|t| \leq k$, and we may assume $p(0) = T(0) = 0$. Hence, for every $x \in X$, we have $|T(f(x)) - p(f(x))| < \varepsilon$, because $|f(x)| \leq k$. Now $p \circ f$ belongs to $B$, and therefore $T \circ f$ belongs to the uniform closure of $B$, that is $B$ itself. \hfill \Box

Corollary 1. Every subalgebra $W \subset C_b(X; \mathbb{R})$ is $\beta$-truncation stable.

Proof. Let $f \in W$ and $M > 0$ be given. Let $B$ be the $\beta$-closure of $W$ in $C_b(X; \mathbb{R})$. We know that $B$ is then a uniformly closed subalgebra. By Lemma 1 applied to $T = T_M$, we see that $T_M \circ f$ belongs to the $\beta$-closure of $W$ as claimed. \hfill \Box

Corollary 2. Every uniformly closed subalgebra of $C_b(X; \mathbb{R})$ is a lattice.

Proof. Since

$$\max(f, g) = \frac{1}{2}[f + g + |f - g|]$$

$$\min(f, g) = \frac{1}{2}[f + g - |f - g|]$$

it suffices to show that $|f| \in B$, for every $f \in B$. This follows from Lemma 1, by taking $T : \mathbb{R} \to \mathbb{R}$ to be the mapping $T(t) = |t|$, for $t \in \mathbb{R}$. \hfill \Box

Theorem 3. Every subalgebra $W \subset C_b(X; \mathbb{R})$ is $\beta$-localizable under itself.

Proof. Let $f \in C_b(X; \mathbb{R})$ and assume that condition (2) of Definition 1 holds with $S = W$. Notice that for every $x \in X$ one has

$$[x]_W = [x]_B$$

where $B$ is the $\beta$-closure of $W$. Let now

$$V = \{\psi \in B; ||\psi||_X \leq 1\} \quad \text{and} \quad A = \{\psi \in B; 0 \leq \psi \leq 1\}.$$
It is easy to see that

\[ [x]_B = [x]_V \subseteq [x]_A, \]

for each \( x \in X \). Notice that, by Corollary 2, every \( \psi \in V \) can be written in the form \( \psi = \psi^+ - \psi^- \), with \( \psi^+ \) and \( \psi^- \) in \( A \). Hence \( [x]_A \subseteq [x]_V \) is also true. Hence \( f \) satisfies condition (2) of Definition 1 with respect to \( S = A \). Now \( A \) is a set of multipliers of \( B \), and the algebra \( B \), by Corollary 1, is \( \beta \)-truncation stable. Hence, by Theorem 3, the function \( f \) belongs to the \( \beta \)-closure of \( B \), that is \( B \) itself. We have proved that \( f \) belongs to the \( \beta \)-closure of \( W \). Hence \( W \) is \( \beta \)-localizable under \( S = W \).

\[ \Box \]

**Corollary 3.** Let \( W \subset C_b(X; \mathbb{R}) \) be a subalgebra, and let \( f \in C_b(X; \mathbb{R}) \) be given. Then \( f \) belongs to the \( \beta \)-closure of \( W \) if, and only if, the following conditions are satisfied:

1. For each pair, \( x \) and \( y \), of elements of \( X \) such that \( f(x) \neq f(y) \), there is some \( g \in W \) such that \( g(x) \neq g(y) \);
2. For each \( x \in X \) such that \( f(x) \neq 0 \) there is some \( g \in W \) such that \( g(x) \neq 0 \).

**Proof.** Clearly, if \( f \in W^\beta \), then (1) and (2) are satisfied. Conversely, assume that conditions (1) and (2) are verified.

Let \( x \in X \) be given. By condition (1) the function \( f \) is constant on \( [x]_W \). Let \( f(x) \) be its value. If \( f(x) = 0 \), then \( g_x = 0 \) belongs to \( W \) and \( f(t) = f(x) = 0 = g_x(t) \) for all \( t \in [x]_W \). If \( f(x) \neq 0 \), by condition (2) there is \( g \in W \) such that \( g(x) \neq 0 \). Define \( g_x = [f(x)/g(x)]g \). Then \( g_x \in W \) and \( g_x(t) = f(x) = f(t) \) for all \( t \in [x]_W \). Hence \( f \) satisfies condition (2) of Definition 1 with respect to \( S = W \). By Theorem 3, we conclude that \( f \) belongs to the \( \beta \)-closure of \( W \).

\[ \Box \]

Corollary 3 implies the following results.
Corollary 4. Let \( A \) be a subalgebra of \( C_b(X; \mathbb{R}) \) which for each \( x \in X \) contains a function \( g \) with \( g(x) \neq 0 \), and let \( f \in C_b(X; \mathbb{R}) \) be given. Then \( f \) belongs to the \( \beta \)-closure of \( A \) if, and only if, for each pair, \( x \) and \( y \), of elements of \( X \) such that \( f(x) \neq f(y) \), there is some \( g \in A \) such that \( g(x) \neq g(y) \).

Corollary 5. Let \( A \) be a subalgebra of \( C_b(X; \mathbb{R}) \) which separates the points of \( X \) and for each \( x \in X \) contains a function \( g \) with \( g(x) \neq 0 \). Then \( A \) is \( \beta \)-dense in \( C_b(X; \mathbb{R}) \).

Corollary 6. If \( X \) is a locally compact Hausdorff space, then \( C_{00}(X; \mathbb{R}) \) is \( \beta \)-dense in \( C_b(X; \mathbb{R}) \).

Lemma 2. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( f(t) \geq 0 \) for all \( t \in \mathbb{R} \) and \( f(0) = 0 \). If \( k > 0 \) and \( \varepsilon > 0 \) are given, there is a real algebraic polynomial \( p \) such that \( p(t) \geq 0 \) for all \( 0 \leq t \leq k \), \( p(0) = 0 \) and \( |p(t) - f(t)| \leq \varepsilon \) for all \( 0 \leq t \leq k \).

Proof. Define \( g : [0,1] \to \mathbb{R} \) by setting \( g(u) = f(ku) \), for each \( u \in [0,1] \). Clearly, \( g(u) \geq 0 \), for all \( 0 \leq u \leq 1 \) and \( g(0) = 0 \). Now, given \( \varepsilon > 0 \), choose \( n \) so that the \( n \)-th Bernstein polynomial of \( g \), written \( B_n g \), is such that

\[
|(B_n g)(u) - g(u)| < \varepsilon
\]

for all \( 0 \leq u \leq 1 \). For \( t \in \mathbb{R} \), define \( p(t) = (B_n g)(t/k) \). Since \( B_n g \geq 0 \) in \( [0,1] \), it follows that \( p(t) \geq 0 \), for \( t \in [0,k] \). Since \( (B_n g)(0) = g(0) = f(0) = 0 \), we see that \( p(0) = 0 \). It remains to notice that, for any \( 0 \leq t \leq k \) we have \( 0 \leq t/k \leq 1 \) and

\[
|p(t) - f(t)| = |(B_n g)(t/k) - g(t/k)| < \varepsilon
\]

Lemma 3. If \( A \subset C_b(X; \mathbb{R}) \) is a subalgebra, then \( A^+ \) is \( \beta \)-truncation stable.
Proof. Let $f \in A^+$ and $M > 0$ be given. We claim that $P_M \circ f$ belongs to the $\beta$-closure of $A^+$. Let $k > 0$ be such that $0 \leq f(x) \leq k$ for all $x \in X$. Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. By Lemma 2 above there exists a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $p(t) \geq 0$ for all $0 \leq t \leq k$, $p(0) = 0$ and $|p(t) - P_M(t)| < \varepsilon$ for all $0 \leq t \leq k$.

Let $x \in X$. Then $\varphi(x) \leq 1$ and so $\varphi(x)|p(f(x)) - P_M(f(x))| < \varepsilon$. Now $p \circ f$ belongs to $A$ (since $p(0) = 0$) and $p(f(x)) \geq 0$ for all $x \in X$, since $0 \leq f(x) \leq k$. Hence $p \circ f \in A^+$. This ends the proof that $P_M \circ f$ belongs to the $\beta$-closure of $A^+$ as claimed.

Theorem 4. If $A \subset C_b(X; \mathbb{R})$ is a subalgebra, then $A^+$ is localizable under itself.

Proof. Let $f \in C_b(X; \mathbb{R})$ be given satisfying condition (2) of Definition 1 with respect to $S = A^+$. Define $B = \{f \in A; 0 \leq f \leq 1\}$. It is easy to see that $[x]_S = [x]_B$, for every $x \in X$. Hence $f$ satisfies condition (2) of Definition 1 with respect to $B$, which is a set of multipliers of $A^+$. By Lemma 3, the set $A^+$ is $\beta$-truncation stable. Therefore $A^+$ is $\beta$-localizable under $B$, by Theorem 2. Hence $f$ belongs to the $\beta$-closure of $A^+$.

Theorem 4. Let $A \subset C_b(X; \mathbb{R})$ be a subalgebra and let $f \in C_b^+(X; \mathbb{R})$ be given. Then $f$ belongs to the $\beta$-closure of $A^+$ if, and only if, the following two conditions hold:

1. for each pair, $x$ and $y$, of elements of $X$ such that $f(x) \neq f(y)$, there is some $g \in A^+$ such that $g(x) \neq g(y)$;
2. for each $x \in X$ such that $f(x) > 0$ there is some $g \in A^+$ such that $g(x) > 0$.

Proof. If $f$ belongs to the $\beta$-closure of $A^+$ the two conditions (1) and (2) above are easily seen to hold. Conversely, assume that conditions (1) and (2) above hold. Let $x \in X$ be given. By condition (1), the function $f$ is constant on $[x]_S$ where $S = A^+$. Let $f(x) \geq 0$ be its constant value. If $f(x) = 0$, then $g_x = 0$ belongs to
A^+ and f(t) = f(x) = 0 = g_x(t) for all t \in [x]_S. If f(x) > 0, then by condition (2) there is g_x \in A^+ such that g(x) > 0. Let g_x = [f(x)/g(x)]g. Then g_x \in A^+ and

g_x(t) = f(x) = f(t) for all t \in [x]_S. Hence f satisfies condition (2) of Definition 1 with respect to W = A^+ and S = A^+. By Theorem 4, we conclude that f belongs to the \beta\text{-closure of } A^+.

\section{The case of uniformly bounded subsets}

\textbf{Theorem 5.} Let \( W \) be a uniformly bounded subset of \( C_b(X; E) \) and let \( A \) be the set of all multipliers of \( W \). Then \( W \) is \( \beta\text{-localizable under } A \).

\textbf{Proof.} Let \( f \in C_b(X; E) \) be given and assume that condition (2) of Definition 1 holds with \( S = A \). Let \( \varepsilon > 0 \) and \( \varphi \in D_0(X) \) be given. Choose \( M > 0 \) so big that \( M > \|f\|_X \) and \( M > k = \sup\{\|g\|_X; g \in W\} \), and the compact set \( K = \{t \in X; \varphi(t) \geq \varepsilon/(2M)\} \) is non-empty. (Without loss of generality we may assume that \( \varphi \) is not identically zero). Consider the non-empty set \( W_K \subset C(K; E) \). Clearly, the set \( A_K \) is a set of multipliers of \( W_K \). Take a point \( x \in K \). By condition (2) applied to \( \varepsilon^2/(2M) \), there exists some \( g_x \in W \) such that

\[ \varphi(t)\|f(t) - g_x(t)\| < \varepsilon^2/(2M) \]

for all \( t \in [x]_A \). Hence \( \|f(t) - g_x(t)\| < \varepsilon \) for all \( t \in [x]_{A_K} \), since \( \varphi(t) \geq \varepsilon/(2M) \) for all \( t \in K \). Let now \( M \) be the set of all multipliers of \( W_K \subset C(K; E) \). Since \( A_K \subset M \), it follows that \( [x]_M \subset [x]_{A_K} \) and so \( \|f(t) - g_x(t)\| < \varepsilon \) for all \( t \in [x]_M \). By Theorem 1, Chapter 4 of Prolla [6] there is \( g \in W \) such that \( \|f(t) - g(t)\| < \varepsilon \) for all \( t \in K \). We claim that \( \varphi_x(t - g) < \varepsilon \). Let \( x \in X \). If \( x \in K \), then \( \varphi(x) \leq 1 \) and

\[ \varphi(t)\|f(t) - g(t)\| \leq \|f(t) - g(x)\| < \varepsilon. \]

If \( x \not\in K \), then

\[ \varphi(x)\|f(x) - g(x)\| \leq \frac{\varepsilon}{2M} [\|f\|_X + \|g\|_X] < \varepsilon. \]
Hence $f$ belongs to the $\beta$-closure of $W$ and so $W$ is $\beta$-localizable under $A$. $\square$

**Theorem 6.** Let $W$ be a uniformly bounded subset of $C_b(X; E)$ and let $B$ be any non-empty set of multipliers of $W$. Then $W$ is $\beta$-localizable under $B$.

**Proof.** Let $A$ be the set of all multipliers of $W$. Since $B \subset A$ and by Theorem 5 the set $W$ is $\beta$-localizable under $A$, it follows that $W$ is also $\beta$-localizable under $B$. $\square$

**Theorem 7.** Let $A$ be a non-empty subset of $D(X)$ with property $V$ and let $f \in D(X)$. Then $f$ belongs to the $\beta$-closure of $A$ if, and only if, the following two conditions hold:

1. For every pair of points, $x$ and $y$, of $X$ such that $f(x) \neq f(y)$, there exists $g \in A$ such that $g(x) \neq g(y)$;
2. For every $x \in X$ such that $0 < f(x) < 1$, there exists $g \in A$ such that $0 < g(x) < 1$.

**Proof.** It is easy to see that conditions (1) and (2) are necessary for $f$ to belong to the $\beta$-closure of $A$. Conversely, assume that $f$ satisfies conditions (1) and (2).

Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. Without loss of generality we may assume that $\varphi$ is not identically zero. Choose $\delta > 0$ so small that $2\delta < \varepsilon$ and the compact set $K = \{t \in X; \varphi(t) \geq \delta\}$ is non-empty. Clearly, $A_K$ has property $V$. Since conditions (1) and (2) hold, we may apply Theorem 1, Chapter 8, Prolla [6] to conclude that $f_K$ belongs to the uniform closure of $A_K$. Hence there is some $g \in A$ such that $|f(t) - g(t)| < \varepsilon$ for all $t \in K$. We claim that $p_\varphi(f - g) < \varepsilon$. Let $x \in X$. If $x \in K$, then $\varphi(x) \leq 1$ and $\varphi(x)|f(x) - g(x)| \leq |f(x) - g(x)| < \varepsilon$.

If $x \not\in K$, then $\varphi(x) < \delta$ and

$$\varphi(x)|f(x) - g(x)| \leq \delta[\|f\|_X + \|g\|_X] \leq 2\delta < \varepsilon.$$ 

Hence $f$ belongs to the $\beta$-closure of $A$. $\square$

**Remark.** We say that a subset $A \subset D(X)$ has property $\text{VN}$ if $fg + (1 - f)h \in A$
for all \( f, g, h \in A \). Clearly, if \( A \) has property \( VN \) and contains 0 and 1, then \( A \) has property \( V \).

**Corollary 6.** Let \( A \) be a non-empty subset of \( D(X) \) with property \( V \), and let \( W \) be its \( \beta \)-closure. Then \( W \) has property \( VN \) and \( W \) is a lattice.

**Proof.** (a) \( W \) has property \( VN \): Let \( f, g, \varphi \) belong to \( W \), and let \( h = \varphi f + (1 - \varphi)g \). Assume \( h(x) \neq h(y) \). Then at least one of the following three equalities is necessarily false: \( \varphi(x) = \varphi(y) \), \( f(x) = f(y) \) and \( g(x) = g(y) \). Since \( \varphi \), \( f \) and \( g \) belong all three to \( W \), there exists \( a \in A \) such that \( a(x) \neq a(y) \). Hence \( h \) satisfies condition (1) of Theorem 7. Suppose now that \( 0 < h(x) < 1 \). If \( 0 < \varphi(x) < 1 \), then \( 0 < a(x) < 1 \) for some \( a \in A \), because \( \varphi \) belongs to the \( \beta \)-closure of \( A \). Assume that \( \varphi(x) = 0 \). Then \( h(x) = g(x) \) and so \( 0 < g(x) < 1 \). Since \( g \in W \), it follows that \( 0 < a(x) < 1 \) for some \( a \in A \). Similarly, if \( \varphi(x) = 1 \) then \( h(x) = f(x) \) and so \( 0 < f(x) < 1 \). Since \( f \in W \), there is \( a \in A \) such that \( 0 < a(x) < 1 \). Hence \( h \) satisfies condition (2) of Theorem 7. By Theorem 7 above, the function \( h \) belongs to \( W \).

(b) \( W \) is lattice: Let \( f \) and \( g \) belong to \( W \). Let \( h = \max(f, g) \). Let \( x \) and \( y \) be a pair of points of \( X \) such that \( h(x) \neq h(y) \). Then at least one of the two equalities \( f(x) = f(y), g(x) = g(y) \) must be false. Since \( f \) and \( g \) both belong to the \( \beta \)-closure of \( A \), there exists \( a \in A \) such that \( a(x) \neq a(y) \). On the other hand, let \( x \in X \) be such that \( 0 < h(x) < 1 \). If \( f(x) \geq g(x) \), then \( h(x) = f(x) \) and so \( 0 < f(x) < 1 \). Since \( f \in W \), there exists \( a \in A \) such that \( 0 < a(x) < 1 \). Assume now \( f(x) < g(x) \). Then \( h(x) = g(x) \) and so \( 0 < g(x) < 1 \). Since \( g \in W \), there exists \( a \in A \) such that \( 0 < a(x) < 1 \). By Theorem 7 above, the function \( h \) belongs to \( W \). Similarly, one shows that the function \( \min(f, g) \) belongs to \( W \).

**Corollary 7.** Let \( A \) be a \( \beta \)-closed non-empty subset of \( D(X) \) with property \( V \). Then \( A \) has property \( VN \) and \( A \) is a lattice.
Proof. Immediate from Corollary 6.

§5. The case of convex subsets

In this section we suppose that $X$ is a completely regular Hausdorff space. We denote its Stone-Čech compactification by $\beta X$, and by $\beta : C_b(X; IR) \to C(\beta X; IR)$ the linear isometry which to each $f \in C_b(X; IR)$ assigns its (unique) continuous extension to $\beta X$. Since $\beta$ is an algebra (and lattice) isomorphism, the image $\beta(A)$ of any subset $A \subset C_b(X, IR)$ with property $V$ is contained in $D(\beta X)$ and has property $V$. If $B = \beta(A)$, then for each $x \in X$ one has

$$[x]_A = [x]_B \cap X.$$ 

If $Y$ denotes the quotient space of $\beta X$ by the equivalence relation $x \equiv y$ if and only if $x = y$ for all $y \in B$, then $Y$ is a compact Hausdorff space.

If $x \in X$ and $K_x \subset X$ is a compact subset disjoint from $[x]_A$, then $\pi(K_x)$ is a compact subset in $Y$ which does not contain the point $\pi(x)$. (Here we have denoted by $\pi$ the canonical projection $\pi : \beta X \to Y$. Indeed, if $x \in K_x$, then $\pi(x) = \pi(y)$ for some $y \in K_x$. Now $y \in [x]_B$ because that $y \in [x]_A$. But $K_x \cap [x]_A = \phi$, and we have reached a contradiction. Hence $\pi(x) \notin \pi(K_x)$. We will apply these remarks in the proof of the following lemma.

Lemma 4. Let $A \subset D(X)$ be a subset with property $V$ and containing some constant $0 < c \leq 1$. Let $x \in X$ and let $K_x \subset X$ be a compact subset, disjoint from $[x]_A$. Then, there exists an open neighborhood $W(x)$ of $[x]_A$ in $X$, disjoint from $K_x$ and such that given $0 < \delta < 1$ there is $\varphi \in A$ such that

1. $\varphi(t) < \delta$, for all $t \in K_x$;
2. $\varphi(t) > 1 - \delta$, for all $t \in W(x)$.

Proof. Let $N(x)$ be the complement of $K_x$ in $\beta X$. Then $N(x)$ is an open neigh-
We know that \( \pi(K_x) \) is a compact subset of \( Y \) which does not contain the point \( y = \pi(x) \). Let \( f \in C(Y; \mathbb{R}) \) be a mapping such that \( 0 \leq f \leq 1, f(y) = 0 \) and \( f(t) = 1 \) for all \( t \in \pi(K_x) \). Let \( g = f \circ \pi \). By Theorem 1, Chapter 8, Prolla [6], the function \( g \) belongs to the uniform closure of \( B \) in \( D(\beta X) \). Notice that \( a(x) = 0 \) and \( g(u) = 1 \), for all \( u \in K_x \). Define \( N(x) = \{ t \in \beta X; g(t) < 1/4 \} \). Clearly, \( [x]_B \subset N(x) \), since \( g(t) = 0 \) for all \( t \in [x]_B \).

It is also clear that \( N(x) \) is disjoint from \( K_x \). Let us define \( W(x) = N(x) \cap X \). Then \( W(x) \) is an open neighborhood of \( [x]_A \) in \( X \), which is disjoint from \( K_x \).

Given \( 0 < \delta < 1 \), let \( p \) be a polynomial determined by Lemma 1, Chapter 1, Prolla [6], applied to \( a = 1/4 \) and \( b = 3/4 \), and \( \varepsilon = \delta/2 \). Let
\[
 h(t) = p(g(t)), \quad \text{for all } t \in \beta X.
\]
Since \( \overline{B} \) has property \( V \), it follows that \( h \in \overline{B} \). If \( t \in K_x \), then \( g(t) = 1 \) and so \( h(t) < \delta/2 \). If \( t \in W(x) \), then \( g(t) < 1/4 \) and so \( h(t) > 1 - \delta/2 \). Choose now \( \psi \in B \) with \( ||\psi - h||_X < \delta/2 \), and let \( \varphi \in A \) be such that \( \beta(\varphi) = \psi \). Then \( \varphi \in A \) satisfies conditions (1) and (2).

---

**Theorem 8.** Let \( W \subset C_b(X; E) \) be a non-empty subset and let \( A \) be a set of multipliers of \( W \) which has property \( V \) and contains some constant \( 0 < c < 1 \). Then \( W \) is \( \beta \)-localizable under \( A \).

**Proof.** Assume that condition (2) of Definition 1 is true with \( S = A \). For each \( x \in X \), there is some \( g_x \in W \) such that, for all \( t \in [x]_A \), one has \( \varphi(t)||f(t) - g_x(t)|| < \varepsilon/2 \). Consider the compact subset \( K_x \) of \( X \) defined by

\[
 K_x = \{ t \in X; \varphi(t)||f(t) - g_x(t)|| \geq \frac{\varepsilon}{2} \}. 
\]

Clearly, \( K_x \) is disjoint from \([x]_A\). Now for each \( x \in X \), select an open neighborhood \( W(x) \) of \([x]_A\), disjoint from \( K_x \), according to Lemma 4.

Select and fix a point \( x_1 \in X \). Let \( K = K_{x_1} \). By compactness of \( K \), there exists a finite set \( \{x_2, \ldots, x_m\} \subset K \) such that

\[
 K \subset W(x_2) \cup W(x_3) \cup \ldots \cup W(x_m)
\]
Let \( k = \sum_{i=1}^{m} p_{\varphi}(f - g_{x_{i}}) \) and let \( 0 < \delta < 1 \) be so small that \( \delta k < \varepsilon / 2 \).

By Lemma 4, there are \( \varphi_{2}, \ldots, \varphi_{m} \in A \) such that

(a) \( \varphi_{i}(t) < \delta, \) for all \( t \in K_{x_{i}} \);

(b) \( \varphi_{i}(t) > 1 - \delta, \) for all \( t \in W(x_{i}) \)

for \( i = 2, \ldots, m \). Define

\[
\psi_{2} = \varphi_{2} \\
\psi_{3} = (1 - \varphi_{2})\varphi_{3} \\
\vdots \\
\psi_{m} = (1 - \varphi_{2})(1 - \varphi_{3})\ldots(1 - \varphi_{m-1})\varphi_{m}.
\]

Clearly, \( \psi_{i} \in A \) for all \( i = 2, \ldots, m \). Now

\[
\psi_{2} + \ldots + \psi_{j} = 1 - (1 - \varphi_{2})(1 - \varphi_{3})\ldots(1 - \varphi_{j})
\]

for all \( j \in \{2, \ldots, m\} \), can be easily seen by induction. Define

\[
\psi_{1} = (1 - \varphi_{2})(1 - \varphi_{3})\ldots(1 - \varphi_{m})
\]

then \( \psi_{1} \in A \) and \( \psi_{1} + \psi_{2} + \ldots + \psi_{m} = 1 \).

Notice that

(c) \( \psi_{i}(t) < \delta \) for all \( t \in K_{x_{i}} \),

for each \( i = 1, 2, \ldots, m \). Indeed, if \( i \geq 2 \) then (c) follows from (a). If \( i = 1 \), then for \( t \in K \), we have \( t \in W(x_{j}) \) for some \( j = 2, \ldots, m \). By (b), one has \( 1 - \varphi_{j}(t) < \delta \) and so

\[
\psi_{1}(t) = (1 - \varphi_{j}(t)) \prod_{i \neq j}(1 - \varphi_{i}(t)) < \delta.
\]

Let us write \( g_{i} = g_{x_{i}} \) for \( i = 1, 2, \ldots, m \).

Define \( g = \psi_{1}g_{1} + \psi_{2}g_{2} + \ldots + \psi_{m}g_{m} \).

Notice that

\[
g = \varphi_{2}g_{2} + (1 - \varphi_{2})[\varphi_{3}g_{3} + (1 - \varphi_{3})[\varphi_{4}g_{4} + \ldots + \\
+(1 - \varphi_{m-1})[\varphi_{m}g_{m} + (1 - \varphi_{m})g_{1}]\ldots]].
\]
Hence \( g \in W \). Let \( x \in X \) be given. Then

\[
\varphi(x)\|f(x) - g(x)\| = \varphi(x)\left\| \sum_{i=1}^{m} \psi_i(x)(f(x) - g_i(x)) \right\|
\leq \varphi(x)\|\sum_{i=1}^{m} \psi_i(x)\|(f(x) - g_i(x))\|
\]

Define \( I = \{1 \leq r \leq m; x \notin K_{x_i}\} \); \( J = \{1 \leq i \leq m; x \in K_{x_i}\} \).

If \( i \in I \), then \( x \notin K_{x_i} \) and

\[
\varphi(x)\|f(x) - g_i(x)\| < \frac{\epsilon}{2}
\]

and therefore

\[
(*) \sum_{i \in I} \varphi(x)\psi_i(x)\|f(x) - g_i(x)\| \leq \frac{\epsilon}{2} \sum_{i \in I} \psi_i(x) \leq \frac{\epsilon}{2}.
\]

If \( i \in J \), then by (c), \( \psi_i(x) < \delta \) and so

\[
(**) \sum_{i \in J} \varphi(x)\psi_i(x)\|f(x) - g_i(x)\| \leq \delta k < \frac{\epsilon}{2}.
\]

From (*) and (**) we get \( \varphi(x)\|f(x) - q(x)\| < \epsilon \). \( \square \)

**Theorem 9.** Let \( W \subset C_b(X; E) \) be a non-empty convex subset and let \( A \) be the set of all multipliers of \( W \). Then \( W \) is \( \beta \)-localizable under \( A \).

**Proof.** The set \( A \) has property \( V \) and, since \( W \) is convex, every constant \( 0 < c < 1 \) belongs to \( A \). \( \square \)

**Theorem 10.** Let \( W \subset C_b(X; E) \) be a non-empty convex subset and let \( B \) be any non-empty set of multipliers of \( W \). Then \( W \) is \( \beta \)-localizable under \( B \).

**Proof.** Similar to that of Theorem 6, using now Theorem 9 instead of Theorem 5.
Corollary 8. Let $W \subset C_b(X; E)$ be a non-empty convex subset such that the set of all multipliers of $W$ separates the points of $X$. Then, for each $f \in C_b(X; \mathbb{R})$ the following are equivalent:

1. $f$ belongs to the $\beta$-closure of $W$;
2. for each $\varepsilon > 0$ and each $x \in X$, there is some $g \in W$ such that $||f(x) - g(x)|| < \varepsilon$.

Proof. Clearly, (1) $\Rightarrow$ (2). Suppose now that (2) holds. Let $\varphi \in D_0(X), \varepsilon > 0$ and $x \in X$ be given. Notice that $[x]_W = \{x\}$. If $\varphi(x) = 0$, for any $g \in W$ one has $\varphi(x)||f(x) - g(x)|| = 0 < \varepsilon$. If $\varphi(x) > 0$, by (2) there is $g \in W$ such that $||f(x) - g(x)|| < \varepsilon / \varphi(x)$. Hence $\varphi(x)||f(x) - g(x)|| < \varepsilon$, and by Theorem 9, (1) is true.

Corollary 9. Let $S \subset X$ be a non-empty closed subset and let $V \subset E$ be a non-empty convex subset. Let $W = \{g \in C_b(X; E); g(S) \subset V\}$. Then, for each $f \in C_b(X; E)$ the following are equivalent:

1. $f$ belongs to the $\beta$-closure of $W$;
2. for each $x \in S$, $f(x)$ belongs to the closure of $V$ in $E$.

Hence, $W^\theta = \{f \in C_b(X; E); f(S) \subset \overline{V}\}$, where $\overline{V}$ is the closure of $V$ in $E$.

Proof. Clearly, (1) $\Rightarrow$ (2). Conversely, assume that (2) holds. Clearly, $W$ is a convex set such that $D(X)$ is the set of all multipliers of $W$. Since $X$ is a completely regular Hausdorff space, $D(X)$ separates the points of $X$. Let $\varepsilon > 0$ and $x \in X$ be given. If $x \in S$ there is $v \in V$ such that $||f(x) - v|| < \varepsilon$, and the constant mapping on $X$ whose value is $v$ belongs to $W$ and $g(x) = v$. If $x \not\in S$, choose $\varphi \in C_b(X; \mathbb{R}), 0 \leq \varphi \leq 1, \varphi(t) = 1$ for all $t \in S$ and $\varphi(x) = 0$; and let $g \in C_b(X; E)$
be defined by \( g = \varphi \otimes v_0 + (1 - \varphi) \otimes f(x) \), where \( v_0 \in V \) is chosen arbitrarily. Then \( g(t) = v_0 \) for all \( t \in S \), and therefore \( g \in W \), and \( g(x) = f(x) \). Hence (2) of Corollary 8 is verified and so \( f \) belongs to the \( \beta \)-closure of \( W \).

**Corollary 10.** Let \( W \subset C_b(X; E) \) be a non-empty convex subset such that the set of all multipliers of \( W \) separates the points of \( X \) and, for each \( x \in X \), the set \( W(x) = \{ g(x) ; g \in W \} \) is dense in \( E \). Then \( W \) is \( \beta \)-dense in \( C_b(X; E) \).

**Proof.** Apply Corollary 8.

**Corollary 11.** The vector subspace \( W = C_b(X; \mathbb{R}) \otimes E \) is \( \beta \)-dense in \( C_b(X; E) \).

**Proof.** The set \( A \) of all multipliers of \( W \) is \( D(X) \), and \( W(x) = E \), for each \( x \in X \). It remains to apply Corollary 10.

**Corollary 12.** If \( X \) is a locally compact Hausdorff space, then \( C_00(X; \mathbb{R}) \otimes E \) is \( \beta \)-dense in \( C_b(X; E) \).

**Proof.** Let \( W = C_00(X; \mathbb{R}) \otimes E \). As in the previous corollary, the set \( A \) of all multipliers of \( W \) is \( D(X) \), and for each \( x \in X, W(x) = E \).

**Theorem 11.** Let \( A \subset C_b(X; \mathbb{R}) \) be a subalgebra and let \( W \subset C_b(X; E) \) be a vector subspace which is an \( A \)-module, i.e., \( AW \subset W \). Then \( W \) is \( \beta \)-localizable under \( A \).

**Proof.** Let \( f \in C_b(X; E) \) be given. Assume that condition (2) of Definition 1 holds with \( S = A \). Without loss of generality we may assume that \( A \) is \( \beta \)-closed and contains the constants. Let \( M \) be the set of all multipliers of \( W \). We claim that, for each \( x \in X \), one has \( [x]_M \subset [x]_A \). Indeed, let \( t \in [x]_M \) and let \( \varphi \in A \). If \( \varphi = 0 \), then \( \varphi \in M \) and \( \varphi(t) = \varphi(x) \). Assume \( \varphi \neq 0 \). Write \( \varphi = \varphi^+ - \varphi^- \),
where $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = \max(-\varphi, 0)$. By Corollary 2, §3, both $\varphi^+$ and $\varphi^-$ belong to $A$. If $\varphi^+ = 0$, then $\varphi^+$ belongs to $M$ and $\varphi^+(t) = \varphi^+(x)$. If $\varphi^+ \neq 0$, let $\psi = \varphi^+/\|\varphi^+\|_X$. Now $\psi$ belongs to $A$ and $0 \leq \psi \leq 1$. Hence $\psi \in M$ and therefore $\psi(t) = \psi(x)$. Consequently, one has $\varphi^+(t) = \varphi^+(x)$. Similarly, one proves that $\varphi^-(t) = \varphi^-(x)$. Hence $\varphi(t) = \varphi(x)$. This ends the proof that $[x]_M \subseteq [x]_A$ for all $x \in X$. Hence condition (2) of Definition 1 is verified with $S = M$. By Theorem 9, $W$ is $\beta$-localizable under $M$. Hence $f$ belongs to the $\beta$-closure of $W$. 

**Corollary 13.** Let $W \subset C_b(X; E)$ be a vector subspace, and let

$$A = \{\psi \in C_b(X; IR); \psi g \in W \text{ for all } g \in W\}.$$

Then $W$ is $\beta$-localizable under $A$.

**Proof.** Clearly $A$ is a subalgebra of $C_b(X; IR)$ and $W$ is an $A$-module. 

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